

# Formal Methods

## Module I: Automated Reasoning

### Ch. 03: **Temporal Logics**

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# Outline

- 1 Transition Systems as Kripke Models
  - Kripke Models
  - Languages for Transition Systems (hints)
- 2 Properties and Temporal Logics
  - Properties
  - Temporal Logics
- 3 Linear Temporal Logic – LTL
  - LTL: Syntax and Semantics
  - Some LTL Model Checking Examples
- 4 Computation Tree Logic – CTL
  - CTL: Syntax and Semantics
  - Some CTL Model Checking Examples
- 5 LTL vs. CTL
- 6 Exercises

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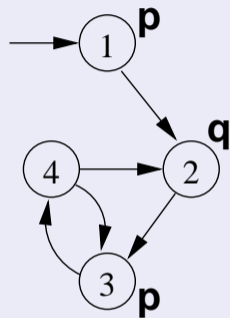
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# Kripke Models

- **Theoretical role:** the semantic framework for a variety of logics
  - Modal Logics
  - Description Logics
  - **Temporal Logics**
  - ...
- **Practical role:** used to describe **reactive systems**:
  - nonterminating systems with **infinite** behaviors (e.g. communication protocols, hardware circuits);
  - represent the **dynamic evolution** of modeled systems;
  - a state includes values to state variables, program counters, content of communication channels.
  - **can be animated and validated before their actual implementation**

# Kripke Model: Formal Definition

- A Kripke model  $\langle S, I, R, AP, L \rangle$  consists of
  - a **finite** set of states  $S$ ;
  - a set of **initial states**  $I \subseteq S$ ;
  - a set of **transitions**  $R \subseteq S \times S$ ;
  - a set of **atomic propositions**  $AP$ ;
  - a **labeling function**  $L : S \mapsto 2^{AP}$ .
- We assume  $R$  **total**: for every state  $s$ , there exists (at least) one state  $s'$  s.t.  $(s, s') \in R$
- Sometimes we use variables with discrete bounded values  $v_i \in \{d_1, \dots, d_k\}$  (can be encoded with  $\lceil \log(k) \rceil$  Boolean variables)

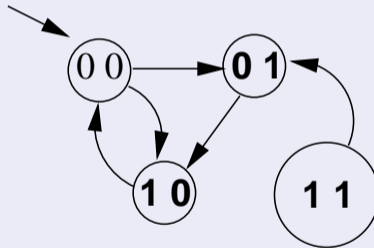
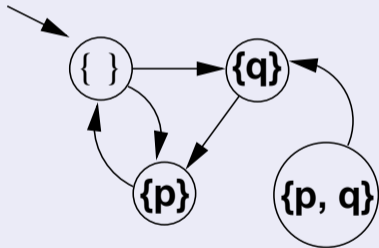


## Remark

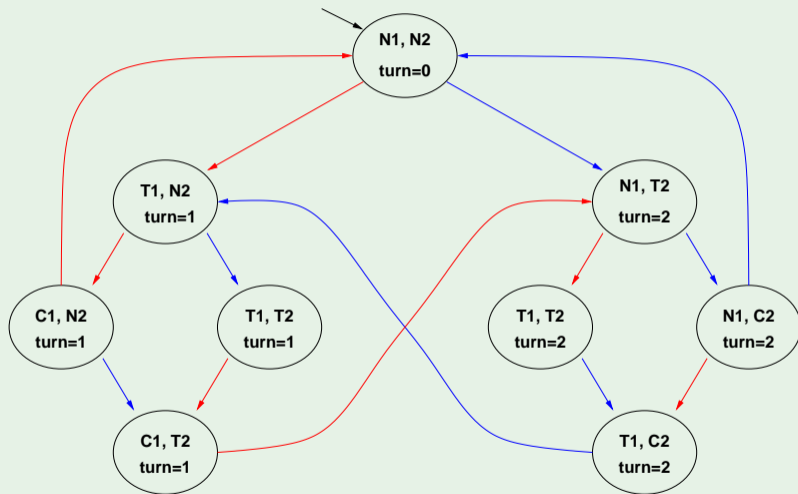
Unlike with other types of Automata (e.g., Buechi), in Kripke models **the values of all variables are always assigned in each state.**

## Kripke Structures: Two Alternative Representations:

- each state identifies univocally the values of the atomic propositions which hold there
- each state is labeled by a bit vector



## Example: a Kripke model for mutual exclusion



N = noncritical, T = trying, C = critical

User 1 User 2

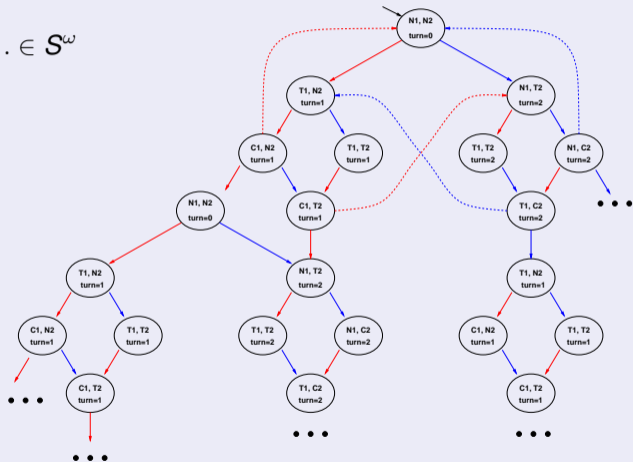


# Path in a Kripke Model

A **path** in a Kripke model  $M$  is an infinite sequence of states

$$\pi = s_0, s_1, s_2, \dots \in S^\omega$$

such that  $s_0 \in I$  and  $(s_i, s_{i+1}) \in R$ .



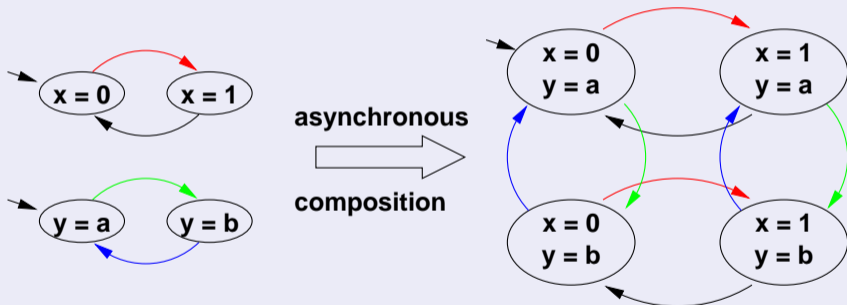
A state  $s$  is **reachable in  $M$**  if there is a path from the initial states to  $s$ .

# Composing Kripke Models

- Complex Kripke Models are typically obtained by composition of smaller ones
- Components can be combined via
  - **asynchronous** composition.
  - **synchronous** composition,

# Asynchronous Composition

- Interleaving of evolution of components.
- At each time instant, one component is selected to perform a transition.



- Typical example: communication protocols.

# Asynchronous Composition/Product: formal definition

## Asynchronous product of Kripke models

Let  $M_1 \stackrel{\text{def}}{=} \langle S_1, I_1, R_1, AP_1, L_1 \rangle$ ,  $M_2 \stackrel{\text{def}}{=} \langle S_2, I_2, R_2, AP_2, L_2 \rangle$ . Then the **asynchronous product**  $M \stackrel{\text{def}}{=} M_1 || M_2$  is  $M \stackrel{\text{def}}{=} \langle S, I, R, AP, L \rangle$ , where

- $S \subseteq S_1 \times S_2$  s.t.,  $\forall \langle s_1, s_2 \rangle \in S, \forall I \in AP_1 \cap AP_2, I \in L_1(s_1)$  iff  $I \in L_2(s_2)$
- $I \subseteq I_1 \times I_2$  s.t.  $I \subseteq S$
- $R(\langle s_1, s_2 \rangle, \langle t_1, t_2 \rangle)$  iff  $(R_1(s_1, t_1)$  **and**  $s_2 = t_2)$  **or**  $(s_1 = t_1$  **and**  $R_2(s_2, t_2))$
- $AP = AP_1 \cup AP_2$
- $L : S \mapsto 2^{AP}$  s.t.  $L(\langle s_1, s_2 \rangle) \stackrel{\text{def}}{=} L_1(s_1) \cup L_2(s_2)$ .

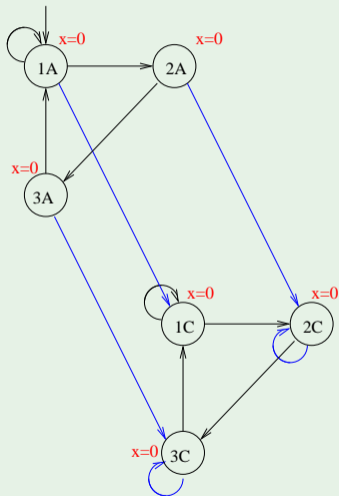
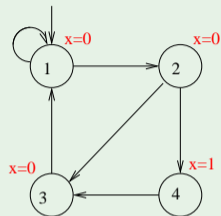
Note: combined states must agree on the values of Boolean variables.

Asynchronous composition is associative:

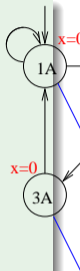
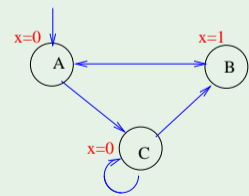
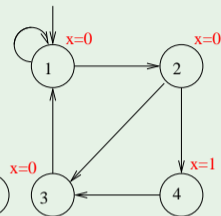
$$(\dots(M_1 || M_2) || \dots) || M_n = (M_1 || (M_2 || (\dots || M_n) \dots)) = M_1 || M_2 || \dots || M_n$$



# Asynchronous Composition: Example 2

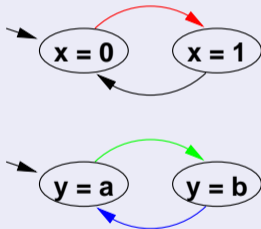


non-reachable state

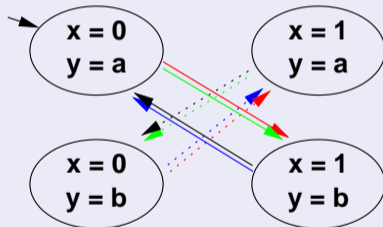


# Synchronous Composition

- Components evolve in parallel.
- At each time instant, every component performs a transition.



synchronous  
composition



- Typical example: sequential hardware circuits.

