

Course Formal Methods

Module I: Automated Reasoning

Ch. 02: **Satisfiability Modulo Theories (SMT)**

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 - Basics on First-order Logic
 - What is a Theory?
 - Satisfiability Modulo Theories
 - Motivations and Goals of SMT
- 2 Efficient SMT solving
 - Combining SAT with Theory Solvers
 - Theory Solvers for Theories of Interest (hints)
 - SMT for Combinations of Theories
- 3 Beyond Solving: Advanced SMT Functionalities
 - Proofs and Unsatisfiable Cores
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First-Order Logic (FOL)

- PL assumes world contains **facts**
 - atomic events
- FOL is **structured**: a world/state includes **objects**, each of which may have **attributes** of its own as well as **relationships** to other objects
- FOL assumes the world contains:
 - **Objects**:
e.g., **people, houses, numbers, theories, Jim Morrison, colors, basketball games, wars, centuries, ...**
 - **Relations**:
e.g., **red, round, bogus, prime, tall ...,**
brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
 - **Functions**:
e.g., **father of, best friend, one more than, end of, ...**
- Allows to **quantify** on objects
 - ex: **“All man are equal”, “some persons are left-handed”, ...**

Syntax of FOL: Basic Elements

- **Constant symbols:** KingJohn, 2, UniversityofTrento,...
- **Predicate symbols:** Man(.), Brother(.,.), (. > .), AllDifferent(...),...
 - may have different arities (1,2,3,...)
 - may be **prefix** (e.g. Brother(.,.)) or **infix** (e.g. (. > .))
- **Function symbols:** Sqrt, LeftLeg, MotherOf
 - may have different arities (1,2,3,...)
 - may be **prefix** (e.g. Sqrt(.)) or **infix** (e.g. (. + .))
- **Variable symbols:** x, y, a, b, ...
- **Propositional Connectives:** $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$
- **Equality:** “=” (also “ \neq ” s.t. “ $a \neq b$ ” shortcut for “ $\neg(a = b)$ ”)
- **Quantifiers:** “ \forall ” (“forall”), “ \exists ” (“exists”, aka “for some”)
- **Punctuation Symbols:** “,”, “(”, “)”

- Constants symbols are 0-ary function symbols
- Propositions are 0-ary predicates \implies PL subcase of FOL
- **Signature:** the set of predicate, function & constant symbols

FOL: Syntax

- Terms:

- constant or variable or $function(term_1, \dots, term_n)$
- ex: KingJohn, x, LeftLeg(Richard), $(z * \log(2))$
- denote objects in the real world (aka domain)

- Atomic sentences (aka atomic formulas):

- \top, \perp
- proposition or $predicate(term_1, \dots, term_n)$ or $term_1 = term_2$
- $(Length(LeftLeg(Richard)) > Length(LeftLeg(KingJohn)))$
- denote facts

- Non-atomic sentences/formulas:

- $\neg\alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta, \alpha \leftrightarrow \beta, \alpha \oplus \beta,$
 $\forall x.\alpha, \exists x.\alpha$ s.t. x (typically) occurs in α
- Ex: $\forall y.(Italian(y) \rightarrow President(Mattarella, y))$
 $\exists x\forall y.President(x, y) \rightarrow \forall y\exists x.President(x, y)$
 $\forall x.(P(x) \wedge Q(x)) \leftrightarrow ((\forall x.P(x)) \wedge (\forall x.Q(x)))$
 $\forall x.(((x \geq 0) \wedge (x \leq \pi)) \rightarrow (sin(x) \geq 0))$
- denote (complex) facts

Truth in FOL: Intuitions

- Sentences are true with respect to a **model**
 - containing a **domain** and an **interpretation**
- The **domain** contains ≥ 1 objects (**domain elements**) and relations and functions over them
- An **interpretation** specifies referents for
 - **variables** \rightarrow objects
 - **constant symbols** \rightarrow objects
 - **predicate symbols** \rightarrow relations
 - **function symbols** \rightarrow functional relations
- An atomic sentence $P(t_1, \dots, t_n)$ is true in an interpretation iff the objects referred to by t_1, \dots, t_n are in the relation referred to by P

FOL: Semantics

FOL Models (aka possible worlds)

- A model \mathcal{M} is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ (\langle *domain, interpretation* \rangle)
- Domain \mathcal{D} : a **non-empty** set of objects (aka **domain elements**)
- Interpretation \mathcal{I} : a (non-injective) map on elements of the signature
 - **constant symbols** \mapsto **domain elements**:
a constant symbol C is mapped into a particular object $[C]^{\mathcal{I}}$ in \mathcal{D}
 - **predicate symbols** \mapsto **domain relations**:
a k -ary predicate $P(\dots)$ is mapped into a subset $[P]^{\mathcal{I}}$ of \mathcal{D}^k
(i.e., the set of object tuples satisfying the predicate in this world)
 - **functions symbols** \mapsto **domain functions**:
a k -ary function f is mapped into a domain function $[f]^{\mathcal{I}} : \mathcal{D}^k \mapsto \mathcal{D}$ ($[f]^{\mathcal{I}}$ must be total)

(we denote by $[\cdot]^{\mathcal{I}}$ the result of the interpretation \mathcal{I})

An **Interpretation** \mathcal{I} is extended to assign domain values to variables, domain values to terms and truth values to formulas.

FOL: Semantics [cont.]

Interpretation of terms

\mathcal{I} maps terms into domain elements

- Variables are assigned domain values
 - **variables** \mapsto **domain elements**:
a variable x is mapped into a particular object $[x]^{\mathcal{I}}$ in \mathcal{D}
- A term $f(t_1, \dots, t_k)$ is mapped by \mathcal{I} into the value $[f(t_1, \dots, t_k)]^{\mathcal{I}}$ returned by applying the domain function $[f]^{\mathcal{I}}$, into which f is mapped, to the values $[t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}}$ obtained by applying recursively \mathcal{I} to the terms t_1, \dots, t_k :
 - $[f(t_1, \dots, t_k)]^{\mathcal{I}} = [f]^{\mathcal{I}}([t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}})$
 - Ex: if “**Me, Mother, Father**” are interpreted as usual, then “**Mother(Father(Me))**” is interpreted as my (paternal) grandmother
 - Ex: if “**+, -, ·, 0, 1, 2, 3, 4**” are interpreted as usual, then “**(3 - 1) · (0 + 2)**” is interpreted as **4**

FOL: Semantics [cont.]

Interpretation of formulas

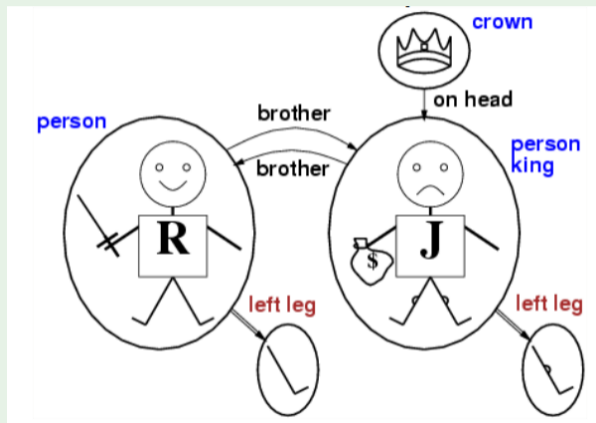
\mathcal{I} maps formulas into truth values

- An atomic formula $P(t_1, \dots, t_k)$ is true in \mathcal{I} iff the objects into which the terms t_1, \dots, t_k are mapped by \mathcal{I} comply to the relation into which P is mapped
 - $[P(t_1, \dots, t_k)]^{\mathcal{I}}$ is true iff $\langle [t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}} \rangle \in [P]^{\mathcal{I}}$
 - Ex: if “Me, Mother, Father, Married” are interpreted as tradition, then “Married(Mother(Me), Father(Me))” is interpreted as true
 - Ex: if “+, -, >, 0, 1, 2, 3, 4” are interpreted as usual, then “ $(4 - 0) > (1 + 2)$ ” is interpreted as true
- An atomic formula $t_1 = t_2$ is true in \mathcal{I} iff the terms t_1, t_2 are mapped by \mathcal{I} into the same domain element
 - $[t_1 = t_2]^{\mathcal{I}}$ is true iff $[t_1]^{\mathcal{I}}$ same as $[t_2]^{\mathcal{I}}$
 - Ex: if “Mother” is interpreted as usual, Richard, John are brothers, then “Mother(Richard)=Mother(John)” is interpreted as true
 - Ex: if “+, -, 0, 1, 2, 3, 4” are interpreted as usual, then “ $(4 - 1) = (1 + 2)$ ” is interpreted as true
- $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$ interpreted by \mathcal{I} as in PL

Models for FOL: Example

Richard Lionheart and John Lackland

- \mathcal{D} : domain at right
- \mathcal{I} : s.t.
 - $[\text{Richard}]^{\mathcal{I}}$: Richard the Lionheart
 - $[\text{John}]^{\mathcal{I}}$: evil King John
 - $[\text{Brother}]^{\mathcal{I}}$: brotherhood
- $[\text{Brother}(\text{Richard}, \text{John})]^{\mathcal{I}}$ is true
- $[\text{LeftLeg}]^{\mathcal{I}}$ maps any individual to his left leg
- ...



(© S. Russell & P. Norwig, Artificial Intelligence: A Modern Approach, III Ed., Pearson)

Equality

- Equality is a special predicate: $t_1 = t_2$ is true under a given interpretation if and only if t_1 and t_2 refer to the same object
 - Ex: $1 = 2$ and $x * x = x$ are satisfiable (!)
 - Ex: $2 = 2$ is valid
- Ex: definition of *Siblings* in terms of *Parent*
 $\forall x, y. (Siblings(x, y) \leftrightarrow [\neg(x = y) \wedge \exists p_1, p_2. (\neg(p_1 = p_2) \wedge Parent(p_1, x) \wedge Parent(p_2, x) \wedge Parent(p_1, y) \wedge Parent(p_2, y))])$)

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Traditional Definition (FOL)

Given a FOL signature Σ , a Σ -Theory \mathcal{T} (hereafter simply “theory”) is a (possibly infinite) set of FOL closed formulas (axioms)

- Typically used to provide some *intended interpretation* to the symbols in the signature Σ
- FOL formulas deduces from these axioms via inference rules
- Definition used by logicians,
- Very low practical use in AR & Formal Verification

Example: A FOL Theory of Positive Integer Numbers (aka “Peano Arithmetic”, \mathcal{P})

- Signature

- (basic) unary predicate symbol: NatNum (“natural number”)
- (basic) unary function symbol: S (“successor”)
- (basic) constant symbol: 0
- (derived) binary function symbols: $+, *$ (infix)
- (derived) constant symbols: $1, 2, 3, 4, 5, 6, \dots$

- Axioms

- 1 $\text{NatNum}(0)$
- 2 $\forall x. (\text{NatNum}(x) \rightarrow \text{NatNum}(S(x)))$
- 3 $\forall x. (\text{NatNum}(x) \rightarrow (0 \neq S(x)))$
- 4 $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
- 5 $\forall x. (\text{NatNum}(x) \rightarrow (x = (0 + x)))$
- 6 $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow (S(x) + y) = S(x + y))$
- 7 $1 = S(0), 2 = S(1), 3 = S(2), \dots$

- Formulas deduced

- ex: $\mathcal{P} \vdash \text{NatNum}(25)$
- ex: $\mathcal{P} \vdash \forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow ((x + y) = (y + x)))$

FOL Theories (cont.)

SMT Definition

Given a FOL signature Σ , a Σ -Theory \mathcal{T} (hereafter simply “theory”) is one (or more) model(s) constraining the interpretations of Σ

- Provides an **intended interpretation** to the symbols in Σ
 - constants mapped into domain elements
 - ex: “1” mapped into the number one
 - predicate symbols mapped into relations on domain elements
 - ex: “. < .” mapped into the arithmetical relation “less than”
 - function symbols mapped into functions on domain elements
 - ex: “S(.)” mapped into the arithmetical function “successor of”

These symbols are called **interpreted**

- Compliant with previous definition: **model(s) satisfying all axioms**
- Ad hoc “ \mathcal{T} -aware” decision procedures for reasoning on formulas
- Very effective in practical applications

Example: Linear Arithmetic on the Integers (\mathcal{LIA})

- Domain: integer numbers
- Numerical constants interpreted as **numbers**
 - ex: “1”, “1346231” mapped directly into the corresponding number
- function and predicates interpreted as **arithmetical operations**
 - “+” as addition, “*” as multiplication, “<” as less-than, . etc.
- **ILP solvers** used to do logical reasoning
 - ex: $(3x - 2y \leq 3) \wedge (4y - 2z < -7) \models (6x - 2z < -1)$

Satisfiability, Validity, Entailment (Modulo a Theory \mathcal{T})

Definitions

- Idea: **We restrict to models satisfying \mathcal{T}** (“ \mathcal{T} -models”)
- A formula is **satisfiable in \mathcal{T}** (aka “ φ is \mathcal{T} -satisfiable”) iff some \mathcal{T} -model satisfies also φ
 - ex: $(x < 3)$ satisfiable in \mathcal{LIA}
 - ex: $(1 = 2)$ **not** satisfiable in \mathcal{LIA} !
- A formula φ is **valid in \mathcal{T}** (aka “ φ is \mathcal{T} -valid” or “ $\models_{\mathcal{T}} \varphi$ ”) iff all \mathcal{T} -models satisfy also φ
 - ex: $(x < 3) \rightarrow (x < 4)$ valid in \mathcal{LIA}
- A formula φ **entails ψ in \mathcal{T}** (aka “ φ \mathcal{T} -entails ψ ” or “ $\varphi \models_{\mathcal{T}} \psi$ ”) iff all \mathcal{T} -models satisfying φ satisfy also ψ
 - ex: $(x < 3) \models_{\mathcal{LIA}} (x < 4)$

Properties

- φ is \mathcal{T} -valid iff $\neg\varphi$ is \mathcal{T} -unsatisfiable
 - $\varphi \models_{\mathcal{T}} \psi$ iff $\varphi \rightarrow \psi$ is \mathcal{T} -valid
- $\implies \varphi \models_{\mathcal{T}} \psi$ iff $\varphi \wedge \neg\psi$ \mathcal{T} -unsatisfiable

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Satisfiability Modulo Theories (SMT(\mathcal{T}))

Satisfiability Modulo Theories (SMT(\mathcal{T}))

The problem of deciding the satisfiability of (typically quantifier-free) formulas in some decidable first-order theory \mathcal{T}

- \mathcal{T} can also be a **combination of theories** $\bigcup_i \mathcal{T}_i$.

SMT(\mathcal{T}): Theories of Interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions (\mathcal{EUF}):
 $((x = y) \wedge (y = f(z))) \rightarrow (g(x) = g(f(z)))$
- Difference logic (\mathcal{DL}): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
- UTVPI (\mathcal{UTVPI}): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x + z \leq 6)$
- Linear arithmetic over the rationals (\mathcal{LRA}):
 $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
- Linear arithmetic over the integers (\mathcal{LIA}): $(x = x_l + 2^{16}x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
- Arrays (\mathcal{AR}): $(i = j) \vee \text{read}(\text{write}(a, i, e), j) = \text{read}(a, j)$
- Bit vectors (\mathcal{BV}): $x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \ll w_{[8]}[3 : 0]$
- Non-Linear arithmetic over the reals ($\mathcal{NLA}(\mathbb{R})$):
 $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = b + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
- ...

Satisfiability Modulo Theories (SMT(\mathcal{T})): Example

Example: SMT($\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$)

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- involves arithmetical, arrays, and uninterpreted function/predicate symbols, plus Boolean operators
 - Is it satisfiable?
 - No:

$$\begin{array}{ll} \implies_{\mathcal{LIA}} & \varphi \\ \implies_{\mathcal{EUF}} & (d = 0) \\ \implies_{\text{Bool}} & (f(d) = f(0)) \\ \implies_{\mathcal{LIA}} & (\text{read}(\text{write}(V, i, x), i + d) = x + 1) \\ \implies_{\mathcal{LIA}} & (\text{read}(\text{write}(V, i, x), i) = x + 1) \\ \implies_{\mathcal{LIA}} & \neg(\text{read}(\text{write}(V, i, x), i) = x) \\ \implies_{\mathcal{AR}} & \perp \end{array}$$

SMT and SMT solvers

Common fact about SMT problems from various applications

SMT requires capabilities for **heavy Boolean reasoning** combined with capabilities for **reasoning in expressive decidable F.O. theories**

- SAT alone not expressive enough
- standard automated theorem proving inadequate (e.g., arithmetic)
- may involve also numerical computation (e.g., simplex)

Modern SMT solvers

- combine **SAT solvers** with \mathcal{T} -specific **decision procedures** (**theory solvers** or **\mathcal{T} -solvers**)
 - contributions from SAT, Automated Theorem Proving (ATP), formal verification (FV) and operational research (OR)

Notational remark (1): most/all examples in \mathcal{LRA}

For better readability, in most/all the examples of this presentation we will use the theory of linear arithmetic on rational numbers (\mathcal{LRA}) because of its intuitive semantics. E.g.:

$$(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$$

Nevertheless, analogous examples can be built with all other theories of interest.

Notational remark (2): “constants” vs. “variables”

- Consider, e.g., the formula:
 $(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$
- How do we call A_1, A_2 ?:
 - (a) Boolean/propositional **variables**?
 - (b) uninterpreted **0-ary predicates**?
- How do we call x_1, x_2, x_3 ?:
 - (a) domain **variables**?
 - (b) uninterpreted Skolem **constants/0-ary uninterpreted functions**?
- Hint:
 - (a) typically used in SAT, CSP and OR communities
 - (b) typically used in logic & ATP communities

Hereafter we call A_1, A_2 “Boolean/propositional **variables**” and x_1, x_2, x_3 “domain **variables**” (logic purists, please forgive me!)

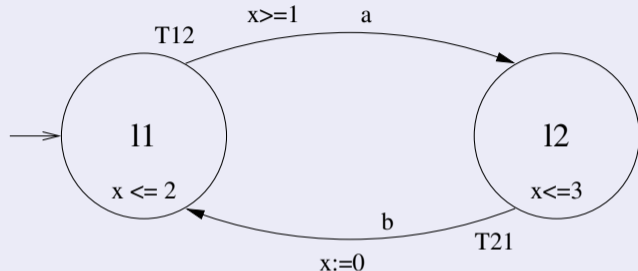
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Some Motivating Applications

Interest in SMT triggered by some real-world applications

- Verification of Hybrid & Timed Systems
- Verification of RTL Circuit Designs & of Microcode
- SW Verification
- Planning with Resources
- Temporal reasoning
- Scheduling
- Compiler optimization
- ...

