Formal Methods
Module I: Automated Reasoning
Ch. 04: **Automata-Theoretic LTL Reasoning**

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Outline

1. Büchi Automata

2. The Automata-Theoretic Approach to LTL Reasoning
   - General Ideas
   - Language-Emptiness Checking of Büchi Automata
   - From Kripke Models to Büchi Automata
   - From LTL Formulas to Büchi Automata
   - Complexity

3. Exercises

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Infinite Word Languages

Modeling infinite computations of reactive systems

Given an Alphabet Σ (e.g. $\Sigma \overset{\text{def}}{=} \{a, b\}$)

- An ω-word α over Σ is an infinite sequence

  $$a_0,\ a_1,\ a_2 \ldots.$$

  Formally, $\alpha : \mathbb{N} \to \Sigma$.
- The set of all infinite words is denoted by Σ^ω.
- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.
- Example: All words over $\{a, b\}$ with infinitely many a's.

Notation:

omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.

omega-languages $L, L_1 \subseteq \Sigma^\omega$

For $u \in \Sigma^+$, let $u^\omega = u.u.u\ldots$

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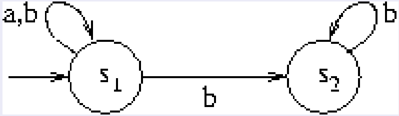
Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\ldots$.
  There are several (infinite) possible runs.
  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
  Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \ldots$
- Acceptance Conditions: Büchi (Muller, Rabin, Street):
  Acceptance is based on states occurring infinitely often
- Notation: Let Q be the set of states. Let $\rho \in Q^\omega$. Then,
  $$Inf(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}.\ \rho(i) = s\}.$$
  (The set of states occurring infinitely many times in ρ.)

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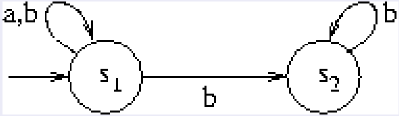

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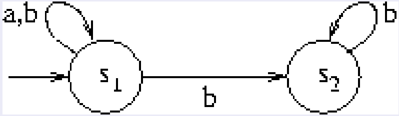

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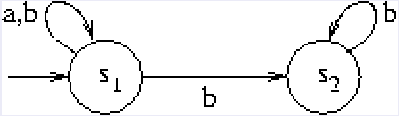

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Büchi Automata

Nondeterministic Büchi Automaton

- A **Nondeterministic Büchi Automaton (NBA)** is $(Q, \Sigma, \delta, I, F)$ s.t.
  - Q Finite set of states.
  - Σ is a finite alphabet
  - $I \subseteq Q$ set of initial states.
  - $F \subseteq Q$ set of accepting states.
  - $\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).
- A **Deterministic Büchi Automaton (DBA)** is an NBA s.t. the transition relation is functional: $\delta : Q \times \Sigma \longmapsto Q$

Runs and Language of NBAs

- A run ρ of A on ω-word $\alpha = a_0, a_1, a_2, \ldots$ is an infinite sequence $\rho = q_0, q_1, q_2, \ldots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
- The run ρ is accepting if

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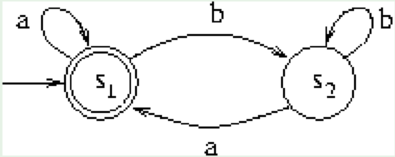
Büchi Automaton: Example

Let $\Sigma = \{a, b\}$.
Let a Deterministic Büchi Automaton (DBA) A_1 be



- With $F = \{s_1\}$ the automaton recognizes words with infinitely many a's.
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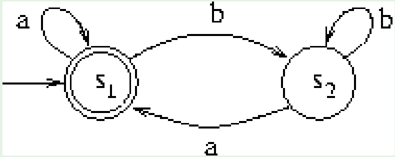

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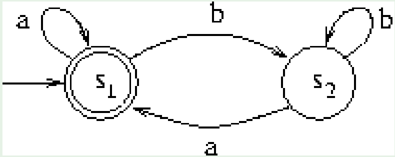

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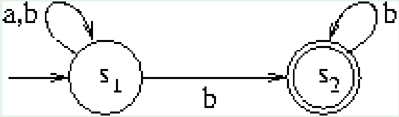
Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

Theorem

*DBA*s are strictly less powerful than *NBA*s.

Remark:

The subset construction of standard Final-State automata does not work!

Let DA_2 be

- DA_2 is not equivalent to A_2 (e.g., it recognizes $(b.a)^\omega$)
- There is no DBA equivalent to A_2

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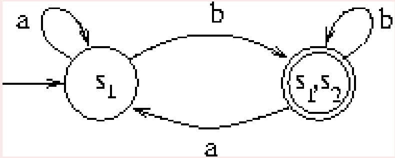

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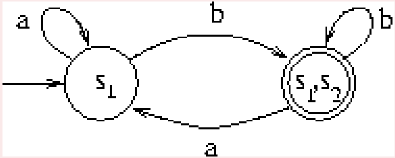

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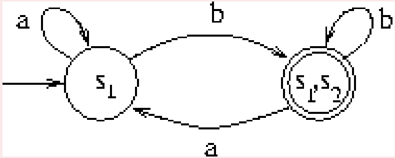

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Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
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Union of two NBAs

Definition: union of NBAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.
Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

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- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
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(same construction as with ordinary automata)

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Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2
(same construction as with ordinary automata)

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
$$I = I_1 \times I_2 \times \{1\}.$$
$$F = F_1 \times Q_2 \times \{1\}.$$

$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \notin F_1$.
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Theorem

- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

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Synchronous Product of NBAs: Intuition

- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track

$\Longrightarrow$ to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2

- Important subcase: If $F_2 = Q_2$, then

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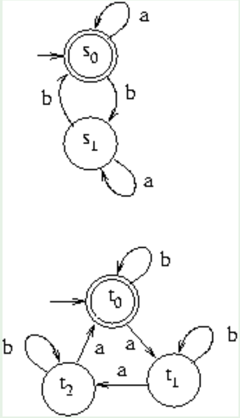

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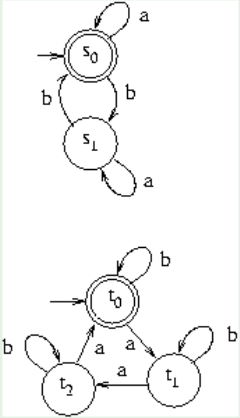

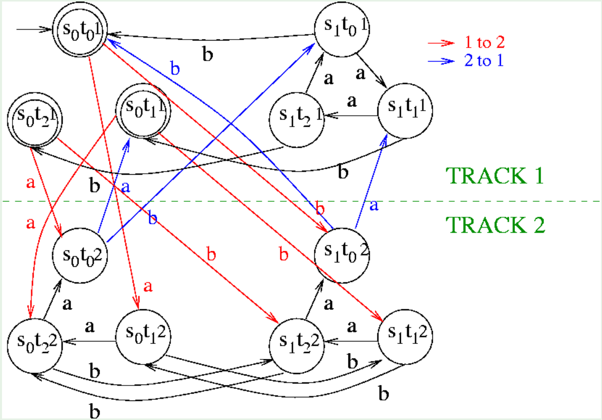

Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
$|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

(i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
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Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
The automaton remains in phase i till it visits a state in F_i. Then, it moves to $(i \ mod \ K) + 1$ mode.

