

Formal Methods

Module I: Automated Reasoning

Ch. 04: Automata-Theoretic LTL Reasoning

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- 1 Büchi Automata
- 2 The Automata-Theoretic Approach to LTL Reasoning
 - General Ideas
 - Language-Emptiness Checking of Büchi Automata
 - From Kripke Models to Büchi Automata
 - From LTL Formulas to Büchi Automata
 - Complexity
- 3 Exercises

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Infinite Word Languages

Modeling infinite computations of reactive systems

Given an **Alphabet** Σ (e.g. $\Sigma \stackrel{\text{def}}{=} \{a, b\}$)

- An ω -word α over Σ is an **infinite** sequence

$a_0, a_1, a_2 \dots$

Formally, $\alpha : \mathbb{N} \rightarrow \Sigma$.

- The set of all infinite words is denoted by Σ^ω .
- A ω -language L is collection of ω -words, i.e. $L \subseteq \Sigma^\omega$.
- Example: **All words over $\{a, b\}$ with infinitely many a 's.**

Notation:

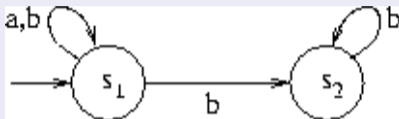
omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.

omega-languages $L, L_1 \subseteq \Sigma^\omega$

For $u \in \Sigma^+$, let $u^\omega = u.u.u \dots$

Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\dots$

There are several (infinite) possible runs.

Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \dots$

Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \dots$

- Acceptance Conditions: Büchi (Muller, Rabin, Street):
Acceptance is based on states occurring infinitely often

- Notation: Let Q be the set of states. Let $\rho \in Q^\omega$. Then,

$$\text{Inf}(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}. \rho(i) = s\}.$$

(The set of states occurring infinitely many times in ρ .)

Büchi Automata

Nondeterministic Büchi Automaton

- A **Nondeterministic Büchi Automaton (NBA)** is $(Q, \Sigma, \delta, I, F)$ s.t.
 - Q Finite set of states.
 - Σ is a finite alphabet
 - $I \subseteq Q$ set of initial states.
 - $F \subseteq Q$ set of accepting states.
 - $\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).
- A **Deterministic Büchi Automaton (DBA)** is an NBA s.t. the transition relation is functional:
 $\delta : Q \times \Sigma \mapsto Q$

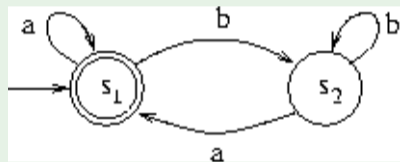
Runs and Language of NBAs

- A **run** ρ of A on ω -word $\alpha = a_0, a_1, a_2, \dots$ is an infinite sequence $\rho = q_0, q_1, q_2, \dots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
- The run ρ is **accepting** if
$$\text{Inf}(\rho) \cap F \neq \emptyset.$$
- The **language accepted by A**
$$\mathcal{L}(A) = \{\alpha \in \Sigma^\omega \mid A \text{ has an accepting run on } \alpha\}$$

Büchi Automaton: Example

Let $\Sigma = \{a, b\}$.

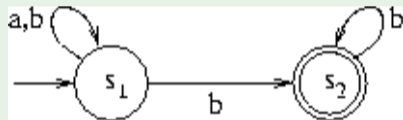
Let a Deterministic Büchi Automaton (DBA) A_1 be



- With $F = \{s_1\}$ the automaton recognizes words with infinitely many a 's.
- With $F = \{s_2\}$ the automaton recognizes words with infinitely many b 's.

Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a . Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

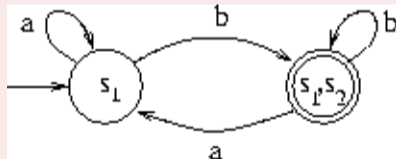
Theorem

DBAs are strictly less powerful than NBAs.

Remark:

The subset construction of standard Final-State automata does not work!

Let DA_2 be



- DA_2 is not equivalent to A_2
(e.g., it recognizes $(b.a)^\omega$)
- There is no DBA equivalent to A_2

Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$. $|A| \leq |A_1| \cdot |A_2| \cdot 2$.

Union of two NBAs

Definition: union of NBAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.

Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') & \text{if } s \in Q_1 \\ R_2(s, s') & \text{if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2
(same construction as with ordinary automata)

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.

Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$

$$I = I_1 \times I_2 \times \{1\}.$$

$$F = F_1 \times Q_2 \times \{1\}.$$

$\langle p, q, 1 \rangle \xrightarrow{a} \langle p', q', 1 \rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \notin F_1$.

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$\langle p, q, 2 \rangle \xrightarrow{a} \langle p', q', 1 \rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \in F_2$.

Theorem

- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

Synchronous Product of NBAs: Intuition

- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track

⇒ to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2

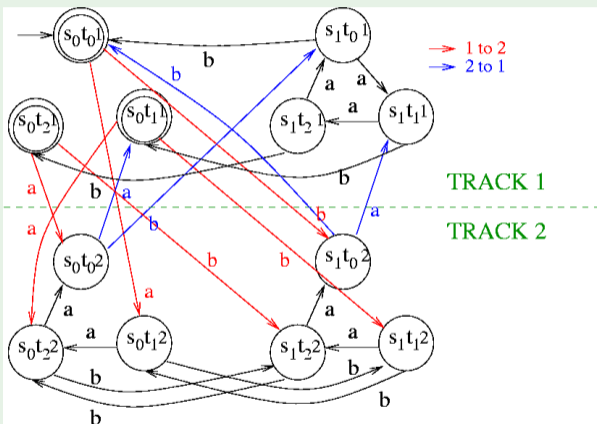
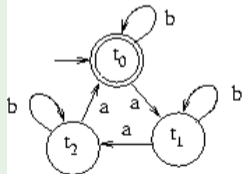
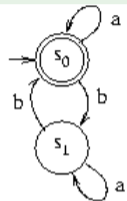
- Important subcase: If $F_2 = Q_2$, then

$$Q = Q_1 \times Q_2.$$

$$I = I_1 \times I_2.$$

$$F = F_1 \times Q_2.$$

Synchronous Product of NBAs: Example



Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
 $|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

- (i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
- (ii) determinize and Complement the Rabin automaton
- (iii) convert the Rabin automaton into a Büchi automaton.

Generalized Büchi Automaton

Definition

- A **Generalized Büchi Automaton** is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \dots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \dots, K\}$.

The automaton remains in phase i till it visits a state in F_i . Then, it moves to $(i \bmod K) + 1$ mode.

De-generalization of a generalized NBA

Definition: De-generalization of a generalized NBA

Let $A \stackrel{\text{def}}{=} (Q, \Sigma, \delta, I, FT)$ a generalized BA s.f. $FT \stackrel{\text{def}}{=} \{F_1, \dots, F_K\}$.

Then a language-equivalent BA $A' \stackrel{\text{def}}{=} (Q', \Sigma, \delta', I', F')$ is built as follows

$$Q' = Q_1 \times \{1, \dots, K\}.$$

$$I' = I \times \{1\}.$$

$$F' = F_1 \times \{1\}.$$

δ' is s.t., for every $i \in [1, \dots, K]$:

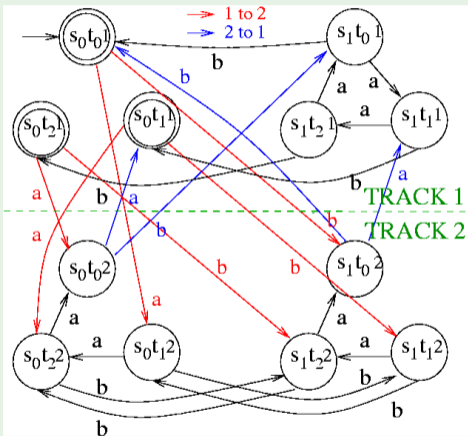
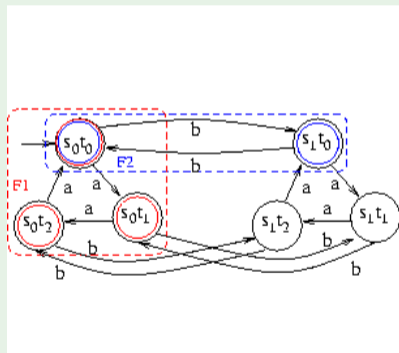
$$\langle p, i \rangle \xrightarrow{a} \langle q, i \rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \notin F_i.$$

$$\langle p, i \rangle \xrightarrow{a} \langle q, (i \bmod K) + 1 \rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \in F_i.$$

Theorem

- $\mathcal{L}(A') = \mathcal{L}(A)$.
- $|A'| \leq K \cdot |A|$.