# Synchronous Composition/Product: formal definition

## Synchronous product of Kripke models

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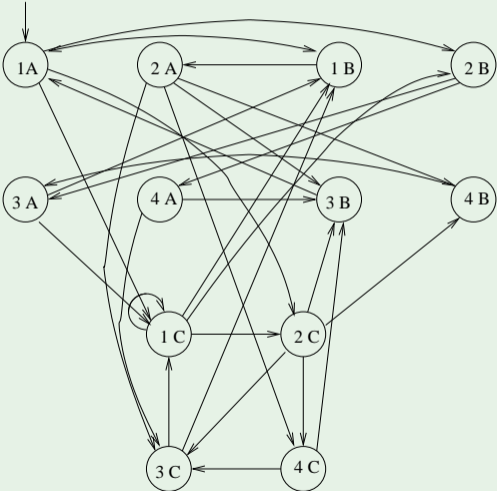
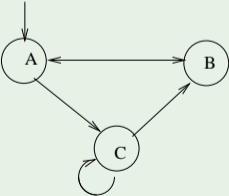
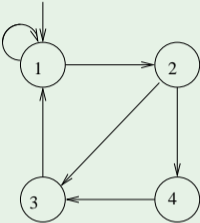
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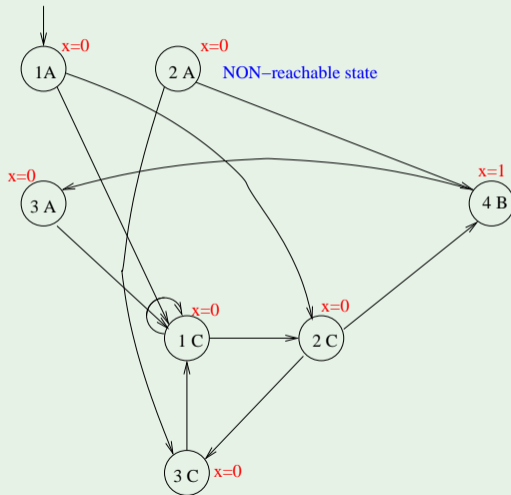
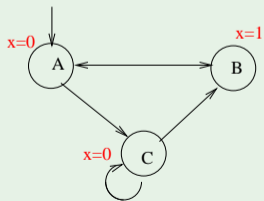
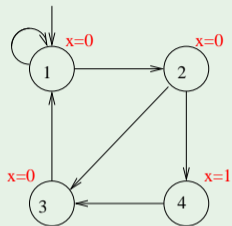
$$(\dots(M_1 \times M_2) \times \dots) \times M_n = (M_1 \times (M_2 \times (\dots \times M_n)\dots)) = M_1 \times M_2 \times \dots \times M_n$$



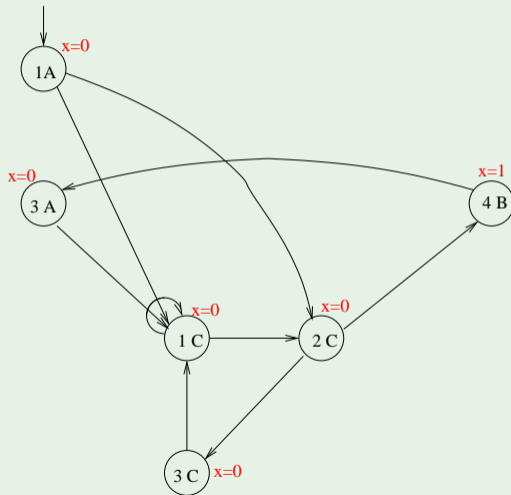
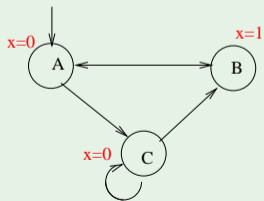
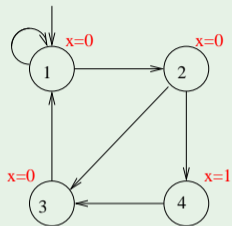
# Synchronous Composition: Example 1



# Synchronous Composition: Example 2



# Synchronous Composition: Example 2 (cont.)



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# Description languages for Kripke Model

- Most often a Kripke model is not given **explicitly** (states, arcs),...
- ... rather it is usually presented in a **structured language** (e.g., SMV, PROMELA, StateCharts, VHDL, ...)
  - even a piece of SW can be seen as a Kripke model!
- Each component is presented by specifying
  - **state variables**: determine the set of atomic propositions  $AP$ , the state space  $S$  and the labeling  $L$ .
  - **initial values of variables  $V$** : determine the set of **initial states  $I$** .
    - described as a relation  $I(V_0)$  in terms of state variables at step 0
  - **instructions**: determine the **transition relation  $R$** .
    - described as a relation  $R(V, V')$  in terms of **current state variables  $V$**  and **next state variables  $V'$**
- Aka as **symbolic representation of a Kripke model**

## Remark

Typically symbolic description are much more compact (and intuitive) than the explicit representation of the Kripke model.

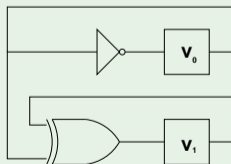
# The SMV language

- The input language of the SMV M.C. (and NuSMV)
- Booleans, enumerative and bounded integers as data types
- now enriched with other constructs, e.g. in NuXMV language
- An SMV program consists of:
  - Declarations of the state variables (e.g., `b0`);
  - Assignments that define the **initial states** (e.g., `init (b0) := 0`).
  - Assignments that define the **transition relation** (e.g., `next (b0) := !b0`).
- Allows for both synchronous and asynchronous composition of modules (though synchronous interaction more natural)

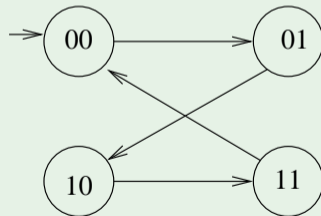
# Example: a Simple Counter Circuit

```
MODULE main
VAR
  v0      : boolean;
  v1      : boolean;
  out     : 0..3;

ASSIGN
  init(v0) := 0;
  next(v0) := !v0;
  init(v1) := 0;
  next(v1) := (v0 xor v1);
  out := toint(v0) + 2*toint(v1);
```



$v_1$	$v_0$	$v'_1$	$v'_0$
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



$$I(V) = (\neg v_0 \wedge \neg v_1)$$

$$R(V, V') = (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \oplus v_1)$$

# Standard Programming Languages

- Standard programming languages are typically sequential

⇒ Transition relation defined in terms also of the **program counter**

- Numbers & values Booleanized

```
...  
10. i = 0;  
11. acc = 0.0;  
12. while (i < dim) {  
13.   acc += V[i];  
14.   i++;  
15. }  
...
```

```
....  
(pc = 10) → ((i' = 0) ∧ (pc' = 11))  
(pc = 11) → ((acc' = 0.0) ∧ (pc' = 12))  
(pc = 12) → ((i < dim) → (pc' = 13))  
(pc = 12) → (¬(i < dim) → (pc' = 16))  
(pc = 13) → ((acc' = acc + read(V, i)) ∧ (pc' = 14))  
(pc = 14) → (i' = i + 1) ∧ (pc' = 15))  
(pc = 15) → (pc' = 12))  
...
```



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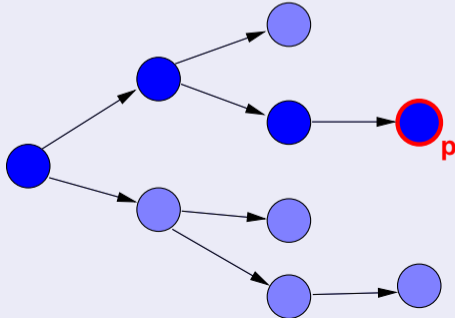
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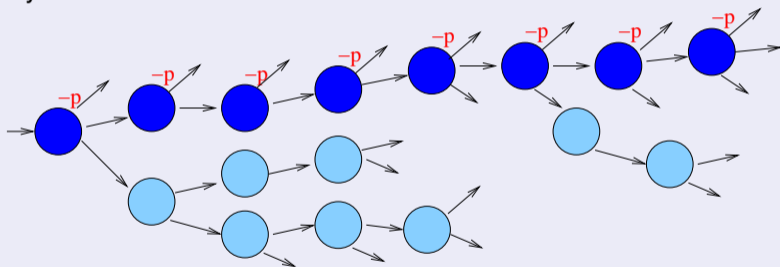
# Safety Properties

- Bad events never happen
  - deadlock: two processes waiting for input from each other, the system is unable to perform a transition.
  - no reachable state satisfies a “bad” condition, e.g. never two processes in critical section at the same time
- Can be refuted by a **finite** behaviour
- Ex.: it is never the case that  $p$ .



# Liveness Properties

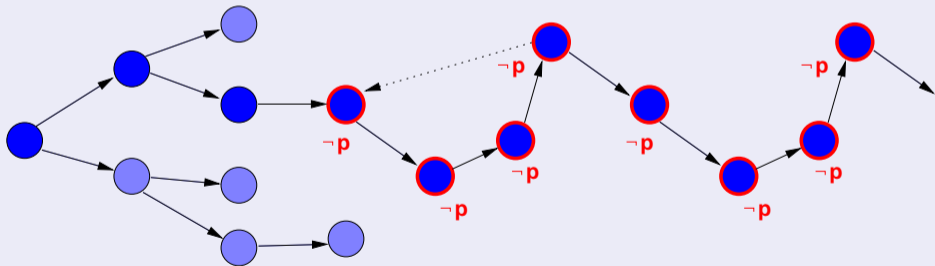
- Something desirable will eventually happen
  - sooner or later this will happen
- Can be refuted by **infinite** behaviour



- an infinite behaviour can be typically presented as a loop

# Fairness Properties

- Something desirable will happen **infinitely often**
  - important subcase of liveness
  - whenever a subroutine takes control, it will always return it (sooner or later)
- Can be refuted by infinite behaviour
  - a subroutine takes control and never returns it



- an infinite behaviour can be typically presented as a loop

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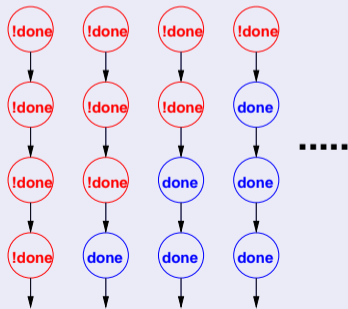
# Computation tree vs. computation paths

- Consider the following Kripke structure:

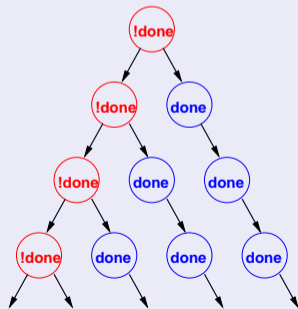


- Its execution can be seen as:

- an infinite set of **computation paths**



- an infinite **computation tree**



# Temporal Logics

- Express properties of “Reactive Systems”
  - nonterminating behaviours,
  - without explicit reference to time.
- **Linear Temporal Logic (LTL)**
  - interpreted over each path of the Kripke structure
  - linear model of time
  - temporal operators
  - “Medieval”: “since birth, one’s destiny is set”.
- **Computation Tree Logic (CTL)**
  - interpreted over computation tree of Kripke model
  - branching model of time
  - temporal operators plus path quantifiers
  - “Humanistic”: “one makes his/her own destiny step-by-step”.