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De-generalization of a generalized NBA

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δ' is s.t., for every $i \in [1, ..., K]$:

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- $\mathcal{L}(A') = \mathcal{L}(A)$.
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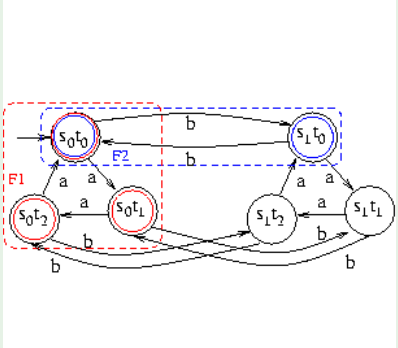
Degeneralizing a Büchi automaton: Example



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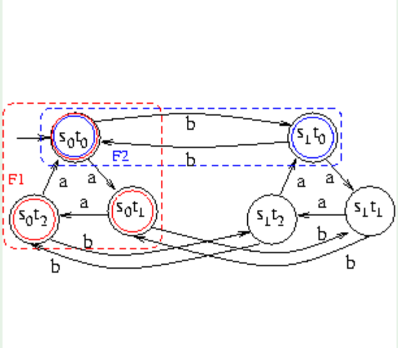

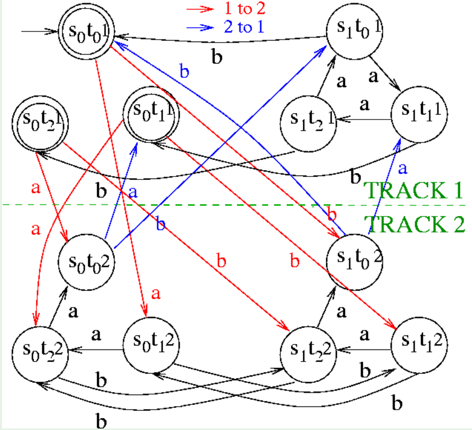

Omega-regular Expressions

Recall:

A finite-word language is called regular if it it is recognizable by some Finite-State-Automaton (FSA).

Definition

An infinite-word language is called ω-regular if it has the form $\cup_{i=1}^{n} \; U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

Theorem

A language L is ω-regular iff it is NBA-recognizable.

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2. The Automata-Theoretic Approach to LTL Reasoning
   - General Ideas
   - Language-Emptiness Checking of Büchi Automata
   - From Kripke Models to Büchi Automata
   - From LTL Formulas to Büchi Automata
   - Complexity

3. Exercises

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Automata-Theoretic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$\begin{aligned} & \models \psi \quad \text{(LTL)} \\ \Longleftrightarrow\ & \neg\psi \text{ unsat} \\ \Longleftrightarrow\ & \mathcal{L}(A_{\neg\psi}) = \emptyset \end{aligned}$$

- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

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Two steps for checking $\models \psi$ [resp. $\varphi \models \psi$]

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Automata-Theoretic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

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Automata-Theoretic LTL Model Checking

Four steps

Let $\varphi \overset{\text{def}}{=} \neg\psi$:

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(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

Outline

1 Büchi Automata

2 The Automata-Theoretic Approach to LTL Reasoning
  - General Ideas
  - Language-Emptiness Checking of Büchi Automata
  - From Kripke Models to Büchi Automata
  - From LTL Formulas to Büchi Automata
  - Complexity

3 Exercises

NBA emptiness checking

- Idea: find an accepting cycle reachable from an initial state
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  (i) a DFS finds the accepting states *f* reachable from an initial state;
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  (i.e., if there exists a loop)

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- Two nested DFSs
    - `DFS1` finds the accepting states f reachable from an initial state
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- Two Hash tables:
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    - the stack of `DFS2` (an accepting, preceded by cycle)
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- DFS1 invokes DFS2 on each f_i only after popping it (postorder)
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(Smart) Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
   stack S1=I; stack S2=∅;
   Hashtable T1=I; Hashtable T2=∅;
   while S1!=∅ {
      v=top(S1);
      if ∃w s.t. w∈ δ(v) && T1(w)==0 {
         hash(w,T1);
         push(w,S1);
      } else {
         pop(S1);
         if (v∈F && !DFS2(v,S2,T2,A))     //test after popping!
            return False;
   }     }
   return True;
}
```

(Smart) Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
    hash(f,T);
    S = {f}
    while S!=$\emptyset$ {
        v=top(S);
        if f$\in\delta$(v) return False;
        if $\exists$w s.t. w$\in\delta$(v) && T(w)==0 {
            hash(w);
            push(w);
        } else pop(S);
    }
    return True;
}
```

Remark: `T` passed by reference (or static) $\Longrightarrow$ is not reset at each call of `DFS2` !

Double nested DFS: Intuition

DFS1 invokes DFS2 on each $f_1, \ldots, f_n$ only after popping it (postorder):

- suppose $DFS2$ is invoked on f_j earlier than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_i, \ldots)$ it is encountered a state S which has already been explored by $DFS2(f_j, \ldots)$ for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

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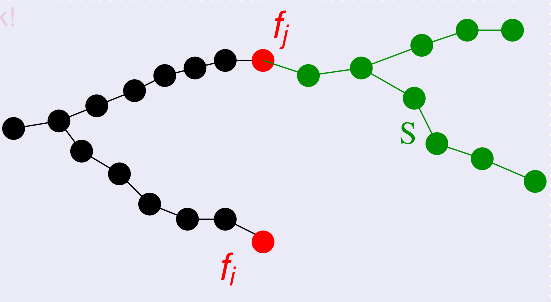

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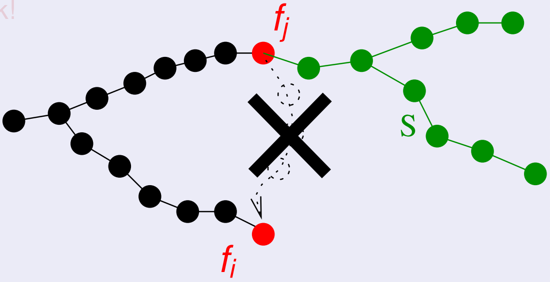

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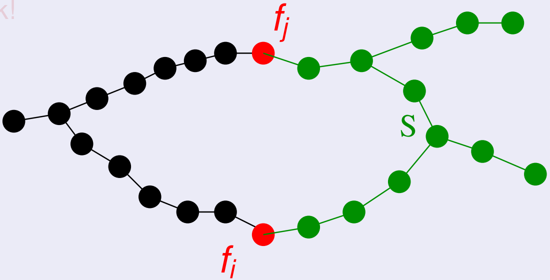

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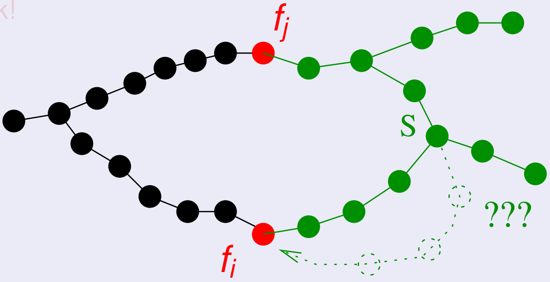

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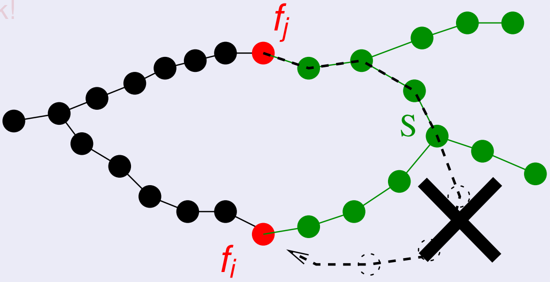

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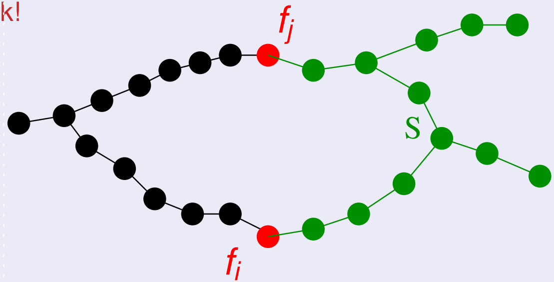

