

# Formal Methods

## Module I: Automated Reasoning

### Ch. 01: **Propositional Satisfiability (SAT)**

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# Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
  - Generalities
  - Resolution
  - Tableaux
  - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers
  - Limitations of Chronological Backtracking
  - Conflict-Driven Clause-Learning SAT solvers
  - Further Improvements
  - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

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# Propositional Logic (aka Boolean Logic)



# Basic Definitions

- **Propositional formula** (aka **Boolean formula**)
  - $\top, \perp$  are formulas
  - a **propositional atom**  $A_1, A_2, A_3, \dots$  is a formula;
  - if  $\varphi_1$  and  $\varphi_2$  are formulas, then  
 $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$   
are formulas.
- Ex:  $\varphi \stackrel{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4)))$
- **Atoms**( $\varphi$ ): the set  $\{A_1, \dots, A_N\}$  of atoms occurring in  $\varphi$ .
  - Ex:  $\text{Atoms}(\varphi) = \{A_1, A_2, A_3, A_4\}$
- **Literal**: a propositional atom  $A_i$  (**positive literal**) or its negation  $\neg A_i$  (**negative literal**)
  - Notation: if  $l := \neg A_i$ , then  $\neg l := A_i$
- **Clause**: a disjunction of literals  $\bigvee_j l_j$  (e.g.,  $(A_1 \vee \neg A_2 \vee A_3 \vee \dots)$ )
- **Cube**: a conjunction of literals  $\bigwedge_j l_j$  (e.g.,  $(A_1 \wedge \neg A_2 \wedge A_3 \wedge \dots)$ )

# Semantics of Boolean operators

## Truth Table

$\alpha$	$\beta$	$\neg\alpha$	$\alpha\wedge\beta$	$\alpha\vee\beta$	$\alpha\rightarrow\beta$	$\alpha\leftarrow\beta$	$\alpha\leftrightarrow\beta$	$\alpha\oplus\beta$
$\perp$	$\perp$	T	$\perp$	$\perp$	T	T	T	$\perp$
$\perp$	T	T	$\perp$	T	T	$\perp$	$\perp$	T
T	$\perp$	$\perp$	$\perp$	T	$\perp$	T	$\perp$	T
T	T	$\perp$	T	T	T	T	T	$\perp$

## English meaning of connectives

English	Logic
A and B	$A \wedge B$
A if B   A when B   A whenever B	$A \leftarrow B$
if A, then B   A implies B   A forces B   A requires B	$A \rightarrow B$
A precisely when B   A if and only if B	$A \leftrightarrow B$
A or B (or both)   A unless B	$A \vee B$ (logical or)
either A or B (but not both)	$A \oplus B$ (exclusive or)

# Semantics of Boolean operators (cont.)

## Note

- $\wedge$ ,  $\vee$ ,  $\leftrightarrow$  and  $\oplus$  are commutative:

$$(\alpha \wedge \beta) \iff (\beta \wedge \alpha)$$

$$(\alpha \vee \beta) \iff (\beta \vee \alpha)$$

$$(\alpha \leftrightarrow \beta) \iff (\beta \leftrightarrow \alpha)$$

$$(\alpha \oplus \beta) \iff (\beta \oplus \alpha)$$

- $\wedge$ ,  $\vee$ ,  $\leftrightarrow$  and  $\oplus$  are associative:

$$((\alpha \wedge \beta) \wedge \gamma) \iff (\alpha \wedge (\beta \wedge \gamma)) \iff (\alpha \wedge \beta \wedge \gamma)$$

$$((\alpha \vee \beta) \vee \gamma) \iff (\alpha \vee (\beta \vee \gamma)) \iff (\alpha \vee \beta \vee \gamma)$$

$$((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) \iff (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) \iff (\alpha \leftrightarrow \beta \leftrightarrow \gamma)$$

$$((\alpha \oplus \beta) \oplus \gamma) \iff (\alpha \oplus (\beta \oplus \gamma)) \iff (\alpha \oplus \beta \oplus \gamma)$$