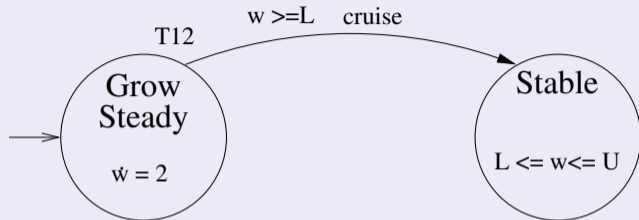
Verification of Timed Systems



- Model checking of Timed Systems [6, 35, 58], ...
- Timed Automata encoded into \mathcal{T} -formulas:
 - discrete information (locations, transitions, events) with Boolean vars.
 - timed information (clocks, elapsed time) with differences ($t_3 - x_3 \leq 2$), equalities ($x_4 = x_3$) and linear constraints ($t_8 - x_8 = t_2 - x_2$) on \mathbb{Q}

⇒ SMT on $\mathcal{DL}(\mathbb{Q})$ or \mathcal{LRA} required

Verification of Hybrid Systems ...

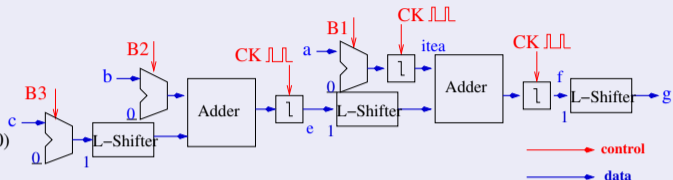


- Model checking of Hybrid Systems [5],...
- Hybrid Automata encoded into \mathcal{L} -formulas:
 - discrete information (locs, trans., events) with Boolean vars.
 - timed information (clocks, elapsed time) with differences ($t_3 - x_3 \leq 2$), equalities ($x_4 = x_3$) and linear constraints ($t_8 - x_8 = t_2 - x_2$) on \mathbb{Q}
 - Evolution of Physical Variables (e.g., speed, pressure) with linear ($\omega_4 = 2\omega_3$) and non-linear constraints ($P_1 V_1 = 4T_1$) on \mathbb{Q}
- Undecidable under simple hypotheses!

⇒ SMT on $\mathcal{DL}(\mathbb{Q})$, \mathcal{LRA} or $\mathcal{NLA}(\mathbb{R})$ required

Verification of HW circuit designs & microcode

$g = 2 * f$
 $f = itea + 2 * e$
 $itea' = ITE(B1; a; 0)$
 $e' = ITE(B2; b; 0) + 2 * ite(B3; c; 0)$



- SAT/SMT-based **Model Checking & Equiv. Checking** of RTL designs, **symbolic simulation** of μ -code [25, 22, 42]
 - **Control paths** handled by Boolean reasoning
 - **Data paths** information abstracted into theory-specific terms
 - **words** (bit-vectors, integers, \mathcal{EUF} vars, ...): $\underline{a}[31 : 0]$, a
 - **word operations**: (\mathcal{BV} , \mathcal{EUF} , \mathcal{AR} , \mathcal{LIA} , $\mathcal{NLA}(\mathbb{Z})$) operators
$$x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \ll w_{[8]}[3 : 0], (a = a_L + 2^{16}a_H), (m_1 = store(m_0, l_0, v_0)),$$

...
 - Trades **heavy Boolean reasoning** ($\approx 2^{64}$ factors) with **\mathcal{T} -solving**
- \Rightarrow SMT on \mathcal{BV} , \mathcal{EUF} , \mathcal{AR} , modulo- \mathcal{LIA} [$\mathcal{NLA}(\mathbb{Z})$] required

Verification of SW systems

```
...  
10. i = 0;  
11. acc = 0.0;  
12. while (i < dim) {  
13.   acc += V[i];  
14.   i++;  
15. }  
...
```

```
...  
(pc = 10) → ((i' = 0) ∧ (pc' = 11))  
(pc = 11) → ((acc' = 0.0) ∧ (pc' = 12))  
(pc = 12) → ((i < dim) → ∧(pc' = 13))  
(pc = 12) → (¬(i < dim) → ∧(pc' = 16))  
(pc = 13) → ((acc' = acc + read(V, i)) ∧ (pc' = 14))  
(pc = 14) → (i' = i + 1) ∧ (pc' = 15))  
(pc = 15) → (pc' = 12))  
...
```

- Verification of SW code

- BMC, K-induction, Check of proof obligations, interpolation-based model checking, symbolic simulation, concolic testing, ...

⇒ SMT on BV , \mathcal{EUF} , \mathcal{AR} , (modulo-) \mathcal{LIA} [$\mathcal{NLA}(\mathbb{Z})$] required

Planning with Resources [80]

- SAT-bases planning augmented with numerical constraints
- Straightforward to encode into into $SMT(\mathcal{LRA})$

Example (sketch) [80]

```
(Deliver)                 $\wedge$  // goal
(MaxLoad)                 $\wedge$  // load constraint
(MaxFuel)                 $\wedge$  // fuel constraint
(Move  $\rightarrow$  MinFuel)    $\wedge$  // move requires fuel
(Move  $\rightarrow$  Deliver)    $\wedge$  // move implies delivery
(GoodTrip  $\rightarrow$  Deliver)  $\wedge$  // a good trip requires
(GoodTrip  $\rightarrow$  AllLoaded)  $\wedge$  // a full delivery
-----
(MaxLoad  $\rightarrow$  (load  $\leq$  30))  $\wedge$  // load limit
(MaxFuel  $\rightarrow$  (fuel  $\leq$  15))  $\wedge$  // fuel limit
(MinFuel  $\rightarrow$  (fuel  $\geq$  7 + 0.5load))  $\wedge$  // fuel constraint
(AllLoaded  $\rightarrow$  (load = 45)) //
```

(Disjunctive) Temporal Reasoning & Scheduling [77, 2]

- Temporal reasoning problems encoded as disjunctions of difference constraints

$$\begin{aligned} & ((x_1 - x_2 \leq 6) \quad \vee \quad (x_3 - x_4 \leq -2)) \quad \wedge \\ & ((x_2 - x_3 \leq -2) \quad \vee \quad (x_4 - x_5 \leq 5)) \quad \wedge \\ & ((x_2 - x_1 \leq 4) \quad \vee \quad (x_3 - x_7 \leq -6)) \quad \wedge \\ & \dots \end{aligned}$$

- Straightforward to encode into into $\text{SMT}(\mathcal{DL})$

Goal

Provide an overview of standard “lazy” SMT:

- foundations
- SMT-solving techniques
- beyond solving: advanced SMT functionalities
- ongoing research

We do **not** cover related approaches like:

- Eager SAT encodings
- Rewrite-based approaches

We refer to [70, 10] for an overview and references.

- 1 Introduction
 - Basics on First-order Logic
 - What is a Theory?
 - Satisfiability Modulo Theories
 - Motivations and Goals of SMT
- 2 **Efficient SMT solving**
 - Combining SAT with Theory Solvers
 - Theory Solvers for Theories of Interest (hints)
 - SMT for Combinations of Theories
- 3 Beyond Solving: Advanced SMT Functionalities
 - Proofs and Unsatisfiable Cores
 - All-SMT & Predicate Abstraction (hints)
 - SMT with Optimization (Optimization Modulo Theories)

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Modern “lazy” SMT(\mathcal{T}) solvers

A prominent “lazy” approach [45, 2, 80, 3, 8, 35] (aka “DPLL(\mathcal{T})”)

- a **CDCL SAT solver** is used to enumerate truth assignments μ_i for (the Boolean abstraction φ^D of) the input formula φ
 - the Boolean abstraction φ^D of φ maps theory atoms in φ into fresh Boolean variables
- a theory-specific solver **\mathcal{T} -solver** checks the \mathcal{T} -satisfiability of the **set of \mathcal{T} -literals** corresponding to each assignment

- Built on top of modern SAT CDCL solvers
 - benefit for free from all modern CDCL techniques (e.g., **Boolean preprocessing, backjumping & learning, restarts,...**)
 - benefit for free from all state-of-the-art data structures and implementation tricks (e.g., **two-watched literals,...**)
- Many techniques to maximize the benefits of integration [70, 10]
- Many lazy SMT tools available (**Barcelogic, CVC5, MathSAT, OpenSMT, Yices, Z3, ...**)

Basic schema: example

$\varphi =$

$$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$$

$$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$$

$$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$$

$$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$$

$$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$$

$$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$$

$$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$$

true, false

$\varphi^p =$

$$\neg B_1 \vee A_1$$

$$\neg A_2 \vee B_2$$

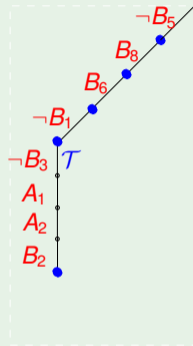
$$B_3 \vee A_2$$

$$\neg B_4 \vee \neg B_5 \vee \neg A_1$$

$$A_1 \vee B_3$$

$$B_6 \vee B_7 \vee \neg A_1$$

$$A_1 \vee B_8 \vee A_2$$



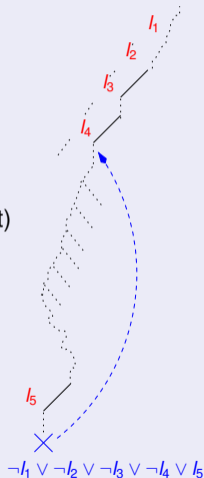
$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$

\implies unsatisfiable in $\mathcal{LR}\mathcal{A} \implies$ backtrack

\mathcal{T} -Backjumping & \mathcal{T} -learning [50, 80, 3, 8, 35]

- Similar to Boolean backjumping & learning
- important property of \mathcal{T} -solver:
 - **extraction of \mathcal{T} -conflict sets**: if μ is \mathcal{T} -unsatisfiable, then \mathcal{T} -solver(μ) returns the subset η of μ causing the \mathcal{T} -unsatisfiability of μ (\mathcal{T} -conflict set)
- If so, the **\mathcal{T} -conflict clause** $C := \neg\eta$ is used to drive the backjumping & learning mechanism of the SAT solver
⇒ lots of search saved
- **the less redundant is η , the more search is saved**

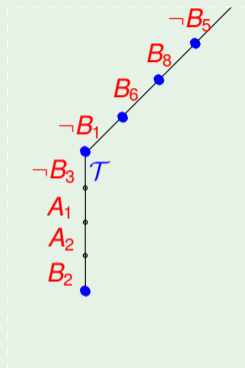


\mathcal{T} -Backjumping & \mathcal{T} -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

true, false



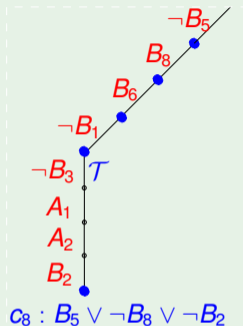
$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$

\mathcal{T} -Backjumping & \mathcal{T} -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

true, false



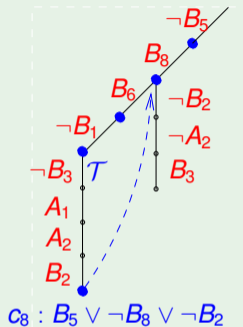
$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$

\mathcal{T} -Backjumping & \mathcal{T} -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

true, false



$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$

\mathcal{T} -Backjumping & \mathcal{T} -learning: example (2)

$\varphi =$

$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$c_8' : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$

$c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$

true, false

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

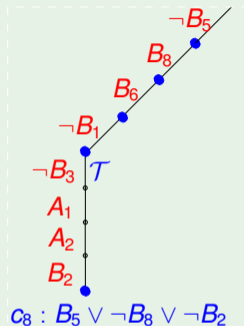
$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

$B_5 \vee \neg B_8 \vee B_1$

$B_5 \vee \neg B_8 \vee \neg B_2$



c_8 : theory conflicting clause

$$\frac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_2} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2} \quad (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3} \quad (\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_T}}{B_5 \vee \neg B_8 \vee B_1} \quad (B_3)$$

c_8' : mixed Boolean+theory conflict clause

\mathcal{T} -Backjumping & \mathcal{T} -learning: example (2)

$\varphi =$

$$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$$

$$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$$

$$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$$

$$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$$

$$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$$

$$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$$

$$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$$

$$c'_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$$

$$c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$$

true, false

$\varphi^p =$

$$\neg B_1 \vee A_1$$

$$\neg A_2 \vee B_2$$

$$B_3 \vee A_2$$

$$\neg B_4 \vee \neg B_5 \vee \neg A_1$$

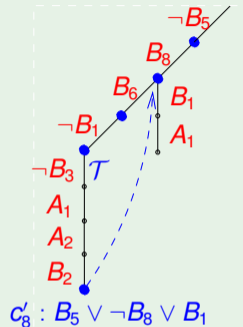
$$A_1 \vee B_3$$

$$B_6 \vee B_7 \vee \neg A_1$$

$$A_1 \vee B_8 \vee A_2$$

$$B_5 \vee \neg B_8 \vee B_1$$

$$B_5 \vee \neg B_8 \vee \neg B_2$$



c_8 : theory conflicting clause

$$\frac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_2} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2} \quad (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3} \quad (\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_T} \quad (B_3)$$

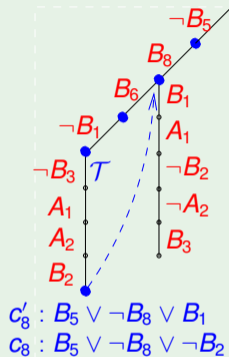
$$\frac{B_5 \vee \neg B_8 \vee B_3}{B_5 \vee \neg B_8 \vee B_1}$$

c'_8 : mixed Boolean+theory conflict clause

\mathcal{T} -Backjumping & \mathcal{T} -learning: example (2)

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8' : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \\ \text{true, false} \end{array}$$

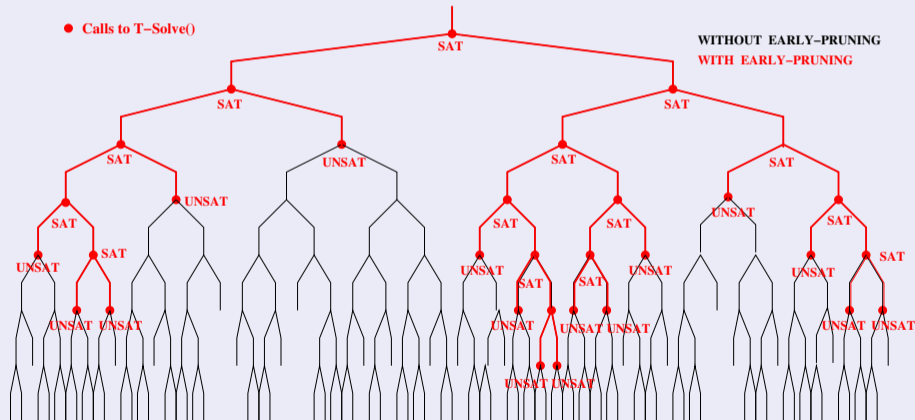
$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee B_1 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$



$$\begin{array}{c} c_8: \text{theory conflicting clause} \\ \frac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_2} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2} \quad (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3} \\ \frac{\overbrace{B_5 \vee \neg B_8 \vee \neg A_2} \quad \overbrace{B_3 \vee A_2}}{B_5 \vee \neg B_8 \vee B_3} \quad (\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_T} \\ \frac{\overbrace{B_5 \vee \neg B_8 \vee B_3} \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}}{B_5 \vee \neg B_8 \vee B_1} \quad (B_3) \\ c_8': \text{mixed Boolean+theory conflict clause} \end{array}$$

Early Pruning [45, 2, 80]

- Introduce a \mathcal{T} -satisfiability test on **intermediate assignments**:
if \mathcal{T} -solver returns UNSAT, the procedure backtracks.
 - benefit: prunes drastically the Boolean search
 - Drawback: possibly **many useless calls to \mathcal{T} -solver**



Early pruning: example

$$\begin{aligned}\varphi = & \{ \neg(2v_2 - v_3 > 2) \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee (2v_1 - 4v_5 > 3) \} \wedge \\ & \{ (3v_1 - 2v_2 \leq 3) \vee A_2 \} \wedge \\ & \{ \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (3v_1 - 2v_2 \leq 3) \} \wedge \\ & \{ (v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \}. \\ \varphi^p = & \{ \neg B_1 \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee B_2 \} \wedge \\ & \{ B_3 \vee A_2 \} \wedge \\ & \{ \neg B_4 \vee \neg B_5 \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee B_3 \} \wedge \\ & \{ B_6 \vee B_7 \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee B_8 \vee A_2 \}.\end{aligned}$$

- Suppose it is built the intermediate assignment:

$$\mu^p = \neg B_1 \wedge \neg A_2 \wedge B_3 \wedge \neg B_5.$$

corresponding to the following set of \mathcal{T} -literals

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

- If \mathcal{T} -solver is invoked on μ' , then it returns UNSAT, and DPLL backtracks **without exploring any extension of μ'** .

Early Pruning [45, 2, 80] (cont.)

- Different strategies for interleaving Boolean search steps and \mathcal{T} -solver calls
 - **Eager E.P.** [80, 11, 78, 44]): invoke \mathcal{T} -solver every time a new \mathcal{T} -atom is added to the assignment (unit propagations included)
 - **Selective E.P.**: Do not call \mathcal{T} -solver if the have been added only literals which hardly cause any \mathcal{T} -conflict with the previous assignment (e.g., Boolean literals, disequalities $(x - y \neq 3)$, \mathcal{T} -literals introducing new variables $(x - z = 3)$)
 - **Weakened E.P.**: for intermediate checks only, use **weaker** but faster versions of \mathcal{T} -solver (e.g., check μ on \mathbb{R} rather than on \mathbb{Z}): $\{(x - y \leq 4), (z - x \leq -6), (z = y), (3x + 2y - 3z = 4)\}$

Early pruning: remark

Incrementality & Backtrackability of \mathcal{T} -solvers

- With early pruning, lots of **incremental calls to \mathcal{T} -solver**:

$\mathcal{T}\text{-solver}(\mu_1)$	$\Rightarrow \text{Sat}$	Undo μ_4, μ_3, μ_2	
$\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$	$\Rightarrow \text{Sat}$	$\mathcal{T}\text{-solver}(\mu_1 \cup \mu'_2)$	$\Rightarrow \text{Sat}$
$\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2 \cup \mu_3)$	$\Rightarrow \text{Sat}$	$\mathcal{T}\text{-solver}(\mu_1 \cup \mu'_2 \cup \mu'_3)$	$\Rightarrow \text{Sat}$
$\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2 \cup \mu_3 \cup \mu_4)$	$\Rightarrow \text{Unsat}$...	