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# Linear Temporal Logic (LTL): Syntax

- An **atomic proposition** is a LTL formula;
- if  $\varphi_1$  and  $\varphi_2$  are LTL formulae, then  $\neg\varphi_1$ ,  $\varphi_1 \wedge \varphi_2$ ,  $\varphi_1 \vee \varphi_2$ ,  $\varphi_1 \rightarrow \varphi_2$ ,  $\varphi_1 \leftrightarrow \varphi_2$ ,  $\varphi_1 \oplus \varphi_2$  are LTL formulae;
- if  $\varphi_1$  and  $\varphi_2$  are LTL formulae, then **X** $\varphi_1$ , **G** $\varphi_1$ , **F** $\varphi_1$ ,  $\varphi_1$ **U** $\varphi_2$  are LTL formulae, where **X**, **G**, **F**, **U** are the “next”, “globally”, “eventually”, “until” temporal operators respectively.
- Another operator **R** “releases” (the dual of **U**) is used sometimes.

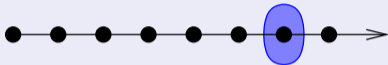
# LTL semantics: intuitions

LTL is given by the standard boolean logic enhanced with the following **temporal operators**, which operate through **paths**  $\langle s_0, s_1, \dots, s_k, \dots \rangle$ :

- “**Next**” **X**:  $\mathbf{X}\varphi$  is true in  $s_t$  iff  $\varphi$  is true in  $s_{t+1}$
  - “**Finally**” (or “eventually”) **F**:  $\mathbf{F}\varphi$  is true in  $s_t$  iff  $\varphi$  is true in **some**  $s_{t'}$  with  $t' \geq t$
  - “**Globally**” (or “henceforth”) **G**:  $\mathbf{G}\varphi$  is true in  $s_t$  iff  $\varphi$  is true in **all**  $s_{t'}$  with  $t' \geq t$
  - “**Until**” **U**:  $\varphi\mathbf{U}\psi$  is true in  $s_t$  iff, for some state  $s_{t'}$  s.t.  $t' \geq t$ :
    - $\psi$  is true in  $s_{t'}$  **and**
    - $\varphi$  is true in all states  $s_{t''}$  s.t.  $t \leq t'' < t'$
  - “**Releases**” **R**:  $\varphi\mathbf{R}\psi$  is true in  $s_t$  iff, for all states  $s_{t'}$  s.t.  $t' \geq t$ :
    - $\psi$  is true **or**
    - $\varphi$  is true in some states  $s_{t''}$  with  $t \leq t'' < t'$
- “ $\psi$  can become false only if  $\varphi$  becomes true first”

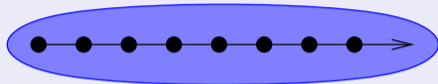
# LTL semantics: intuitions

finally  $P$



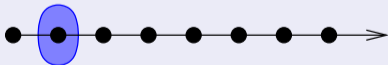
$F P$

globally  $P$



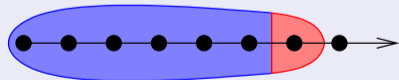
$G P$

next  $P$



$X P$

$P$  until  $q$



$P U q$

## LTL: Some Noteworthy Examples

- **Safety:** “it never happens that a train is arriving and the bar is up”

$$\mathbf{G}(\neg(\text{train\_arriving} \wedge \text{bar\_up}))$$

- **Liveness:** “if input, then eventually output”

$$\mathbf{G}(\text{input} \rightarrow \mathbf{F}\text{output})$$

- **Releases:** “the device is not working if you don’t first repair it”

$$(\text{repair\_device} \mathbf{R} \neg\text{working\_device})$$

- **Fairness:** “infinitely often send ”

$$\mathbf{GF}\text{send}$$

- **Strong fairness:** “infinitely often send implies infinitely often recv.”

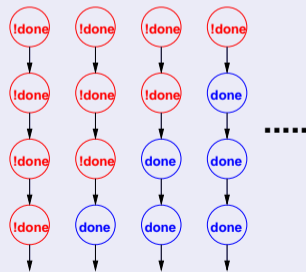
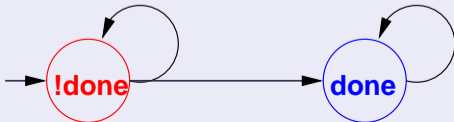
$$\mathbf{GF}\text{send} \rightarrow \mathbf{GF}\text{recv}$$

# LTL Formal Semantics

$\pi, s_j \models a$	iff	$a \in L(s_j)$	
$\pi, s_j \models \neg\varphi$	iff	$\pi, s_j \not\models \varphi$	
$\pi, s_j \models \varphi \wedge \psi$	iff	$\pi, s_j \models \varphi$	and
		$\pi, s_j \models \psi$	
$\pi, s_j \models \mathbf{X}\varphi$	iff	$\pi, s_{j+1} \models \varphi$	
$\pi, s_j \models \mathbf{F}\varphi$	iff	for some $j \geq i : \pi, s_j \models \varphi$	
$\pi, s_j \models \mathbf{G}\varphi$	iff	for all $j \geq i : \pi, s_j \models \varphi$	
$\pi, s_j \models \varphi \mathbf{U}\psi$	iff	for some $j \geq i : (\pi, s_j \models \psi$	and
		for all $k$ s.t. $i \leq k < j : \pi, s_k \models \varphi)$	
$\pi, s_j \models \varphi \mathbf{R}\psi$	iff	for all $j \geq i : (\pi, s_j \models \psi$	or
		for some $k$ s.t. $i \leq k < j : \pi, s_k \models \varphi)$	

# LTL Formal Semantics (cont.)

- LTL properties are evaluated over paths, i.e., over infinite, linear sequences of states:  
 $\pi = s_0 \rightarrow s_1 \rightarrow \dots \rightarrow s_t \rightarrow s_{t+1} \rightarrow \dots$
- Given an infinite sequence  $\pi = s_0, s_1, s_2, \dots$ 
  - $\pi, s_i \models \phi$  if  $\phi$  is true in state  $s_i$  of  $\pi$ .
  - $\pi \models \phi$  if  $\phi$  is true in the initial state  $s_0$  of  $\pi$ .
- The LTL model checking problem  $\mathcal{M} \models \phi$ 
  - check if  $\pi \models \phi$  for every path  $\pi$  of the Kripke structure  $\mathcal{M}$  (e.g.,  $\phi = \mathbf{Fdone}$ )





## The LTL model checking problem $\mathcal{M} \models \phi$ : remark

The LTL model checking problem  $\mathcal{M} \models \phi$

$\pi \models \phi$  for every path  $\pi$  of the Kripke structure  $\mathcal{M}$

### Important Remark

$\mathcal{M} \not\models \phi \not\Rightarrow \mathcal{M} \models \neg\phi$  (!!)

- E.g. if  $\phi$  is a LTL formula and two paths  $\pi_1$  and  $\pi_2$  are s.t.  $\pi_1 \models \phi$  and  $\pi_2 \models \neg\phi$ .

# Example: $\mathcal{M} \not\models \phi \not\Rightarrow \mathcal{M} \models \neg\phi$

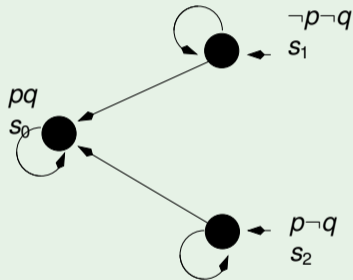
Let  $\pi_1 \stackrel{\text{def}}{=} \{s_1\}^\omega$ ,  $\pi_2 \stackrel{\text{def}}{=} \{s_2\}^\omega$ .

•  $\mathcal{M} \not\models \mathbf{G}p$ , in fact:

- $\pi_1 \not\models \mathbf{G}p$
- $\pi_2 \models \mathbf{G}p$

•  $\mathcal{M} \not\models \neg\mathbf{G}p$ , in fact:

- $\pi_1 \models \neg\mathbf{G}p$
- $\pi_2 \not\models \neg\mathbf{G}p$



# Syntactic properties of LTL operators

$$\begin{aligned}\varphi_1 \vee \varphi_2 &\iff \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \dots & \\ \mathbf{F}\varphi_1 &\iff \top\mathbf{U}\varphi_1 \\ \mathbf{G}\varphi_1 &\iff \perp\mathbf{R}\varphi_1 \\ \mathbf{F}\varphi_1 &\iff \neg\mathbf{G}\neg\varphi_1 \\ \mathbf{G}\varphi_1 &\iff \neg\mathbf{F}\neg\varphi_1 \\ \neg\mathbf{X}\varphi_1 &\iff \mathbf{X}\neg\varphi_1 \\ \varphi_1\mathbf{R}\varphi_2 &\iff \neg(\neg\varphi_1\mathbf{U}\neg\varphi_2) \\ \varphi_1\mathbf{U}\varphi_2 &\iff \neg(\neg\varphi_1\mathbf{R}\neg\varphi_2)\end{aligned}$$

## Note

LTL can be defined in terms of  $\wedge$ ,  $\neg$ ,  $\mathbf{X}$ ,  $\mathbf{U}$  only

## Exercise

Prove that  $\varphi_1\mathbf{R}\varphi_2 \iff \mathbf{G}\varphi_2 \vee \varphi_2\mathbf{U}(\varphi_1 \wedge \varphi_2)$

# Proof of $\varphi R\psi \Leftrightarrow (\mathbf{G}\psi \vee \psi\mathbf{U}(\varphi \wedge \psi))$

[Solution proposed by the student Samuel Valentini, 2016]

(All state indexes below are implicitly assumed to be  $\geq 0$ .)