- $\rightarrow$ ,  $\leftarrow$  are neither commutative nor associative:

$$(\alpha \rightarrow \beta) \not\iff (\beta \rightarrow \alpha)$$

$$((\alpha \rightarrow \beta) \rightarrow \gamma) \not\iff (\alpha \rightarrow (\beta \rightarrow \gamma))$$

# Equivalences with Boolean Operators

$\neg\neg\alpha$	$\iff$	$\alpha$
$(\alpha \vee \beta)$	$\iff$	$\neg(\neg\alpha \wedge \neg\beta)$
$\neg(\alpha \vee \beta)$	$\iff$	$(\neg\alpha \wedge \neg\beta)$
$(\alpha \wedge \beta)$	$\iff$	$\neg(\neg\alpha \vee \neg\beta)$
$\neg(\alpha \wedge \beta)$	$\iff$	$(\neg\alpha \vee \neg\beta)$
$(\alpha \rightarrow \beta)$	$\iff$	$(\neg\alpha \vee \beta)$
$\neg(\alpha \rightarrow \beta)$	$\iff$	$(\alpha \wedge \neg\beta)$
$(\alpha \leftarrow \beta)$	$\iff$	$(\alpha \vee \neg\beta)$
$\neg(\alpha \leftarrow \beta)$	$\iff$	$(\neg\alpha \wedge \beta)$
$(\alpha \leftrightarrow \beta)$	$\iff$	$((\alpha \rightarrow \beta) \wedge (\alpha \leftarrow \beta))$
	$\iff$	$((\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta))$
$\neg(\alpha \leftrightarrow \beta)$	$\iff$	$(\neg\alpha \leftrightarrow \beta)$
	$\iff$	$(\alpha \leftrightarrow \neg\beta)$
	$\iff$	$((\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta))$
$(\alpha \oplus \beta)$	$\iff$	$\neg(\alpha \leftrightarrow \beta)$

Boolean logic can be expressed in terms of  $\{\neg, \wedge\}$  (or  $\{\neg, \vee\}$ ) only!



# Exercises

1 For every pair of formulas  $\alpha \iff \beta$  below, show that  $\alpha$  and  $\beta$  can be rewritten into each other by either (i) applying the truth tables or by (ii) applying the syntactic properties of the previous slides

- $(A_1 \wedge A_2) \vee A_3 \iff (A_1 \vee A_3) \wedge (A_2 \vee A_3)$
- $(A_1 \vee A_2) \wedge A_3 \iff (A_1 \wedge A_3) \vee (A_2 \wedge A_3)$
- $A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow A_4)) \iff (A_1 \wedge A_2 \wedge A_3) \rightarrow A_4$
- $A_1 \rightarrow (A_2 \wedge A_3) \iff (A_1 \rightarrow A_2) \wedge (A_1 \rightarrow A_3)$
- $(A_1 \vee A_2) \rightarrow A_3 \iff (A_1 \rightarrow A_3) \wedge (A_2 \rightarrow A_3)$
- $A_1 \oplus A_2 \iff (A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$
- $\neg A_1 \leftrightarrow \neg A_2 \iff A_1 \leftrightarrow A_2$
- $A_1 \leftrightarrow A_2 \leftrightarrow A_3 \iff A_1 \oplus A_2 \oplus A_3$

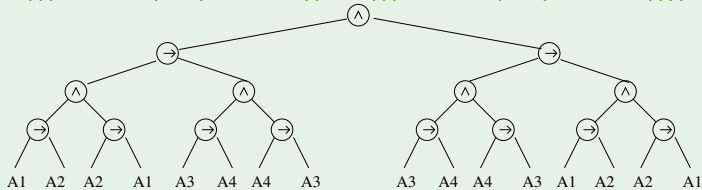
# Tree & DAG Representations of Formulas

- Formulas can be represented either as **trees** or as **DAGS** (Directed Acyclic Graphs)
- **DAG representation can be up to exponentially smaller**
  - in particular, when  $\leftrightarrow$ 's are involved