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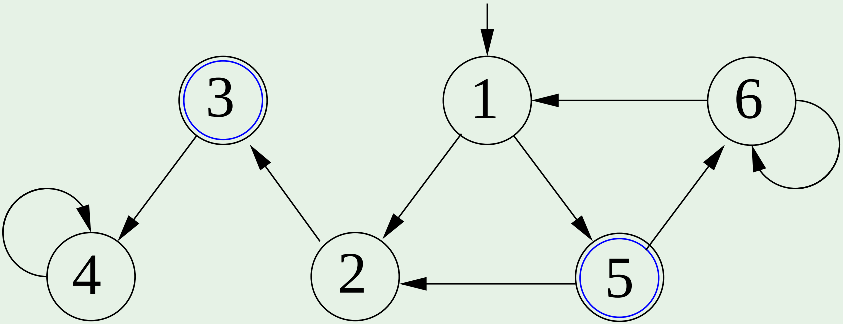

T1

S1

T2

S2

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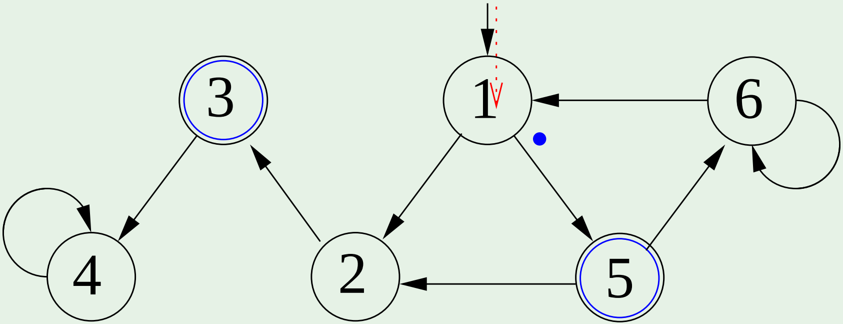

T1 1

S1 1

T2

S2

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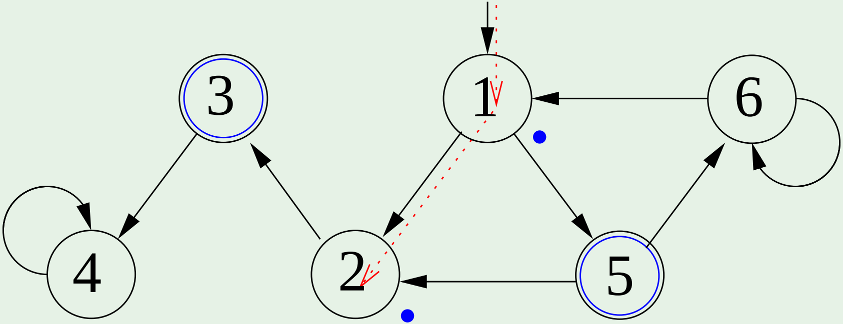

T1 1 2 T2

S1 1 2 S2

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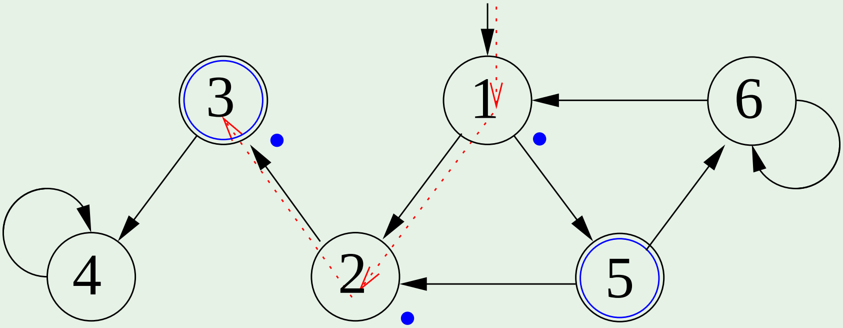

T1 1 2 3

S1 1 2 3

T2

S2

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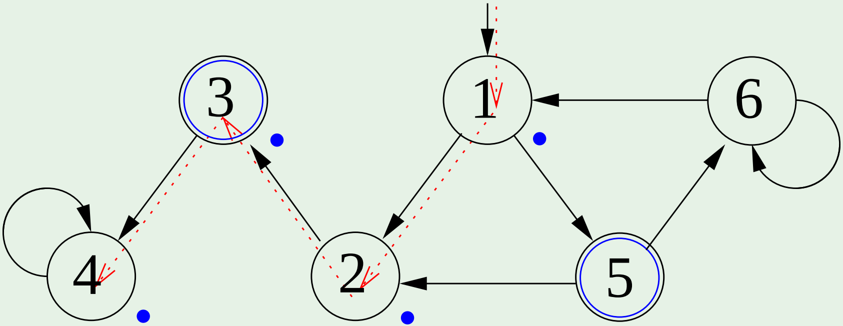

T1 1 2 3 4 T2

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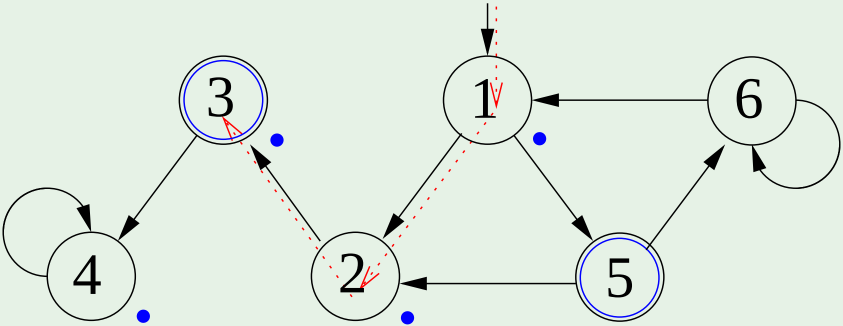

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S1 1 2 3

T2

S2

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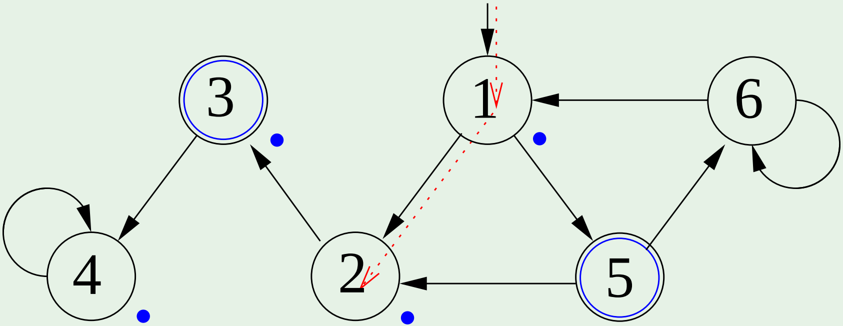

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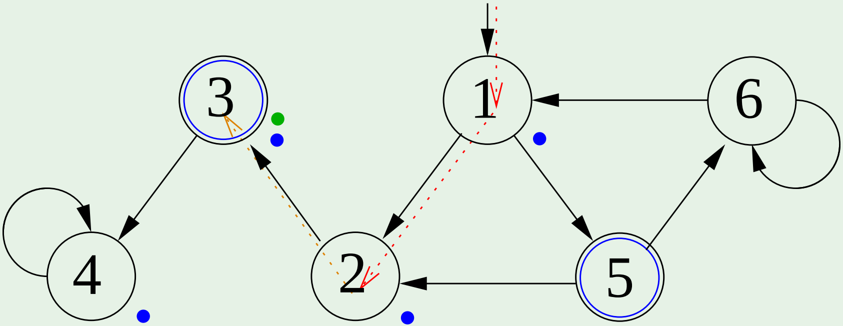