$\Rightarrow$ : Let  $\pi$  be s.t.  $\pi, s_0 \models \varphi R\psi$

- If  $\forall j, \pi, s_j \models \psi$ , then  $\pi, s_0 \models \mathbf{G}\psi$ .
- Otherwise, let  $s_k$  be the **first** state s.t.  $\pi, s_k \not\models \psi$ .
- Since  $\pi, s_0 \models \varphi R\psi$ , then  $k > 0$  and exists  $k' < k$  s.t.  $\pi, s_{k'} \models \varphi$
- By construction,  $\pi, s_{k'} \models \varphi \wedge \psi$  and, for every  $w < k'$ ,  $\pi, s_w \models \psi$ , so that  $\pi, s_0 \models \psi\mathbf{U}(\varphi \wedge \psi)$ .
- Thus,  $\pi, s_0 \models \mathbf{G}\psi \vee \psi\mathbf{U}(\varphi \wedge \psi)$

$\Leftarrow$ : Let  $\pi$  be s.t.  $\pi, s_0 \models \mathbf{G}\psi \vee \psi\mathbf{U}(\varphi \wedge \psi)$

- If  $\pi, s_0 \models \mathbf{G}\psi$ , then  $\forall j, \pi, s_j \models \psi$ , so that  $\pi, s_0 \models \varphi R\psi$ .
- Otherwise,  $\pi, s_0 \models \psi\mathbf{U}(\varphi \wedge \psi)$ .
- Let  $s_k$  be the **first** state s.t.  $\pi, s_k \not\models \psi$ .
- by construction,  $\exists k'$  such that  $\pi, s_{k'} \models \varphi \wedge \psi$
- by the definition of  $k$ , we have that  $k' < k$  and  $\forall w < k, \pi, s_w \models \psi$ .
- Thus  $\pi, s_0 \models \varphi R\psi$

# Strength of LTL operators

- $\mathbf{G}\varphi \models \varphi \models \mathbf{F}\varphi$
- $\mathbf{G}\varphi \models \mathbf{X}\varphi \models \mathbf{F}\varphi$
- $\mathbf{G}\varphi \models \mathbf{XX}\dots\mathbf{X}\varphi \models \mathbf{F}\varphi$
- $\varphi \mathbf{U}\psi \models \mathbf{F}\psi$
- $\mathbf{G}\psi \models \varphi \mathbf{R}\psi$

# LTL tableaux rules

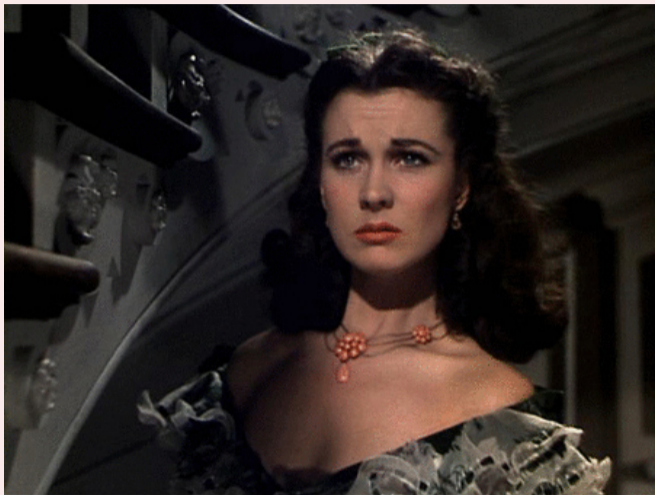
- Let  $\varphi_1$  and  $\varphi_2$  be LTL formulae:

$$\begin{aligned}\mathbf{F}\varphi_1 &\iff (\varphi_1 \vee \mathbf{X}\mathbf{F}\varphi_1) \\ \mathbf{G}\varphi_1 &\iff (\varphi_1 \wedge \mathbf{X}\mathbf{G}\varphi_1) \\ \varphi_1 \mathbf{U}\varphi_2 &\iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X}(\varphi_1 \mathbf{U}\varphi_2))) \\ \varphi_1 \mathbf{R}\varphi_2 &\iff (\varphi_2 \wedge (\varphi_1 \vee \mathbf{X}(\varphi_1 \mathbf{R}\varphi_2)))\end{aligned}$$

- If applied recursively, rewrite an LTL formula in terms of atomic and  $\mathbf{X}$ -formulas:

$$(p\mathbf{U}q) \wedge (\mathbf{G}\neg p) \implies (q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))) \wedge (\neg p \wedge \mathbf{X}\mathbf{G}\neg p)$$

## Tableaux Rules: a Quote



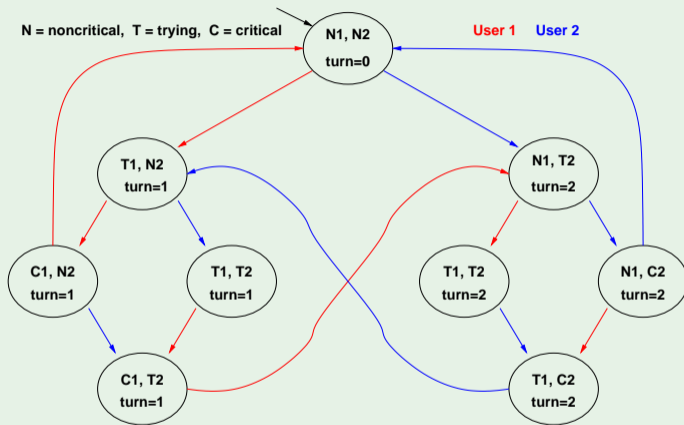
*"After all... tomorrow is another day."  
[Scarlett O'Hara, "Gone with the Wind"]*

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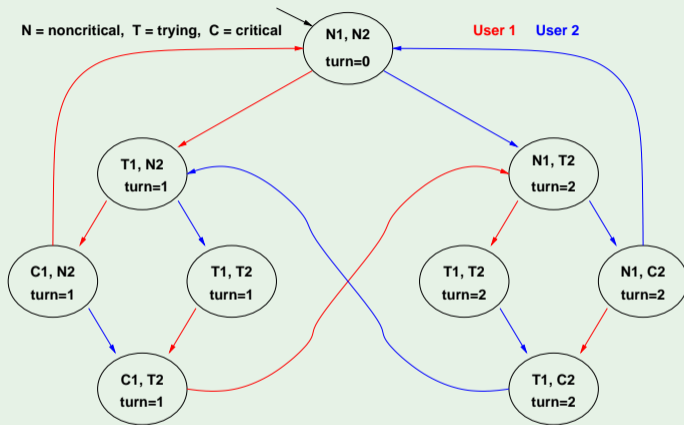
## Example 1: mutual exclusion (safety)



$$M \models \mathbf{G}\neg(C_1 \wedge C_2) ?$$

**YES:** There is no reachable state in which  $(C_1 \wedge C_2)$  holds!

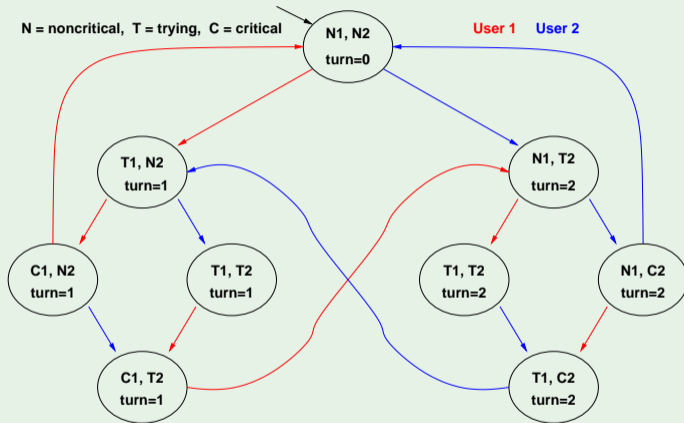
## Example 2: liveness



$M \models FC_1 ?$

**NO:** there is an infinite cyclic solution in which  $C_1$  never holds!

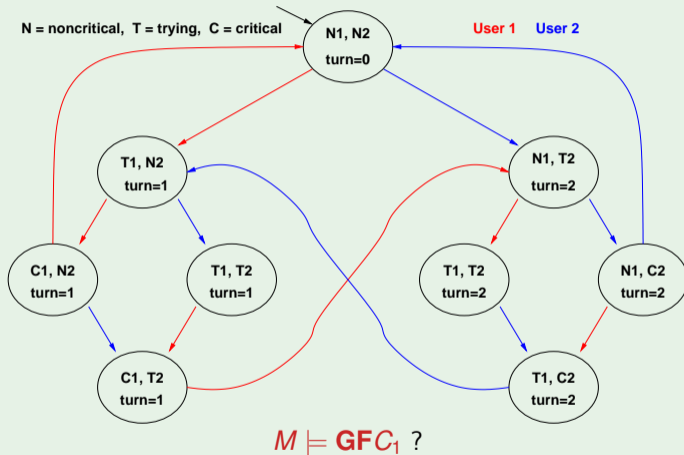
## Example 3: liveness



$$M \models \mathbf{G}(T_1 \rightarrow \mathbf{F}C_1) ?$$

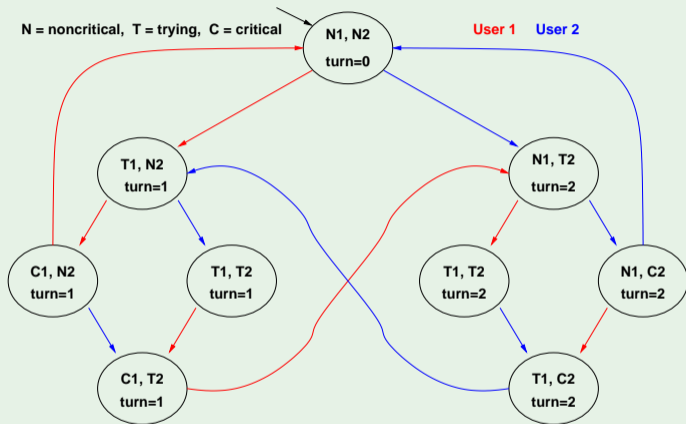
YES: every path starting from each state where  $T_1$  holds passes through a state where  $C_1$  holds.