\Rightarrow Desirable features of \mathcal{T} -solvers:

- **incrementality**: $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}\text{-solver}(\mu_1)$ without restarting from scratch
- **backtrackability (resettability)**: $\mathcal{T}\text{-solver}$ can efficiently undo steps and return to a previous status on the stack

\Rightarrow \mathcal{T} -solver requires a **stack-based interface**

\mathcal{T} -Propagation [2, 3, 44]

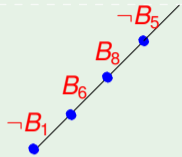
- strictly related to early pruning
- important property of \mathcal{T} -solver:
 - **\mathcal{T} -deduction**: when a partial assignment μ is \mathcal{T} -satisfiable, \mathcal{T} -solver may be able to return also an assignment η to some unassigned atom occurring in φ s.t. $\mu \models_{\mathcal{T}} \eta$.
- If so:
 - the literal η is then unit-propagated;
 - optionally, a **\mathcal{T} -deduction clause** $C := \neg\mu' \vee \eta$ can be learned, μ' being the subset of μ which caused the deduction ($\mu' \models_{\mathcal{T}} \eta$)
 - **lazy explanation**: compute C only if needed for conflict analysis

⇒ may prune drastically the search

Both \mathcal{T} -deduction clauses and \mathcal{T} -conflict clauses are called **\mathcal{T} -lemmas** since they are valid in \mathcal{T}

\mathcal{T} -propagation: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \end{array} \quad \begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \end{array}$$



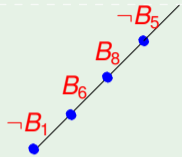
true, false

$$\begin{aligned} \mu^p &= \{\neg B_5, B_8, B_6, \neg B_1\} \\ \mu &= \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2)\} \\ &\models_{\mathcal{LR}, \mathcal{A}} \underbrace{\neg(3v_1 - 2v_2 \leq 3)}_{\neg B_3} \end{aligned}$$

\Rightarrow propagate $\neg B_3$ [and learn the deduction clause $B_5 \vee B_1 \vee \neg B_3$]

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$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \end{array} \quad \begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \end{array}$$



true, false

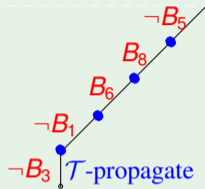
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\implies propagate $\neg B_3$ [and learn the deduction clause $B_5 \vee B_1 \vee \neg B_3$]

Pure-literal filtering [80, 3, 17]

Property

If we have non-Boolean \mathcal{T} -atoms occurring only positively [negatively] in the original formula φ (learned clauses are not considered), we can drop every negative [positive] occurrence of them from the assignment to be checked by \mathcal{T} -solver (and from the \mathcal{T} -deducible ones).

- increases the chances of finding a model
- reduces the effort for the \mathcal{T} -solver
- eliminates unnecessary “nasty” negated literals (e.g. negative equalities like $\neg(3v_1 - 9v_2 = 3)$ in \mathcal{LIA} force splitting: $(3v_1 - 9v_2 > 3) \vee (3v_1 - 9v_2 < 3)$).
- may weaken the effect of early pruning.

Pure literal filtering: example

$$\begin{aligned}\varphi = & \{ \neg(2v_2 - v_3 > 2) \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee (2v_1 - 4v_5 > 3) \} \wedge \\ & \{ (3v_1 - 2v_2 \leq 3) \vee A_2 \} \wedge \\ & \{ \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq -2) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (3v_1 - 2v_2 \leq 3) \} \wedge \\ & \{ (v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \} \wedge \\ & \{ (2v_2 - v_3 > 2) \vee \neg(3v_1 - 2v_2 \leq 3) \vee (3v_1 - v_3 \leq -2) \} \text{ learned}\end{aligned}$$

$$\mu' = \{ \neg(2v_2 - v_3 > 2), \neg A_2, (3v_1 - 2v_2 \leq 3), \neg A_1, (v_3 = 3v_5 + 4), (3v_1 - v_3 \leq -2) \}.$$

\implies Sat: $v_1 = v_2 = v_3 = 0, v_5 = -4/3$ is a solution

Note

- $(3v_1 - v_3 \leq -2)$ “filtered out” from μ' because it occurs only negatively in the original formula φ
- $\mu' \cup \{(3v_1 - v_3 \leq -2)\}$ is \mathcal{LRA} -unsatisfiable

Preprocessing atoms [45, 50, 4]

Source of inefficiency:

Semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other]

⇒ they may be assigned different [resp. identical] truth values.

⇒ lots of redundant unsatisfiable assignment generated

Solution

Rewrite a priori trivially-equivalent atoms/literals into the same atom/literal.

Preprocessing atoms (cont.)

- **Sorting:** $(v_1 + v_2 \leq v_3 + 1), (v_2 + v_1 \leq v_3 + 1), (v_1 + v_2 - 1 \leq v_3) \implies (v_1 + v_2 - v_3 \leq 1)$;
- **Rewriting dual operators:**
 $(v_1 < v_2), (v_1 \geq v_2) \implies (v_1 < v_2), \neg(v_1 < v_2)$
- **Exploiting associativity:**
 $(v_1 + (v_2 + v_3) = 1), ((v_1 + v_2) + v_3) = 1 \implies (v_1 + v_2 + v_3 = 1)$;
- **Factoring** $(v_1 + 2.0v_2 \leq 4.0), (-2.0v_1 - 4.0v_2 \geq -8.0), \implies (0.25v_1 + 0.5v_2 \leq 1.0)$;
- **Exploiting properties of \mathcal{T} :**
 $(v_1 \leq 3), (v_1 < 4) \implies (v_1 \leq 3)$ if $v_1 \in \mathbb{Z}$;
- ...

Static Learning [2, 4]

- Often possible to quickly detect a priori short and “obviously unsatisfiable” pairs or triplets of literals occurring in φ .
 - mutual exclusion $\{x = 0, x = 1\}$,
 - congruence $\{(x_1 = y_1), (x_2 = y_2), \neg(f(x_1, x_2) = f(y_1, y_2))\}$,
 - transitivity $\{(x - y = 2), (y - z \leq 4), \neg(x - z \leq 7)\}$,
 - substitution $\{(x = y), (2x - 3z \leq 3), \neg(2y - 3z \leq 3)\}$
 - ...
 - Preprocessing step: detect these literals and add blocking clauses to the input formula:
(e.g., $\neg(x = 0) \vee \neg(x = 1)$)
- ⇒ **No assignment including one such group of literals is ever generated:** as soon as all but one literals are assigned, the remaining one is immediately assigned false by unit-propagation.

Other optimization techniques

- \mathcal{T} -deduced-literal filtering
- Ghost-literal filtering
- \mathcal{T} -solver layering
- \mathcal{T} -solver clustering
- ...

(see [70, 10] for an overview)

Other SAT-solving techniques for SMT?

Frequently-asked question:

Are CDCL SAT solvers the only suitable Boolean Engines for SMT?

Some previous attempts:

- Ordered Binary Decision Diagrams (OBDDs) [81, 60, 1]
- Stochastic Local Search [49]

CDCL based currently much more efficient.

SMT formulas = “partially-invisible” SAT formulas

An SMT problem φ from the perspective of a SAT solver:

- a “partially-invisible” Boolean CNF formula $\varphi^p \wedge \tau^p$:
 - φ^p : the Boolean abstraction of the input formula φ
 - τ^p : (the Boolean abstraction of) the set τ of all \mathcal{T} -lemmas on atoms in φ .

φ \mathcal{T} -satisfiable iff $\varphi^p \wedge \tau^p$ satisfiable.

- the SAT solver:
 - “sees” only φ^p
 - finds μ^p s.t. $\mu^p \models \varphi^p$
 - cannot state if $\mu^p \models \tau^p$
 - invokes \mathcal{T} -solver on μ^p
- the \mathcal{T} -solver:
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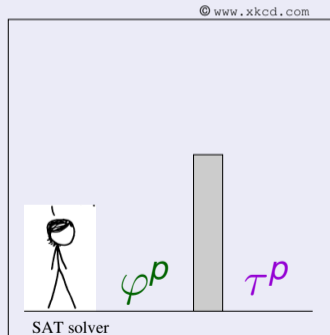
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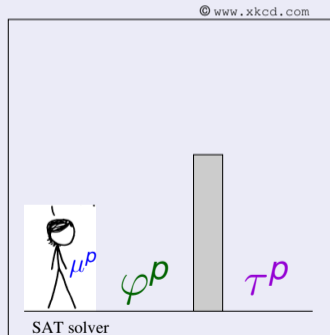
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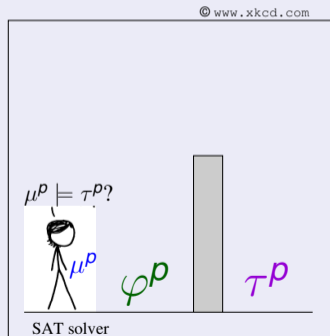
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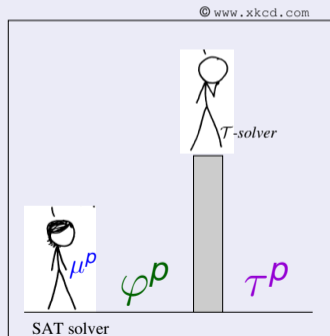
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 - if no, returns UNSAT and some falsified clauses $c_1^p, \dots, c_k^p \in \tau^p$



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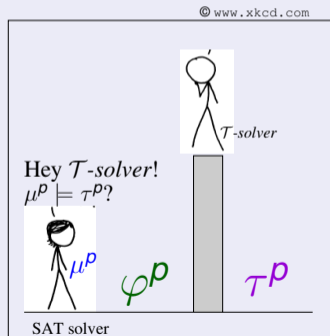
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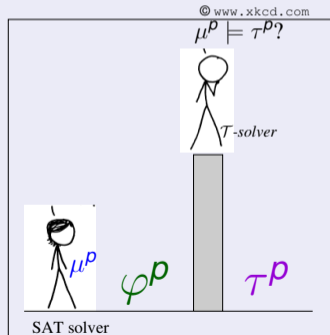
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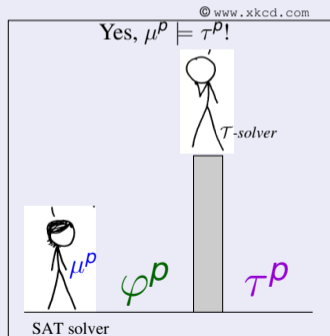
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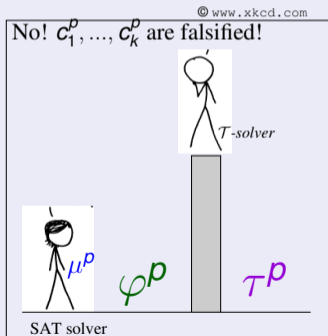
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Example

φ :

c_1 : $\{A_1\}$

c_2 : $\{\neg A_1 \vee (x - z > 4)\}$

c_3 : $\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$

c_4 : $\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$

c_5 : $\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$

c_6 : $\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$

c_7 : $\{A_3 \vee \neg(x + y = 0) \vee A_2\}$

c_8 : $\{\neg A_3 \vee (z + y = 2)\}$

τ : (all possible \mathcal{T} -lemmas on the \mathcal{T} -atoms of φ)

c_9 : $\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$

c_{10} : $\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$

c_{11} : $\{(x - z > 4) \vee (x - y \leq 3) \vee (y - z \leq 1)\}$

c_{12} : $\{\neg(x - z > 4) \vee \neg(x + y = 1) \vee \neg(z + y = 2)\}$

c_{13} : $\{\neg(x - z > 4) \vee \neg(x + y = 0) \vee \neg(z + y = 2)\}$

...

μ_1^p : $\{A_1, B_1, \neg A_2, A_3, \neg A_4, \neg A_5, \neg B_6, B_5, B_3, B_4, B_7, \neg B_2\}$

μ_1 : $\{\underline{(x - z > 4)}, \neg(x + y = 0), \underline{(x + y = 1)}, \underline{(x - y \leq 3)}, \underline{(y - z \leq 1)}, \underline{(z + y = 2)}, \neg(y \geq 1)\}$

satisfies φ^p , but violates both c_{10} and c_{12} in τ^p .

φ^p :

c_1 : $\{A_1\}$

c_2 : $\{\neg A_1 \vee B_1\}$

c_3 : $\{\neg A_3 \vee A_1 \vee B_2\}$

c_4 : $\{\neg A_2 \vee \neg B_1 \vee \neg A_1\}$

c_5 : $\{B_3 \vee \neg A_4 \vee A_5\}$

c_6 : $\{\neg B_4 \vee B_5 \vee \neg A_5\}$

c_7 : $\{A_3 \vee \neg B_6 \vee A_2\}$

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τ^p :

c_9 : $\{\neg B_6 \vee \neg B_5\}$

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c_{13} : $\{\neg B_1 \vee \neg B_6 \vee \neg B_7\}$

...

Exercise

Consider the following formula in the theory \mathcal{EUF} .

$$\begin{aligned}\varphi = & \{(f(x) = f(f(y))) \vee A_2\} \wedge \\ & \{\neg(h(x, f(y)) = h(g(x), y)) \vee \neg(h(x, g(z) = h(f(x), y))) \vee \neg A_1\} \wedge \\ & \{A_1 \vee (h(x, y) = h(y, x))\} \wedge \\ & \{\underline{x = f(x)} \vee A_3 \vee \neg A_1\} \wedge \\ & \{\neg(w(x) = g(f(y))) \vee A_1\} \wedge \\ & \{\underline{\neg A_2} \vee (w(g(x)) = w(f(x)))\} \wedge \\ & \{A_1 \vee \underline{y = g(z)} \vee A_2\}\end{aligned}$$

and consider the partial truth assignment μ given by the underlined literals above:

$$\{\neg(w(x) = g(f(y))), \neg A_2, \neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\}.$$

- 1 Does (the Boolean abstraction of) μ propositionally satisfy (the Boolean abstraction of) φ ?
- 2 Is μ satisfiable in \mathcal{EUF} ?
 - 1 If no, find a minimal conflict set for μ and the corresponding conflict clause C .
 - 2 If yes, show one unassigned literal which can be deduced from μ , and show the corresponding deduction clause C .

- 1 Introduction
 - Basics on First-order Logic
 - What is a Theory?
 - Satisfiability Modulo Theories
 - Motivations and Goals of SMT
- 2 **Efficient SMT solving**
 - Combining SAT with Theory Solvers
 - **Theory Solvers for Theories of Interest (hints)**
 - SMT for Combinations of Theories
- 3 Beyond Solving: Advanced SMT Functionalities
 - Proofs and Unsatisfiable Cores
 - All-SMT & Predicate Abstraction (hints)
 - SMT with Optimization (Optimization Modulo Theories)

Summary: desirable properties for \mathcal{T} -solver

- Correctness & Completeness: be correct & complete
- Time efficiency: be fast
- Incrementality & backtrackability: $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}\text{-solver}(\mu_1)$
- Diagnosis capabilities: $\mathcal{T}\text{-solver}$ able to produce conflict sets
- Deduction capabilities: $\mathcal{T}\text{-solver}$ able to deduce assignments

\mathcal{T} -solvers for Equality and Uninterpreted Functions (\mathcal{EUF})

- Typically used as a “core” \mathcal{T} -solver
- \mathcal{EUF} polynomial: $O(n \cdot \log(n))$
- Fully incremental and backtrackable (stack-based)
- Uses a congruence closure data structures (**E-Graphs**) [39, 64, 34],
 - based on the Union-Find data-structure for equivalence classes
- Supports efficient \mathcal{T} -propagation
 - Exhaustive for positive equalities
 - Incomplete for disequalities
- Supports Lazy explanations and conflict generation
 - However, minimality not guaranteed
- Supports efficient extensions
(e.g., Integer offsets, Bit-vector slicing and concatenation)

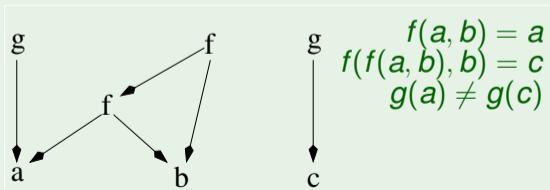
\mathcal{T} -solvers for \mathcal{EUF} : Example

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given the set of terms occurring in the formula represented as nodes in a DAG (aka [term bank](#)):

- if $(t = s)$, then merge the eq. classes of t and s
 - e.g. use the [union-find](#) data structure
- if $\forall i \in 1 \dots k, t_i$ and s_i pairwise belong to the same eq. classes, then merge the eq. classes of $f(t_1, \dots, t_k)$ and $f(s_1, \dots, s_k)$
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E-Graph:



Example borrowed from [39].

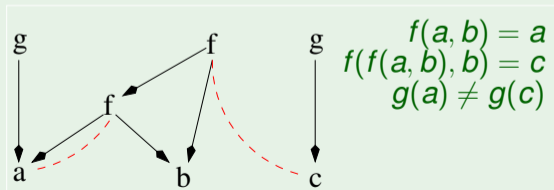
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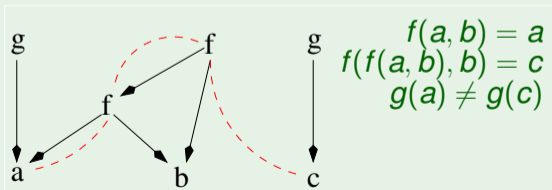
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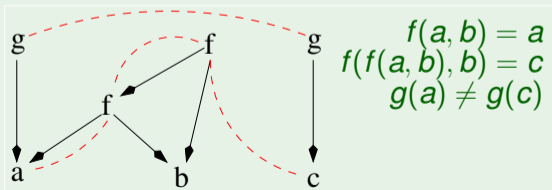
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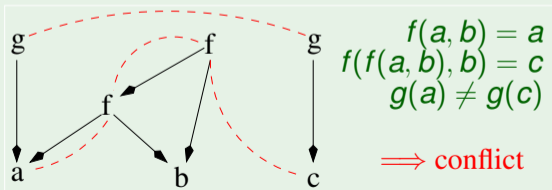
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Idea (sketch), based on **E-Graphs**:

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E-Graph:



Example borrowed from [39].