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T2 3

S1 1 2

S2 3

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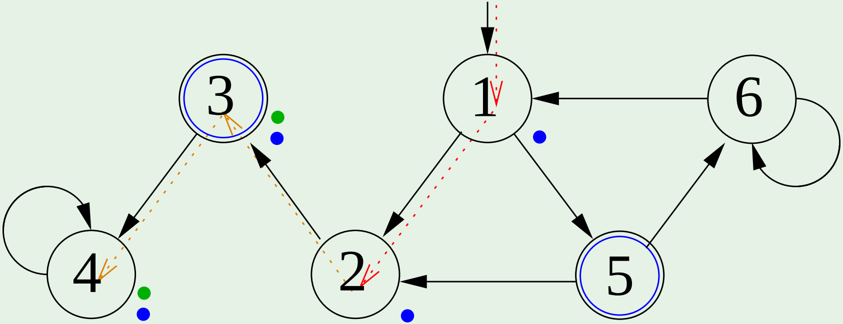

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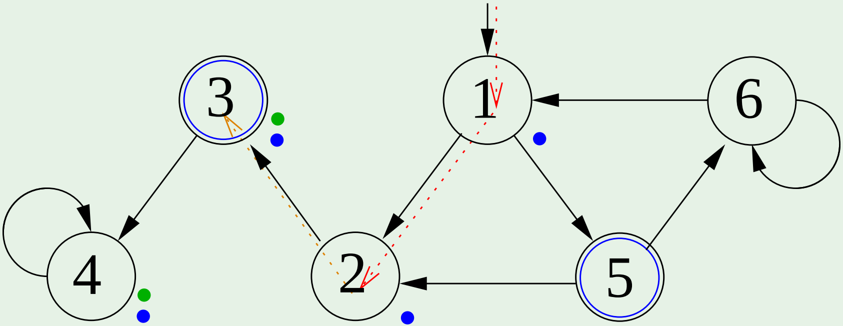

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S2 3

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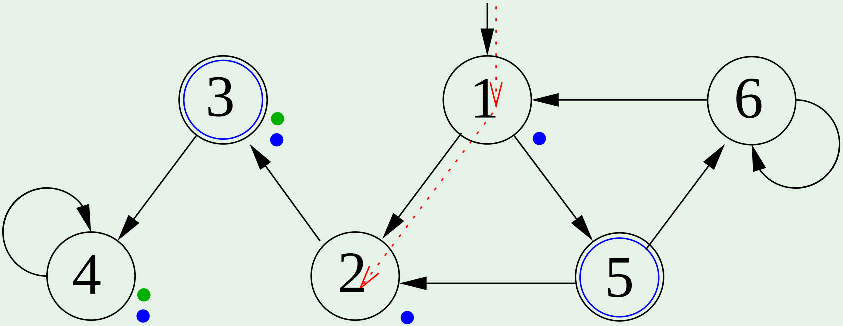

T1 1 2 3 4

S1 1 2

T2 3 4

S2

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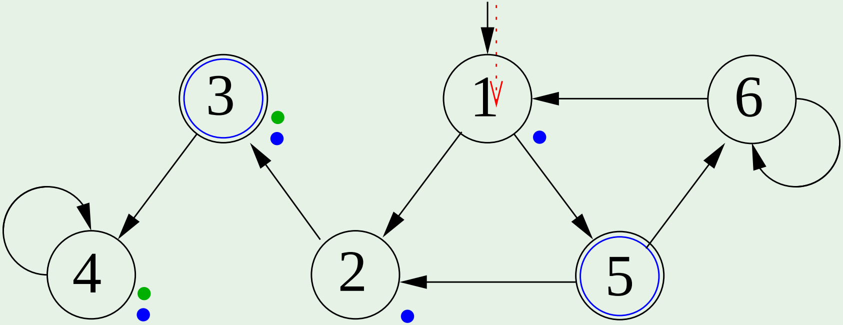

T1 1 2 3 4

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S1 1

S2

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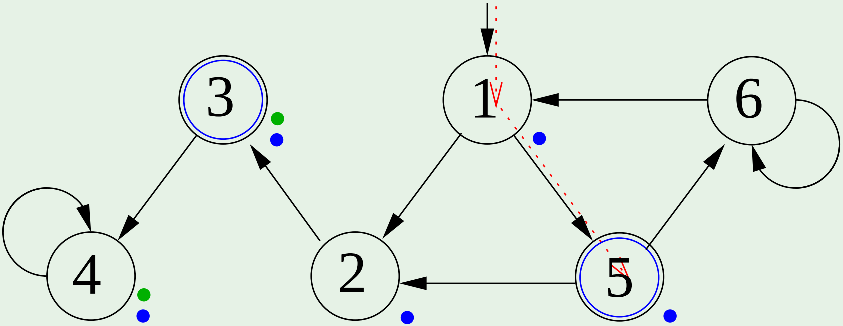

T1 1 2 3 4 5

T2 3 4

S1 1 5

S2

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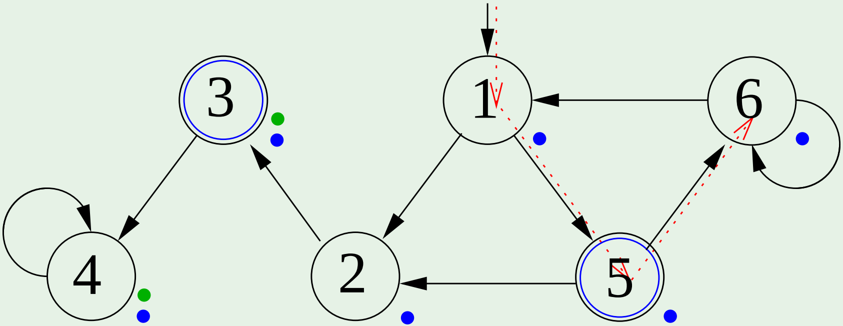

T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

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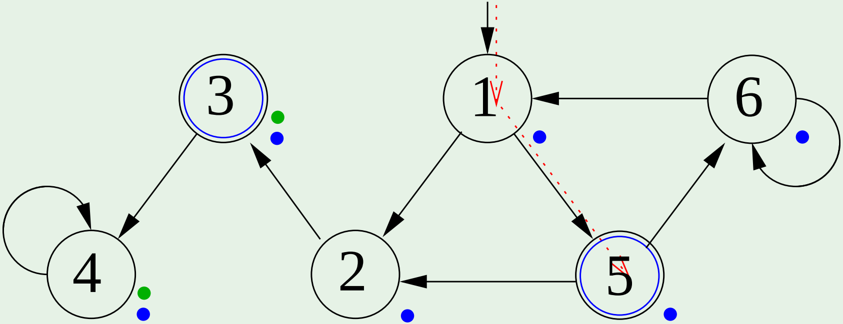

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S1 1 5

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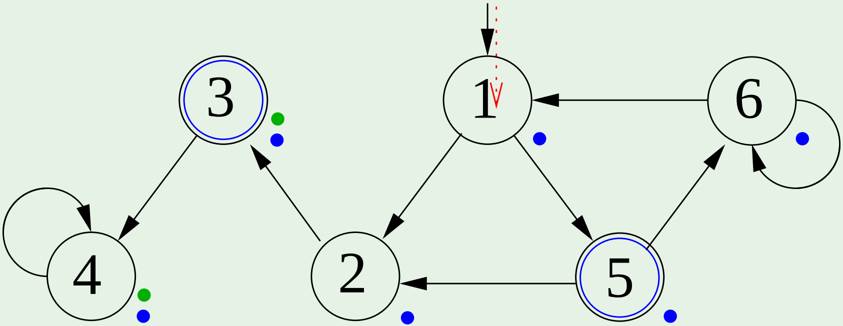