## Example 4: fairness



**NO:** e.g., in the initial state, there is an infinite cyclic solution in which  $C_1$  never holds!

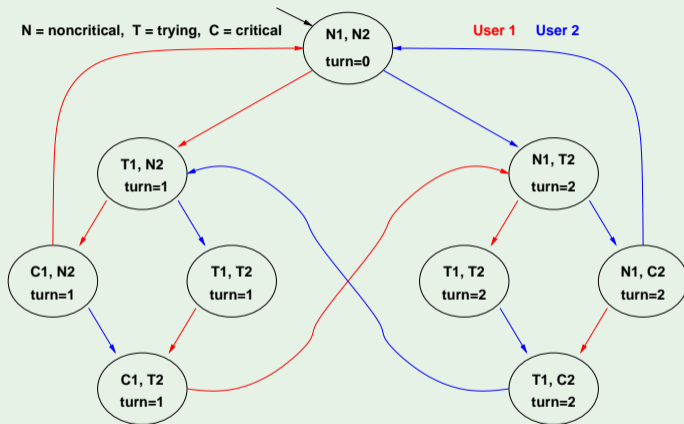
## Example 5: strong fairness



$$M \models \mathbf{GFT}_1 \rightarrow \mathbf{GFC}_1 ?$$

**YES:** every path which visits  $T_1$  infinitely often also visits  $C_1$  infinitely often (see liveness property of previous example).

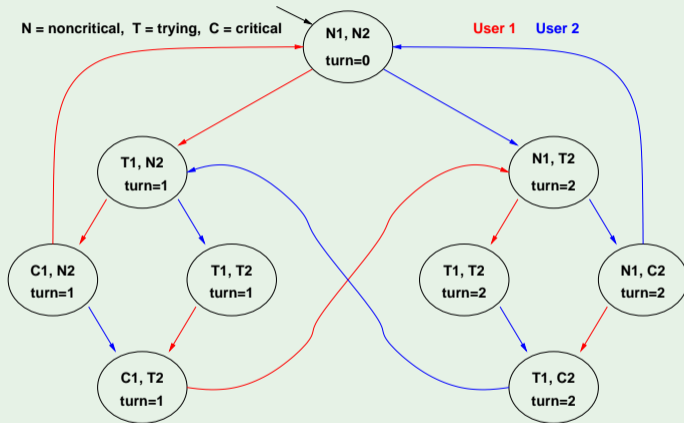
## Example 6: blocking



$$M \models \mathbf{G}(N_1 \rightarrow \mathbf{F} T_1) ?$$

**NO:** e.g., in the initial state, there is an infinite cyclic solution in which  $N_1$  holds and  $T_1$  never holds!

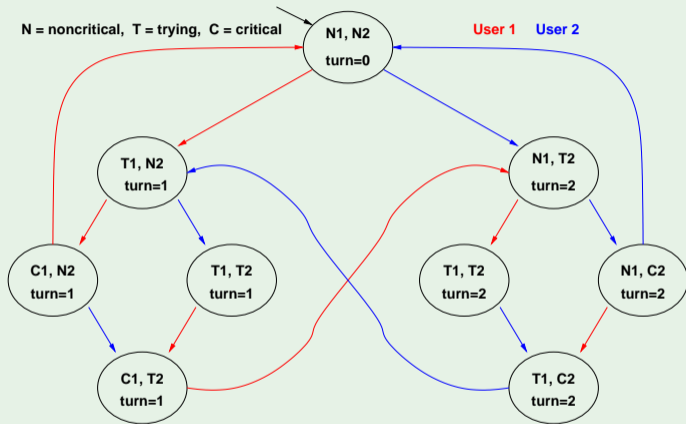
## Example 7: Releases



$M \models T_1 R \neg C_1 ?$

**YES:**  $C_1$  in paths only strictly after  $T_1$  has occurred.

## Example 8: XF



$M \models \mathbf{XF}(\text{turn} = 0) ?$

**NO:** a counter-example is the  $\infty$ -shaped loop:

$(N1, N2), \{(T1, N2), (C1, N2), (C1, T2), (N1, T2), (N1, C2), (T1, C2)\}^\omega$



## Exercise: $\mathbf{G}(T \rightarrow \mathbf{FC})$ vs. $\mathbf{GFT} \rightarrow \mathbf{GFC}$

- Prove that  $\mathbf{G}(T \rightarrow \mathbf{FC}) \implies \mathbf{GFT} \rightarrow \mathbf{GFC}$ , or produce a counterexample
- Prove that  $\mathbf{GFT} \rightarrow \mathbf{GFC} \implies \mathbf{G}(T \rightarrow \mathbf{FC})$ , or produce a counterexample

## Example: $\mathbf{G}(T \rightarrow \mathbf{FC})$ vs. $\mathbf{GFT} \rightarrow \mathbf{GFC}$

- $\mathbf{G}(T \rightarrow \mathbf{FC}) \implies \mathbf{GFT} \rightarrow \mathbf{GFC}$  ?
- YES: if  $M \models \mathbf{G}(T \rightarrow \mathbf{FC})$ , then  $M \models \mathbf{GFT} \rightarrow \mathbf{GFC}$  !
- let  $M \models \mathbf{G}(T \rightarrow \mathbf{FC})$ .
  - let  $\pi \in M$  s.t.  $\pi \models \mathbf{GFT}$
  - $\implies \pi, s_i \models \mathbf{FT}$  for each  $s_i \in \pi$
  - $\implies \pi, s_j \models T$  for each  $s_i \in \pi$  and for some  $s_j \in \pi$  s.t.  $j \geq i$
  - $\implies \pi, s_j \models \mathbf{FC}$  for each  $s_i \in \pi$  and for some  $s_j \in \pi$  s.t.  $j \geq i$
  - $\implies \pi, s_k \models C$  for each  $s_i \in \pi$ , for some  $s_j \in \pi$  s.t.  $j \geq i$  and for some  $k \geq j$
  - $\implies \pi, s_k \models C$  for each  $s_i \in \pi$  and for some  $k \geq i$
  - $\implies \pi \models \mathbf{GFC}$
  - $\implies M \models \mathbf{GFT} \rightarrow \mathbf{GFC}$ .

# Example: $\mathbf{G}(T \rightarrow \mathbf{FC})$ vs. $\mathbf{GFT} \rightarrow \mathbf{GFC}$

- $\mathbf{G}(T \rightarrow \mathbf{FC}) \iff \mathbf{GFT} \rightarrow \mathbf{GFC}$  ?
- NO!
- Counter example:



- $\mathbf{GFT} \rightarrow \mathbf{GFC}$  is satisfied
- $\mathbf{G}(T \rightarrow \mathbf{FC})$  is not satisfied

(Counter-example proposed by the student Vaishak Belle, 2008)

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# Computational Tree Logic (CTL): Syntax

- An **atomic proposition** is a CTL formula;
- if  $\varphi_1$  and  $\varphi_2$  are CTL formulae, then  $\neg\varphi_1$ ,  $\varphi_1 \wedge \varphi_2$ ,  $\varphi_1 \vee \varphi_2$ ,  $\varphi_1 \rightarrow \varphi_2$ ,  $\varphi_1 \leftrightarrow \varphi_2$  are CTL formulae;
- if  $\varphi_1$  and  $\varphi_2$  are CTL formulae, then **AX** $\varphi_1$ , **A**( $\varphi_1$ **U** $\varphi_2$ ), **AG** $\varphi_1$ , **AF** $\varphi_1$ , **EX** $\varphi_1$ , **E**( $\varphi_1$ **U** $\varphi_2$ ), **EG** $\varphi_1$ , **EF** $\varphi_1$ , are CTL formulae.  
(**E**( $\varphi_1$ **R** $\varphi_2$ ) and **A**( $\varphi_1$ **R** $\varphi_2$ ) never used in practice.)

## CTL semantics: intuitions

CTL is given by the standard boolean logic enhanced with the operators **AX**, **AG**, **AF**, **AU**, **EX**, **EG**, **EF**, **EU**:

- “Necessarily Next” **AX**: **AX** $\varphi$  is true in  $s_t$  iff  $\varphi$  is true in every successor state  $s_{t+1}$
- “Possibly Next” **EX**: **EX** $\varphi$  is true in  $s_t$  iff  $\varphi$  is true in one successor state  $s_{t+1}$
- “Necessarily in the future” (or “Inevitably”) **AF**: **AF** $\varphi$  is true in  $s_t$  iff  $\varphi$  is inevitably true in **some**  $s_{t'}$  with  $t' \geq t$
- “Possibly in the future” (or “Possibly”) **EF**: **EF** $\varphi$  is true in  $s_t$  iff  $\varphi$  may be true in **some**  $s_{t'}$  with  $t' \geq t$

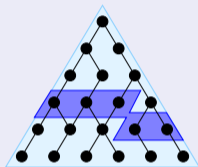
## CTL semantics: intuitions [cont.]

- “Globally” (or “always”) **AG**: **AG** $\varphi$  is true in  $s_t$  iff  $\varphi$  is true in **all**  $s_{t'}$  with  $t' \geq t$
- “Possibly henceforth” **EG**: **EG** $\varphi$  is true in  $s_t$  iff  $\varphi$  is possibly true henceforth
- “Necessarily Until” **AU**: **A**( $\varphi$ **U** $\psi$ ) is true in  $s_t$  iff necessarily  $\varphi$  holds until  $\psi$  holds.
- “Possibly Until” **EU**: **E**( $\varphi$ **U** $\psi$ ) is true in  $s_t$  iff possibly  $\varphi$  holds until  $\psi$  holds.