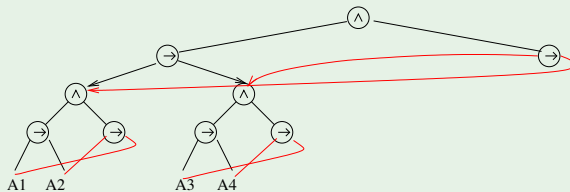
$$\begin{aligned} & (A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\ & \quad \Downarrow \\ & (((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge \\ & ((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2))) \\ & \quad \Downarrow \\ & (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge \\ & (((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)))) \end{aligned}$$

# Tree & DAG Representations of Formulas: Example

$$(((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge$$
$$(((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1))))$$



*Tree Representation*



*DAG Representation*

# Semantics: Basic Definitions

- **Total truth assignment**  $\mu$  for  $\varphi$ :  
 $\mu : \mathit{Atoms}(\varphi) \mapsto \{\top, \perp\}$ .
  - represents a **possible world** or a **possible state of the world**
- **Partial Truth assignment**  $\mu$  for  $\varphi$ :  
 $\mu : \mathcal{A} \mapsto \{\top, \perp\}, \mathcal{A} \subset \mathit{Atoms}(\varphi)$ .
  - represents  $2^k$  total assignments,  $k$  is # unassigned variables
- **Notation: set and formula representations of an assignment**
  - $\mu$  can be represented **as a set of literals**:  
EX:  $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies \{A_1, \neg A_2\}$
  - $\mu$  can be represented **as a formula (cube)**:  
EX:  $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies (A_1 \wedge \neg A_2)$

## Semantics: Basic Definitions [cont.]

- A **total** truth assignment  $\mu$  **satisfies**  $\varphi$  ( $\mu$  is a model of  $\varphi$ ,  $\mu \models \varphi$ ):

$$\mu \models A_i \iff \mu(A_i) = \top$$

$$\mu \models \neg\varphi \iff \text{not } \mu \models \varphi$$

$$\mu \models \alpha \wedge \beta \iff \mu \models \alpha \text{ and } \mu \models \beta$$

$$\mu \models \alpha \vee \beta \iff \mu \models \alpha \text{ or } \mu \models \beta$$

$$\mu \models \alpha \rightarrow \beta \iff \text{if } \mu \models \alpha, \text{ then } \mu \models \beta$$

$$\mu \models \alpha \leftrightarrow \beta \iff \mu \models \alpha \text{ iff } \mu \models \beta$$

$$\mu \models \alpha \oplus \beta \iff \mu \models \alpha \text{ iff not } \mu \models \beta$$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$  (the set of models of  $\varphi$ )
- A **partial** truth assignment  $\mu$  **satisfies**  $\varphi$  iff all total assignments extending  $\mu$  satisfy  $\varphi$ 
  - Ex:  $\{A_1\} \models (A_1 \vee A_2)$  because both  $\{A_1, A_2\} \models (A_1 \vee A_2)$  and  $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$
- $\varphi$  is **satisfiable** iff  $\mu \models \varphi$  for some  $\mu$  (i.e.  $M(\varphi) \neq \emptyset$ )
- $\alpha$  **entails**  $\beta$  ( $\alpha \models \beta$ ):  $\alpha \models \beta$  iff  $\mu \models \alpha \implies \mu \models \beta$  for all  $\mu$ s (i.e.,  $M(\alpha) \subseteq M(\beta)$ )
- $\varphi$  is **valid** ( $\models \varphi$ ):  $\models \varphi$  iff  $\mu \models \varphi$  for all  $\mu$ s (i.e.,  $\mu \in M(\varphi)$  for all  $\mu$ s)

# Properties & Results

## Property

$\varphi$  is valid iff  $\neg\varphi$  is not satisfiable

## Deduction Theorem

$\alpha \models \beta$  iff  $\alpha \rightarrow \beta$  is valid ( $\models \alpha \rightarrow \beta$ )