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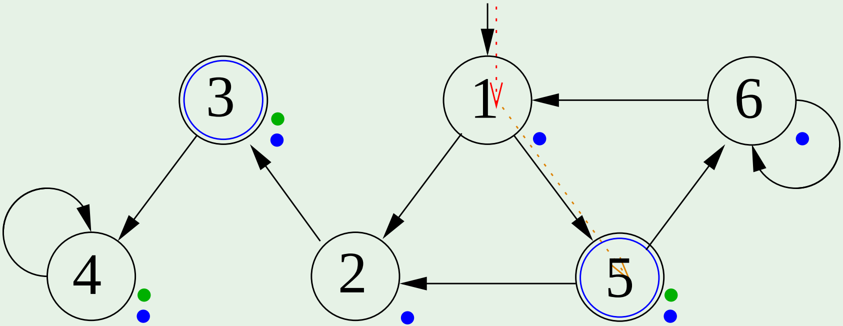

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S1 1

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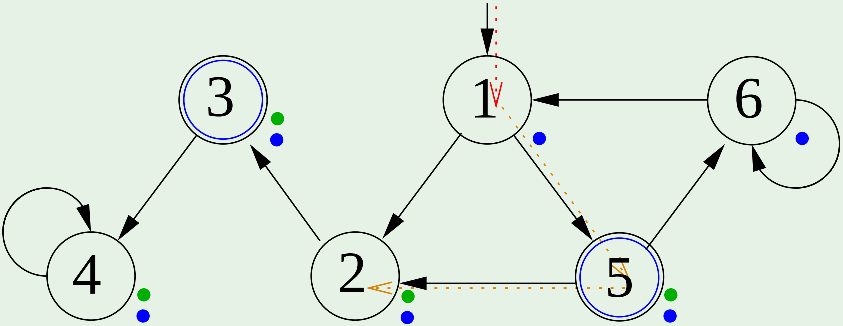

T1 1 2 3 4 5 6

T2 3 4 5 2

S1 1

S2 5 2

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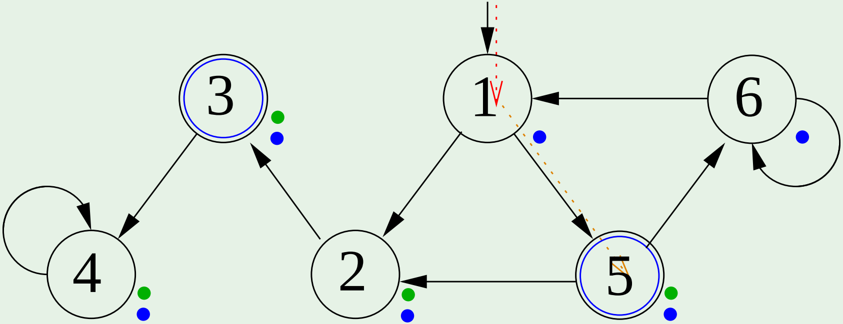

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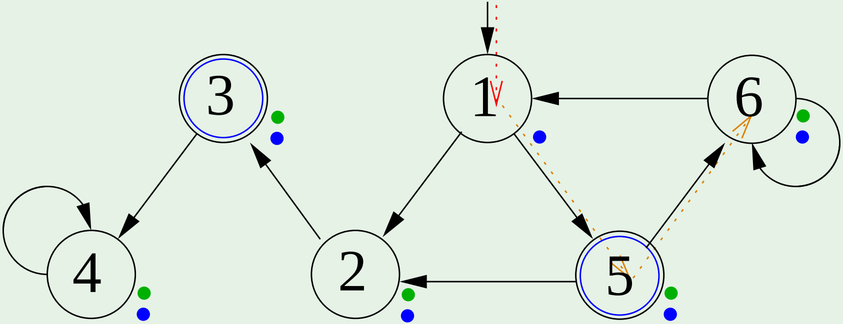

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T2 3 4 5 2 6

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S2 5 6

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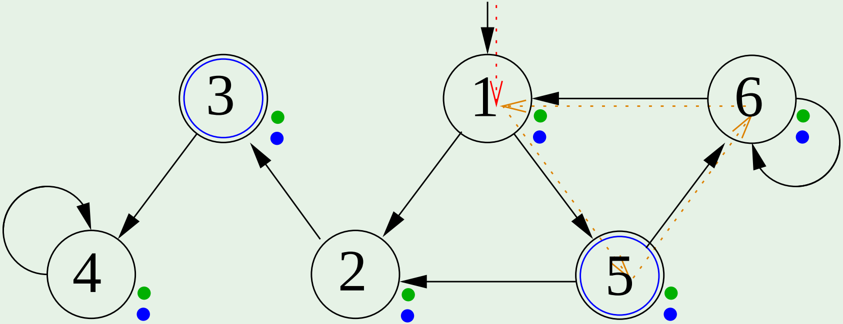

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T2 3 4 5 2 6 1

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Outline

1. Büchi Automata

2. The Automata-Theoretic Approach to LTL Reasoning
   - General Ideas
   - Language-Emptiness Checking of Büchi Automata
   - From Kripke Models to Büchi Automata
   - From LTL Formulas to Büchi Automata
   - Complexity

3. Exercises

Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke model $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
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  - Transitions:

$$\begin{aligned} \delta: \quad & q \xrightarrow{a} q' \text{ iff } (q, q') \in R \; and \; L(q') = a \\ & init \xrightarrow{a} q \text{ iff } q \in S_0 \; and \; L(q) = a \end{aligned}$$

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- $|A_M| = |M| + 1$

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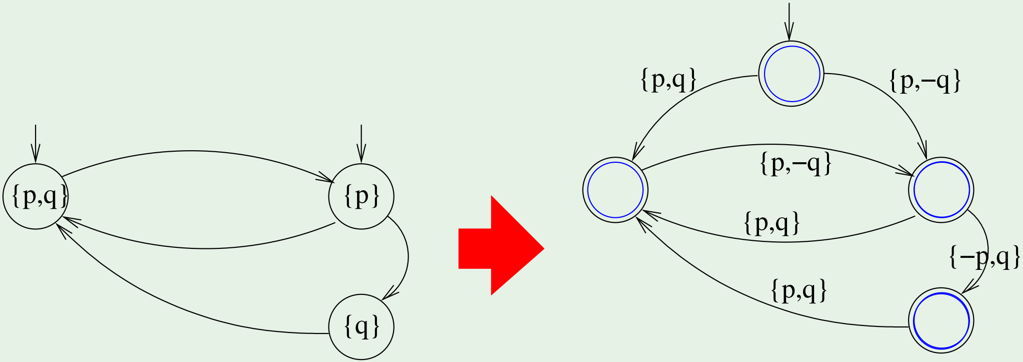


Kripke Structure

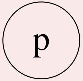
Buechi Automaton

$\Longrightarrow$ Substantially:

1. add one initial state,
2. move labels from states to incoming edges,
3. set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



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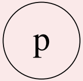

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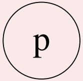

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Translation problem

Problem

Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL validity/satisfiability/entailment and model checking
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LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, $\mathbf{X}$, $\mathbf{U}$, $\mathbf{R}$ on propositional literals.
- Done by pushing negations down to literal level:

$$\begin{array}{lcl}
\neg\neg\varphi_1 & \Longrightarrow & \varphi_1 \\
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$\Longrightarrow$ The resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).