Exercises

For each of the following sets of \mathcal{EUF} -literals

- 1 build the E-graph
- 2 say if the set is \mathcal{EUF} -satisfiable or not
- 3 if \mathcal{EUF} -unsatisfiable, return a \mathcal{EUF} -conflict set

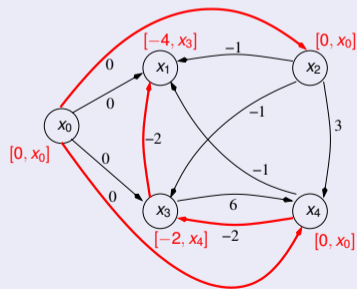
Sets of \mathcal{EUF} -literals:

- (a) $\{g(a) = b, h(b) = a, h(a) = c, g(h(b)) = c, h(h(g(a))) \neq c\}$
- (b) $\{g(a) = b, h(b) = a, h(a) = c, f(h(g(a)), g(h(b))) \neq f(a, b)\}$
- (c) [... invent your own sets of \mathcal{EUF} -literals]

\mathcal{T} -solvers for Difference logic (\mathcal{DL})

- \mathcal{DL} polynomial: $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 33]
- Ex:

$$\{(x_1 - x_2 \leq -1), (x_1 - x_4 \leq -1), (x_1 - x_3 \leq -2), (x_2 - x_1 \leq 2), \\ (x_3 - x_4 \leq -2), (x_3 - x_2 \leq -1), (x_4 - x_2 \leq 3), (x_4 - x_3 \leq 6)\}$$

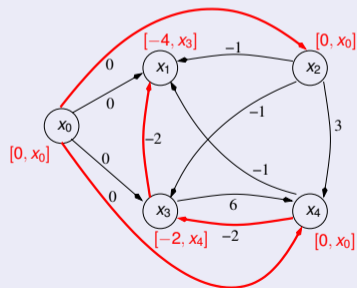


\Rightarrow Sat

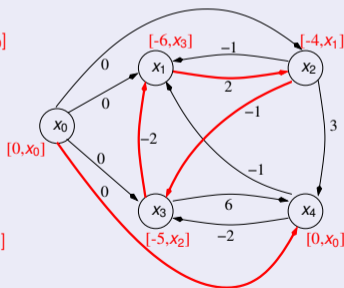
T-solvers for Difference logic (\mathcal{DL})

- \mathcal{DL} polynomial: $O(\#vars \cdot \#constraints)$
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$$\{(x_1 - x_2 \leq -1), (x_1 - x_4 \leq -1), (x_1 - x_3 \leq -2), (x_2 - x_1 \leq 2), \\ (x_3 - x_4 \leq -2), (x_3 - x_2 \leq -1), (x_4 - x_2 \leq 3), (x_4 - x_3 \leq 6)\}$$



\Rightarrow Sat



\Rightarrow Unsat

Exercises

For each of the following sets of \mathcal{DL} -literals

- 1 build the minimum-path graph
- 2 say if the set is \mathcal{DL} -satisfiable or not
- 3 if \mathcal{DL} -unsatisfiable, return a \mathcal{DL} -conflict set

Sets of \mathcal{DL} -literals:

$$(a) \left\{ \begin{array}{l} (x_1 - x_3 \leq 5), (x_1 - x_2 \leq -2), (x_1 - x_4 \leq -3), (x_2 - x_4 \leq 0), (x_3 - x_5 \leq -2), \\ (x_4 - x_5 \leq 8), (x_5 - x_1 \leq -4) \end{array} \right\}$$

$$(b) \left\{ \begin{array}{l} (x_1 - x_2 \leq 3), (x_1 - x_4 \leq -1), (x_1 - x_5 \leq 4), (x_2 - x_3 \leq 0), (x_2 - x_5 \leq -1) \\ (x_3 - x_1 \leq 1), (x_3 - x_2 \leq 2), (x_4 - x_3 \leq 1), (x_5 - x_3 \leq -2), (x_5 - x_4 \leq -5) \end{array} \right\}$$

(c) [... invent your own sets of \mathcal{DL} -literals]

\mathcal{T} -solvers for Linear arithmetic over the rationals (\mathcal{LRA})

- EX: $\{(s_1 - s_2 \leq 5.2), (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0), \neg(s_1 = s_0)\}$
- \mathcal{LRA} polynomial
- variants of the simplex LP algorithm [41]
- [41] allows for detecting conflict sets & performing \mathcal{T} -propagation
- strict inequalities $t < 0$ rewritten as $t + \epsilon \leq 0$, ϵ treated symbolically

$$\begin{array}{c} \mathcal{B} \\ \left[\begin{array}{c} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{array} \right] \end{array} = \begin{array}{c} \left[\begin{array}{ccc} \dots & A_{1j} & \dots \\ & \vdots & \\ A_{i1} & \dots & A_{ij} & \dots & A_{iM} \\ & \vdots & & & \\ \dots & A_{Nj} & \dots \end{array} \right] \end{array} \begin{array}{c} \mathcal{N} \\ \left[\begin{array}{c} x_{N+1} \\ \vdots \\ x_j \\ \vdots \\ x_{N+M} \end{array} \right] \end{array} ;$$

Invariant: $\beta(x_j) \in [l_j, u_j] \forall x_j \in \mathcal{N}$

Remark: infinite precision arithmetic

In order to avoid incorrect results due to numerical errors and to overflows, all \mathcal{T} -solvers for \mathcal{LRA} , \mathcal{LIA} and their subtheories which are based on numerical algorithms must be implemented on top of infinite-precision-arithmetic software packages.

\mathcal{T} -solvers for Linear arithmetic over the integers (\mathcal{LIA})

- EX: $\{(x := x_l + 2^{16}x_h), (x \geq 0), (x \leq 2^{16} - 1)\}$
- \mathcal{LIA} NP-complete
- combination of many techniques: simplex, branch&bound, cutting planes, ... [41, 47]

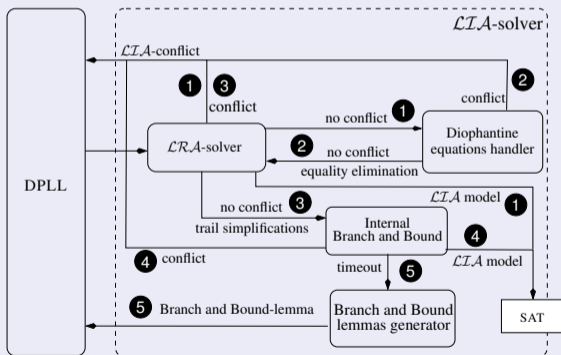


Figure courtesy of A. Griggio [47]

\mathcal{T} -solvers for Arrays (\mathcal{AR})

- EX: $(write(A, i, v) = write(B, i, w)) \wedge \neg(v = w)$
- NP-complete
- congruence closure (\mathcal{EUF}) plus on-the-fly instantiation of array's axioms:

$$\forall a. \forall i. \forall e. (read(write(a, i, e), i) = e), \quad (1)$$

$$\forall a. \forall i. \forall j. \forall e. ((i \neq j) \rightarrow read(write(a, i, e), j) = read(a, j)), \quad (2)$$

$$\forall a. \forall b. (\forall i. (read(a, i) = read(b, i)) \rightarrow (a = b)). \quad (3)$$

- EX:

$$\text{Input : } (write(A, i, v) = write(B, i, w)) \wedge \neg(v = w)$$

$$\text{inst. (1) : } (read(write(A, i, v), i) = v) \\ (read(write(B, i, w), i) = w)$$

$$\models_{\mathcal{EUF}} (v = w)$$

$$\models_{Bool} \perp$$

- many strategies discussed in the literature (e.g., [39, 46, 20, 38])

Exercises

For each of the following sets of \mathcal{AR} -literals, try to infer \perp by instantiating the \mathcal{AR} axioms and applying \mathcal{EUF} -inference and \mathcal{BOOL} -inference.

Sets of \mathcal{AR} -literals:

- (a) $\{ A_1 = \text{Write}(A_0, i, x), A_2 = \text{Write}(A_0, j, y), (i = j), \neg(k = i), \neg(\text{Read}(A_2, k) = \text{Read}(A_1, k)) \}$
- (b) $\{ A_1 = \text{Write}(A_0, i, x), A_2 = \text{Write}(A_0, i, y), \neg(x = y), (A_1 = A_2) \}$
- (c) $\{ A_1 = \text{Write}(A_0, i, x), \text{Read}(A_0, i) = \text{Read}(A_1, i), \neg(A_1 = A_0) \}$
- (d) [... invent your own sets of \mathcal{AR} -literals]

\mathcal{T} -solvers for Bit vectors (\mathcal{BV})

Bit vectors (\mathcal{BV})

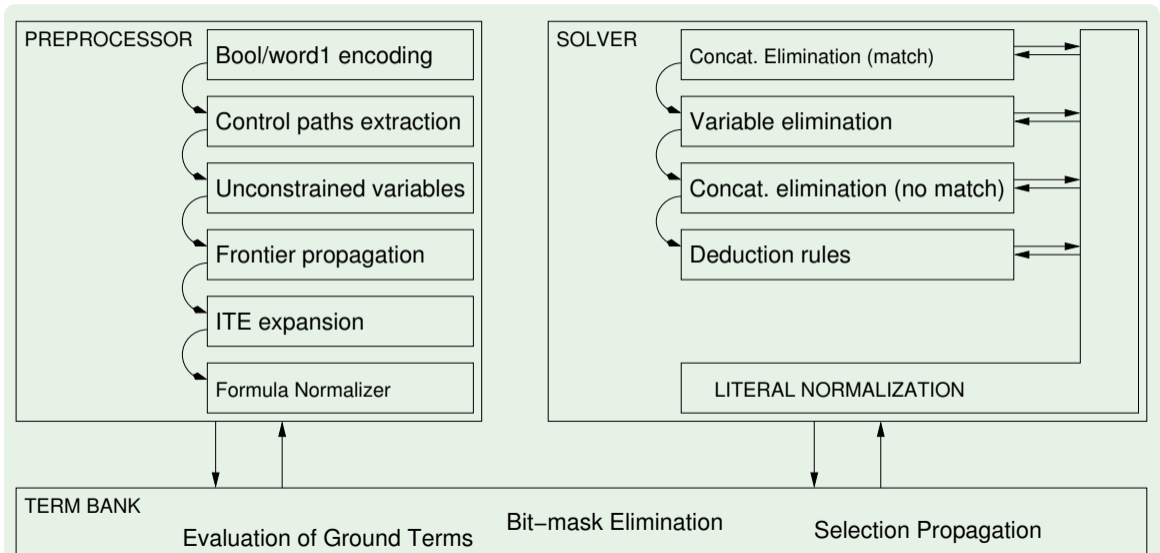
- EX: $\{(x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \ll w_{[16]}[3 : 0]), \dots\}$
- NP-hard
- involve complex word-level operations: word partition/concat, modulo- 2^N arithmetic, shifts, bitwise-operations, multiplexers, ...
- \mathcal{T} -solving: combination of rewriting & simplification techniques with either:
 - final encoding into \mathcal{LIA} [19, 22]
 - final encoding into SAT (**lazy bit-blasting**) [25, 43, 21, 42]

Eager approach

Most solvers use an **eager** approach for \mathcal{BV} (e.g., [21]):

- Heavy preprocessing, based on rewriting rules
- bit-blasting

\mathcal{T} -solvers for Bit vectors (BV) [cont.]



Example borrowed from [22]

\mathcal{T} -solvers for Bit vectors (\mathcal{BV}) [cont.]

Lazy bit-blasting

- Two nested SAT solvers

- bit-blast each \mathcal{BV} atom ψ_i

$$\implies \Phi \stackrel{\text{def}}{=} \bigwedge_i (A_i \leftrightarrow \text{BB}(\psi_i)),$$

A_i fresh variables labeling \mathcal{BV} -atoms ψ_i in φ

$\implies \varphi$ \mathcal{BV} -satisfiable iff $\varphi^p \wedge \Phi$ satisfiable

- Exploit SAT under assumptions

- let μ^p an assignment for φ^p , s.t. $\mu^p \stackrel{\text{def}}{=} \{[\neg]A_1, \dots, [\neg]A_n\}$

- \mathcal{T} -solver for \mathcal{BV} : $\text{SAT}_{\text{assumption}}(\Phi, \mu^p)$

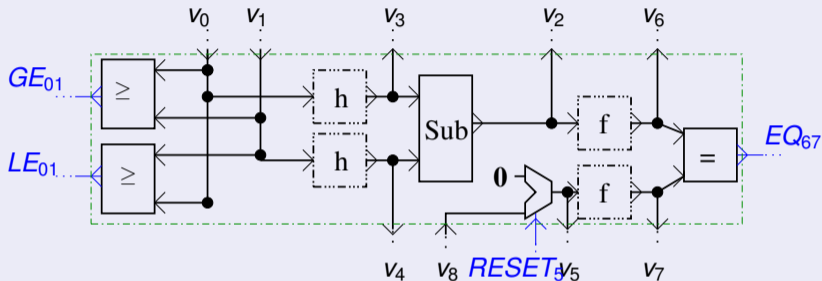
- If UNSAT, generate the **unsat core** $\eta^p \subseteq \mu^p$

$\implies \neg\eta^p$ used as blocking clause

- 1 Introduction
 - Basics on First-order Logic
 - What is a Theory?
 - Satisfiability Modulo Theories
 - Motivations and Goals of SMT
- 2 **Efficient SMT solving**
 - Combining SAT with Theory Solvers
 - Theory Solvers for Theories of Interest (hints)
 - **SMT for Combinations of Theories**
- 3 Beyond Solving: Advanced SMT Functionalities
 - Proofs and Unsatisfiable Cores
 - All-SMT & Predicate Abstraction (hints)
 - SMT with Optimization (Optimization Modulo Theories)

SMT for combined theories: $SMT(\cup_i T_i)$

Problem: Many problems can be expressed as SMT problems only in combination of theories



$\cup_i T_i - SMT(\cup_i T_i)$

$\mathcal{L}IA$: $(GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge$

$\mathcal{E}UF$: $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge$

$\mathcal{L}IA$: $(v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

$\mathcal{E}UF$ or $\mathcal{L}IA$: $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge$

$\mathcal{E}UF$: $(v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{E}UF$ or $\mathcal{L}IA$: $(EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \dots$

SMT for combined theories: $\text{SMT}(\mathcal{T}_1 \cup \mathcal{T}_2)$

- Combined theories may be much harder to decide [Pratt'77]
- Solvers have to be combined
- Standard approach for combining \mathcal{T}_i -solvers:
(deterministic) Nelson-Oppen/Shostak (N.O.) [61, 63, 75]
 - based on deduction and exchange of equalities on shared variables
 - combined \mathcal{T}_i -solvers integrated with a SAT tool
- SMT-specific approaches: Delayed Theory Combination [15, 14] and Model-Based Theory Combination [36]
 - based on Boolean search on equalities on shared variables
 - \mathcal{T}_i -solvers integrated directly with a SAT tool

Background: Pure Formulas

Consider two theories $\mathcal{T}_1, \mathcal{T}_2$ with equality and disjoint signatures Σ_1, Σ_2

- W.l.o.g. we assume all input formulas $\phi \in \mathcal{T}_1 \cup \mathcal{T}_2$ are **pure**.
 - A formula ϕ is **pure** iff every atom in ϕ is i -pure for some $i \in \{1, 2\}$.
 - An atom/literal ψ in ϕ is i -**pure** if only $=$, variables and symbols from Σ_i can occur in ψ

Purification:

Maps a formula into an equisatisfiable pure formula by labeling terms with fresh variables

$$(f(\underbrace{x + 3y}_w) = g(\underbrace{2x - y}_t)) \quad [not\ pure]$$

$$\Downarrow$$
$$(w = x + 3y) \wedge (t = 2x - y) \wedge (f(w) = g(t)) \quad [pure]$$

Exercise

- Purify the following $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

Background: Interface equalities

Interface variables & equalities

- A variable v occurring in a pure formula ϕ is an **interface variable** iff it occurs in both 1-pure and 2-pure atoms of ϕ .
- An equality $(v_i = v_j)$ is an **interface equality** for ϕ iff v_i, v_j are interface variables for ϕ .
- We denote the interface equality $v_i = v_j$ by “ e_{ij} ”

Example:

$$\begin{aligned} \mathcal{LIA} : & \quad (GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge \\ \mathcal{EUF} : & \quad (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge \\ \mathcal{LIA} : & \quad (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ \mathcal{EUF} \text{ or } \mathcal{LIA} : & \quad (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \\ \mathcal{EUF} : & \quad (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{EUF} \text{ or } \mathcal{LIA} : & \quad (EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \dots \end{aligned}$$

$v_0, v_1, v_2, v_3, v_4, v_5$ are interface variables, v_6, v_7, v_8 are not
 $\implies (v_0 = v_1)$ is an interface equality, $(v_0 = v_6)$ is not.

Background: Stably-infinite & Convex Theories

Stably-infinite Theories

A Σ -theory \mathcal{T} is **stably-infinite** iff every quantifier-free \mathcal{T} -satisfiable formula is satisfiable in an infinite model of \mathcal{T} .

- \mathcal{EUF} , \mathcal{DL} , \mathcal{LRA} , \mathcal{LIA} are stably-infinite
- (fixed-width) bit-vector theories are not stably-infinite

Intuition: **a variable can be given an infinite amount of distinct values**

Convex Theories

A Σ -theory \mathcal{T} is **convex** iff, for every collection l_1, \dots, l_k, l', l'' of literals in \mathcal{T} s.t. l', l'' are in the form $(x = y)$, x, y being variables, we have that:

$$\{l_1, \dots, l_k\} \models_{\mathcal{T}} (l' \vee l'') \iff \{l_1, \dots, l_k\} \models_{\mathcal{T}} l' \text{ or } \{l_1, \dots, l_k\} \models_{\mathcal{T}} l''$$

- \mathcal{EUF} , \mathcal{DL} , \mathcal{LRA} are convex
- \mathcal{LIA} is not convex:
 - $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \models ((v = v_0) \vee (v = v_1))$,
 - $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \not\models (v = v_0)$
 - $\{(v_0 = 0), (v_1 = 1), (v \geq 0), (v \leq v_1)\} \not\models (v = v_1)$

Intuition: **non-convexity produces “case splits”**

SMT($\bigcup_i \mathcal{T}_i$) via “classic” Nelson-Oppen

Main Problem

- Given $\mu \stackrel{\text{def}}{=} \bigcup_i \mu_i$ s.t. each μ_i contains i-pure literals
 - distinct \mathcal{T}_i -solver can be invoked separately on each μ_i ...
 - ...producing distinct \mathcal{T}_i -specific models \mathcal{M}_i
- If the theories share no constant, function or predicate symbol \implies trivial
- Problem: One predicate (nearly) always shared between distinct theories \mathcal{T}_i : equality “=”

\implies all models must agree on interface equalities:

$$\mathcal{M}_i \models_{\mathcal{T}_i} (v_k = v_l) \text{ iff } \mathcal{M}_j \models_{\mathcal{T}_j} (v_k = v_l),$$

for every pair of shared variables v_k, v_l

Main idea

Combine two or more \mathcal{T}_i -solvers into one $(\bigcup_i \mathcal{T}_i)$ -solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 76]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij} s)
- important improvements and evolutions [68, 7, 39]

Schema of N.O. combination of \mathcal{T} -solvers: $\text{no}(\mathcal{T}_1, \mathcal{T}_2)$

For $i \in \{1, 2\}$, let \mathcal{T}_i be a stably infinite theory admitting a satisfiability \mathcal{T}_i -solver, and μ_i a set of i -pure literals.