## Corollary

$\alpha \models \beta$  iff  $\alpha \wedge \neg\beta$  is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

# Equivalence and Equi-Satisfiability

- $\alpha$  and  $\beta$  are **equivalent** iff, for every  $\mu$ ,  $\mu \models \alpha$  iff  $\mu \models \beta$   
(i.e., if  $M(\alpha) = M(\beta)$ )
- $\alpha$  and  $\beta$  are **equi-satisfiable** iff exists  $\mu_1$  s.t.  $\mu_1 \models \alpha$  iff exists  $\mu_2$  s.t.  $\mu_2 \models \beta$   
(i.e., if  $M(\alpha) \neq \emptyset$  iff  $M(\beta) \neq \emptyset$ )
- $\alpha, \beta$  equivalent  
     $\Downarrow \Updownarrow$   
     $\alpha, \beta$  equi-satisfiable
- EX:  $A_1 \vee A_2$  and  $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$  are equi-satisfiable, not equivalent.  
     $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$ , but  $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when  $\beta$  is the result of applying some transformation  $T$  to  $\alpha$ :  $\beta \stackrel{\text{def}}{=} T(\alpha)$ :
  - $T$  is **validity-preserving** [resp. **satisfiability-preserving**] iff  
     $T(\alpha)$  and  $\alpha$  are equivalent [resp. equi-satisfiable]

# Boolean Quantification

## Shannon's expansion:

- If  $v$  is a Boolean variable and  $f$  is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- $v$  does not occur in  $\exists v.\varphi$  and  $\forall v.\varphi$  !!
- Multi-variable quantification:  $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

- Intuition:

- $\mu \models \exists v.\varphi$  iff exists *truthvalue*  $\in \{\top, \perp\}$  s.t.  $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$  iff forall *truthvalue*  $\in \{\top, \perp\}$ ,  $\mu \cup \{v := \text{truthvalue}\} \models \varphi$

- Example:  $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

## Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae



## NP-Completeness of SAT

- For  $N$  variables, there are up to  $2^N$  truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is **NP-complete**

⇒ The most important logical problems (**validity**, **inference**, **entailment**, **equivalence**, ...) can be straightforwardly reduced to **(un)satisfiability**, and are thus **(co)NP-complete**.



**No existing worst-case-polynomial algorithm.**

# POLARITY of subformulas

**Polarity:** the number of nested negations modulo 2.

- **Positive/negative occurrences**

- $\varphi$  occurs positively in  $\varphi$ ;
- if  $\neg\varphi_1$  occurs positively [negatively] in  $\varphi$ ,  
then  $\varphi_1$  occurs negatively [positively] in  $\varphi$
- if  $\varphi_1 \wedge \varphi_2$  or  $\varphi_1 \vee \varphi_2$  occur positively [negatively] in  $\varphi$ ,  
then  $\varphi_1$  and  $\varphi_2$  occur positively [negatively] in  $\varphi$ ;
- if  $\varphi_1 \rightarrow \varphi_2$  occurs positively [negatively] in  $\varphi$ ,  
then  $\varphi_1$  occurs negatively [positively] in  $\varphi$  and  $\varphi_2$  occurs positively [negatively] in  $\varphi$ ;
- if  $\varphi_1 \leftrightarrow \varphi_2$  or  $\varphi_1 \oplus \varphi_2$  occurs in  $\varphi$ ,  
then  $\varphi_1$  and  $\varphi_2$  occur positively and negatively in  $\varphi$ ;

# Negative Normal Form (NNF)

- $\varphi$  is in **Negative normal form** iff it is given only by the recursive applications of  $\wedge, \vee$  to literals.
- **every  $\varphi$  can be reduced into NNF:**
  - (i) substituting all  $\rightarrow$ 's and  $\leftrightarrow$ 's:

$$\begin{aligned}\alpha \rightarrow \beta &\implies \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta &\implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)\end{aligned}$$