  - the encoding is linear if a DAG representation is used

- In the construction of A_φ we now assume that φ is in NNF.
  $\Longrightarrow$ every non-atomic subformula occurs positively in φ
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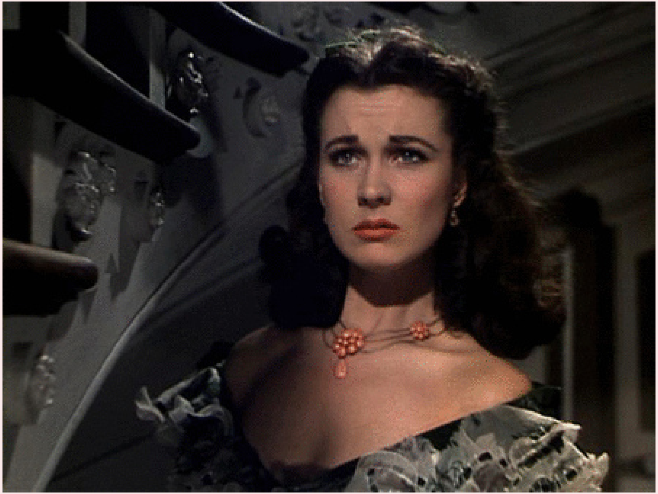
On-the-fly Construction of A_φ (Intuition)

(Implicitly) Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ:
$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$ [and $\mathbf{F}\psi \Longrightarrow \psi \vee \mathbf{XF}\psi$]
$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$ [and $\mathbf{G}\psi \Longrightarrow \psi \wedge \mathbf{XG}\psi$]
until we get a Boolean combination of elementary subformulas of φ
(An elementary formula is a proposition or a **X**-formula.)

Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, by:
(i) applying recursively the DeMorgan rule: $\varphi_1 \wedge (\varphi_2 \vee \varphi_3) \Longrightarrow (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$, and then
(ii) pushing the conjunctions inside the next operator:

$$\varphi \xRightarrow{(i)} \bigvee_i (\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X} \psi_{ik}) \xRightarrow{(ii)} \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}).$$

- Each disjunct $(\overbrace{\bigwedge_j l_{ij}}^{labels} \wedge \overbrace{\mathbf{X} \bigwedge_k \psi_{ik}}^{next\ part})$ represents a state:
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  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ
    (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
  - $\mathbf{X} \bigwedge_k \psi_{ik}$ represents the next part of the state
    (obbligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, by:
(i) applying recursively the DeMorgan rule: $\varphi_1 \wedge (\varphi_2 \vee \varphi_3) \implies (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$, and then
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Step 3: For every state S_i represented by $(\bigwedge_j l_{ij} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\varphi_i})$

- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
- mark that the state S_i satisfies φ
- apply recursively steps 1-2-3 to $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$,
  - rewrite φ_i into $\bigvee_{i'}(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$
  - from each disjunct $(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$ generate a new state $S_{ii'}$ (if not already present) and label it as satisfying $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$
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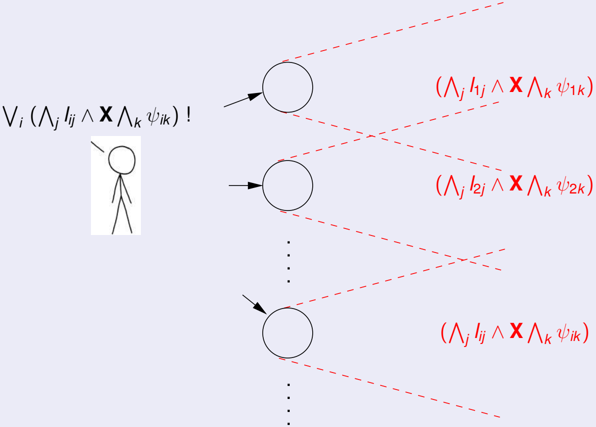
φ ??

On-the-fly Construction of A_φ (Intuition) [cont.]

$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

On-the-fly Construction of A_φ (Intuition) [cont.]



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On-the-fly Construction of A_φ (Intuition) [cont.]

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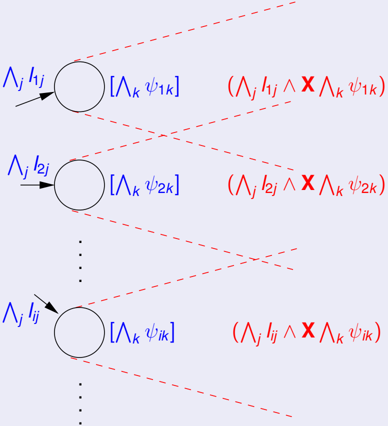
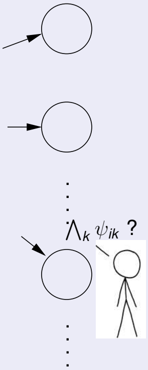
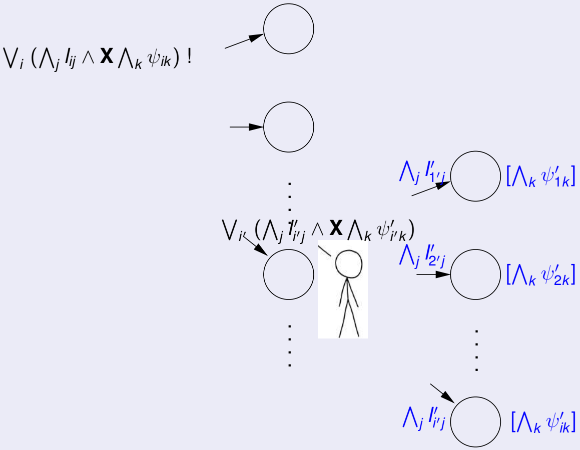


On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



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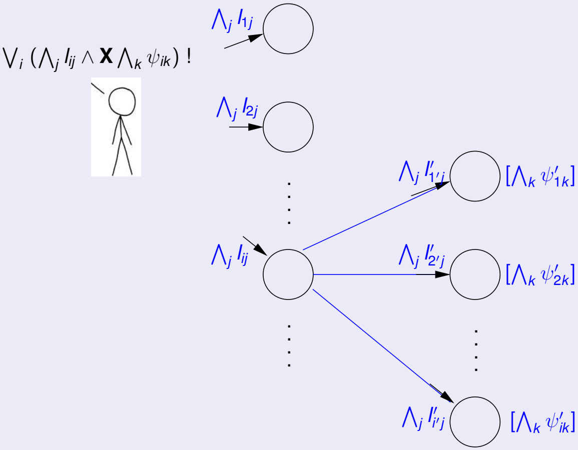

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When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j, mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no **U**-subformulas, then mark all states with F_1 —i.e., $FT \overset{\text{def}}{=} \{Q\}$).

Remark

The fact that we initially converted the formula into NNF guarantees that only original positive **U**/**F**-subformulas and negative **R**-/**G**-subformulas are considered in step 4

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Dealing with $\mathbf{U}$-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \Longleftrightarrow (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X} \varphi_1 \mathbf{U} \varphi_2))$
  are a property, not a definition of $\mathbf{U}$:
  $\Longrightarrow$ they implicitly admit a "weaker" semantics of $\varphi_1 \mathbf{U} \varphi_2$, in which $\varphi_1 \mathbf{U} \varphi_2$ always holds and φ_2 never holds
- It cannot happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.