Degeneralizing a Büchi automaton: Example



Omega-regular Expressions

Recall:

A finite-word language is called **regular** if it is recognizable by some Finite-State-Automaton (FSA).

Definition

An infinite-word language is called **ω -regular** if it has the form $\cup_{i=1}^n U_i \cdot (V_i)^\omega$ where U_i, V_i are regular languages.

Theorem

A language L is **ω -regular** iff it is **NBA-recognizable**.

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Automata-Theoretic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$\models \psi \quad (\text{LTL})$$

$$\iff \neg\psi \text{ \textit{unsat}}$$

$$\iff \mathcal{L}(A_{\neg\psi}) = \emptyset$$

- $A_{\neg\psi}$ is a **Büchi Automaton** which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

$$\varphi \models \psi \quad (\text{LTL})$$

$$\models \varphi \rightarrow \psi \quad (\text{LTL})$$

$$\iff \varphi \wedge \neg\psi \text{ \textit{unsat}}$$

$$\iff \mathcal{L}(A_{\varphi \wedge \neg\psi}) = \emptyset$$

- $A_{\varphi \wedge \neg\psi}$ is a **Büchi Automaton** which represents all and only the paths that satisfy $\varphi \wedge \neg\psi$ (satisfy φ and do not satisfy ψ)

Automata-Theoretic LTL Satisfiability and Entailment

Two steps for checking $\models \psi$ [resp. $\varphi \models \psi$]

- (i) Compute $A_{\neg\psi}$ [resp. $A_{\varphi \wedge \neg\psi}$]
- (ii) Check the emptiness of $\mathcal{L}(A_{\neg\psi})$ [resp. $\mathcal{L}(A_{\varphi \wedge \neg\psi})$]

Automata-Theoretic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$M \models \psi \quad (\text{LTL})$$

$$\iff \mathcal{L}(M) \subseteq \mathcal{L}(\psi)$$

$$\iff \mathcal{L}(M) \cap \mathcal{L}(\psi) = \mathcal{L}(M)$$

$$\iff \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset$$

$$\iff \mathcal{L}(A_M) \cap \mathcal{L}(A_{\neg\psi}) = \emptyset$$

$$\iff \mathcal{L}(A_M \times A_{\neg\psi}) = \emptyset$$

- A_M is a **Büchi Automaton** equivalent to M (which represents all and only the executions of M)

- $A_{\neg\psi}$ is a **Büchi Automaton** which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\implies A_M \times A_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ .

Four steps

Let $\varphi \stackrel{\text{def}}{=} \neg\psi$:

- (i) Compute A_M
- (ii) Compute A_φ
- (iii) Compute the product $A_M \times A_\varphi$
- (iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

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NBA emptiness checking

- Idea: **find an accepting cycle reachable from an initial state**

- accepting cycle: a cycle containing some accepting state f

- A naive algorithm (Naive Double Nested DFS algorithm):

- (i) a DFS finds the accepting states f reachable from an initial state;
- (ii) for each f , a second DFS finds if it can reach f
(i.e., if there exists a loop)

Complexity: $O(n^2)$

- SCC-based algorithm:

- (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
- (ii) drop all SCCs which do not have at least one arc, and which do not contain at least one accepting state f
- (iii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.

Complexity: $O(n)$

- Drawbacks: it stores too much information and does not find directly a counterexample.

(Smart) Double Nested DFS algorithm

(Smart) Double Nested DFS

- Two nested DFSs
 - DFS_1 finds the accepting states f reachable from an initial state
 - for each f , DFS_2 finds if it can reach f (i.e., if there exists a loop)
 - Two Hash tables:
 - T_1 : reachable states
 - T_2 : states reachable from a reachable accepting state
 - Two stacks:
 - S_1 : current branch of states reachable
 - S_2 : current branch of states reachable from accepting state f
 - It stops as soon as it finds a counterexample.
 - The counterexample is given by
 - the stack of DFS_2 (an accepting, preceded by cycle)
 - the stack of DFS_1 (a path from an initial state to the cycle)
-
- DFS_1 invokes DFS_2 on each f_i **only after popping it (postorder)**
 - T_2 passed by reference (or static) \implies is not reset at each call of DFS_2 !

(Smart) Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
    stack S1=I; stack S2=∅;
    Hashtable T1=I; Hashtable T2=∅;
    while S1!=∅ {
        v=top(S1);
        if ∃w s.t. w∈δ(v) && T1(w)==0 {
            hash(w,T1);
            push(w,S1);
        } else {
            pop(S1);
            if (v∈F && !DFS2(v,S2,T2,A)) //test after popping!
                return False;
        }
    }
    return True;
}
```

(Smart) Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
    hash(f, T);
    S = {f}
    while S !=  $\emptyset$  {
        v = top(S);
        if  $f \in \delta(v)$  return False;
        if  $\exists w$  s.t.  $w \in \delta(v)$  && T(w) == 0 {
            hash(w);
            push(w);
        } else pop(S);
    }
    return True;
}
```

Remark: T passed by reference (or static) \implies is not reset at each call of DFS2 !

Double nested DFS: Intuition

DFS1 invokes DFS2 on each f_1, \dots, f_n only after popping it (postorder):

- suppose $DFS2$ is invoked on f_j earlier than on f_i

⇒ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_j, \dots)$ it is encountered a state S which has already been explored by $DFS2(f_j, \dots)$ for some f_j ,
 - can we reach f_i from S ?
 - No, because f_i is not reachable from f_j !

⇒ It is safe to backtrack!

Double nested DFS: Intuition

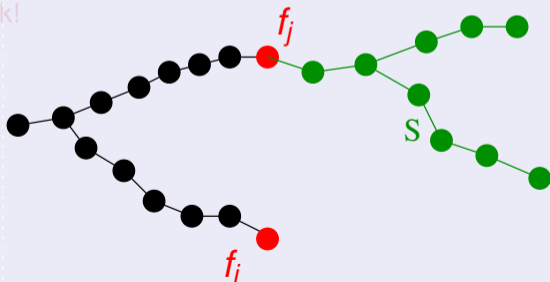
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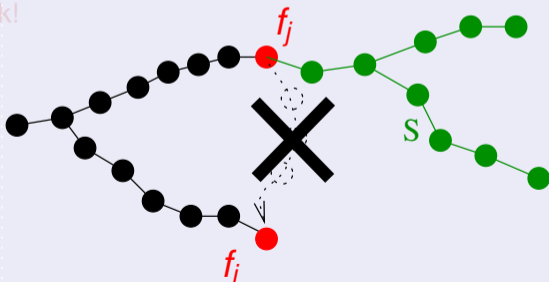
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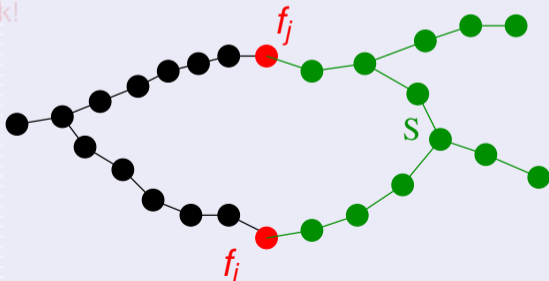
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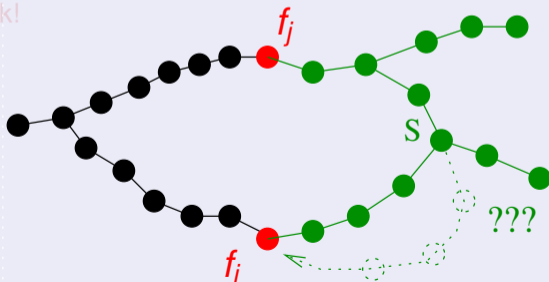
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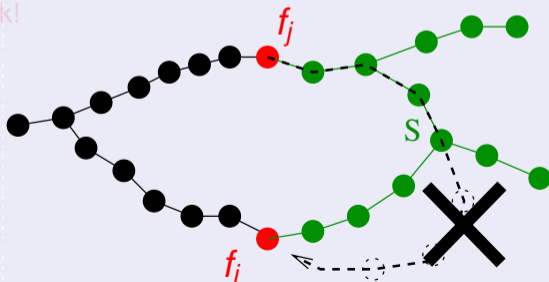
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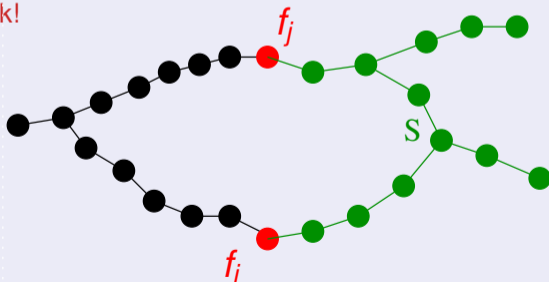
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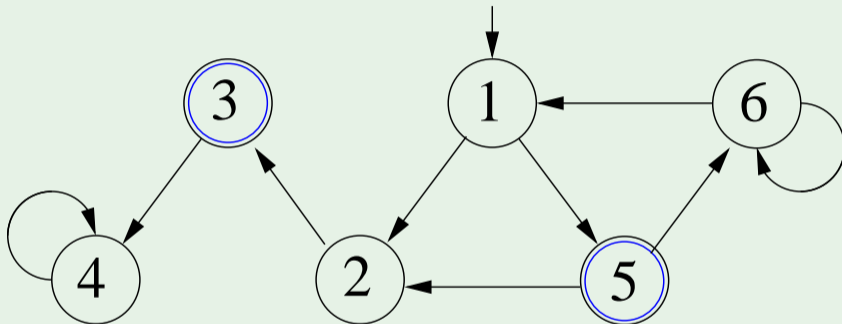
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(Smart) Double Nested DFS: example