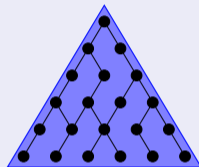
# CTL semantics: intuitions [cont.]

finally  $P$



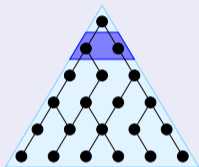
$AF P$

globally  $P$



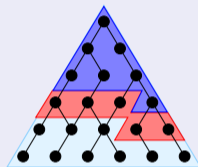
$AG P$

next  $P$

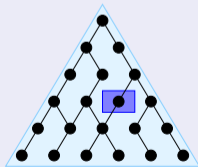


$AX P$

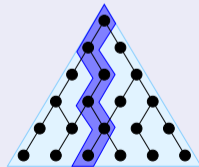
$P$  until  $q$



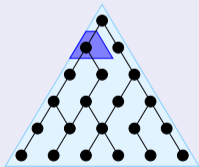
$A[P U q]$



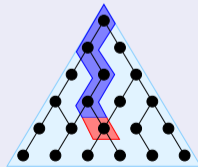
$EF P$



$EG P$



$EX P$



$E[P U q]$

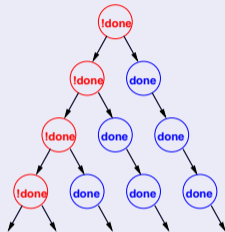
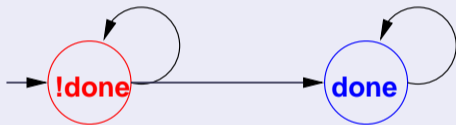
# CTL Formal Semantics

Let  $(s_i, s_{i+1}, \dots)$  be a path outgoing from state  $s_i$  in  $M$

$M, s_i \models a$	iff	$a \in L(s_i)$	
$M, s_i \models \neg\varphi$	iff	$M, s_i \not\models \varphi$	
$M, s_i \models \varphi \vee \psi$	iff	$M, s_i \models \varphi$ or $M, s_i \models \psi$	
$M, s_i \models AX\varphi$	iff	for all $(s_i, s_{i+1}, \dots)$ ,	$M, s_{i+1} \models \varphi$
$M, s_i \models EX\varphi$	iff	for some $(s_i, s_{i+1}, \dots)$ ,	$M, s_{i+1} \models \varphi$
$M, s_i \models AG\varphi$	iff	for all $(s_i, s_{i+1}, \dots)$ ,	for all $j \geq i. M, s_j \models \varphi$
$M, s_i \models EG\varphi$	iff	for some $(s_i, s_{i+1}, \dots)$ ,	for all $j \geq i. M, s_j \models \varphi$
$M, s_i \models AF\varphi$	iff	for all $(s_i, s_{i+1}, \dots)$ ,	for some $j \geq i. M, s_j \models \varphi$
$M, s_i \models EF\varphi$	iff	for some $(s_i, s_{i+1}, \dots)$ ,	for some $j \geq i. M, s_j \models \varphi$
$M, s_i \models A(\varphi U\psi)$	iff	for all $(s_i, s_{i+1}, \dots)$ ,	for some $j \geq i.$ $(M, s_j \models \psi$ and forall $k$ s.t. $i \leq k < j. M, s_k \models \varphi)$
$M, s_i \models E(\varphi U\psi)$	iff	for some $(s_i, s_{i+1}, \dots)$ ,	for some $j \geq i.$ $(M, s_j \models \psi$ and forall $k$ s.t. $i \leq k < j. M, s_k \models \varphi)$

## Formal Semantics (cont.)

- CTL properties (e.g. **AF***done*) are evaluated over trees.



- Every temporal operator (**F**, **G**, **X**, **U**) is preceded by a **path quantifier** (**A** or **E**).
- **Universal modalities** (**AF**, **AG**, **AX**, **AU**): the temporal formula is true in **all** the paths starting in the current state.
- **Existential modalities** (**EF**, **EG**, **EX**, **EU**): the temporal formula is true in **some** path starting in the current state.

# The CTL model checking problem $\mathcal{M} \models \phi$

The CTL model checking problem  $\mathcal{M} \models \phi$

$\mathcal{M}, s \models \phi$  for every initial state  $s \in I$  of the Kripke structure

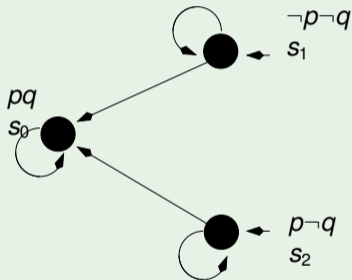
## Important Remark

$\mathcal{M} \not\models \phi \not\Rightarrow \mathcal{M} \models \neg\phi$  (!!)

- E.g. if  $\phi$  is a universal formula **A**... and two initial states  $s_0, s_1$  are s.t.  $\mathcal{M}, s_0 \models \phi$  and  $\mathcal{M}, s_1 \not\models \phi$
- $\mathcal{M} \not\models \phi \Rightarrow \mathcal{M} \models \neg\phi$  if  $\mathcal{M}$  has only one initial state

# Example: $\mathcal{M} \not\models \phi \not\Rightarrow \mathcal{M} \models \neg\phi$

- $\mathcal{M} \not\models \mathbf{AG}p$ , in fact:
  - $\mathcal{M}, s_1 \not\models \mathbf{AG}p$   
(e.g.,  $\{s_1, \dots\}$  is a counter-example)
  - $\mathcal{M}, s_2 \models \mathbf{AG}p$
- $\mathcal{M} \not\models \neg\mathbf{AG}p$ , in fact:
  - $\mathcal{M}, s_1 \models \neg\mathbf{AG}p$   
(i.e.,  $\mathcal{M}, s_1 \models \mathbf{EF}\neg p$ )
  - $\mathcal{M}, s_2 \not\models \neg\mathbf{AG}p$   
(i.e.,  $\mathcal{M}, s_2 \not\models \mathbf{EF}\neg p$ )



# Syntactic properties of CTL operators

$$\varphi_1 \vee \varphi_2 \iff \neg(\neg\varphi_1 \wedge \neg\varphi_2)$$

...

$$\mathbf{A}(\varphi_1 \mathbf{U} \varphi_2) \iff \neg \mathbf{E}(\neg\varphi_2 \mathbf{U} (\neg\varphi_1 \wedge \neg\varphi_2)) \wedge \neg \mathbf{EG} \neg\varphi_2$$

$$\mathbf{EF} \varphi_1 \iff \mathbf{E}(\mathbf{TU} \varphi_1)$$

$$\mathbf{AG} \varphi_1 \iff \neg \mathbf{EF} \neg\varphi_1$$

$$\mathbf{AF} \varphi_1 \iff \neg \mathbf{EG} \neg\varphi_1$$

$$\mathbf{AX} \varphi_1 \iff \neg \mathbf{EX} \neg\varphi_1$$

## Note

CTL can be defined in terms of  $\wedge$ ,  $\neg$ , **EX**, **EG**, **EU** only

## Exercise:

prove that  $\mathbf{A}(\varphi_1 \mathbf{U} \varphi_2) \iff \neg \mathbf{EG} \neg\varphi_2 \wedge \neg \mathbf{E}(\neg\varphi_2 \mathbf{U} (\neg\varphi_1 \wedge \neg\varphi_2))$

# Strength of CTL operators

- $\mathbf{A}[\mathbf{OP}]\varphi \models \mathbf{E}[\mathbf{OP}]\varphi$ , s.t.  $[\mathbf{OP}] \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U}\}$
- $\mathbf{AG}\varphi \models \varphi \models \mathbf{AF}\varphi$ ,  $\mathbf{EG}\varphi \models \varphi \models \mathbf{EF}\varphi$
- $\mathbf{AG}\varphi \models \mathbf{AX}\varphi \models \mathbf{AF}\varphi$ ,  $\mathbf{EG}\varphi \models \mathbf{EX}\varphi \models \mathbf{EF}\varphi$
- $\mathbf{AG}\varphi \models \mathbf{AX}\dots\mathbf{AX}\varphi \models \mathbf{AF}\varphi$ ,  $\mathbf{EG}\varphi \models \mathbf{EX}\dots\mathbf{EX}\varphi \models \mathbf{EF}\varphi$
- $\mathbf{A}(\varphi\mathbf{U}\psi) \models \mathbf{AF}\psi$ ,  $\mathbf{E}(\varphi\mathbf{U}\psi) \models \mathbf{EF}\psi$

# CTL tableaux rules

- Let  $\varphi_1$  and  $\varphi_2$  be CTL formulae:

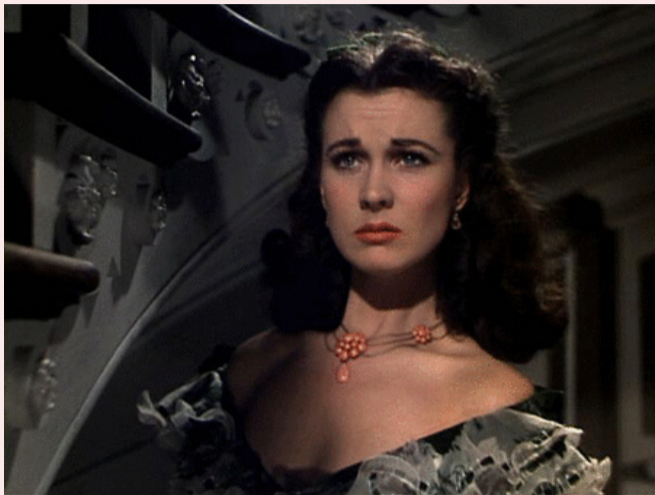
$$\begin{aligned}\mathbf{AF}\varphi_1 &\iff (\varphi_1 \vee \mathbf{AXAF}\varphi_1) \\ \mathbf{AG}\varphi_1 &\iff (\varphi_1 \wedge \mathbf{AXAG}\varphi_1) \\ \mathbf{A}(\varphi_1 \mathbf{U}\varphi_2) &\iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{AXA}(\varphi_1 \mathbf{U}\varphi_2))) \\ \mathbf{EF}\varphi_1 &\iff (\varphi_1 \vee \mathbf{EXEF}\varphi_1) \\ \mathbf{EG}\varphi_1 &\iff (\varphi_1 \wedge \mathbf{EXEG}\varphi_1) \\ \mathbf{E}(\varphi_1 \mathbf{U}\varphi_2) &\iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{EXE}(\varphi_1 \mathbf{U}\varphi_2)))\end{aligned}$$

- Recursive definitions of **AF**, **AG**, **AU**, **EF**, **EG**, **EU**.
- If applied recursively, rewrite a CTL formula in terms of atomic, **AX**- and **EX**-formulas:

$$\mathbf{A}(p\mathbf{U}q) \wedge (\mathbf{EG}\neg p) \implies (q \vee (p \wedge \mathbf{AXA}(p\mathbf{U}q))) \wedge (\neg p \wedge \mathbf{EXEG}\neg p)$$