- (ii) pushing down negations recursively:

$$\begin{aligned}\neg(\alpha \wedge \beta) &\implies \neg\alpha \vee \neg\beta \\ \neg(\alpha \vee \beta) &\implies \neg\alpha \wedge \neg\beta \\ \neg\neg\alpha &\implies \alpha\end{aligned}$$

- Every non-atomic subformula in  $NNF(\varphi)$  **occurs with positive polarity only**  
 $\implies$  a subformula  $\psi$  occurring with both polarities in  $\varphi$  is encoded as both  $NNF(\psi)$  and  $NNF(\neg\psi)$
- The reduction is **linear** if a DAG representation is used.
- Preserves the **equivalence** of formulas.

# NNF: Example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

↓

$$\begin{aligned} & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \wedge \\ & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \end{aligned}$$

↓

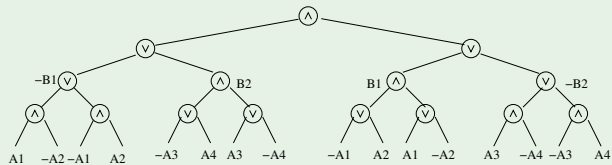
$$\begin{aligned} & ((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2))) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \end{aligned}$$

↓

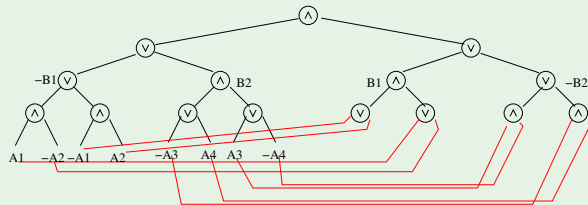
$$\begin{aligned} & (((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))) \end{aligned}$$

# NNF: Example [cont.]

## Note



*Tree Representation*



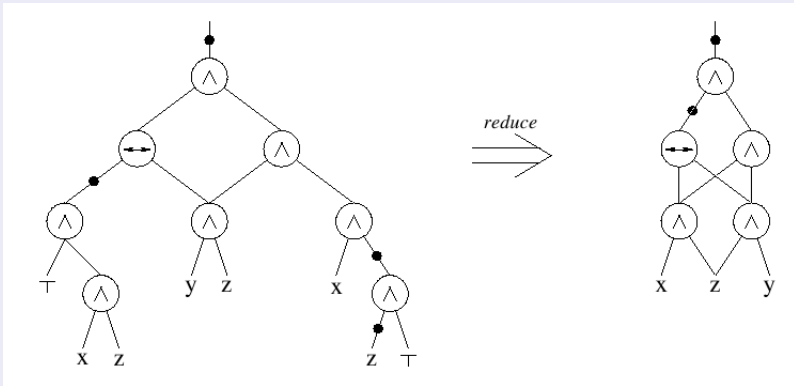
*DAG Representation*

For each non-literal subformula  $\varphi$ ,  $\varphi$  and  $\neg\varphi$  have different representations  $\implies$  they are not shared.

# Optimized polynomial representations

## And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations:  
{ $\wedge$ ,  $\leftrightarrow$ ,  $\neg$ }-only, negations on arcs, sorting of subformulae, lifting of  $\neg$ 's over  $\leftrightarrow$ 's,...



# Conjunctive Normal Form (CNF)

- $\varphi$  is in **Conjunctive normal form** iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^L \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals  $\bigvee_{j_i=1}^{K_i} l_{j_i}$  are called **clauses**
- Easier to handle: list of lists of literals.  
 $\implies$  no reasoning on the recursive structure of the formula

# Classic CNF Conversion $CNF(\varphi)$

- Every  $\varphi$  can be reduced into CNF by, e.g.,

(i) expanding implications and equivalences:

$$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

(ii) pushing down negations recursively:

$$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$$

$$\neg\neg\alpha \implies \alpha$$

$\implies$  Negation Normal Form, NNF (see previous slides)

(iii) applying recursively the DeMorgan's Rule:  $(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$

- Resulting formula worst-case **exponential**:

- ex:  $\|CNF(\bigvee_{i=1}^N (l_{i1} \wedge l_{i2}))\| = \| (l_{11} \vee l_{21} \vee \dots \vee l_{N1}) \wedge (l_{12} \vee l_{22} \vee \dots \vee l_{N2}) \wedge \dots \wedge (l_{1N} \vee l_{2N} \vee \dots \vee l_{NN}) \| = 2^N$

- $Atoms(CNF(\varphi)) = Atoms(\varphi)$

- $CNF(\varphi)$  is **equivalent** to  $\varphi$ .