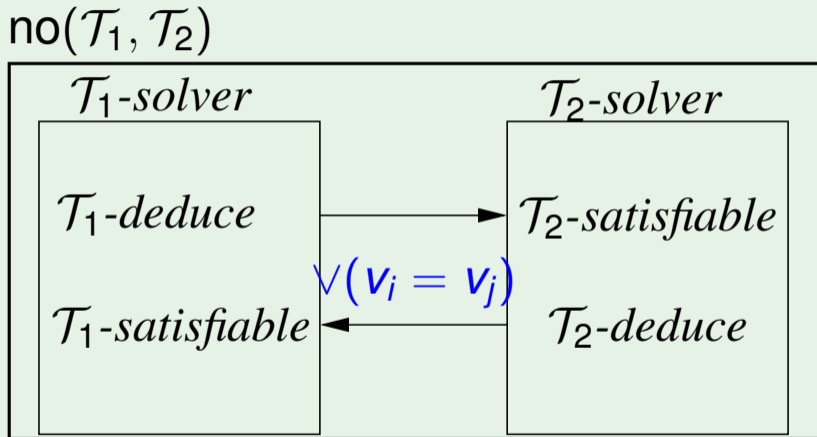
We want to decide the $\mathcal{T}_1 \cup \mathcal{T}_2$ -satisfiability of $\mu_1 \cup \mu_2$

- each \mathcal{T}_i -solver, in turn
 - checks the \mathcal{T}_i -satisfiability of μ_i ,
 - deduces all the (disjunctions of) interface equalities which derive from μ_i
 - passes them to \mathcal{T}_j -solve, $j \neq i$, which adds them to μ_j

until either:

- one \mathcal{T}_i -solver detects unsatisfiability ($\mu_1 \cup \mu_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$ -unsat)
- no more deductions are possible ($\mu_1 \cup \mu_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$ -sat)
- disjunctions of literals (due to non-convexity) force case-splitting

Schema of N.O. combination of T-solvers: $\text{no}(\mathcal{T}_1, \mathcal{T}_2)$



N.O. Example (Convex Theory)

\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

\mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

Both : $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$

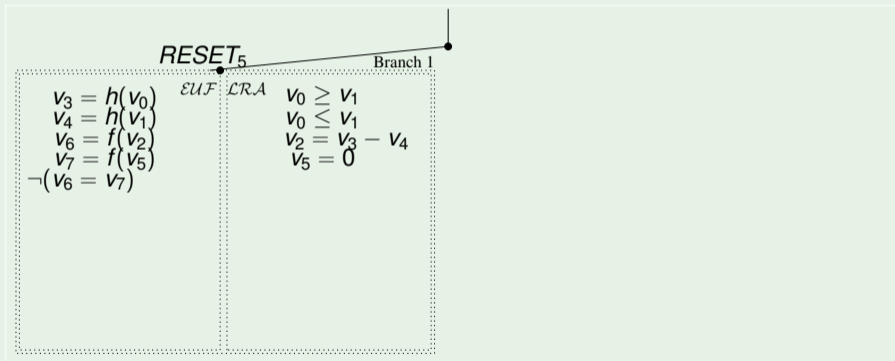


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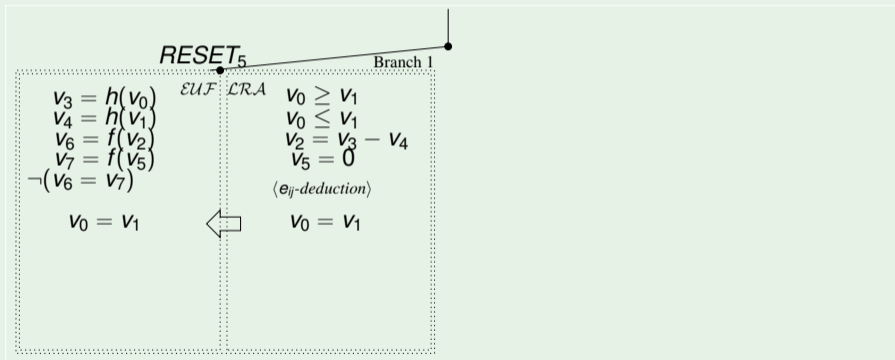


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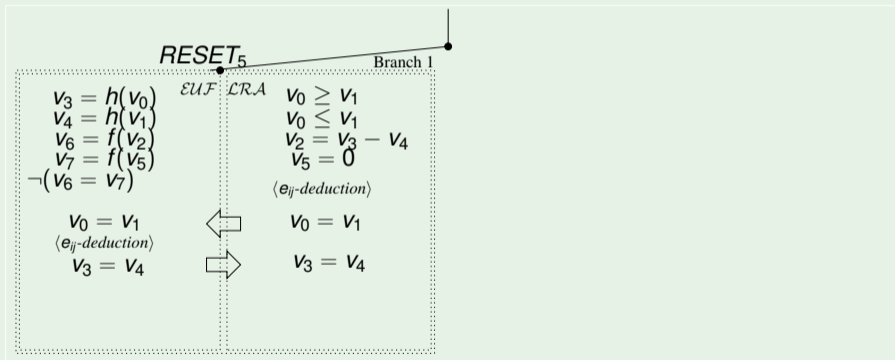


N.O. Example (Convex Theory)

$$\mathcal{EUF} : (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

$$\mathcal{LRA} : (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$$

$$\text{Both} : (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$$

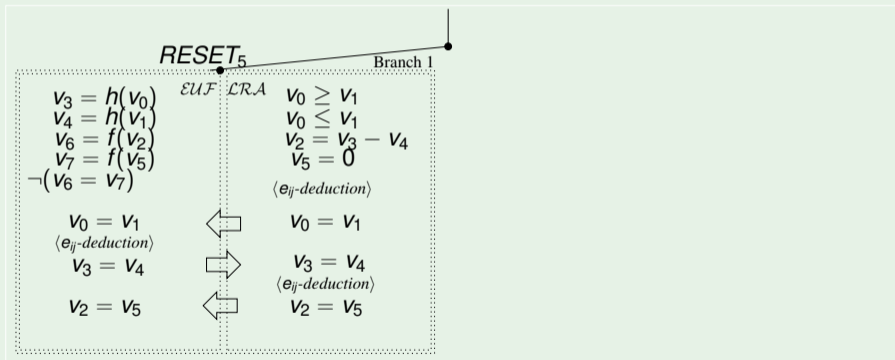


N.O. Example (Convex Theory)

$$\mathcal{EUF} : (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

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$$\text{Both} : (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$$

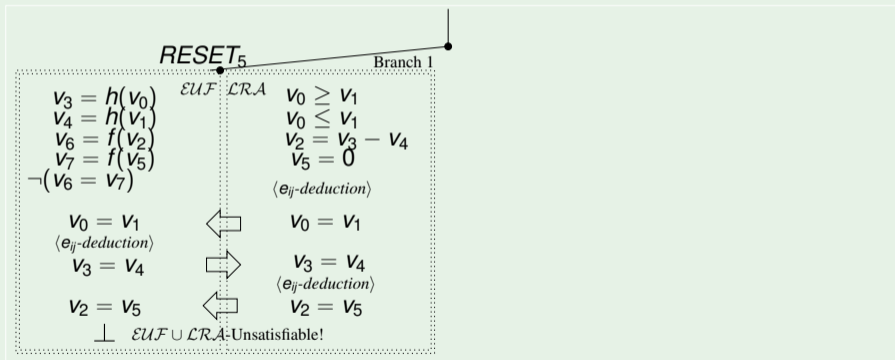


N.O. Example (Convex Theory)

$$\mathcal{EUF}: (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

$$\mathcal{LRA}: (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$$

$$\text{Both}: (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$$

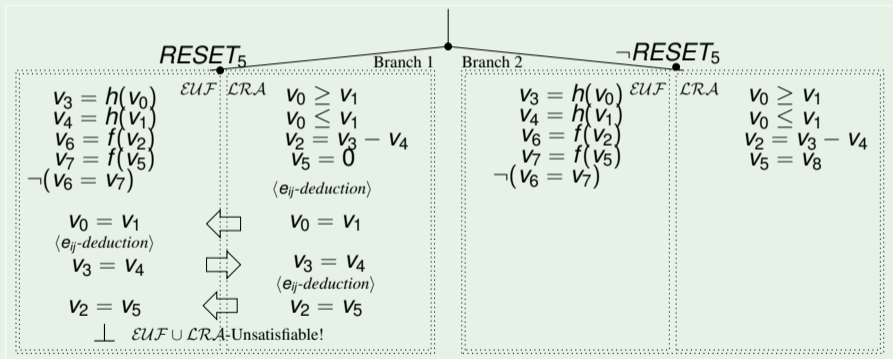


N.O. Example (Convex Theory)

\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

\mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$

Both: $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$.

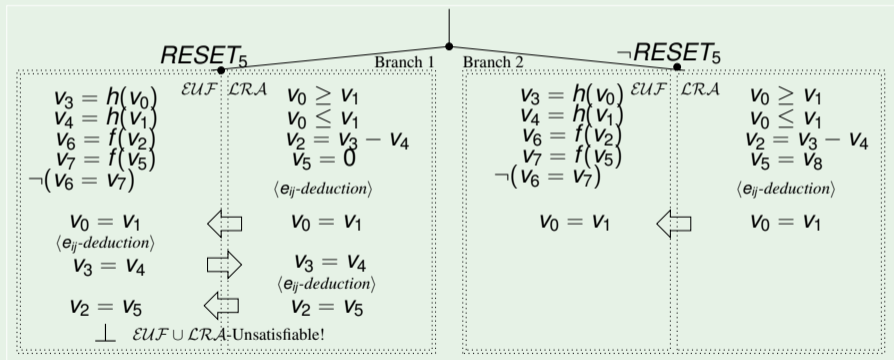


N.O. Example (Convex Theory)

\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

\mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$

Both: $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$.

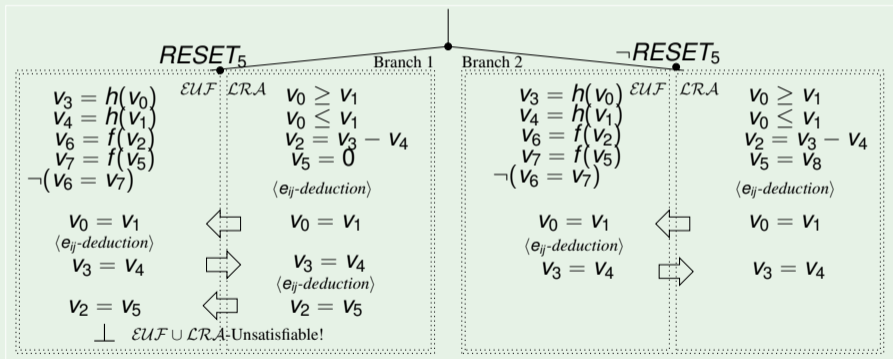


N.O. Example (Convex Theory)

\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

\mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$

Both: $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$.

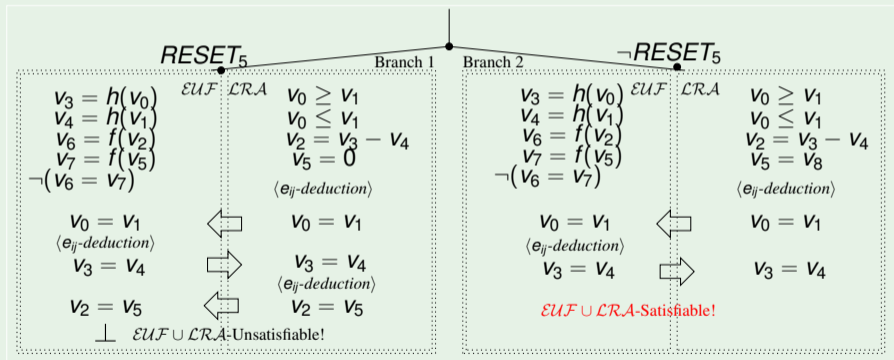


N.O. Example (Convex Theory)

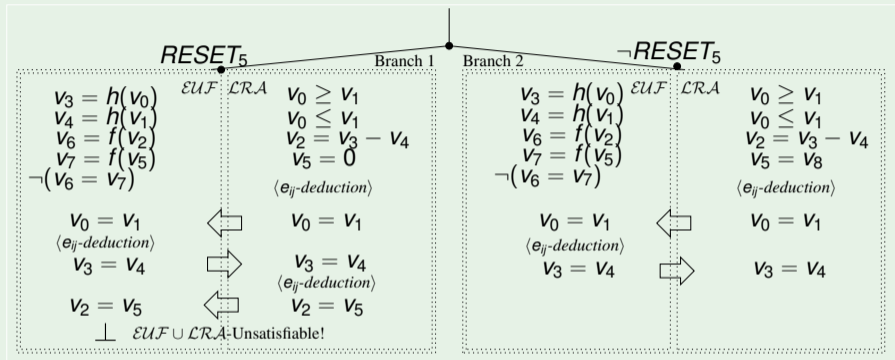
\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

\mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$

Both: $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$.



N.O.: example (convex theory) [cont.]



$EUF\text{-conflict} : ((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_5)) \rightarrow \perp$
 $LRA\text{-deduction} : ((v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$
 $EUF\text{-deduction} : ((v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$
 $LRA\text{-deduction} : ((v_0 \geq v_1) \wedge (v_0 \leq v_1)) \rightarrow (v_0 = v_1)$
 \Rightarrow
 $EUF \cup LRA\text{-conflict} : ((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 \geq v_1)) \rightarrow \perp.$

For the previous N.O. example:

- write the (minimal) clauses corresponding to each e_{ij} -deduction
- find the final conflict clauses by resolving the e_{ij} -deduction clauses

N.O.: example (non-convex theory)

μ_{LIA}

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

μ_{EUF}

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

N.O.: example (non-convex theory)

μ_{LIA}

$$v_1 \geq 0 \quad v_5 = v_4 - 1$$

$$v_1 \leq 1 \quad v_3 = 0$$

$$v_2 \geq v_6 \quad v_4 = 1$$

$$v_2 \leq v_6 + 1$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

μ_{EUF}

$$\neg(f(v_1) = f(v_2))$$

$$\neg(f(v_2) = f(v_4))$$

$$f(v_3) = v_5$$

$$f(v_1) = v_6$$

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$



$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(θ_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$



$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

(θ_{ij} -deduction)

$$v_5 = v_6$$

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

(e_{ij} -deduction)

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

(e_{ij} -deduction)

$$v_5 = v_6$$

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

(e_{ij} -deduction)

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

(e_{ij} -deduction)

$$v_5 = v_6$$

$$v_2 = v_3$$

\perp

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

(e_{ij} -deduction)

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

(e_{ij} -deduction)

$$v_5 = v_6$$

$$\begin{array}{ll} v_2 = v_3 & v_2 = v_4 \\ \perp & \perp \end{array}$$

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(e_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

(e_{ij} -deduction)

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

$$v_1 = v_4$$

(e_{ij} -deduction)

SAT!

$$v_5 = v_6$$

$$v_2 = v_3$$

$$v_2 = v_4$$

\perp

\perp

N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

(\mathfrak{e}_{ij} -deduction)

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

(\mathfrak{e}_{ij} -deduction)

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

(\mathfrak{e}_{ij} -deduction)

$$v_5 = v_6$$

$$v_1 = v_4$$

SAT!

3 \mathfrak{e}_{ij} -deductions,

3 branches

$$v_2 = v_3$$

\perp

$$v_2 = v_4$$

\perp

SMT($\bigcup_i \mathcal{T}_i$) via “classic” Nelson-Oppen

Main idea

Combine two or more \mathcal{T}_i -solvers into one ($\bigcup_i \mathcal{T}_i$)-solver via **Nelson-Oppen/Shostak (N.O.) combination procedure** [62, 76]

- based on the deduction and exchange of equalities between shared variables/terms (**interface equalities, e_{ij} s**)
- important improvements and evolutions [68, 7, 39]
- drawbacks [23, 24]:
 - require (possibly expensive) deduction capabilities from \mathcal{T}_i -solvers
 - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij} 's
 - generate (typically long) ($\bigcup_i \mathcal{T}_i$)-lemmas, without interface equalities
 \implies **no backjumping & learning from e_{ij} -reasoning**

Exercise (cont.)

- Purify the following $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- Solve it via Nelson-Oppen procedure

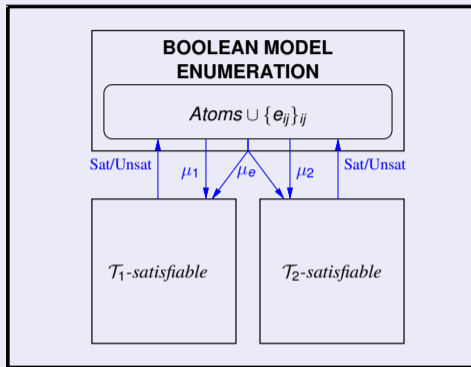
SMT($\bigcup_i \mathcal{T}_i$) via Delayed Theory Combination (DTC)

Main idea

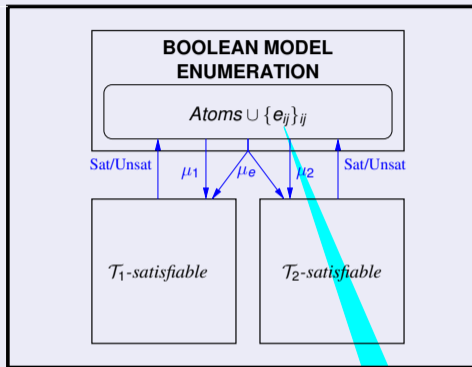
Delegate to the CDCL SAT solver part/most of the (possibly very expensive) reasoning effort on interface equalities previously due to the \mathcal{T}_i -solvers (e_{ij} -deduction, case-split). [15, 16, 24]

- based on Boolean reasoning on interface equalities via CDCL (plus \mathcal{T} -propagation)
- important improvements and evolutions [36, 9]
- feature wrt N.O. [23, 24]
 - do not require (possibly expensive) deduction capabilities from \mathcal{T}_i -solvers
 - with non-convex theories, case-splits on e_{ij} 's handled by SAT solver
 - generate \mathcal{T}_i -lemmas with interface equalities
 \implies **backjumping & learning from e_{ij} -reasoning**

DTC: Basic schema



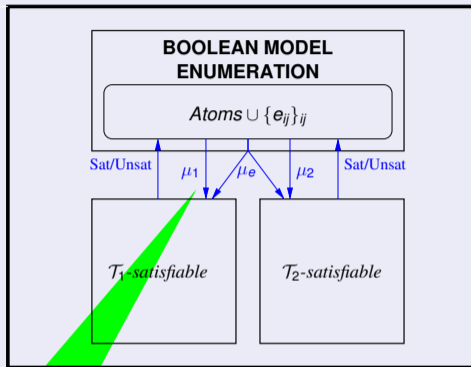
DTC: Basic schema



The boolean solver assigns values not only to atoms in $Atoms(\phi)$, but also to interface equalities $\{(v_i = v_j)\}_{ij}$:

$$\mu = \mu_1 \cup \mu_2 \cup \mu_e, \quad \mu_e := \{[\neg](v_i = v_j) \mid v_i, v_j \in \mu_1 \cup \mu_2\}$$

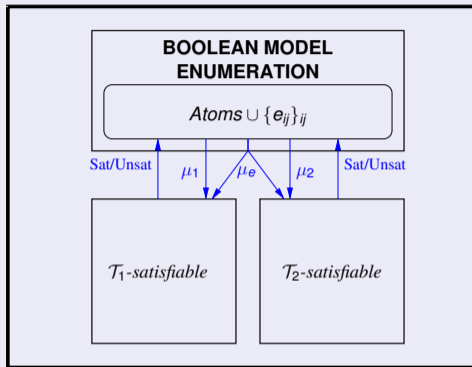
DTC: Basic schema



Each \mathcal{T}_i -solver interacts only with the boolean solver

- receives $\mu'_i := \mu_i \cup \mu_e$ from Bool
- checks the \mathcal{T}_i -satisfiability of μ'_i

DTC: Basic schema



...until either:

- some μ propositionally satisfies ϕ and both $\mu'_i := \mu_i \cup \mu_e$ are \mathcal{T}_i -consistent
 $\implies (\phi \text{ is } \mathcal{T}_1 \cup \mathcal{T}_2\text{-sat})$
- no more assignment μ are available
 $\implies (\phi \text{ is } \mathcal{T}_1 \cup \mathcal{T}_2\text{-unsat})$

DTC: enhanced schema

- **CDCL-based assignment enumeration** on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$,
⇒ benefits of state-of-the-art SAT techniques
- **Early pruning**: invoke the \mathcal{T}_i -solver's before every Boolean decision
⇒ total assignments generated only when strictly necessary
- **Branching**: branching on e_{ij} 's postponed
⇒ Boolean search on e_{ij} 's performed only when strictly necessary
- **Theory-Backjumping & Learning**: e_{ij} 's are involved in conflicts
⇒ e_{ij} 's can be assigned by unit propagation
- **Theory-deduction & learning**: if \mathcal{T}_i -solver deduces unassigned literals I on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$
 - I is passed back to the Boolean solver, which unit-propagates it
 - the deduction $\mu' \models I$ is learned as a clause $\mu' \rightarrow I$ (deduction clause)
- ...