## Tableaux Rules: a Quote

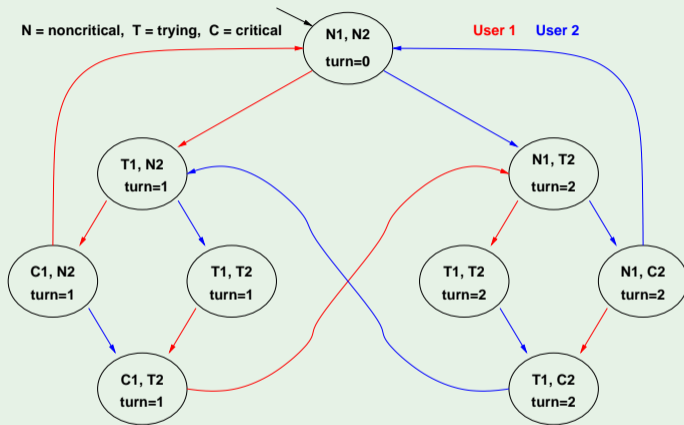


*"After all... tomorrow is another day."  
[Scarlett O'Hara, "Gone with the Wind"]*

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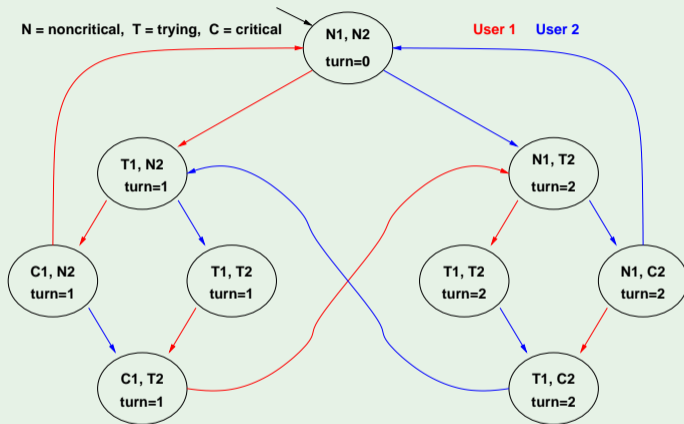
## Example 1: mutual exclusion (safety)



$$M \models \mathbf{AG}\neg(C_1 \wedge C_2) ?$$

**YES:** There is no reachable state in which  $(C_1 \wedge C_2)$  holds!  
(Same as the  $\mathbf{G}\neg(C_1 \wedge C_2)$  in LTL.)

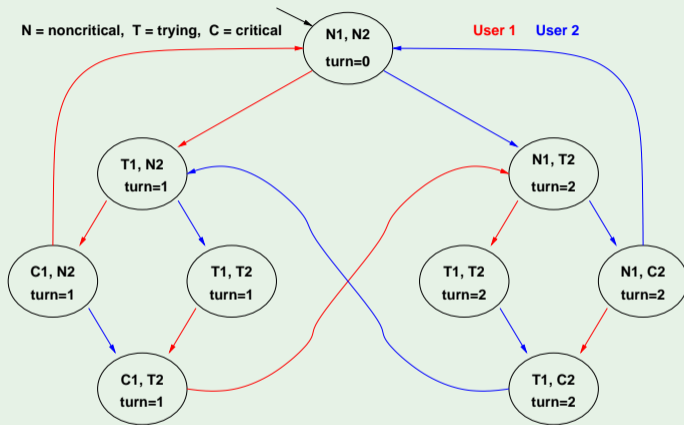
## Example 2: liveness



$M \models \text{AF } C_1 ?$

No: there is an infinite cyclic solution in which  $C_1$  never holds!  
(Same as  $\text{FC}_1$  in LTL.)

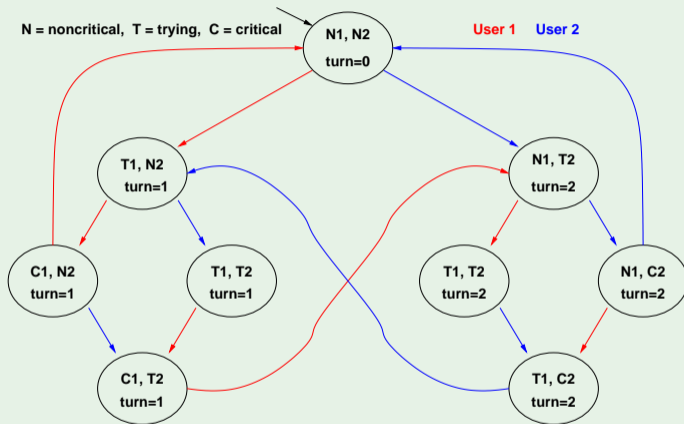
## Example 3: liveness



$$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF} C_1) ?$$

**YES:** every path starting from each state where  $T_1$  holds passes through a state where  $C_1$  holds  
(Same as  $\mathbf{G}(T_1 \rightarrow \mathbf{FC}_1)$  in LTL.)

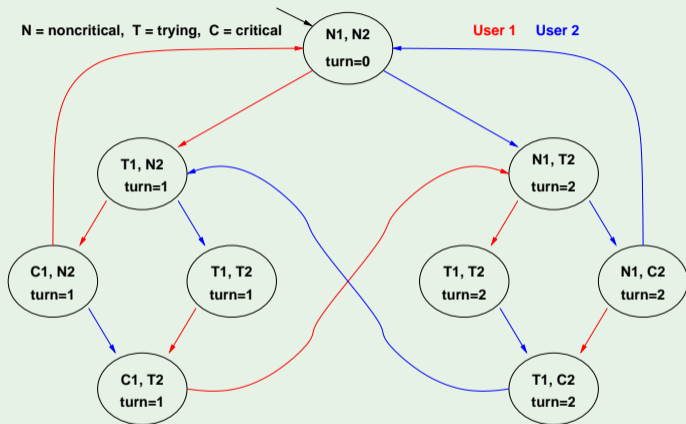
## Example 4: fairness



$M \models \mathbf{AGAF}C_1 ?$

**NO:** e.g., in the initial state, there is an infinite cyclic solution in which  $C_1$  never holds!  
(Same as  $\mathbf{GFC}_1$  in LTL.)

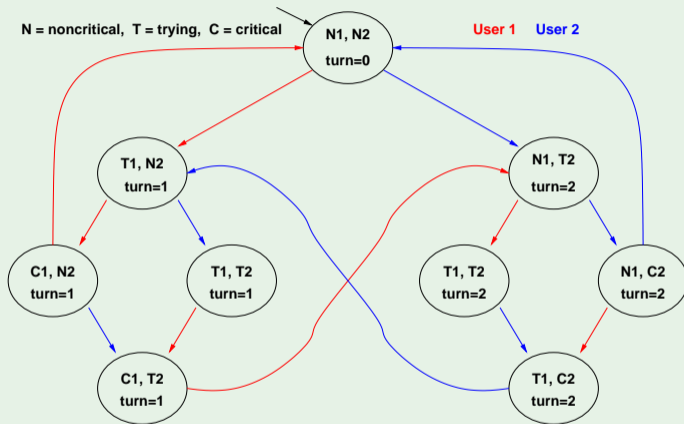
## Example 5: fairness (2)



$M \models \mathbf{AGAF}(\text{turn} = 0) ?$

**NO:** there is an infinite 8-shaped cyclic solution in which ( $\text{turn} = 0$ ) never holds!

## Example 6: blocking

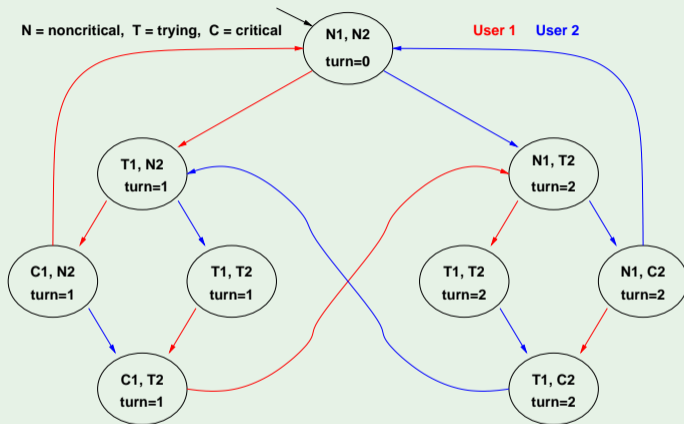


$$M \models \mathbf{AG}(N_1 \rightarrow \mathbf{EF} T_1) ?$$

**YES:** from each state where  $N_1$  holds there is a path leading to a state where  $T_1$  holds  
(No corresponding LTL formula.)



## Example 7: blocking (2)

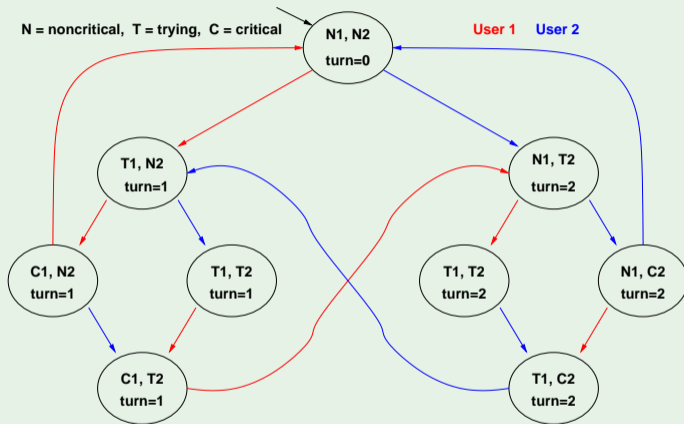


$$M \models \mathbf{AG}(N_1 \rightarrow \mathbf{AF} T_1) ?$$

**NO:** e.g., in the initial state, there is an infinite cyclic solution in which  $N_1$  holds and  $T_1$  never holds!

(Same as LTL formula  $\mathbf{G}(N_1 \rightarrow \mathbf{F}T_1)$ .)