- Rarely used in practice.



# Labeling CNF conversion $CNF_{label}(\varphi)$

## Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Tseitin's CNF-ization)

- Every  $\varphi$  can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$

$$\varphi \implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j))$$

$$\varphi \implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$$

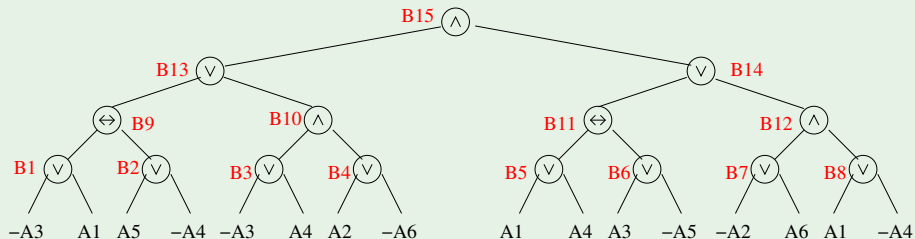
$l_i, l_j$  being literals and  $B$  being a “new” variable.

- Worst-case **linear**!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$  is **equi-satisfiable** (but not equivalent) to  $\varphi$ .
  - moreover:  $\exists B_1, \dots, B_k. CNF_{label}(\varphi)$  equivalent to  $\varphi$ , s.t.  $B_1, \dots, B_k$  all fresh variables introduced
- Much more used than classic conversion in practice

## Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	$\iff$	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	$\iff$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	$\iff$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

# Labeling CNF Conversion $CNF_{label}$ – Example



$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \wedge$$

... $\wedge$

$$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \wedge$$

$$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \wedge$$

... $\wedge$

$$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \wedge$$

$$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \wedge$$

$$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \wedge$$

$$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \wedge$$

$B_{15}$

$$(\neg B_1 \vee \neg A_3 \vee A_1) \wedge (B_1 \vee A_3) \wedge (B_1 \vee \neg A_1) \wedge$$

... $\wedge$

$$(\neg B_8 \vee A_1 \vee \neg A_4) \wedge (B_8 \vee \neg A_1) \wedge (B_8 \vee A_4) \wedge$$

$$(\neg B_9 \vee \neg B_1 \vee B_2) \wedge (\neg B_9 \vee B_1 \vee \neg B_2) \wedge$$

$$(B_9 \vee B_1 \vee B_2) \wedge (B_9 \vee \neg B_1 \vee \neg B_2) \wedge$$

= ... $\wedge$

$$(B_{12} \vee \neg B_7 \vee \neg B_8) \wedge (\neg B_{12} \vee B_7) \wedge (\neg B_{12} \vee B_8) \wedge$$

$$(\neg B_{13} \vee B_9 \vee B_{10}) \wedge (B_{13} \vee \neg B_9) \wedge (B_{13} \vee \neg B_{10}) \wedge$$

$$(\neg B_{14} \vee B_{11} \vee B_{12}) \wedge (B_{14} \vee \neg B_{11}) \wedge (B_{14} \vee \neg B_{12}) \wedge$$

$$(B_{15} \vee \neg B_{13} \vee \neg B_{14}) \wedge (\neg B_{15} \vee B_{13}) \wedge (\neg B_{15} \vee B_{14}) \wedge$$