DTC: example w.out \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l} \mu_{\mathcal{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

DTC: example w.out \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l} \mu_{\mathcal{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{\mathcal{LIA}}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

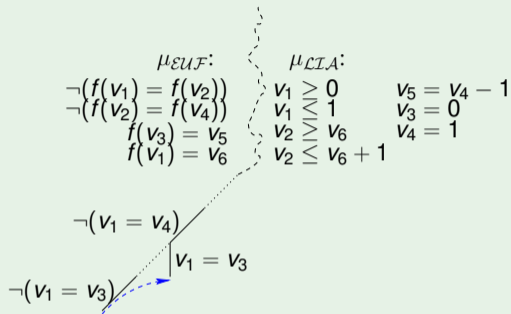
$$\neg(v_1 = v_4)$$

$$\neg(v_1 = v_3)$$

\mathcal{LIA} -unsat, C_{13}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example w.out \mathcal{T} -prop. (non-convex theory)



$$G_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example w.out \mathcal{T} -prop. (non-convex theory)

$\mu_{\mathcal{EUF}}:$	$\mu_{\mathcal{LIA}}:$	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$$\neg(v_1 = v_4)$$

$$\neg(v_1 = v_3)$$

$$\neg(v_5 = v_6)$$

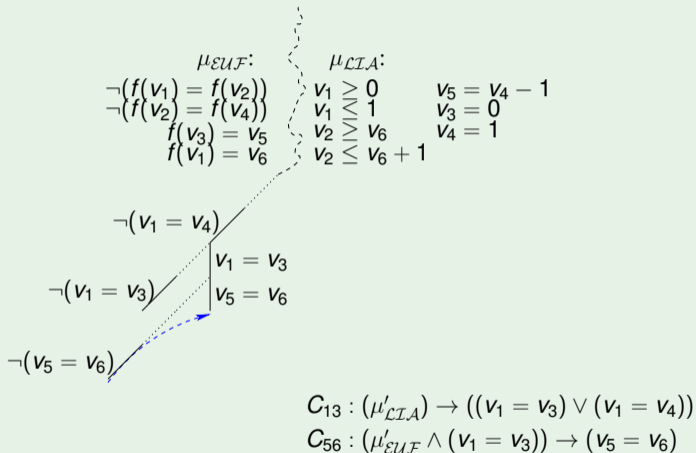
$$v_1 = v_3$$

\mathcal{EUF} -unsat, C_{56}

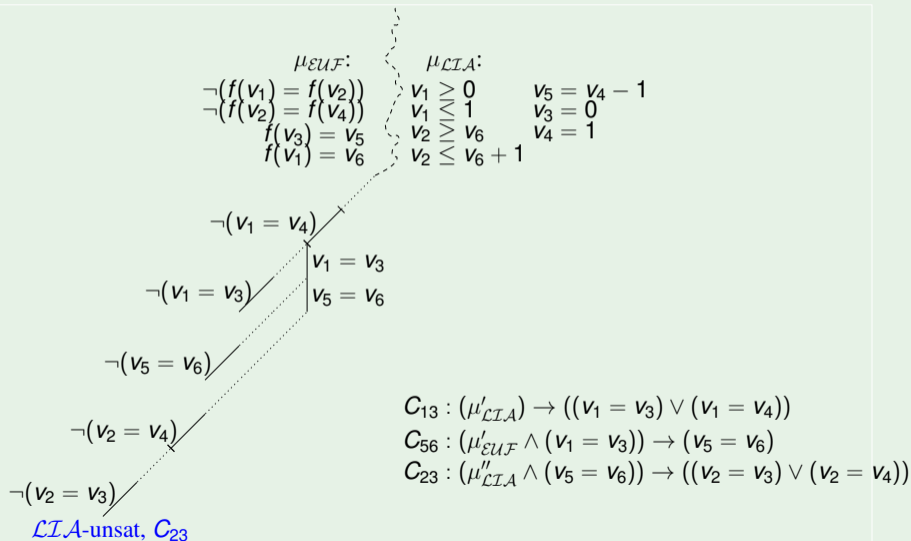
$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

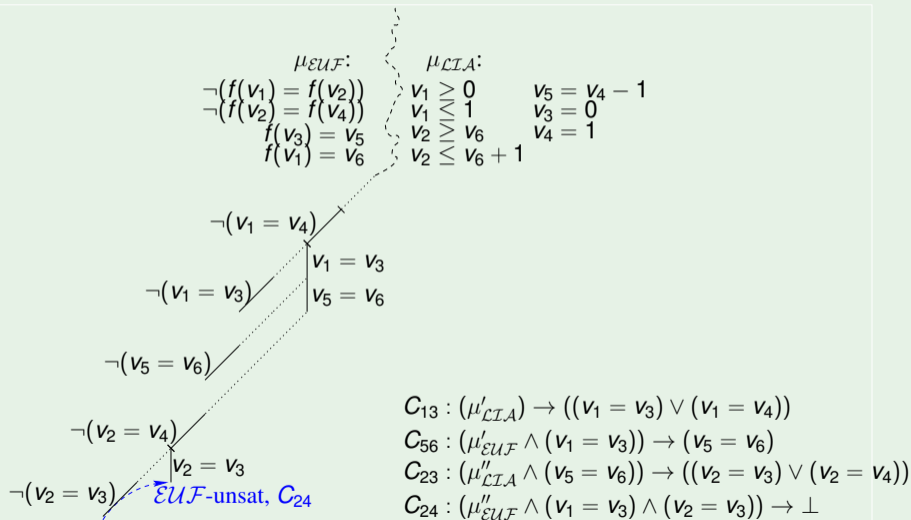
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



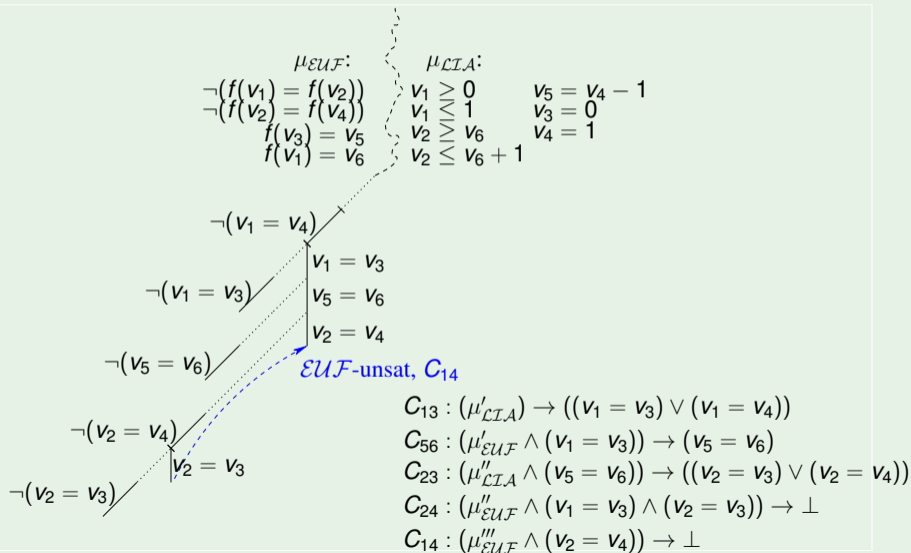
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



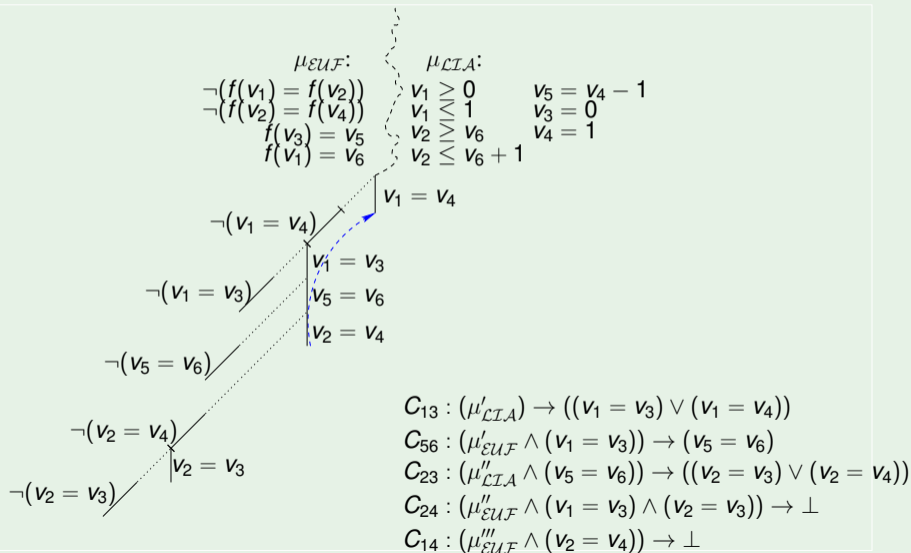
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



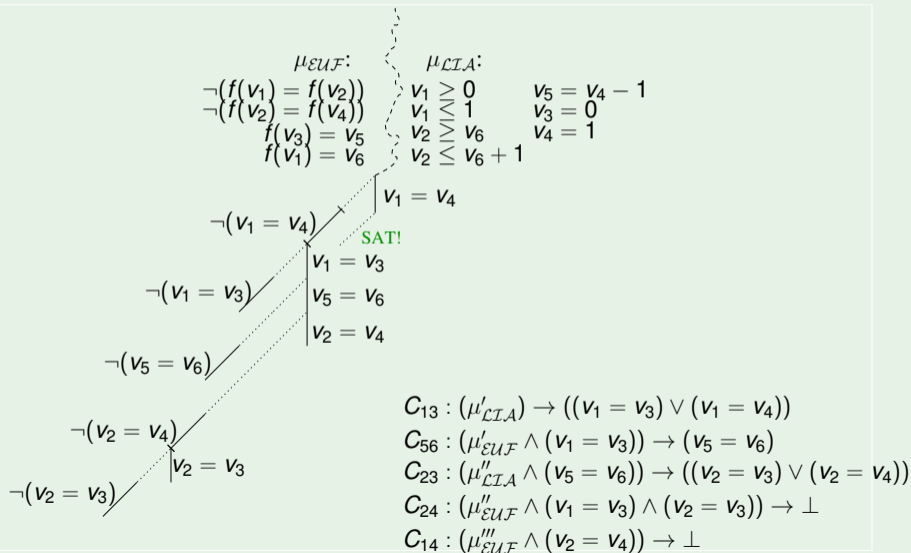
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



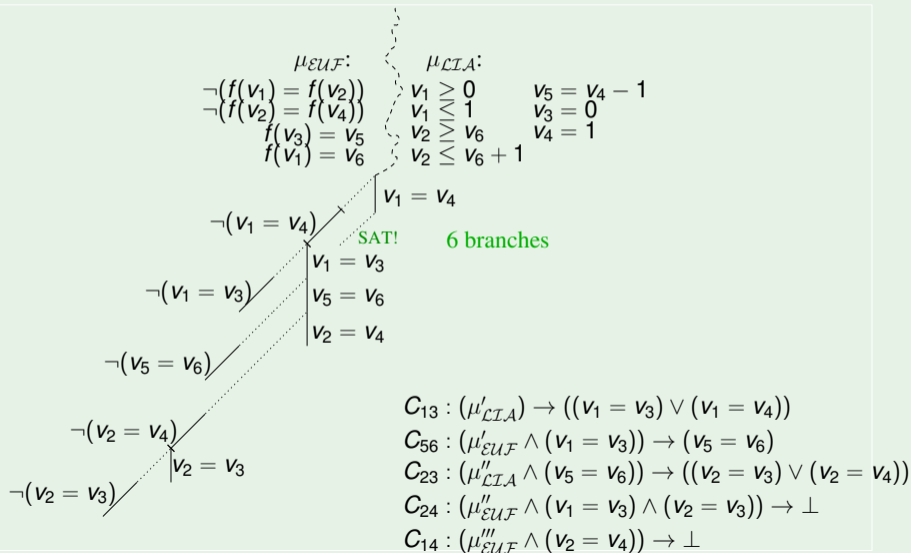
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



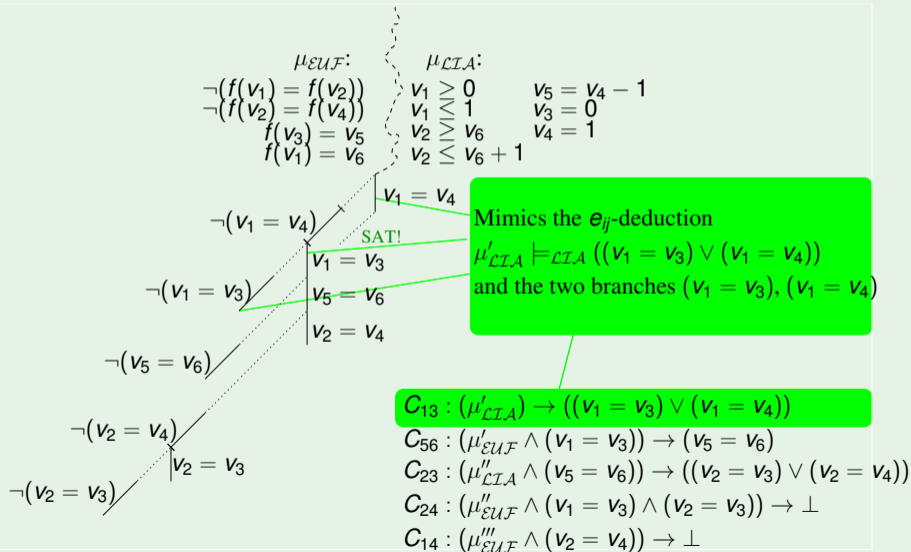
DTC: example w.out \mathcal{T} -prop. (non-convex theory)



DTC: example w.out \mathcal{T} -prop. (non-convex theory)



DTC: example w.out \mathcal{T} -prop. (non-convex theory)



DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l} \mu_{EUF}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l} \mu_{EUF}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

\mathcal{LIA} -deduce $(v_1 = v_4) \vee (v_1 = v_3)$, C_{13}

$$C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l} \mu_{EUF}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

$$\begin{array}{l} \neg(v_1 = v_4) \\ v_1 = v_3 \end{array}$$

$$C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{\mathcal{EUF}}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{\mathcal{LIA}}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1
 \end{array}$$

$$\begin{array}{l}
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6
 \end{array}
 \left| \begin{array}{l}
 \mathcal{EUF}\text{-deduce } (v_5 = v_6), C_{56}
 \end{array} \right.$$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{\mathcal{EUF}}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{\mathcal{LIA}}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1 \\
 \\
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6 \quad \mathcal{LIA}\text{-deduce } (v_2 = v_4) \vee (v_2 = v_3), C_{23}
 \end{array}$$

$$\begin{array}{l}
 C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\
 C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\
 C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))
 \end{array}$$

DTC: example with \mathcal{T} -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{\mathcal{EUF}}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{\mathcal{LIA}}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1
 \end{array}$$

$$\begin{array}{l}
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6 \\
 \neg(v_2 = v_4) \\
 v_2 = v_3
 \end{array}$$

\mathcal{EUF} -unsat, C_{24}

$$\begin{array}{l}
 C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\
 C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\
 C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4)) \\
 C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp
 \end{array}$$

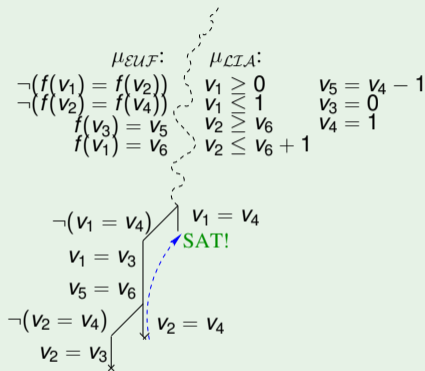
DTC: example with \mathcal{T} -prop. (non-convex theory)

$\mu_{EUF}:$	$\mu_{LIA}:$	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$$\begin{array}{l} \neg(v_1 = v_4) \\ v_1 = v_3 \\ v_5 = v_6 \\ \neg(v_2 = v_4) \\ v_2 = v_3 \end{array} \quad \begin{array}{l} v_2 = v_4 \\ \text{EUF-unsat, } C_{14} \end{array}$$

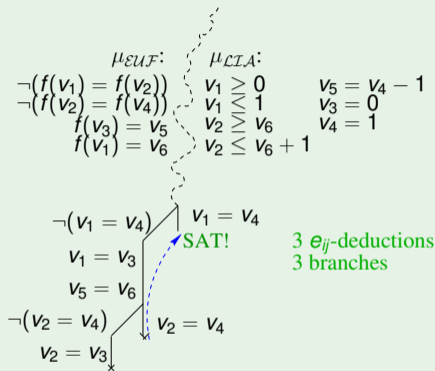
$$\begin{aligned} C_{13} &: (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\ C_{56} &: (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\ C_{23} &: (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4)) \\ C_{24} &: (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp \\ C_{14} &: (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp \end{aligned}$$

DTC: example with \mathcal{T} -prop. (non-convex theory)



- $C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
 $C_{56} : (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
 $C_{23} : (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
 $C_{24} : (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp$
 $C_{14} : (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp$

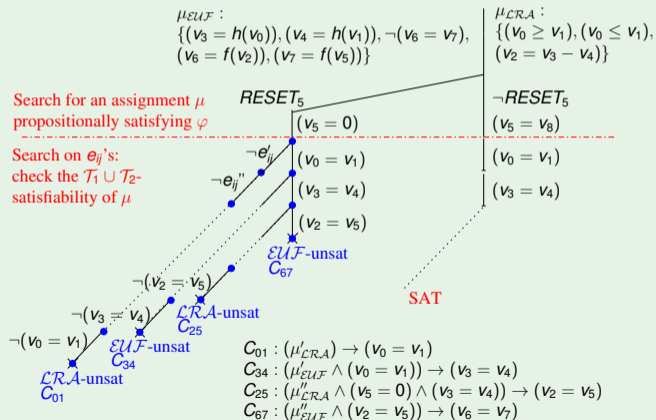
DTC: example with \mathcal{T} -prop. (non-convex theory)



- $C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
 $C_{56} : (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
 $C_{23} : (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
 $C_{24} : (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp$
 $C_{14} : (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp$

DTC: example without \mathcal{T} -propagation (convex theory)

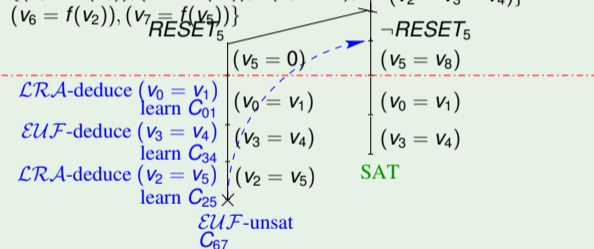
$$\begin{aligned} \mathcal{EUF} : & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA} : & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge \\ \text{Both} : & (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{aligned}$$



DTC: example with \mathcal{T} -propagation (convex theory)

\mathcal{EUF} : $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$
 \mathcal{LRA} : $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$
 Both : $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$

$\mu_{\mathcal{EUF}}$:
 $\{(v_3 = h(v_0)), (v_4 = h(v_1)), \neg(v_6 = v_7),$
 $(v_6 = f(v_2)), (v_7 = f(v_5))\}$
 $\mu_{\mathcal{LRA}}$:
 $\{(v_0 \geq v_1), (v_0 \leq v_1),$
 $(v_2 = v_3 - v_4)\}$



$C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1)$
 $C_{34} : (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$
 $C_{25} : (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$
 $C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)$

DTC + Model-based heuristic (aka Model-Based Theory Combination) [36]

- Initially, no interface equalities generated
- When a model is found, check against all the possible interface equalities
 - If \mathcal{T}_1 and \mathcal{T}_2 agree on the implied equalities, then return SAT
 - Otherwise, branch on equalities **implied by \mathcal{T}_1 -model but not by \mathcal{T}_2 -model**
- “Optimistic” approach, similar to axiom instantiation

Exercises

For each of the previous DTC examples:

- write the (minimal) clauses corresponding to each e_{ij} -deduction (as clauses rather than as implications)
- compute the conflict-analysis steps leading to the backjumping steps in the figures.