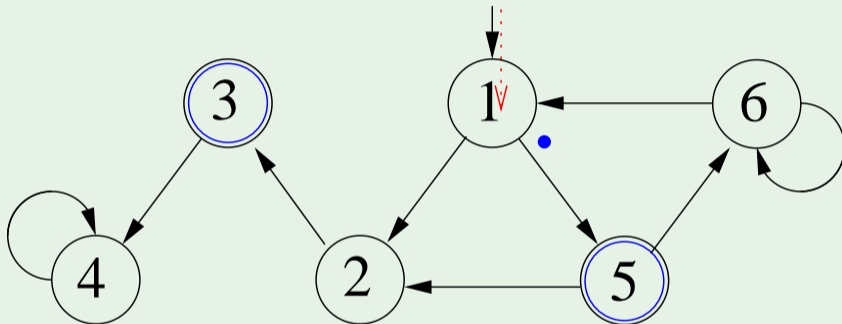
T1

S1

T2

S2

(Smart) Double Nested DFS: example



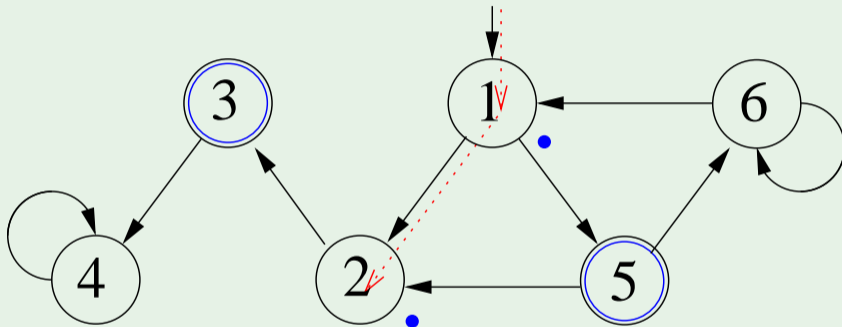
T1 1

S1 1

T2

S2

(Smart) Double Nested DFS: example



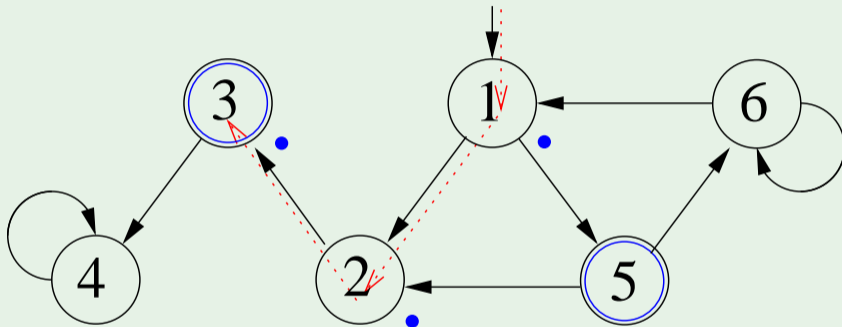
T1 12

S1 12

T2

S2

(Smart) Double Nested DFS: example



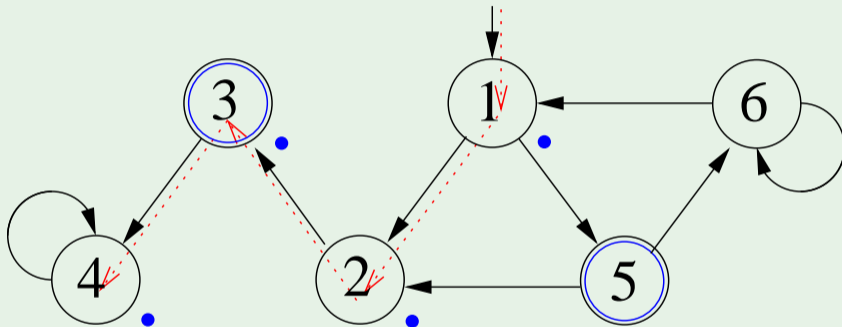
T1 1 2 3

S1 1 2 3

T2

S2

(Smart) Double Nested DFS: example



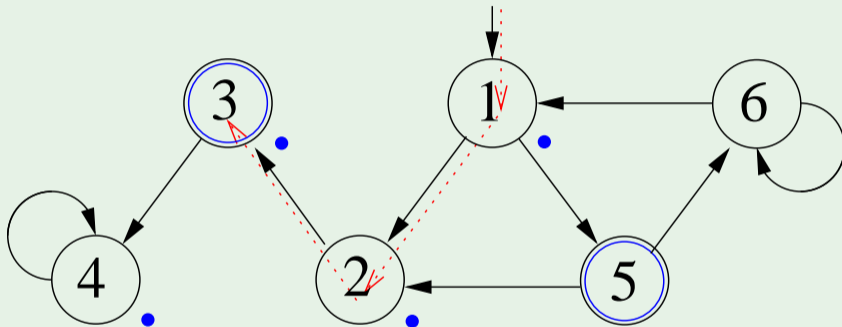
T1 1 2 3 4

S1 1 2 3 4

T2

S2

(Smart) Double Nested DFS: example



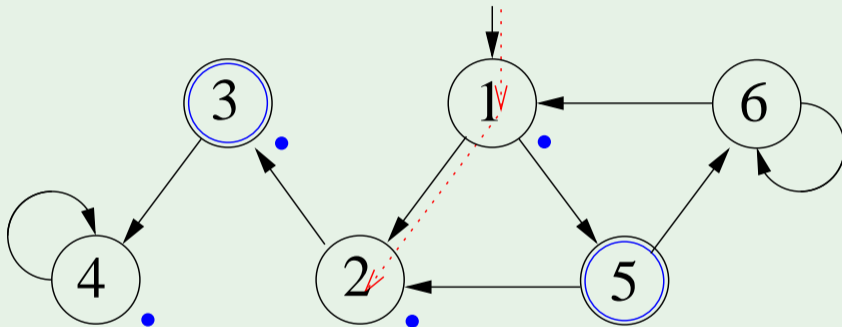
T1 1 2 3 4

S1 1 2 3

T2

S2

(Smart) Double Nested DFS: example



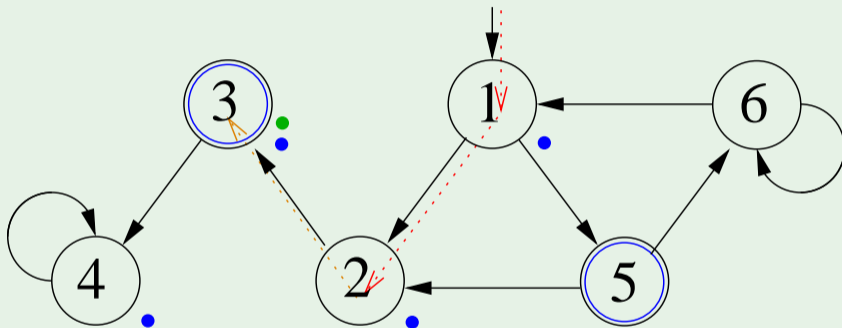
T1 1 2 3 4

S1 1 2

T2

S2

(Smart) Double Nested DFS: example



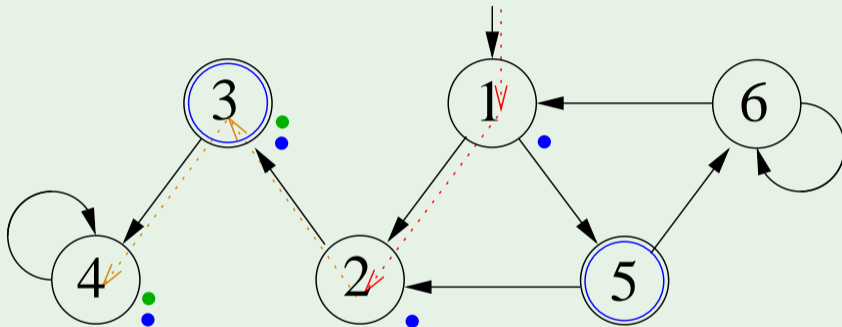
T1 1 2 3 4

T2 3

S1 1 2

S2 3

(Smart) Double Nested DFS: example



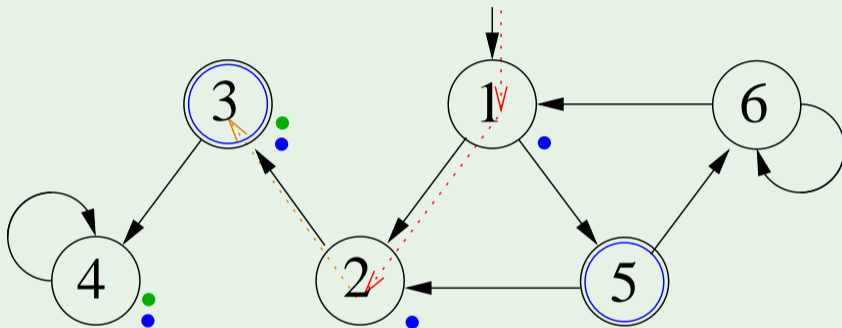
T1 1 2 3 4

S1 1 2

T2 3 4

S2 3 4