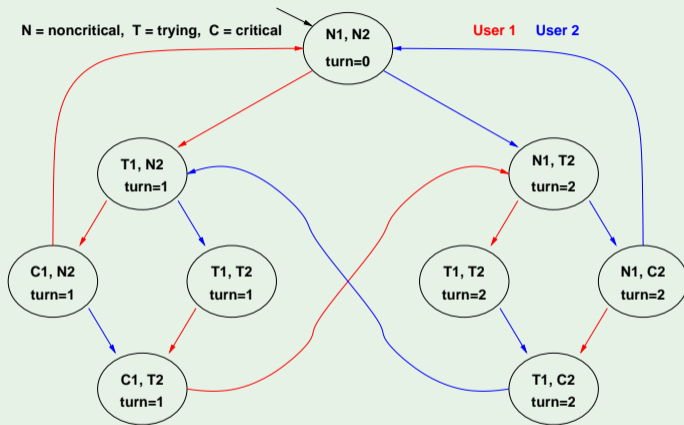
## Example 8:



$$M \models \text{EG}N_1 ?$$

**YES:** there is an infinite cyclic solution where  $N_1$  always holds  
(No corresponding LTL formula.)

## Example 9:



$M \models \mathbf{AFEGN}_1 ?$

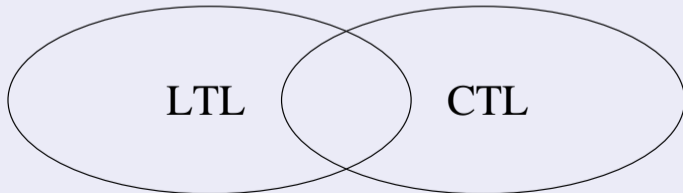
**YES:** there is an infinite cyclic solution where  $N_1$  always holds, and from every state you necessarily reach one state of such cycle  
(No corresponding LTL formula.)

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## LTL vs. CTL: expressiveness

- Many CTL formulas cannot be expressed in LTL (e.g., those containing existentially quantified subformulas)  
E.g.,  $\mathbf{AG}(N_1 \rightarrow \mathbf{EFT}_1)$ ,  $\mathbf{AFAG}\varphi$
- Many LTL formulas cannot be expressed in CTL (e.g. fairness LTL formulas)  
E.g.,  $\mathbf{GFT}_1 \rightarrow \mathbf{GFC}_1$ ,  $\mathbf{FG}\varphi$
- Some formulas can be expressed both in LTL and in CTL (typically LTL formulas with operators of nesting depth 1, and/or with operators occurring positively)  
E.g.,  $\mathbf{G}\neg(C_1 \wedge C_2)$ ,  $\mathbf{FC}_1$ ,  $\mathbf{G}(T_1 \rightarrow \mathbf{FC}_1)$ ,  $\mathbf{GFC}_1$





## LTL vs. CTL: M.C. Algorithms

- LTL M.C. problems are typically handled with **automata-based M.C.** approaches (Wolper & Vardi)
- CTL M.C. problems are typically handled with **symbolic M.C.** approaches (Clarke & McMillan)
- LTL M.C. problems can be reduced to CTL M.C. problems under **fairness constraints** (Clarke et al.)

- Syntax: let  $p$ 's,  $\varphi$ 's,  $\psi$ 's being propositions, state formulae and path formulae respectively:
  - $p, \neg\varphi, \varphi_1 \wedge \varphi_2, \mathbf{A}\psi, \mathbf{E}\psi$  are **state formulae**  
(properties of the set of paths starting from a state)
  - $\varphi, \neg\psi, \psi_1 \wedge \psi_2, \mathbf{X}\psi, \mathbf{G}\psi, \mathbf{F}\psi, \psi_1 \mathbf{U}\psi_2$  are **path formulae**  
(properties of a path)
- Semantics: **A, E, X, G, F, U** as in CTL
  - **A, E**: quantify on paths (as in CTL)
  - **X, G, F, U**: (as in LTL)
  - as in CTL, but **X, G, F, U** not necessarily preceded by **A, E**

## Remark

In principle in CTL\* one may have sequences of nested path quantifiers.  
In such case, the most internal one dominates:

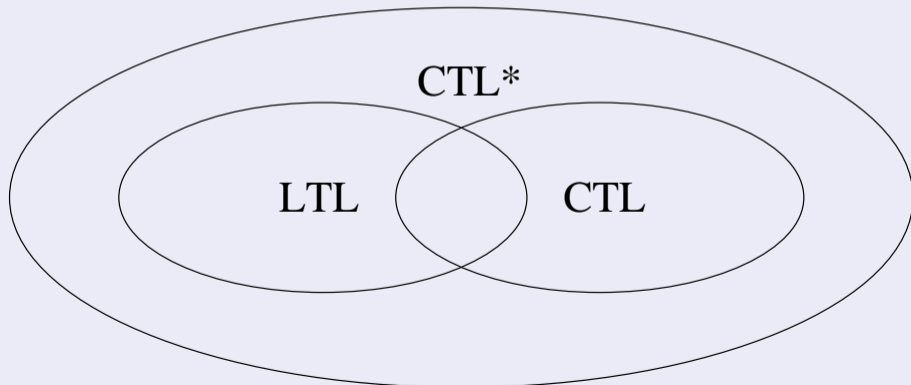
$$M, s \models \mathbf{AE}\psi \text{ iff } M, s \models \mathbf{E}\psi, \quad M, s \models \mathbf{EA}\psi \text{ iff } M, s \models \mathbf{A}\psi.$$



# CTL\* vs LTL & CTL

CTL\* subsumes both CTL and LTL

- $\varphi$  in CTL  $\implies \varphi$  in CTL\* (e.g., **AG**( $N_1 \rightarrow$  **EFT** $_1$ ))
- $\varphi$  in LTL  $\implies \mathbf{A}\varphi$  in CTL\* (e.g., **A**(**GFT** $_1 \rightarrow$  **GFC** $_1$ ))
- $\text{LTL} \cup \text{CTL} \subset \text{CTL}^*$  (e.g., **E**(**GFp**  $\rightarrow$  **GFq))**



“You have no respect for logic. (...)

I have no respect for those who have no respect for logic.”

<https://www.youtube.com/watch?v=uGstM8QMCjQ>



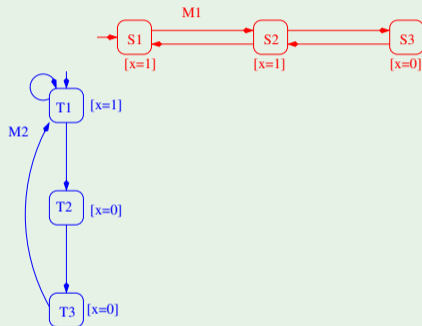
(Arnold Schwarzenegger in "Twins")

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## Exercise: Products of Kripke Models

Consider the following two Kripke models  $M1$  and  $M2$ , which share the variable  $x$ :



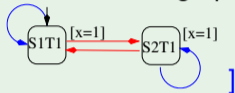
1. Compute and draw the graph of the asynchronous product of  $M1$  and  $M2$ .
2. Compute and draw the graph of the synchronous product of  $M1$  and  $M2$ .

Note: unreachable and deadend states should be removed.

# Exercise: Products of Kripke Models (cont.)

## 1. Asynchronous product

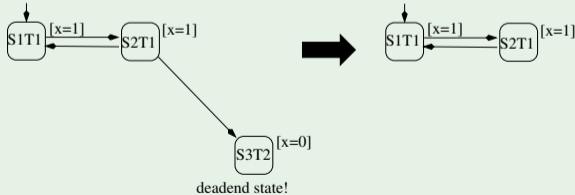
1. Compute and draw the graph of the asynchronous product of  $M1$  and  $M2$ .



[ Solution: ]

## 2. Synchronous product

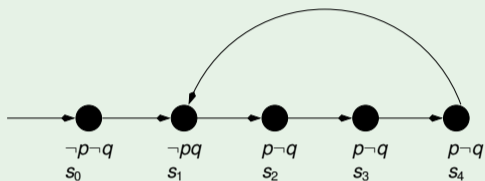
2. Compute and draw the graph of the synchronous product of  $M1$  and  $M2$ .



[ Solution: ]

## Exercise: LTL Model Checking (path)

Consider the following path  $\pi$ :

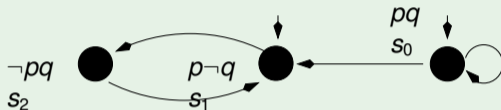


For each of the following facts, say if it is true or false in LTL.

- (a)  $\pi, s_0 \models \mathbf{GF}q$   
[ Solution: true ]
- (b)  $\pi, s_0 \models \mathbf{FG}(q \leftrightarrow \neg p)$   
[ Solution: true ]
- (c)  $\pi, s_2 \models \mathbf{G}p$   
[ Solution: false ]
- (d)  $\pi, s_2 \models p\mathbf{U}q$   
[ Solution: true ]

# Ex: LTL Model Checking

Consider the following Kripke Model  $M$ :

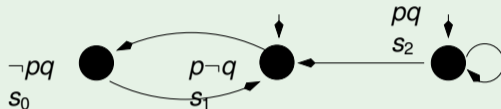


For each of the following facts, say if it is true or false in LTL.

- (a)  $M \models (p \mathbf{U} q)$   
[ Solution: true ]
- (b)  $M \models \mathbf{G}(\neg p \rightarrow F \neg q)$   
[ Solution: true ]
- (c)  $M \models \mathbf{G}p \rightarrow \mathbf{G}q$   
[ Solution: true ]
- (d)  $M \models \mathbf{F} \mathbf{G} p$   
[ Solution: false ]

# Ex: CTL Model Checking

Consider the following Kripke Model  $M$ :



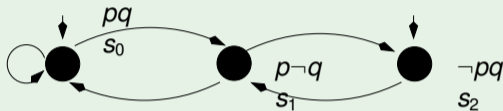
For each of the following facts, say if it is true or false in CTL.

- (a)  $M \models \mathbf{AF} \neg p$   
[ Solution: false ]
- (b)  $M \models \mathbf{EG} p$   
[ Solution: false ]
- (c)  $M \models \mathbf{A}(p \mathbf{U} q)$   
[ Solution: true ]
- (d)  $M \models \mathbf{E}(p \mathbf{U} \neg q)$   
[ Solution: true ]



# Ex: CTL Model Checking

Consider the following Kripke Model  $M$ :



For each of the following facts, say if it is true or false in CTL.

- (a)  $M \models \mathbf{AF} \neg q$   
[ Solution: false ]
- (b)  $M \models \mathbf{EG} q$   
[ Solution: false ]
- (c)  $M \models ((\mathbf{AGAF} p \vee \mathbf{AGAF} q) \wedge (\mathbf{AGAF} \neg p \vee \mathbf{AGAF} \neg q)) \rightarrow q$   
[ Solution: true ]
- (d)  $M \models \mathbf{AFEG}(p \wedge q)$   
[ Solution: false ]