For each of the previous DTC examples, draw the case in which the \mathcal{EUF} -solver has deduction capabilities and the \mathcal{LRA} -solver (resp. the \mathcal{LIA} -solver) does not.

Exercise (cont.)

- Purify the following $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- Solve it via the DTC procedure
 - assuming no \mathcal{T} -deduction capabilities of the \mathcal{T} -solvers
 - assuming \mathcal{T} -deduction capabilities of the \mathcal{T} -solvers

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 - Theory Solvers for Theories of Interest (hints)
 - SMT for Combinations of Theories
- 3 **Beyond Solving: Advanced SMT Functionalities**
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Beyond Solving: advanced SAT & SMT functionalities

Advanced SMT functionalities (very important in FV):

- Building **proofs of \mathcal{T} -unsatisfiability**
- Extracting **\mathcal{T} -unsatisfiable Cores**
- [Computing **Craig interpolants**]
- Performing **All-SMT and Predicate Abstraction**
- Deciding/optimizing **SMT problems with costs**

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Building (Resolution) Proofs of \mathcal{T} -Unsatisfiability

Resolution proof of \mathcal{T} -unsatisfiability

Very similar to building proofs with plain SAT:

- resolution proofs whose leaves are original clauses and \mathcal{T} -lemmas returned by the \mathcal{T} -solver (i.e., \mathcal{T} -conflict and \mathcal{T} -deduction clauses)
- built by backward traversal of implication graphs, as in CDCL SAT
- Sub-proofs of \mathcal{T} -lemmas can be built in some \mathcal{T} -specific deduction framework if requested

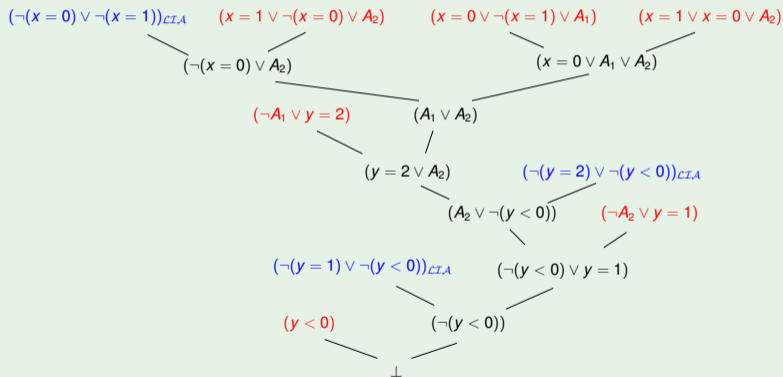
Important for:

- certifying \mathcal{T} -unsatisfiability results
- computing unsatisfiable cores
- computing interpolants

Building Proofs of \mathcal{T} -Unsatisfiability: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$

$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$



relevant original clauses, irrelevant original clauses, \mathcal{T} -lemmas

Example: proof on non-strict \mathcal{LRA} inequalities

- A proof of unsatisfiability for a set of non-strict \mathcal{LRA} inequalities can be obtained by building a linear combination (with positive coefficients) of such inequalities, each time eliminating one or more variables, until you get a contradictory inequality on constant values.
- Example:

$$\varphi \stackrel{\text{def}}{=} (0 \leq x_1 - 3x_2 + 1), (0 \leq x_1 + x_2), (0 \leq x_3 - 2x_1 - 3), (0 \leq 1 - 2x_3).$$

A proof of unsatisfiability P for φ is the following:

$$\frac{\frac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\text{COMB } (0 \leq 4x_1 + 1) \text{ with coefficients } 1 \text{ and } 3} \quad \frac{(0 \leq x_3 - 2x_1 - 3) \quad (0 \leq 1 - 2x_3)}{\text{COMB } (0 \leq -4x_1 - 5) \text{ with coefficients } 2 \text{ and } 1}}{\text{COMB } (0 \leq -4) \text{ with coefficients } 1 \text{ and } 1}$$

- It is possible to produce such proof from an unsatisfiable tableau in Simplex procedure for \mathcal{LRA} [29, 31]
- It is straightforward to produce such proof from a negative cycle in the graph-based procedure for \mathcal{DL} [29, 31]

Exercises

- Given the following \mathcal{LRA} -unsatisfiable sets of \mathcal{LRA} -inequalities, produce a proof of unsatisfiability by means of the technique described in the previous slides.

$$\bullet \left\{ \begin{array}{l} (0 \leq +2x_1 - 2x_2 \quad \quad \quad +6x_4 + 1) \\ (0 \leq -x_1 + 2x_2 \quad \quad \quad -x_4 - 2) \\ (0 \leq \quad \quad -2x_2 + 2x_3 \quad \quad \quad + 3) \\ (0 \leq \quad \quad \quad -x_3 - x_4 - 2) \\ (0 \leq \quad \quad +2x_2 \quad \quad \quad +x_4 + 2) \end{array} \right\}$$

- [create your own \mathcal{LRA} -unsatisfiable set of \mathcal{LRA} -inequalities]
- Given the following \mathcal{DL} -unsatisfiable sets of \mathcal{DL} -inequalities, produce a proof of unsatisfiability by means of the technique described in the previous slides.
 - the \mathcal{DL} example in the \mathcal{T} -solver section and its negative cycle
 - [create your own \mathcal{DL} -unsatisfiable set of \mathcal{LRA} -inequalities]

Extraction of \mathcal{T} -unsatisfiable cores

The problem

Given a \mathcal{T} -unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) \mathcal{T} -unsatisfiable subset (\mathcal{T} -unsatisfiable core)

- Wide literature in SAT
- Some implementations, very few literature for SMT [28, 56]
- We recognize three approaches:
 - **Proof-based** approach (CVC4, MathSAT):
byproduct of finding a resolution proof
 - **Assumption-based** approach (Yices):
use extra variables labeling clauses, as in the plain Boolean case
 - **Lemma-Lifting** approach [28] :
use an external (possibly-optimized) Boolean unsat-core extractor

The proof-based approach to \mathcal{T} -unsat cores

Idea (adapted from [82])

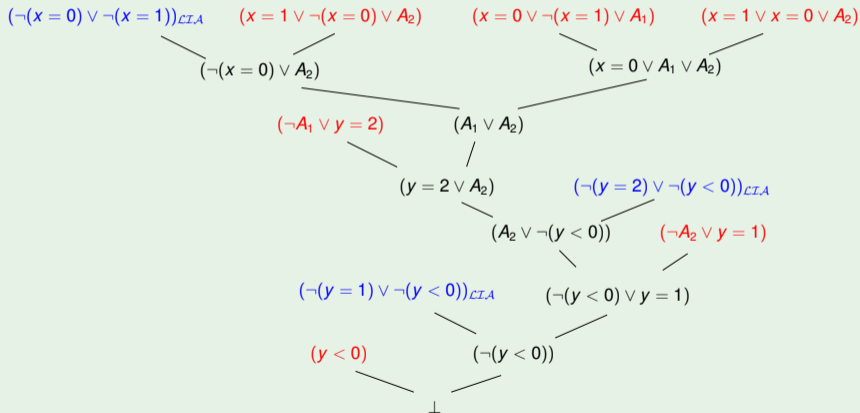
Unsatisfiable core of φ :

- in SAT: the set of leaf clauses of a resolution proof of unsatisfiability of φ
- in $\text{SMT}(\mathcal{T})$: the set of leaf clauses of a resolution proof of \mathcal{T} -unsatisfiability of φ , minus the \mathcal{T} -lemmas

The proof-based approach to \mathcal{T} -unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$

$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$



The Assumption-based approach to \mathcal{T} -unsat cores

Idea (adapted from [57])

Let φ be $\bigwedge_{i=1}^n C_i$ s.t. φ unsatisfiable.

- 1 each clause C_i in φ is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh “selector” variable
- 2 the resulting formula is checked for **satisfiability under the assumption of all S_i 's**
- 3 final conflict clause at dec. level 0: $\bigvee_j \neg S_j$
 $\implies \{C_j\}_j$ is the unsat core

Extends straightforwardly to $\text{SMT}(\mathcal{T})$.

The assumption-based approach to \mathcal{T} -unsat cores: Example

$$\begin{aligned} & (\mathcal{S}_1 \rightarrow (x = 0 \vee \neg(x = 1) \vee A_1)) \wedge (\mathcal{S}_2 \rightarrow (x = 0 \vee x = 1 \vee A_2)) \wedge \\ & \quad (\mathcal{S}_3 \rightarrow (\neg(x = 0) \vee x = 1 \vee A_2)) \wedge (\mathcal{S}_4 \rightarrow (\neg A_2 \vee y = 1)) \wedge \\ & \quad (\mathcal{S}_5 \rightarrow (\neg A_1 \vee x + y > 3)) \wedge (\mathcal{S}_6 \rightarrow y < 0) \wedge \\ & \quad (\mathcal{S}_7 \rightarrow (A_2 \vee x - y = 4)) \wedge (\mathcal{S}_8 \rightarrow (y = 2 \vee \neg A_1)) \wedge (\mathcal{S}_9 \rightarrow x \geq 0) \end{aligned}$$

Conflict analysis (Yices 1.0.6) returns:

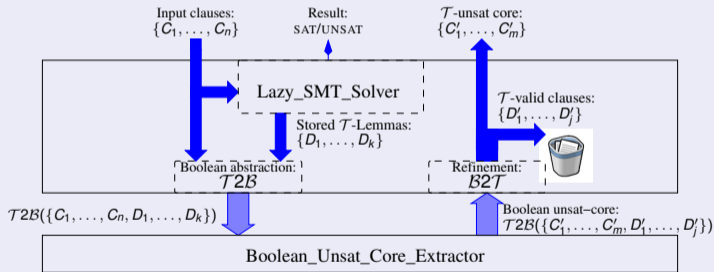
$$\neg \mathcal{S}_1 \vee \neg \mathcal{S}_2 \vee \neg \mathcal{S}_3 \vee \neg \mathcal{S}_4 \vee \neg \mathcal{S}_6 \vee \neg \mathcal{S}_7 \vee \neg \mathcal{S}_8,$$

corresponding to the unsat core in red.

The lemma-lifting approach to \mathcal{T} -unsat cores

Idea [28, 32]

- (i) The \mathcal{T} -lemmas D_i are valid in \mathcal{T}
- (ii) The conjunction of φ with all the \mathcal{T} -lemmas D_1, \dots, D_k is propositionally unsatisfiable:
 $\mathcal{T}2\mathcal{B}(\varphi \wedge \bigwedge_{i=1}^n D_i) \models \perp$.



- interfaces with an external Boolean Unsat-core Extractor

⇒ benefits for free of all state-of-the-art size-reduction techniques

The lemma-lifting approach to \mathcal{T} -unsat cores (cont.)

```
<SatValue, Clause_set>  $\mathcal{T}$ -Unsat_Core(Clause_set  $\varphi$ ) {  
  //  $\varphi$  is  $\{C_1, \dots, C_n\}$   
  if (Lazy_SMT_Solver( $\varphi$ ) == SAT)  
    then return  $\langle \text{SAT}, \emptyset \rangle$ ;  
  //  $D_1, \dots, D_k$  are the  $\mathcal{T}$ -lemmas stored by Lazy_SMT_Solver  
   $\psi^p = \text{Boolean\_Core\_Extractor}(\mathcal{T}2\mathcal{B}(\{C_1, \dots, C_n, D_1, \dots, D_k\}))$ ;  
  //  $\psi^p$  is  $\mathcal{T}2\mathcal{B}(\{C'_1, \dots, C'_m, D'_1, \dots, D'_j\})$ ;  
  return  $\langle \text{UNSAT}, \{C'_1, \dots, C'_m\} \rangle$ ;  
}
```

The lemma-lifting approach to \mathcal{T} -unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge \\ (\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$

1 The SMT solver generates the following set of \mathcal{LIA} -lemmas:

$$\{(\neg(x = 1) \vee \neg(x = 0)), (\neg(y = 2) \vee \neg(y < 0)), (\neg(y = 1) \vee \neg(y < 0))\}.$$

2 The following formula is passed to the external Boolean core extractor

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (B_6 \vee \neg A_1) \wedge B_7 \wedge \\ (\neg B_1 \vee \neg B_0) \wedge (\neg B_6 \vee \neg B_4) \wedge (\neg B_2 \vee \neg B_4)$$

which returns the unsat core in red.

3 The unsat-core is mapped back, the three \mathcal{T} -lemmas are removed
 \implies the final \mathcal{T} -unsat core (in red above).

Exercise

Consider the following set of clauses φ in \mathcal{EUF} .

$$\left\{ \begin{array}{l} (\neg(x = y) \vee (f(x) = f(y))), \\ (\neg(x = y) \vee \neg(f(x) = f(y))), \\ ((x = y) \vee (f(x) = f(y))), \\ ((x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

Find a minimal \mathcal{EUF} -unsatisfiable core.

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 - SMT with Optimization (Optimization Modulo Theories)

All-SAT/All-SMT (hints)

- **All-SAT**: enumerate all truth assignments satisfying φ
- **All-SMT**: enumerate all \mathcal{T} -satisfiable truth assignments propositionally satisfying φ
- **All-SMT over an “important” subset of atoms $\Gamma \stackrel{\text{def}}{=} \{\gamma_i\}_i$** : enumerate all assignments over Γ which can be extended to \mathcal{T} -satisfiable truth assignments propositionally satisfying φ
 \implies can compute **predicate abstraction**
- Algorithms:
 - **BCLT** [53]
each time a \mathcal{T} -satisfiable assignment $\{l_1, \dots, l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \Gamma$ belonged to the clause set
 - **MathSAT/NuSMV** [26]
As above, plus the Boolean search of the SMT solver is driven by an OBDD.

Predicate Abstraction

Predicate abstraction

if $\varphi(\mathbf{v})$ is a SMT formula over the domain variables $\mathbf{v} \stackrel{\text{def}}{=} \{v_j\}_j$, $\{\gamma_i\}_i$ is a set of “relevant” predicates over \mathbf{v} , and $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$ a set of fresh Boolean labels, then:

$$\begin{aligned} & \text{PredAbs}_{\mathbf{P}}(\varphi) \\ \stackrel{\text{def}}{=} & \exists \mathbf{v}. (\varphi(\mathbf{v}) \wedge \bigwedge_i P_i \leftrightarrow \gamma_i(\mathbf{v})) \\ = & \bigvee \left\{ \mu \mid \begin{array}{l} \mu \text{ truth assignment on } \mathbf{P} \\ \text{s.t. } \mu \wedge \varphi \wedge \bigwedge_i (P_i \leftrightarrow \gamma_i) \text{ is } \mathcal{T}\text{-satisfiable} \end{array} \right\} \end{aligned}$$

- projection of φ over (the Boolean abstraction of) the set $\{\gamma_i\}_i$.
- important step in FV: extracts finite-state abstractions from a infinite state space

Predicate Abstraction: example

$$\varphi \stackrel{\text{def}}{=} (v_1 + v_2 > 12)$$

$$\gamma_1 \stackrel{\text{def}}{=} (v_1 + v_2 = 2)$$

$$\gamma_2 \stackrel{\text{def}}{=} (v_1 - v_2 < 10)$$

↓

$$\begin{aligned} \text{PreAbs}(\varphi)_{\{P_1, P_2\}} &\stackrel{\text{def}}{=} \exists v_1 v_2 . \left(\begin{array}{l} (v_1 + v_2 > 12) \\ (P_1 \leftrightarrow (v_1 + v_2 = 2)) \\ (P_2 \leftrightarrow (v_1 - v_2 < 10)) \end{array} \wedge \right) \\ &= (\neg P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \\ &= \neg P_1. \end{aligned}$$

- 1 Introduction
 - Basics on First-order Logic
 - What is a Theory?
 - Satisfiability Modulo Theories
 - Motivations and Goals of SMT
- 2 Efficient SMT solving
 - Combining SAT with Theory Solvers
 - Theory Solvers for Theories of Interest (hints)
 - SMT for Combinations of Theories
- 3 **Beyond Solving: Advanced SMT Functionalities**
 - Proofs and Unsatisfiable Cores
 - All-SMT & Predicate Abstraction (hints)
 - **SMT with Optimization (Optimization Modulo Theories)**

Optimization Modulo Theories: General Case

Ingredients: $\langle \varphi, cost \rangle$

- a **SMT formula** φ in some background theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$
 - $\bigcup_i \mathcal{T}_i$ may be empty
 - \mathcal{T}_{\preceq} has a predicate \preceq representing a **total order**
- a \mathcal{T}_{\preceq} -**variable/term** “*cost*” occurring in φ

Optimization Modulo $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ (OMT($\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$))

The problem of finding a model \mathcal{M} for φ whose value of *cost* is minimum according to \preceq .

- maximization is dual

Note

The cost term can be rewritten as a variable

$$\langle \varphi, term \rangle \implies \langle \varphi \wedge (cost = term), cost \rangle, \quad cost \text{ fresh}$$

Optimization Modulo Theories with $\mathcal{L}\mathcal{A}$ $\mathcal{L}\mathcal{R}\mathcal{A}$ costs

Ingredients

- an **SMT formula** φ on $\mathcal{L}\mathcal{A} \cup \mathcal{T}$ $\mathcal{L}\mathcal{R}\mathcal{A} \cup \mathcal{T}$
 - $\mathcal{L}\mathcal{A}$ can be $\mathcal{L}\mathcal{R}\mathcal{A}$, $\mathcal{L}\mathcal{I}\mathcal{A}$ or a combination of both
 - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
 - $\mathcal{L}\mathcal{A}\mathcal{L}\mathcal{R}\mathcal{A}$ and \mathcal{T}_i Nelson-Oppen theories
(i.e. signature-disjoint infinite-domain theories)
- a $\mathcal{L}\mathcal{A}$ $\mathcal{L}\mathcal{R}\mathcal{A}$ variable [term] “*cost*” occurring in φ
- (optionally) two constant numbers lb (**lower bound**) and ub (**upper bound**) s.t.
 $\text{lb} \leq \text{cost} < \text{ub}$ (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{L}\mathcal{A}$ $\mathcal{L}\mathcal{R}\mathcal{A}$ costs (OMT($\mathcal{L}\mathcal{A} \cup \mathcal{T}$) OMT($\mathcal{L}\mathcal{R}\mathcal{A} \cup \mathcal{T}$))

Find a model for φ whose value of *cost* is minimum.