(Smart) Double Nested DFS: example



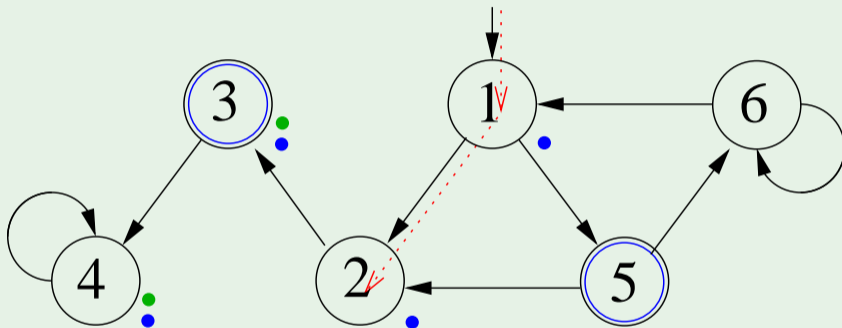
T1 1 2 3 4

S1 1 2

T2 3 4

S2 3

(Smart) Double Nested DFS: example



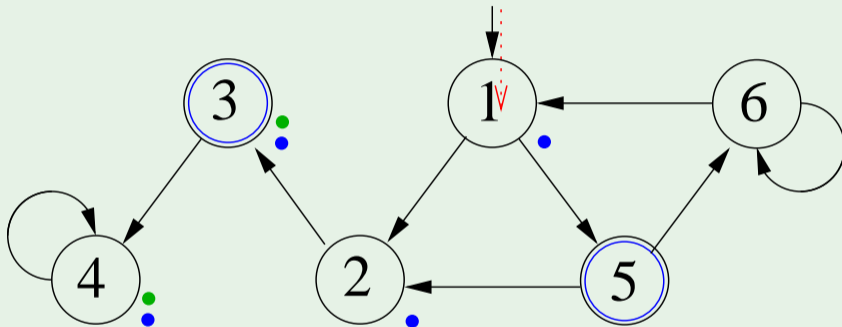
T1 1 2 3 4

S1 1 2

T2 3 4

S2

(Smart) Double Nested DFS: example



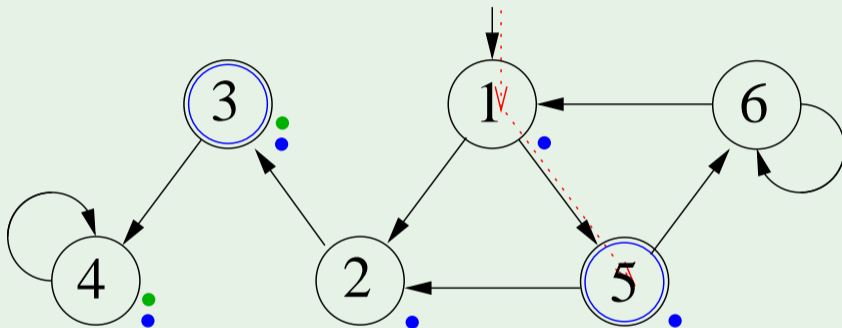
T1 1 2 3 4

T2 3 4

S1 1

S2

(Smart) Double Nested DFS: example



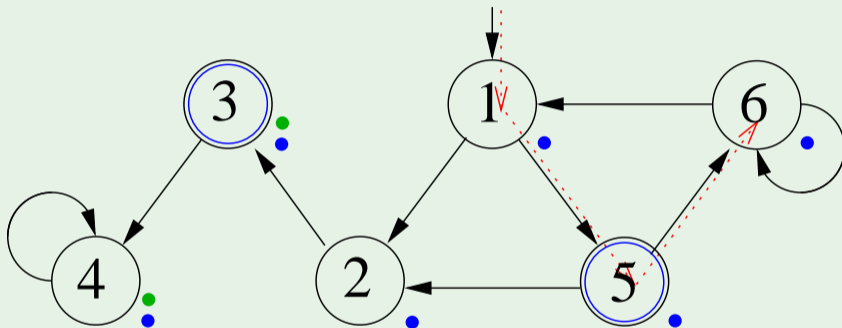
T1 12345

S1 15

T2 34

S2

(Smart) Double Nested DFS: example



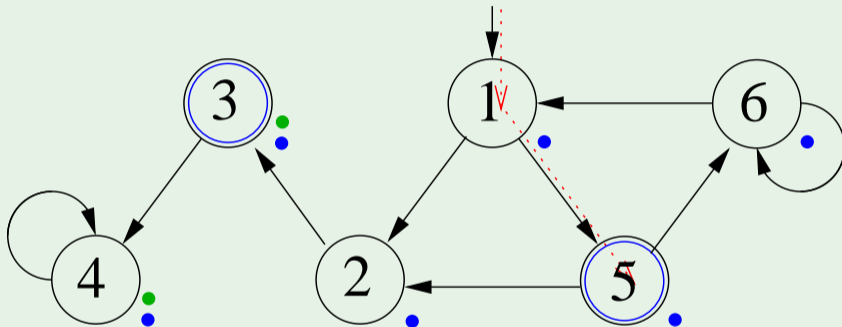
T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

(Smart) Double Nested DFS: example



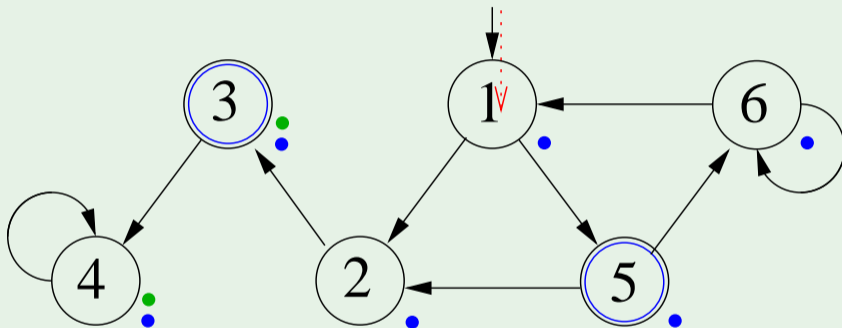
T1 1 2 3 4 5 6

S1 1 5

T2 3 4

S2

(Smart) Double Nested DFS: example



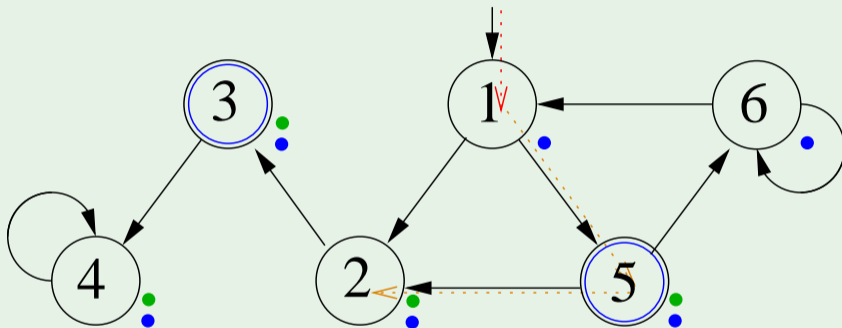
T1 1 2 3 4 5 6

T2 3 4

S1 1

S2

(Smart) Double Nested DFS: example



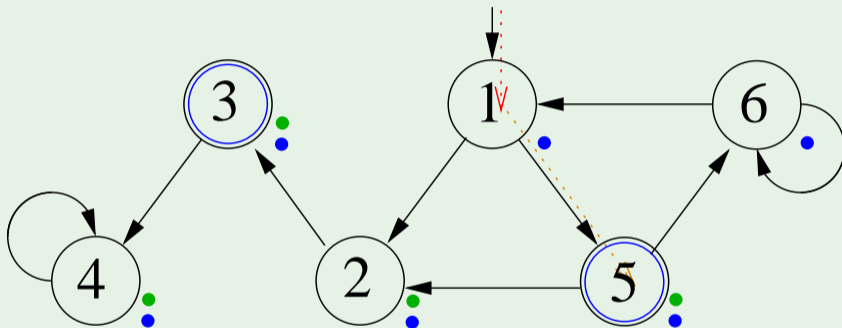
T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2