$B_{15}$

# Improved Labeling CNF conversion $CNF_{label}$

## Polarity-based Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Plaisted&Greenbaum CNF-ization)

- As in the previous case, applying instead the rules:

$$\begin{aligned}\varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \rightarrow (l_i \vee l_j)) && \text{if } (l_i \vee l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF((l_i \vee l_j) \rightarrow B) && \text{if } (l_i \vee l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \rightarrow (l_i \wedge l_j)) && \text{if } (l_i \wedge l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF((l_i \wedge l_j) \rightarrow B) && \text{if } (l_i \wedge l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \rightarrow (l_i \leftrightarrow l_j)) && \text{if } (l_i \leftrightarrow l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF((l_i \leftrightarrow l_j) \rightarrow B) && \text{if } (l_i \leftrightarrow l_j) \text{ neg.}\end{aligned}$$

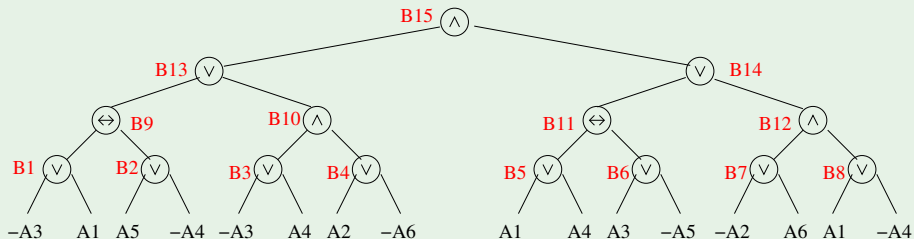
- Smaller in size:

$$\begin{aligned}CNF(B \rightarrow (l_i \vee l_j)) &= (\neg B \vee l_i \vee l_j) \\ CNF(((l_i \vee l_j) \rightarrow B)) &= (\neg l_i \vee B) \wedge (\neg l_j \vee B)\end{aligned}$$

# Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	$\iff$	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	$\iff$	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	$\iff$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	$\iff$	$(B \vee \neg l_i \neg l_j)$
$CNF(B \rightarrow (l_i \leftrightarrow l_j))$	$\iff$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j)$
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	$\iff$	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

# Improved Labeling CNF conversion $CNF_{label}$ – example



Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$   
 $\dots \quad \wedge$   
 $CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \quad \wedge$   
 $CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$   
 $\dots \quad \wedge$   
 $CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \quad \wedge$   
 $CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \quad \wedge$   
 $CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \quad \wedge$   
 $CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \quad \wedge$   
 $B_{15}$

Improved

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$   
 $\dots \quad \wedge$   
 $CNF(B_8 \rightarrow (A_1 \vee \neg A_4)) \quad \wedge$   
 $CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$   
 $\dots \quad \wedge$   
 $CNF(B_{12} \rightarrow (B_7 \wedge B_8)) \quad \wedge$   
 $CNF(B_{13} \rightarrow (B_9 \vee B_{10})) \quad \wedge$   
 $CNF(B_{14} \rightarrow (B_{11} \vee B_{12})) \quad \wedge$   
 $CNF(B_{15} \rightarrow (B_{13} \wedge B_{14})) \quad \wedge$   
 $B_{15}$

## Labeling CNF conversion $CNF_{label}$ – further improvements

- Do not apply  $CNF_{label}$  when not necessary:  
(e.g.,  $CNF_{label}(\varphi_1 \wedge \varphi_2) \implies CNF_{label}(\varphi_1) \wedge \varphi_2$ , if  $\varphi_2$  already in CNF)
- Apply DeMorgan's rules where it is more effective:  
(e.g.,  $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \implies CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$ )
- Exploit the associativity of  $\wedge$ 's and  $\vee$ 's:  
$$\dots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \dots \implies \dots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3)) \dots$$
- Before applying  $CNF_{label}$ , rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

# Exercises

- 1 Consider the following Boolean formula  $\varphi$ :

$$\neg(((\neg A_1 \rightarrow A_2) \wedge (\neg A_3 \rightarrow A_4)) \vee ((A_5 \rightarrow A_6) \wedge (A_7 \rightarrow \neg A_8)))$$