$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ holds

- In LTL: $\neg \mathbf{FG}((\varphi_1 \mathbf{U} \varphi_2) \wedge \neg \varphi_2)$, i.e., $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ ("avoid bad loops")

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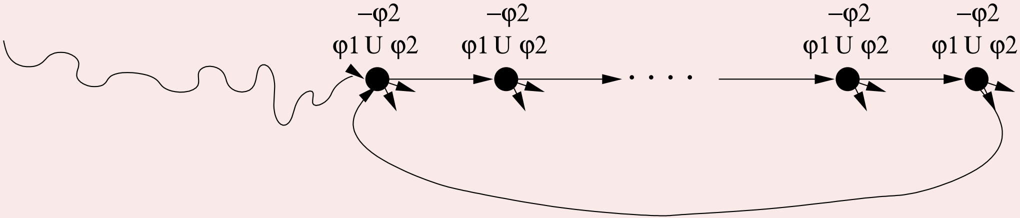

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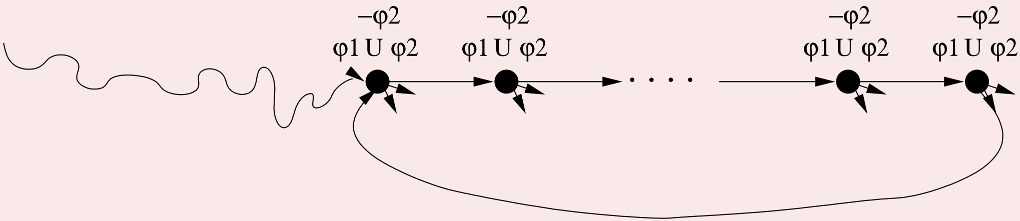

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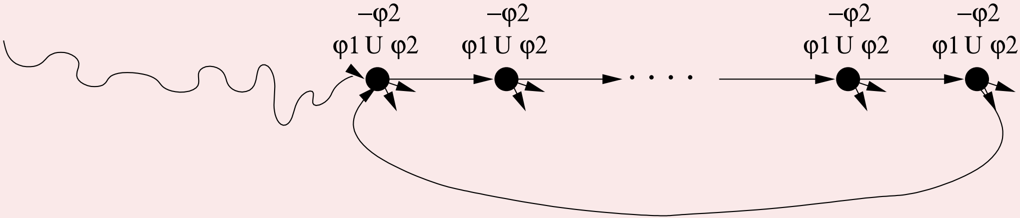

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On-the-fly Construction of A_φ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of φ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define *Cover*$(\Psi) \stackrel{\text{def}}{=}$ *Expand*$(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - *Expand*(Ψ, s) takes as input:
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    and returns a set of states $\{\langle \lambda_i, \chi_i, \sigma_i \rangle\}_i$ representing te expansion of Ψ
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On-the-fly Construction of A_φ - Expand

Given $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$, we define *Expand*$(\Psi, s)$ recursively as follows:

- if $\Psi = \emptyset$, *Expand*$(\Psi, s) = \{s\}$
- if $\bot \in \Psi$, *Expand*$(\Psi, s) = \emptyset$
- if $\top \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  Expand$(\Psi, s) =$ *Expand*$(\Psi \backslash \{\top\}, \langle \lambda, \chi, \sigma \cup \{\top\} \rangle)$
- if $l \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$, l propositional literal
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On-the-fly Construction of A_φ - Expand

Given $\Psi \overset{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \overset{\text{def}}{=} \langle\lambda, \chi, \sigma\rangle$, we define $Expand(\Psi, s)$ recursively as follows:

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- if $\psi_1 \vee \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\begin{aligned}\mathit{Expand}(\Psi, s) = {} & \mathit{Expand}(\Psi \cup \{\psi_1\} \backslash \{\psi_1 \vee \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\}\rangle) \\ \cup {} & \mathit{Expand}(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \vee \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\}\rangle)\end{aligned}$$
  (split *s* into two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
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On-the-fly Construction of A_φ - Expand

Two relevant subcases: $\mathbf{F}\psi \overset{\text{def}}{=} \top\mathbf{U}\psi$ and $\mathbf{G}\psi \overset{\text{def}}{=} \bot\mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
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Definition of A_φ

Given a set of LTL formulas Ψ, we define $\mathit{Cover}(\Psi) \overset{\text{def}}{=} \mathit{Expand}(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.
For an LTL formula φ, we construct a Generalized NBA $A_\varphi = (Q, \Sigma, \delta, I, FT)$ as follows:

- $\Sigma \;=\; 3^{vars(\varphi)}$ ($v \in \{\top, \bot, *\}$, "$*$" is "don't care")
- Q is the smallest set such that
  - $\mathit{Cover}(\{\varphi\}) \subseteq Q$
  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $\mathit{Cover}(\chi) \in Q$
- $Q_0 = \mathit{Cover}(\{\varphi\})$.
- $s \xrightarrow{\lambda'} s' \in \delta$ iff, $s = \langle \lambda, \chi, \sigma \rangle$, $s' = \langle \lambda', \chi', \sigma' \rangle$ and $s' \in \mathit{Cover}(\chi)$
- $FT = \langle F_1, F_2, ..., F_k \rangle$ where, for all $(\psi_i \mathbf{U} \varphi_i)$ occurring positively in φ,
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For an LTL formula φ, we construct a Generalized NBA $A_\varphi = (Q, \Sigma, \delta, I, FT)$ as follows:

- $\Sigma = 3^{vars(\varphi)}$ ($v \in \{\top, \bot, *\}$, "$*$" is "don't care")
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  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
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Example: $\varphi = \mathbf{FG}p$

- $$\begin{aligned}&\mathit{Cover}(\{\mathbf{FG}p\})\\ &= \mathit{Expand}(\{\mathbf{FG}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle) \cup \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\{\mathbf{FG}p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle\}\end{aligned}$$
- $$\begin{aligned}\mathit{Cover}(\{\mathbf{G}p\}) &= \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\}\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle)\\ &= \{\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle\}\end{aligned}$$
- Optimization:
  merge $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle$ and $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle$

Example: $\varphi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
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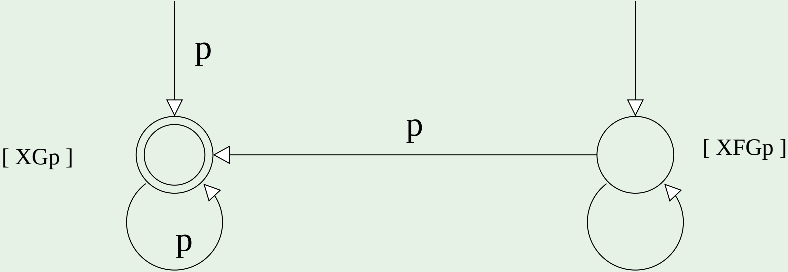

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Example: $\varphi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle$, $s_2 =_{def} \langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\} \rangle$, $s_3 =_{def} \langle \emptyset, \{\top\}, \{\top\} \rangle$.
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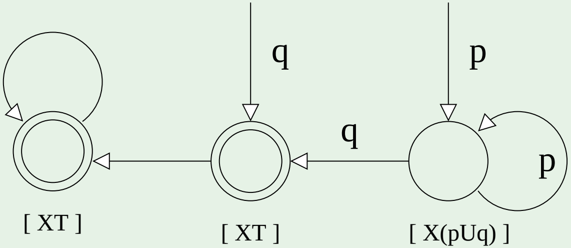

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$$
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\end{aligned}
$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \iff \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: **GF**p

- Let $s_1 =_{def} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$, $s_2 =_{def} \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle$,
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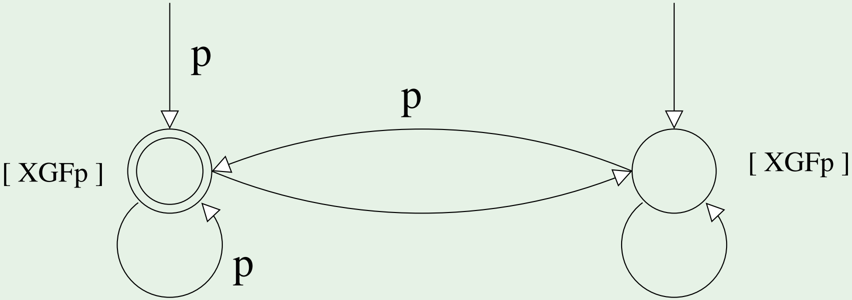