S2 5 2

(Smart) Double Nested DFS: example



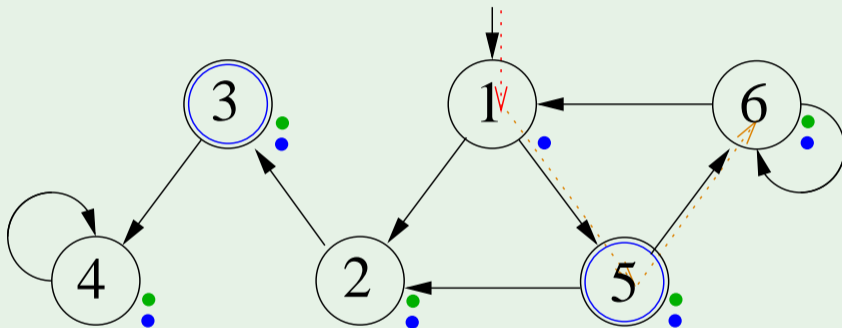
T1 1 2 3 4 5 6

T2 3 4 5 2

S1 1

S2 5

(Smart) Double Nested DFS: example



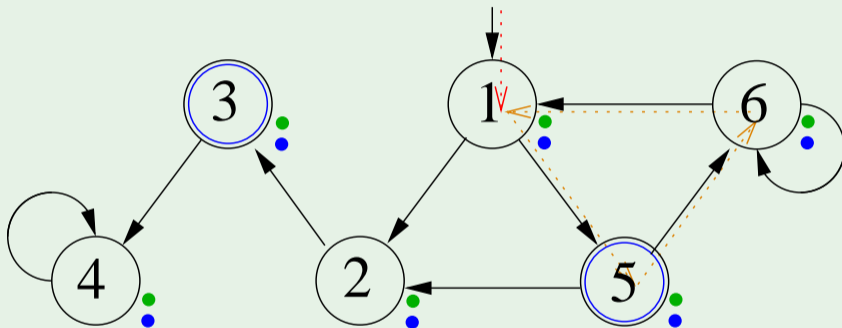
T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2 6

S2 5 6

(Smart) Double Nested DFS: example



T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2 6 1

S2 5 6 1

- 1 Büchi Automata
- 2 **The Automata-Theoretic Approach to LTL Reasoning**
 - General Ideas
 - Language-Emptiness Checking of Büchi Automata
 - **From Kripke Models to Büchi Automata**
 - From LTL Formulas to Büchi Automata
 - Complexity
- 3 Exercises

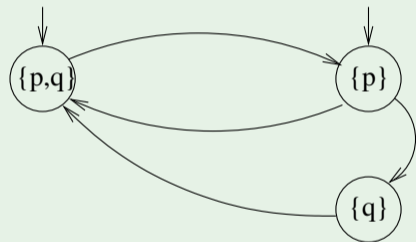
Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke model $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
 - States: $Q := S \cup \{init\}$, $init$ being a new initial state
 - Alphabet: $\Sigma := 2^{AP}$ (total truth-assignments as alphabet symbols!)
 - Initial State: $I := \{init\}$
 - Accepting States: $F := Q = S \cup \{init\}$
 - Transitions:

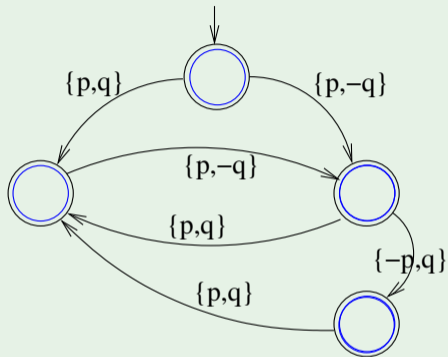
$$\delta : \quad q \xrightarrow{a} q' \text{ iff } (q, q') \in R \text{ and } L(q') = a \\ \quad \quad \quad \text{init} \xrightarrow{a} q \text{ iff } q \in S_0 \text{ and } L(q) = a$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a NBA A_M from a Kripke Structure M : Example



Kripke Structure



Buechi Automaton

\implies Substantially:

1. add one initial state,
2. move labels from states to incoming edges,
3. set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



- in a Kripke Structure, it means that p is true and all other propositions are false;
- in a Büchi Automaton, it means that p is true and all other propositions are irrelevant (“don’t care”), i.e. they can be either true or false.

- 1 Büchi Automata
- 2 The Automata-Theoretic Approach to LTL Reasoning**
 - General Ideas
 - Language-Emptiness Checking of Büchi Automata
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 - From LTL Formulas to Büchi Automata**
 - Complexity
- 3 Exercises

Translation problem

Problem

Given an LTL formula ϕ , find a Büchi Automaton that accepts the same language of ϕ .

- It is a fundamental problem in LTL validity/satisfiability/entailment and model checking
- We translate an LTL formula into a Generalized Büchi Automata (GBA), then into an NBA

LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators \wedge , \vee , **X**, **U**, **R** on propositional literals.

- Done by pushing negations down to literal level:

$$\begin{aligned}\neg\neg\varphi_1 &\implies \varphi_1 \\ \neg(\varphi_1 \vee \varphi_2) &\implies (\neg\varphi_1 \wedge \neg\varphi_2) \\ \neg(\varphi_1 \wedge \varphi_2) &\implies (\neg\varphi_1 \vee \neg\varphi_2) \\ \neg\mathbf{X}\varphi_1 &\implies \mathbf{X}\neg\varphi_1 \\ \neg(\varphi_1 \mathbf{U}\varphi_2) &\implies (\neg\varphi_1 \mathbf{R}\neg\varphi_2) \\ \neg(\varphi_1 \mathbf{R}\varphi_2) &\implies (\neg\varphi_1 \mathbf{U}\neg\varphi_2)\end{aligned}$$

\implies The resulting formula is expressed in terms of \vee , \wedge , **X**, **U**, **R** and literals (Negative Normal Form, NNF).

- the encoding is linear if a DAG representation is used
- In the construction of A_φ we now assume that φ is in NNF.
 \implies every non-atomic subformula occurs positively in φ
- For convenience, we still use **F**'s and **G**'s as shortcuts: **F** φ for $\top\mathbf{U}\varphi$ and **G** φ for $\perp\mathbf{R}\varphi$

On-the-fly Construction of A_φ (Intuition)

(Implicitly) Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ :

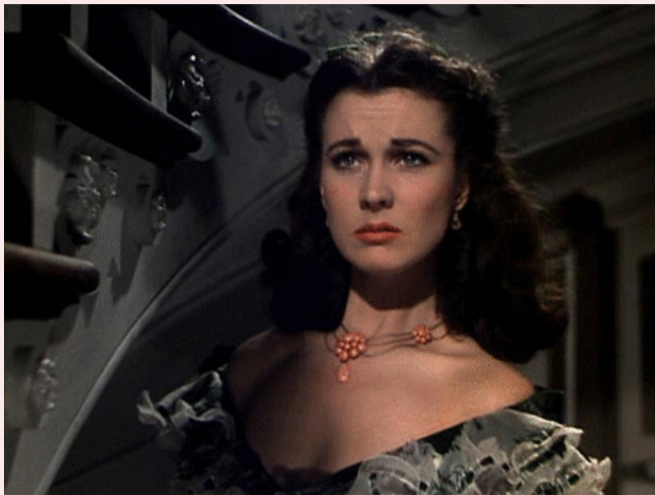
$\psi_1 \mathbf{U} \psi_2 \implies \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$ [and $\mathbf{F}\psi \implies \psi \vee \mathbf{X}\mathbf{F}\psi$]

$\psi_1 \mathbf{R} \psi_2 \implies \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$ [and $\mathbf{G}\psi \implies \psi \wedge \mathbf{X}\mathbf{G}\psi$]

until we get a Boolean combination of **elementary subformulas** of φ

(An elementary formula is a proposition or a \mathbf{X} -formula.)