- maximization dual

We first restrict to the case $\mathcal{L}\mathcal{A} = \mathcal{L}\mathcal{R}\mathcal{A}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{L}\mathcal{R}\mathcal{A}$)).

Solving OMT(\mathcal{LRA}) [71, 72]

General idea

Combine standard SMT and LP minimization techniques.

Offline Schema

- Minimizer: based on the Simplex \mathcal{LRA} -solver by [40]
 - Handles strict inequalities
- Search Strategies:
 - Linear-Search strategy
 - Mixed Linear/Binary strategy

A toy example (linear search)

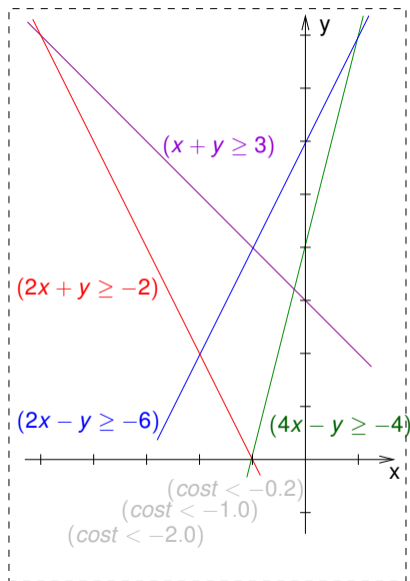
[w. pure-literal filt. \implies partial assignments]

- OMT(\mathcal{LRA}) problem:

$$\begin{aligned} \varphi &\stackrel{\text{def}}{=} (\neg A_1 \vee (2x + y \geq -2)) \\ &\wedge (A_1 \vee (x + y \geq 3)) \\ &\wedge (\neg A_2 \vee (4x - y \geq -4)) \\ &\wedge (A_2 \vee (2x - y \geq -6)) \\ &\wedge (\text{cost} < -0.2) \\ &\wedge (\text{cost} < -1.0) \\ &\wedge (\text{cost} < -2.0) \end{aligned}$$

$$\text{cost} \stackrel{\text{def}}{=} x$$

$$\bullet \mu = \left\{ \begin{array}{l} A_1, \neg A_1, A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$$



A toy example (linear search)

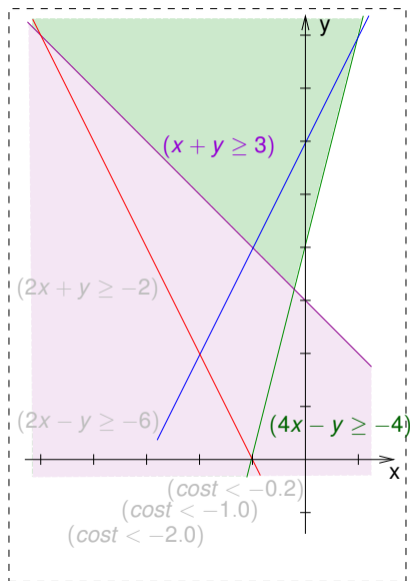
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 $\implies \text{SAT}, \min = -0.2$



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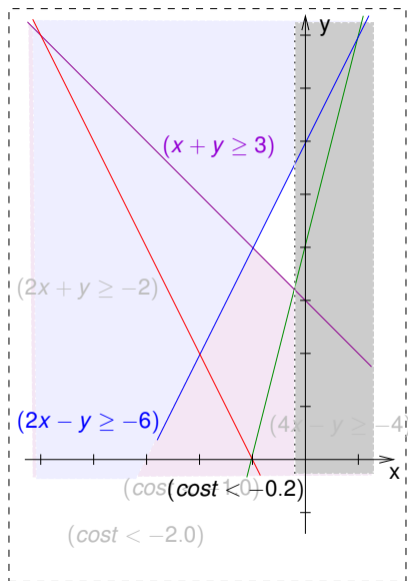
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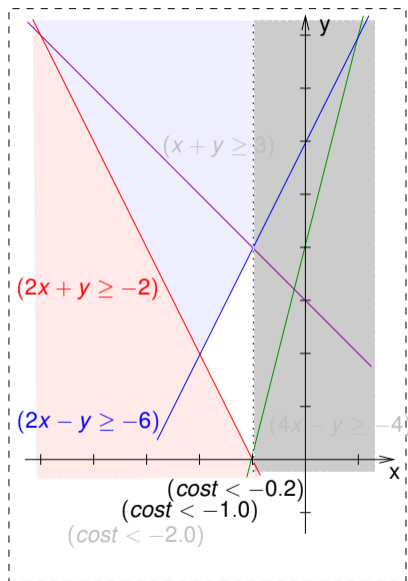
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 $\implies \text{SAT}, \min = -2.0$



A toy example (linear search)

[w. pure-literal filt. \implies partial assignments]

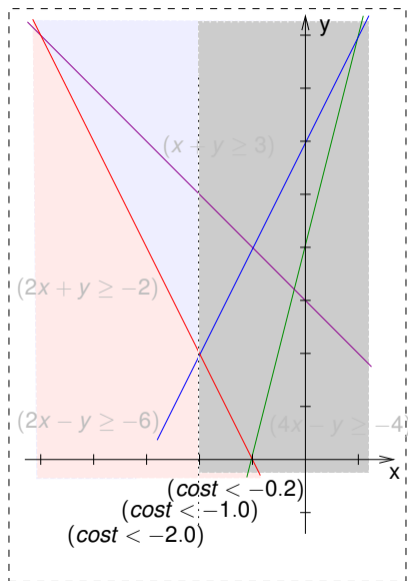
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\implies UNSAT, $\min = -2.0$

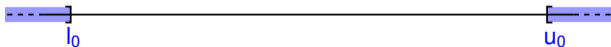


Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, lb, ub \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{ \neg(cost < lb), (cost < ub) \};$

while ($l < u$) **do**



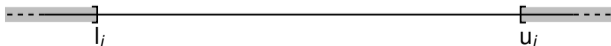
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while ($l < u$) **do**

if (BinSearchMode()) **then** // Binary-search Mode

else // Linear-search Mode



Offline Schema: Mixed Linear/Binary-Search Strategy

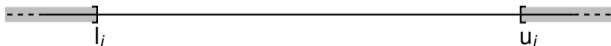
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$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$



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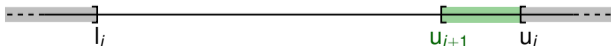
$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$

if ($res = \text{SAT}$) **then**

$\langle \mathcal{M}, u \rangle \leftarrow \text{LRA-Solver.Minimize}(cost, \mu);$

$\varphi \leftarrow \varphi \cup \{(cost < u)\};$

else { $res = \text{UNSAT}$ }



Offline Schema: Mixed Linear/Binary-Search Strategy

```
Input:  $\langle \varphi, cost, lb, ub \rangle$  // lb can be  $-\infty$ , ub can be  $+\infty$   
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$   
while ( $l < u$ ) do  
  if (BinSearchMode()) then // Binary-search Mode  
  else // Linear-search Mode  
     $\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$   
    if ( $res = \text{SAT}$ ) then  
    else { $res = \text{UNSAT}$ }  
       $l \leftarrow u;$   
return  $\langle \mathcal{M}, u \rangle$ 
```



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, lb, ub \rangle$ // lb can be $-\infty$, ub can be $+\infty$
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$

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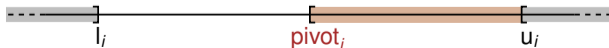
if (BinSearchMode()) **then** // Binary-search Mode

$pivot \leftarrow \text{ComputePivot}(l, u);$

$\varphi \leftarrow \varphi \cup \{(cost < pivot)\};$

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$

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Offline Schema: Mixed Linear/Binary-Search Strategy

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$\langle res, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

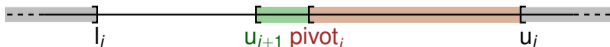
else // Linear-search Mode

if ($res = SAT$) **then**

$\langle \mathcal{M}, u \rangle \leftarrow$ \mathcal{LRA} -Solver.Minimize($cost, \mu$);

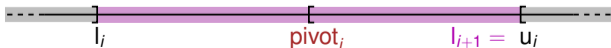
$\varphi \leftarrow \varphi \cup \{(cost < u)\};$

else { $res = UNSAT$ }



Offline Schema: Mixed Linear/Binary-Search Strategy

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while ( $l < u$ ) do  
  if (BinSearchMode()) then // Binary-search Mode  
     $pivot \leftarrow \text{ComputePivot}(l, u);$   
     $\varphi \leftarrow \varphi \cup \{(cost < pivot)\};$   
     $\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$   
  else // Linear-search Mode  
    if ( $res = \text{SAT}$ ) then  
      return  $\langle \mathcal{M}, u \rangle$   
    else { $res = \text{UNSAT}$ }  
      if  $((cost < pivot) \notin \text{SMT.ExtractUnsatCore}(\varphi))$  then  
         $l \leftarrow u;$   
      else
```



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, lb, ub \rangle$ // lb can be $-\infty$, ub can be $+\infty$
 $l \leftarrow lb$; $u \leftarrow ub$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\}$;

while ($l < u$) **do**

if (BinSearchMode()) **then** // Binary-search Mode

$pivot \leftarrow \text{ComputePivot}(l, u)$;

$\varphi \leftarrow \varphi \cup \{(cost < pivot)\}$;

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi)$;

else // Linear-search Mode

if ($res = \text{SAT}$) **then**

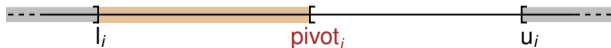
else { $res = \text{UNSAT}$ }

if ($(cost < pivot) \notin \text{SMT.ExtractUnsatCore}(\varphi)$) **then**

else

$l \leftarrow pivot$;

$\varphi \leftarrow (\varphi \setminus \{(cost < pivot)\}) \cup \{\neg(cost < pivot)\}$;



OMT with Independent Objectives (aka Boxed OMT) [55, 74]

The problem: $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$ [55]

Given $\langle \varphi, \mathcal{C} \rangle$ s.t.:

- φ is the input formula
- $\mathcal{C} \stackrel{\text{def}}{=} \{cost_1, \dots, cost_k\}$ is a set of \mathcal{LA} -terms on variables in φ ,

$\langle \varphi, \mathcal{C} \rangle$ is the problem of finding a set of independent \mathcal{LA} -models $\mathcal{M}_1, \dots, \mathcal{M}_k$ s.t. s.t. each \mathcal{M}_i makes $cost_i$ minimum.

Notes

- derives from SW verification problems [55]
- equivalent to k independent problems $\langle \varphi, cost_1 \rangle, \dots, \langle \varphi, cost_k \rangle$
- intuition: share search effort for the different objectives
- generalizes to $OMT(\mathcal{LA} \cup \mathcal{T})$ straightforwardly

OMT with Multiple Objectives [55, 13, 74]

Solution

- Intuition: when a \mathcal{T} -satisfiable satisfying assignment μ is found,

```
  foreach  $cost_i$ 
```

```
     $min_i := \min\{min_i, \mathcal{T}solver.minimize(\mu, cost_i)\};$ 
```

```
  learn  $\bigvee_i (cost_i < min_i);$  //  $(cost_i < -\infty) \equiv \perp$ 
```

```
  proceed until UNSAT;
```

- Notice:

- for each μ , guaranteed improvement of at least one min_i
- in practice, for each μ , multiple $cost_i$ minima are improved

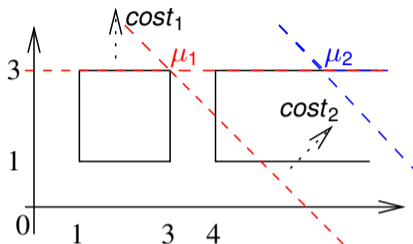
- Implemented improvements:

(a) drop previous clauses $\bigvee_i (cost_i < min_i)$

(b) $(cost_i < min_i)$ pushed in μ first: if \mathcal{T} -unsatisfiable, skip minimization

(c) learn $\neg(cost_i < min_i) \vee (cost_i < min_i^{old})$, s.t. min_i^{old} previous min_i
 \implies reuse previously-learned clauses like $\neg(cost_i < min_i^{old}) \vee C$

Boxed OMT: Example [55, 74]



$$\begin{aligned} \varphi &= (1 \leq y) \wedge (y \leq 3) \wedge (((1 \leq x) \wedge (x \leq 3)) \vee (x \geq 4)) \\ &\wedge (cost_1 = -y) \wedge (cost_2 = -x - y) \end{aligned}$$

$$\begin{aligned} \mu_1 &= \{(1 \leq y), (y \leq 3), (1 \leq x), (x \leq 3)\} \implies \text{SAT} \implies [-3, -6] \\ &\implies \text{learn} \quad \{(cost_1 < -3) \vee (cost_2 < -6)\} \end{aligned}$$

$$\begin{aligned} \mu_2 &= \{(1 \leq y), (y \leq 3), (x \geq 4)\} \implies \text{SAT} \implies [-3, -\infty] \\ &\implies \text{learn} \quad \{(cost_1 < -3)\} \\ &\implies \text{UNSAT} \end{aligned}$$

OMT with Lexicographic Combination of Objectives [13]

The problem

Find one optimal model \mathcal{M} minimizing $\underline{c} \stackrel{\text{def}}{=} cost_1, cost_2, \dots, cost_k$ lexicographically.

Solution

- Intuition:

- { minimize $cost_1$ }*

- when UNSAT*

- { substitute unit clause ($cost_1 < min_1$) with ($cost_1 = min_1$) }*

- { minimize $cost_2$ }*

- ...

- improvement:

- each time UNSAT is found, add $\bigwedge_i (cost_i \leq \mathcal{M}_i(cost_i))$ to φ

Optimization problems encoded into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ I

SMT with Pseudo-Boolean Constraints & Weighted MaxSMT

$$\text{OMT} + \text{PB} : \quad \sum_j w_j \cdot A_j, \quad w_i > 0 \quad // (\sum_j \text{ite}(A_j, w_j, 0))$$

\Downarrow

$$\begin{aligned} & \sum_j x_j, \quad x_j \text{ fresh} \\ \text{s.t.} \quad & \dots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

$$\text{MaxSMT} : \quad \langle \varphi_h, \bigwedge_j \psi_j \rangle \quad \text{s.t. } \psi_j \text{ soft}, \quad w_j = \text{weight}(\psi_j), \quad w_i > 0$$

\Downarrow

$$\begin{aligned} & \text{minimize } \sum_j x_j, \quad x_j, A_j \text{ fresh} \\ & \varphi_h \wedge \bigwedge_j (A_j \vee \psi_j) \wedge \bigwedge_j (\neg A_j \vee (x_j = w_j)) \wedge (A_j \vee (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

Remark: range constraints “ $(x_j \geq 0) \wedge (x_j \leq w_j)$ ”

$$\begin{aligned} OMT + PB : \quad & \sum_j w_j \cdot A_j, w_i > 0 \quad //(\sum_j \text{ite}(A_j, w_j, 0)) \\ & \downarrow \\ & \sum_j x_j, x_j \text{ fresh} \\ \text{s.t.} \quad & \dots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

Range constraints “ $(x_j \geq 0) \wedge (x_j \leq w_j)$ ” logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound **only after all A_i 's are assigned** :
Ex: $w_1 = 4, w_2 = 7, \sum_{i=1} x_i < 10, A_1 = A_2 = \top, A_i = * \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i 's violate a bound
 \implies drastic pruning of the search
- same for weighted MaxSMT

Optimization problems encoded into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ II

OMT with Min-Max [Max-Min] optimization

Given $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$, find a solution which minimizes the maximum value among $\{cost_1, \dots, cost_k\}$. (Max-Min dual.)

- Frequent in some applications (e.g. [72, 79])

\Rightarrow encode into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ problem $\{\varphi \wedge \bigwedge_i (cost_i \leq cost), cost\}$ s.t. $cost$ fresh.

OMT with linear combinations of costs

Given $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$ and a set of weights $\{w_1, \dots, w_k\}$, find a solution which minimizes $\sum_i w_i \cdot cost_i$.

\Rightarrow encode into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ problem $\{\varphi \wedge (cost = \sum_i w_i \cdot cost_i), cost\}$ s.t. $cost$ fresh.

These objectives can be composed with other $\text{OMT}(\mathcal{L}\mathcal{A})$ objectives.

Other OMT Functionalities [hints]

Incremental interface [13, 74]

Allows for pushing/popping sub-formulas into a stack, and then run OMT incrementally over them, reusing previous search.

- useful in some applications (e.g., BMC with parametric systems)
- straightforward variant of incremental SAT and SMT solvers

Pareto Fronts [13, 12]

- Given $cost_1, cost_2$, compute $\mathcal{M}_1, \dots, \mathcal{M}_i, \dots, \mathcal{M}_j, \dots$ s.t.:
 - either $\mathcal{M}_i(cost_1) > \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) > \mathcal{M}_j(cost_2)$ and $\mathcal{M}_i(cost_1) < \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) < \mathcal{M}_j(cost_2)$
 - for each \mathcal{M}_i , no \mathcal{M}' dominates \mathcal{M}_i
- no objective can be improved without degrading some other one

Some OMT tools

- **BCLT** [66, 54]
<https://www.cs.upc.edu/~oliveras/bclt-main.html>
- **OPTIMATHSAT** [71, 72, 74, 73], on top of MATHSAT [27]
<https://optimathsat.disi.unitn.it>
- **SYMBA** [55], on top of Z3 [37]
<https://bitbucket.org/arieg/symba/src>
- **ν Z** [13, 12], on top of Z3 [37]
<https://z3.codeplex.com>

- survey papers:

- Roberto Sebastiani: "Lazy Satisfiability Modulo Theories".
Journal on Satisfiability, Boolean Modeling and Computation, JSAT. Vol. 3, 2007. Pag 141–224, ©IOS Press.
- Clark Barrett, Roberto Sebastiani, Sanjit Seshia, Cesare Tinelli "Satisfiability Modulo Theories".
Part II, Chapter 33, The Handbook of Satisfiability, II ed. 2021. ©IOS press.
- Leonardo de Moura and Nikolaj Bjørner. "Satisfiability modulo theories: introduction and applications".
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- web links:

- The SMT library SMT-LIB: <https://goedel.cs.uiowa.edu/smtlib/>
- The SMT Competition SMT-COMP: <https://www.smtcomp.org/>
- The SAT/SMT Schools <https://satassociation.org/sat-smt-school.html>

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The list of references above is by no means intended to be all-inclusive. I apologize both with the authors and with the readers for all the relevant works which are not cited here.