NBAs of disjunctions of formulas

Remark

If $\varphi \overset{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \overset{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \overset{\text{def}}{=} (\mathbf{GF}p \rightarrow \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.
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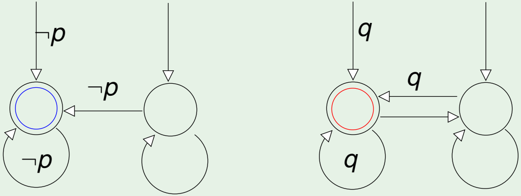

Suggested Exercises:

- Find an NBA encoding:
  - p
  - $(p \wedge q) \vee (\neg p \wedge \neg q)$
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \to \mathbf{G}r$

Outline

1 Büchi Automata

2 The Automata-Theoretic Approach to LTL Reasoning
  - General Ideas
  - Language-Emptiness Checking of Büchi Automata
  - From Kripke Models to Büchi Automata
  - From LTL Formulas to Büchi Automata
  - Complexity

3 Exercises

Automata-Theoretic LTL Model Checking: Complexity

Four steps:

(i) Compute A_M:

$|A_M| = O(|M|)$

(ii) Compute A_φ:

$|A_\varphi| = O(2^{|\varphi|})$

(iii) Compute the product $A_M \times A_\varphi$:

$|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$

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Final Remarks

- Büchi automata are in general more expressive than LTL!

⟹ some tools (e.g., Spin) allow specifications to be expressed directly as NBAs

⟹ complementation of NBA relevanant in general

- For every LTL formula, there are many possible equivalent NBAs

⟹ lots of research for finding “the best” conversion algorithm

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Outline

Ex: Product of Büchi automata

Given the following two Büchi automata (doubly-circled states represent accepting states, *a*, *b* are labels):

Write the product Büchi automaton $BA1 \times BA2$.

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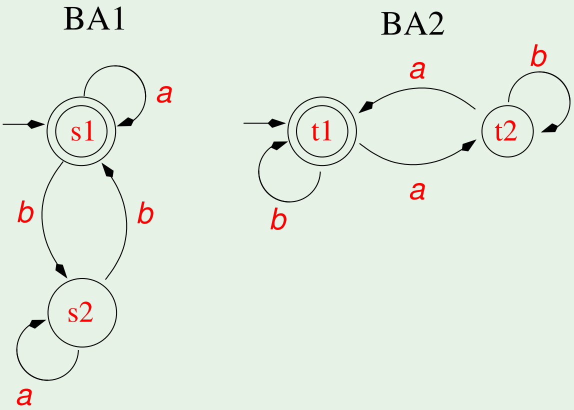

Write the product Büchi automaton *BA*1 × *BA*2.

Ex: Product of Büchi automata

[Solution: The product is:

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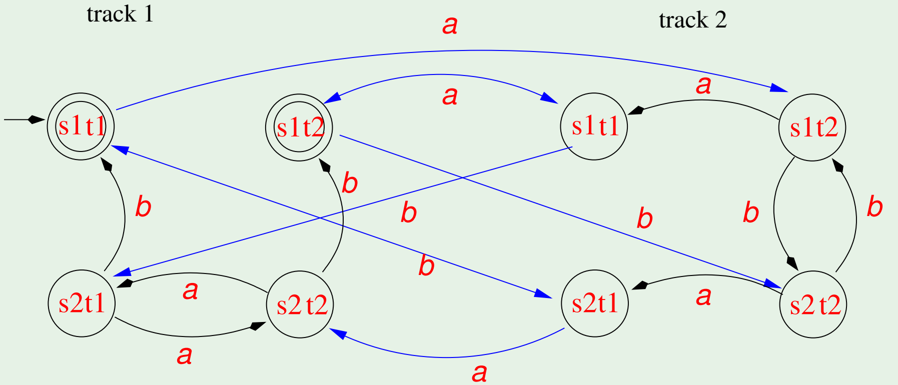

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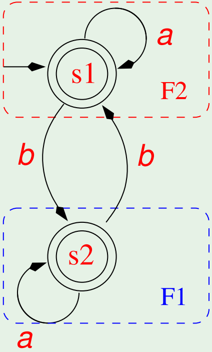
Ex: De-generalization of Büchi Automata

Given the following generalized Büchi automaton $A \overset{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \overset{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \overset{\text{def}}{=} \{s2\}, F2 \overset{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

Ex: De-generalization of Büchi Automata

[Solution: The result is:

]

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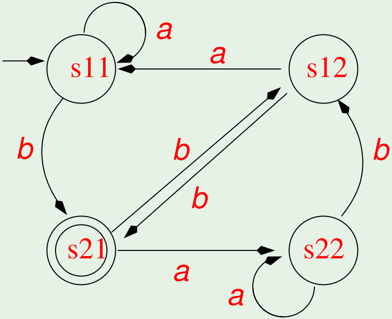

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Ex: Construction of Büchi Automata

Consider the LTL formula $\varphi \overset{\text{def}}{=} (\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q)$.

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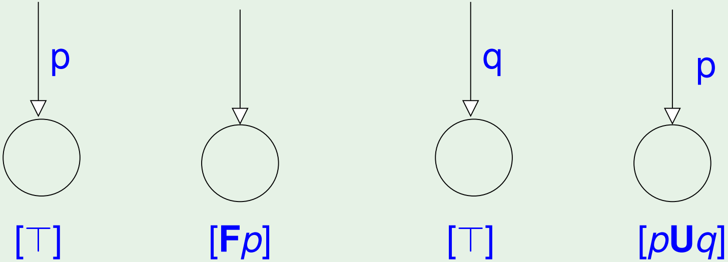

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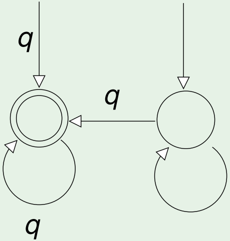

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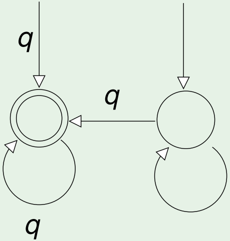

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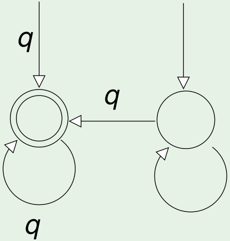

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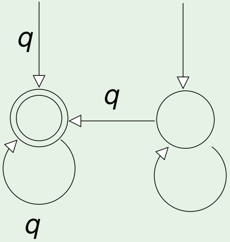

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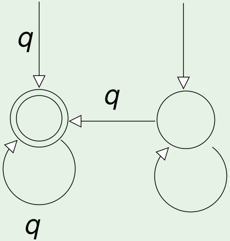

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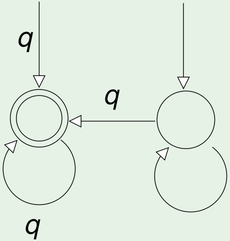

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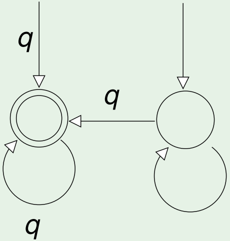

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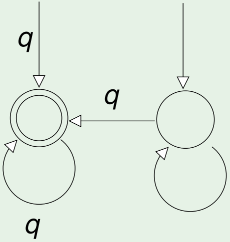

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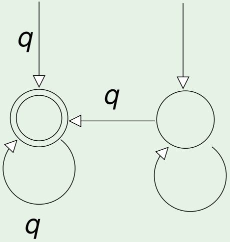

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