Tableaux Rules: a Quote



*"After all... tomorrow is another day."
[Scarlett O'Hara, "Gone with the Wind"]*

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into **Disjunctive Normal Form**, by:

- (i) applying recursively the **DeMorgan rule**: $\varphi_1 \wedge (\varphi_2 \vee \varphi_3) \implies (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$, and then
- (ii) pushing the conjunctions inside the next operator:

$$\varphi \xrightarrow{(i)} \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}) \xrightarrow{(ii)} \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}).$$

- Each disjunct $\left(\overbrace{\bigwedge_j l_{ij}}^{\text{labels}} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\text{next part}} \right)$ represents a state:
 - the conjunction of literals $\bigwedge_j l_{ij}$ represents a **set of labels** in Σ
(e.g., if $\text{Vars}(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
 - $\mathbf{X} \bigwedge_k \psi_{ik}$ represents the **next part** of the state
(obligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 3: For every state S_i represented by $(\bigwedge_j l_{ij} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\varphi_i})$

- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
- mark that the state S_i satisfies φ
- apply recursively steps 1-2-3 to $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$,
 - rewrite φ_i into $\bigvee_{i'} (\bigwedge_j l'_{ij} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$
 - from each disjunct $(\bigwedge_j l'_{ij} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$ generate a new state $S_{i'}$ (if not already present) and label it as satisfying $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$
- draw an edge from S_i to all states $S_{i'}$ which satisfy $\bigwedge_k \psi_{ik}$
- (if no next part occurs, $\mathbf{X}\top$ is implicitly assumed, so that an edge to a “true” node is drawn)

On-the-fly Construction of A_φ (Intuition) [cont.]

φ ??



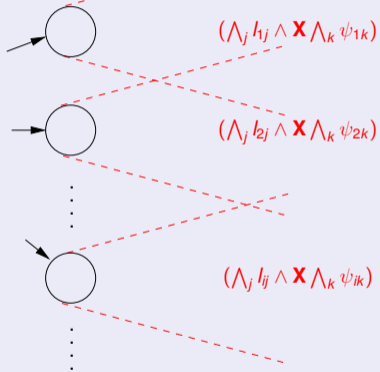
On-the-fly Construction of A_φ (Intuition) [cont.]

$$\forall_i (\bigwedge_j l_{ij} \wedge \mathbf{x} \bigwedge_k \psi_{ik}) !$$



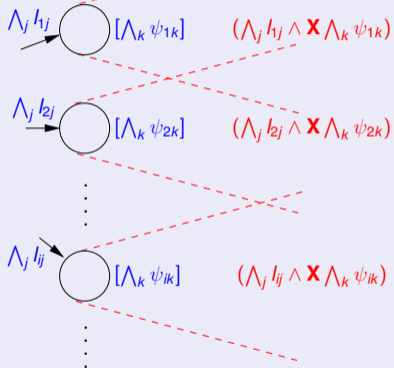
On-the-fly Construction of A_φ (Intuition) [cont.]

$$\forall_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}) !$$

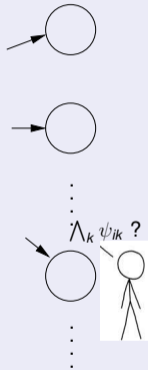


On-the-fly Construction of A_φ (Intuition) [cont.]

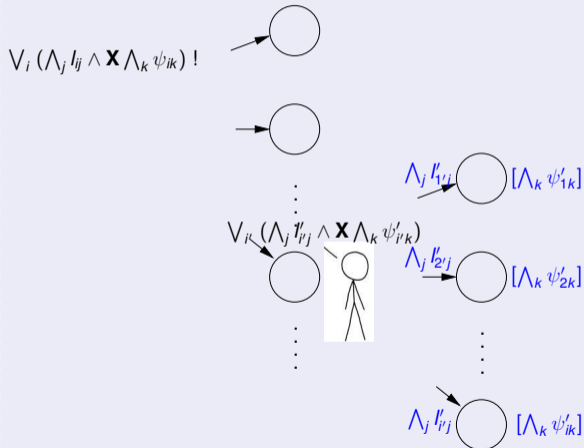
$\forall_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}) !$



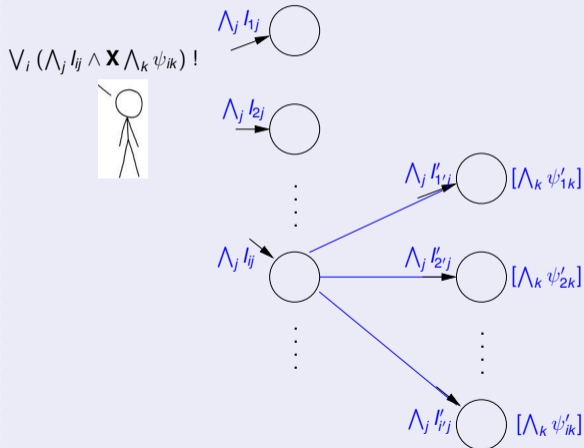
On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

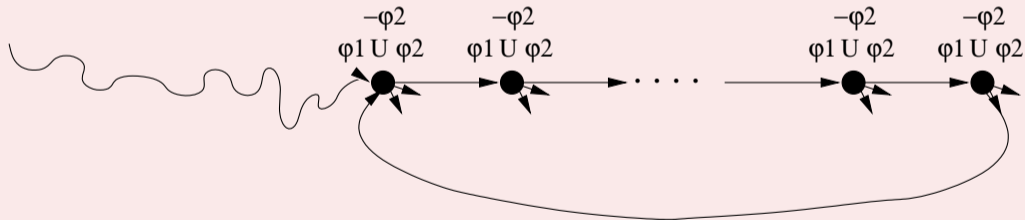
Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j , mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ or $\varphi_i \in q_j$
(If there is no \mathbf{U} -subformulas, then mark all states with F_1 —i.e., $FT \stackrel{\text{def}}{=} \{Q\}$).

Remark

The fact that we initially converted the formula into NNF guarantees that only original positive \mathbf{U}/\mathbf{F} -subformulas and negative \mathbf{R}/\mathbf{G} -subformulas are considered in step 4

Dealing with U-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X} \varphi_1 \mathbf{U} \varphi_2))$
are a **property**, not a **definition** of **U**:
 \implies they implicitly admit a “weaker” semantics of $\varphi_1 \mathbf{U} \varphi_2$, in which $\varphi_1 \mathbf{U} \varphi_2$ always holds and φ_2 never holds
- It cannot happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.



\implies every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2$ holds

- In LTL: $\neg \mathbf{FG}((\varphi_1 \mathbf{U} \varphi_2) \wedge \neg \varphi_2)$, i.e., $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ (“avoid bad loops”)

On-the-fly Construction of A_φ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
 - λ is the set of labels
 - χ is the next part, i.e. the set of X -formulas satisfied by s
 - σ is the set of the subformulas of φ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, \dots, \psi_k\}$, we define $\text{Cover}(\Psi) \stackrel{\text{def}}{=} \text{Expand}(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
 - $\text{Expand}(\Psi, s)$ takes as input:
 - a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, \dots, \psi_k\}$ to be expanded
 - a state $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$ under constructionand returns a set of states $\{\langle \lambda_i, \chi_i, \sigma_i \rangle\}_i$ representing the expansion of Ψ
 - Combines steps 1. and 2. of previous slides

On-the-fly Construction of A_φ - Expand

Given $\Psi \stackrel{\text{def}}{=} \{\psi_1, \dots, \psi_k\}$ and $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$, we define *Expand*(Ψ, s) recursively as follows:

- if $\Psi = \emptyset$, *Expand*(Ψ, s) = $\{s\}$
- if $\perp \in \Psi$, *Expand*(Ψ, s) = \emptyset
- if $\top \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
Expand(Ψ, s) = *Expand*($\Psi \setminus \{\top\}, \langle \lambda, \chi, \sigma \cup \{\top\} \rangle$)
- if $I \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$, I propositional literal
Expand(Ψ, s) = *Expand*($\Psi \setminus \{I\}, \langle \lambda \cup \{I\}, \chi, \sigma \cup \{I\} \rangle$)
(add I to the labels of s and to set of satisfied formulas)
- if $\mathbf{X}\psi \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
Expand(Ψ, s) = *Expand*($\Psi \setminus \{\mathbf{X}\psi\}, \langle \lambda, \chi \cup \{\psi\}, \sigma \cup \{\mathbf{X}\psi\} \rangle$)
(add ψ to the next part of s and $\mathbf{X}\psi$ to set of satisfied formulas)
- if $\psi_1 \wedge \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
Expand(Ψ, s) = *Expand*($\Psi \cup \{\psi_1, \psi_2\} \setminus \{\psi_1 \wedge \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \wedge \psi_2\} \rangle$)
(process both ψ_1 and ψ_2 and add $\psi_1 \wedge \psi_2$ to σ)
- ...

On-the-fly Construction of A_φ - Expand

Given $\Psi \stackrel{\text{def}}{=} \{\psi_1, \dots, \psi_k\}$ and $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$, we define $\text{Expand}(\Psi, s)$ recursively as follows:

- ...
- if $\psi_1 \vee \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
$$\text{Expand}(\Psi, s) = \text{Expand}(\Psi \cup \{\psi_1\} \setminus \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle)$$
$$\cup \text{Expand}(\Psi \cup \{\psi_2\} \setminus \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle)$$

(split s into two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
- if $\psi_1 \mathbf{U} \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
$$\text{Expand}(\Psi, s) = \text{Expand}(\Psi \cup \{\psi_1\} \setminus \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{U} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle)$$
$$\cup \text{Expand}(\Psi \cup \{\psi_2\} \setminus \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle)$$

(split s into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{U} \psi_2$ to σ)
- if $\psi_1 \mathbf{R} \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
$$\text{Expand}(\Psi, s) = \text{Expand}(\Psi \cup \{\psi_2\} \setminus \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{R} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle)$$
$$\cup \text{Expand}(\Psi \cup \{\psi_1, \psi_2\} \setminus \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle)$$

(split s into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{R} \psi_2$ to σ)

On-the-fly Construction of A_φ - Expand

Two relevant subcases: $\mathbf{F}\psi \stackrel{\text{def}}{=} \top \mathbf{U}\psi$ and $\mathbf{G}\psi \stackrel{\text{def}}{=} \perp \mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,

$$\begin{aligned} \text{Expand}(\Psi, s) = & \text{Expand}(\Psi \setminus \{\mathbf{F}\psi\}, \langle \lambda, \chi \cup \{\mathbf{F}\psi\}, \sigma \cup \{\mathbf{F}\psi\} \rangle) \\ & \cup \text{Expand}(\Psi \cup \{\psi\} \setminus \{\mathbf{F}\psi\}, \langle \lambda, \chi, \sigma \cup \{\mathbf{F}\psi\} \rangle) \end{aligned}$$

- if $\mathbf{G}\psi \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,

$$\text{Expand}(\Psi, s) = \text{Expand}(\Psi \cup \{\psi\} \setminus \{\mathbf{G}\psi\}, \langle \lambda, \chi \cup \{\mathbf{G}\psi\}, \sigma \cup \{\mathbf{G}\psi\} \rangle)$$

(Note: $\text{Expand}(\Psi \cup \{\perp, \psi\} \setminus \{\mathbf{G}\psi\}, \dots) = \emptyset$.)

Definition of A_φ

Given a set of LTL formulas Ψ , we define $Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.

For an LTL formula φ , we construct a Generalized NBA $A_\varphi = (Q, \Sigma, \delta, I, FT)$ as follows:

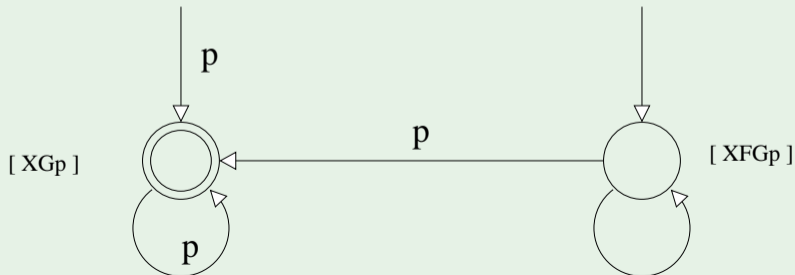
- $\Sigma = 3^{vars(\varphi)}$ ($v \in \{T, \perp, *\}$, “*” is “don’t care”)
- Q is the smallest set such that
 - $Cover(\{\varphi\}) \subseteq Q$
 - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
- $Q_0 = Cover(\{\varphi\})$.
- $s \xrightarrow{\lambda'} s' \in \delta$ iff, $s = \langle \lambda, \chi, \sigma \rangle$, $s' = \langle \lambda', \chi', \sigma' \rangle$ and $s' \in Cover(\chi)$
- $FT = \langle F_1, F_2, \dots, F_k \rangle$ where, for all $(\psi_i \mathbf{U} \varphi_i)$ occurring positively in φ ,
 $F_i = \{ \langle \lambda, \chi, \sigma \rangle \in Q \mid (\psi_i \mathbf{U} \varphi_i) \notin \sigma \text{ or } \varphi_i \in \sigma \}$.
(If there is no \mathbf{U} -subformulas, then $FT \stackrel{\text{def}}{=} \{Q\}$).

Example: $\varphi = \mathbf{FG}p$

- $Cover(\{\mathbf{FG}p\})$
= $Expand(\{\mathbf{FG}p\}, \langle \emptyset, \emptyset, \emptyset \rangle)$
= $Expand(\emptyset, \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle) \cup Expand(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \{\mathbf{FG}p\} \rangle)$
= $\{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup Expand(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\} \rangle)$
= $\{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup Expand(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle)$
= $\{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle\}$
- $Cover(\{\mathbf{G}p\})$ = $Expand(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \emptyset \rangle)$
= $Expand(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\} \rangle)$
= $Expand(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle)$
= $\{\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle\}$
- Optimization:
merge $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$ and $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle$

Example: $\varphi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
- $Q_0 = \{s_1, s_2\}$.
- T : $s_1 \rightarrow \{s_1, s_2\}$,
 $s_2 \rightarrow \{s_2\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2\}$.

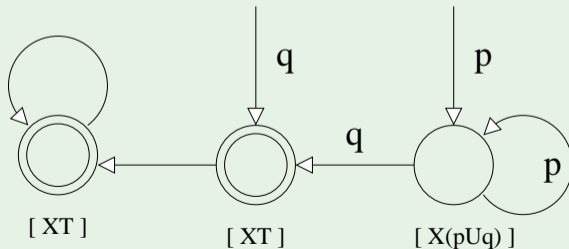


Example: $\varphi = p \mathbf{U} q$

- $Cover(\{p \mathbf{U} q\})$
= $Expand(\{p \mathbf{U} q\}, \langle \emptyset, \emptyset, \emptyset \rangle)$
= $Expand(\{p\}, \langle \emptyset, \{p \mathbf{U} q\}, \{p \mathbf{U} q\} \rangle) \cup Expand(\{q\}, \langle \emptyset, \emptyset, \{p \mathbf{U} q\} \rangle)$
= $Expand(\emptyset, \langle \{p\}, \{p \mathbf{U} q\}, \{p \mathbf{U} q, p\} \rangle) \cup Expand(\emptyset, \langle \{q\}, \emptyset, \{p \mathbf{U} q, q\} \rangle)$
= $\{\langle \{p\}, \{p \mathbf{U} q\}, \{p \mathbf{U} q, p\} \rangle\} \cup \{\langle \{q\}, \{T\}, \{p \mathbf{U} q, q\} \rangle\}$
- $Cover(\{T\}) = \{\langle \emptyset, \{T\}, \{T\} \rangle\}$

Example: $\varphi = pUq$

- Let $s_1 =_{def} \langle \{p\}, \{pUq\}, \{pUq, p\} \rangle$, $s_2 =_{def} \langle \{q\}, \{T\}, \{pUq, q\} \rangle$, $s_3 =_{def} \langle \emptyset, \{T\}, \{T\} \rangle$.
- $Q = \{s_1, s_2, s_3\}$,
- $Q_0 = \{s_1, s_2\}$,
- T :
 - $s_1 \rightarrow \{s_1, s_2\}$,
 - $s_2 \rightarrow \{s_3\}$
 - $s_3 \rightarrow \{s_3\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2, s_3\}$.



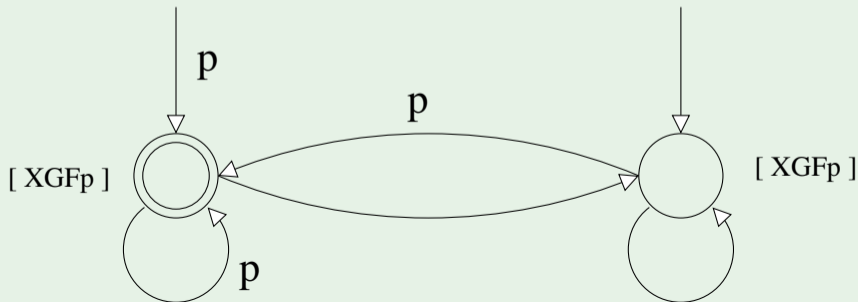
Example: $\varphi = \mathbf{GF}p$

$$\begin{aligned} \text{Cover}(\{\mathbf{GF}p\}) &= \text{Expand}(\{\mathbf{GF}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\ &= \text{Expand}(\{\mathbf{F}p\}, \langle \emptyset, \{\mathbf{GF}p\}, \{\mathbf{GF}p\} \rangle) \\ &= \text{Expand}(\{\}, \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \cup \text{Expand}(\{p\}, \langle \{\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \\ &= \text{Expand}(\{\}, \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle) \cup \text{Expand}(\{\}, \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle) \\ &= \{ \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle \} \cup \{ \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle \} \end{aligned}$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \iff \mathbf{GF}p$, s.t. $\text{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \text{Cover}(\mathbf{GF}p)$

Example: $\mathbf{GF}p$

- Let $s_1 =_{\text{def}} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$, $s_2 =_{\text{def}} \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle$,
- $Q = \{s_1, s_2\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T : \begin{array}{l} s_1 \rightarrow \{s_1, s_2\}, \\ s_2 \rightarrow \{s_1, s_2\} \end{array}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_1\}$.



NBAs of disjunctions of formulas

Remark

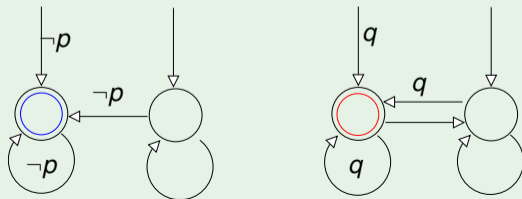
If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_{\varphi} \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_{φ} non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \stackrel{\text{def}}{=} (\mathbf{GF}p \rightarrow \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.

Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ :



Suggested Exercises:

- Find an NBA encoding:
 - p
 - $(p \wedge q) \vee (\neg p \wedge \neg q)$
 - $\mathbf{F}p$
 - $\mathbf{G}p$
 - $p\mathbf{R}q$
 - $(\mathbf{G}p \wedge \mathbf{G}q) \rightarrow \mathbf{G}r$

- 1 Büchi Automata
- 2 The Automata-Theoretic Approach to LTL Reasoning**
 - General Ideas
 - Language-Emptiness Checking of Büchi Automata
 - From Kripke Models to Büchi Automata
 - From LTL Formulas to Büchi Automata
 - Complexity**
- 3 Exercises

Automata-Theoretic LTL Model Checking: Complexity

Four steps:

- (i) Compute A_M :
 $|A_M| = O(|M|)$
 - (ii) Compute A_φ :
 $|A_\varphi| = O(2^{|\varphi|})$
 - (iii) Compute the product $A_M \times A_\varphi$:
 $|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$
 - (iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$:
 $O(|A_M \times A_\varphi|) = O(|M| \cdot 2^{|\varphi|})$
- \implies The complexity of LTL M.C. grows linearly wrt. the size of the model M and exponentially wrt. the size of the property φ

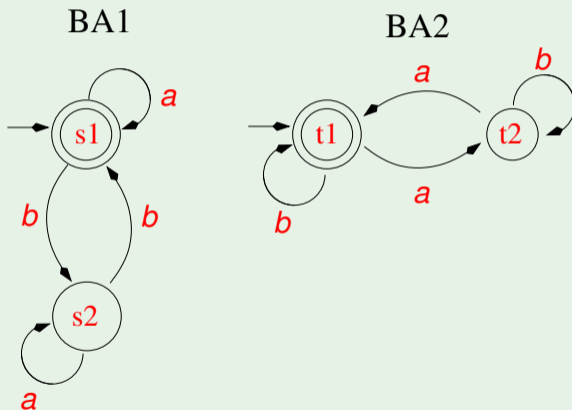
Final Remarks

- Büchi automata are in general more expressive than LTL!
- ⇒ some tools (e.g., Spin) allow specifications to be expressed directly as NBAs
- ⇒ complementation of NBA relevant in general
 - For every LTL formula, there are many possible equivalent NBAs
- ⇒ lots of research for finding “the best” conversion algorithm
 - Performing the product and checking emptiness very relevant
- ⇒ lots of techniques developed (e.g., partial order reduction)
- ⇒ lots on ongoing research

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Ex: Product of Büchi automata

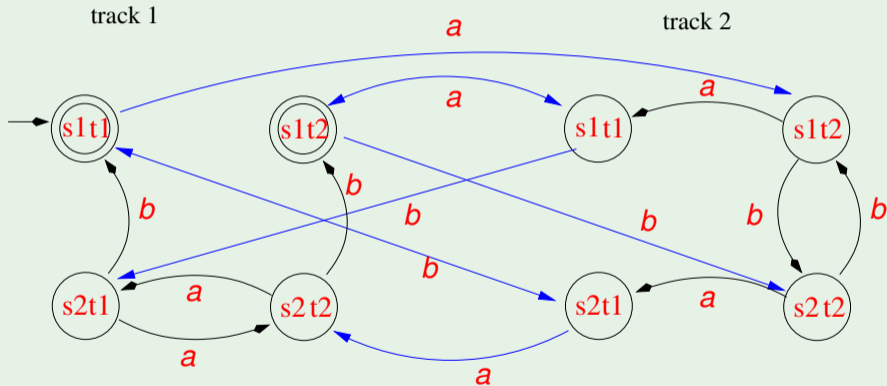
Given the following two Büchi automata (doubly-circled states represent accepting states, a , b are labels):



Write the product Büchi automaton $BA1 \times BA2$.

Ex: Product of Büchi automata

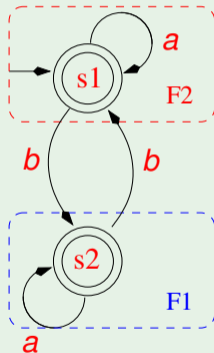
[Solution: The product is:



]

Ex: De-generalization of Büchi Automata

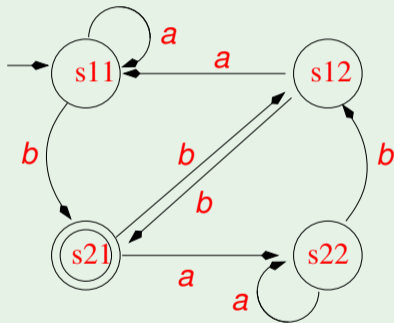
Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$
s.t. $F1 \stackrel{\text{def}}{=} \{s2\}$, $F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

Ex: De-generalization of Büchi Automata

[Solution: The result is:



]

Ex: Construction of Büchi Automata

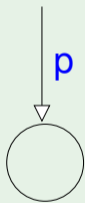
Consider the LTL formula $\varphi \stackrel{\text{def}}{=} (\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q)$.

(a) rewrite φ into Negative Normal Form

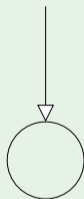
[Solution: $(\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q) \implies (\neg\mathbf{G}\neg p) \vee (p\mathbf{U}q) \implies (\mathbf{F}p) \vee (p\mathbf{U}q)$]

(b) find the initial states of a corresponding Buchi automaton (for each state, define the labels of the incoming arcs and the “next” section.)

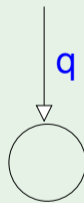
[Solution: Applying tableaux rules we obtain: $p \vee \mathbf{X}\mathbf{F}p \vee q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$, which is already in disjunctive normal form. This correspond to the following four initial states:



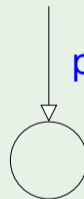
[\top]



[$\mathbf{F}p$]



[\top]

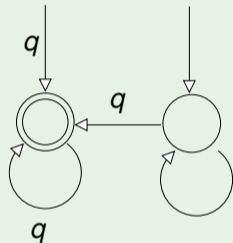


[$p\mathbf{U}q$]

]

Ex: Büchi automaton

Given the following Büchi automaton BA (doubly-circled states represent accepting states):



Say which of the following sentences are true and which are false.

- (a) BA accepts all and only the paths verifying $\mathbf{GF}q$. [Solution: false]
- (b) BA accepts all and only the paths verifying $\mathbf{FG}q$. [Solution: true]
- (c) BA accepts only paths verifying $\mathbf{F}q$, but not all of them. [Solution: true]
- (d) BA accepts all the paths verifying $\mathbf{F}q$, but not only them. [Solution: false]