Compute the Negative Normal Form of  $\varphi$

- 2 Consider the following Boolean formula  $\varphi$ :

$$((\neg A_1 \wedge A_2) \vee (A_7 \wedge A_4) \vee (\neg A_3 \wedge \neg A_2) \vee (A_5 \wedge \neg A_4))$$

- 1 Produce the CNF formula  $CNF(\varphi)$ .
- 2 Produce the CNF formula  $CNF_{label}(\varphi)$ .
- 3 Produce the CNF formula  $CNF_{label}(\varphi)$  (improved version)



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# Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
  - AI, formal verification, circuit synthesis, operational research,....
- Important in AI:  $KB \models \alpha$ : entail fact  $\alpha$  from knowledge base  $KB$  (aka **Model Checking**:  $M(KB) \subseteq M(\alpha)$ )
  - typically  $|KB| \gg |\alpha|$
- All propositional reasoning tasks reduced to **satisfiability (SAT)**
  - $KB \models \alpha \implies \text{SAT}(KB \wedge \neg\alpha) = \text{false}$
  - input formula CNF-ized and fed to a **SAT solver**
- **Current SAT solvers dramatically efficient:**
  - handle industrial problems with  $10^6 - 10^7$  variables & clauses!
  - used as backend engines in a variety of systems

# Truth Tables

- Exhaustive evaluation of all subformulas:

$\varphi_1$	$\varphi_2$	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\perp$	$\perp$	$\perp$	$\perp$	$\top$	$\top$
$\perp$	$\top$	$\perp$	$\top$	$\top$	$\perp$
$\top$	$\perp$	$\perp$	$\top$	$\perp$	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space (draw one line at a time).
- Requires analyzing  $2^{|\text{Atoms}(\varphi)|}$  lines.
- Never used in practice.

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# The Resolution Rule

- **Resolution**: deduction of a new clause from a pair of clauses with exactly one incompatible variable (**resolvent**):

$$\frac{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l}_{\text{resolvent}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'}) \quad (\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{\neg l}_{\text{resolvent}} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}$$

- Ex: 
$$\frac{(A \vee B \vee C \vee D \vee E) \quad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$$

- Note: many standard inference rules subcases of resolution:  
(recall that  $\alpha \rightarrow \beta \iff \neg\alpha \vee \beta$ )

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \text{ (trans.)} \quad \frac{A \quad A \rightarrow B}{B} \text{ (m. ponens)} \quad \frac{\neg B \quad A \rightarrow B}{\neg A} \text{ (m. tollens)}$$

# Improvements: Subsumption & Unit Propagation

Alternative “set” notation ( $\Gamma$  clause set):

$$\frac{\Gamma, \phi_1, \dots, \phi_n}{\Gamma, \phi'_1, \dots, \phi'_n} \quad \left( \text{e.g., } \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2} \right)$$

- Removal of valid clauses:

$$\frac{\Gamma \wedge (p \vee \neg p \vee C)}{\Gamma}$$

- Clause Subsumption ( $C$  clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

$$\frac{\Gamma \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma \wedge (l) \wedge (\bigvee_i l_i)}$$

- Unit Subsumption:

$$\frac{\Gamma \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma \wedge (l)}$$

- Unit Propagation = Unit Resolution + Unit Subsumption

“Deterministic” rule: applied **before** other “non-deterministic” rules!

## Remark

### What happens with more than 1 resolvent?

- Common mistake: the following is not a correct application of the resolution rule:

$$\frac{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2), (C_1 \vee C_2)}$$

- Rather, a correct application would be:

$$\frac{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2), (C_1 \vee l_2 \vee C_2 \vee \neg l_2)}$$

... but  $(C_1 \vee l_2 \vee C_2 \vee \neg l_2)$  is valid and should be removed

⇒ no clause is produced



## Basic Propositional Inference: Resolution [33, 10]

- Assume input formula in CNF
  - if not, apply Tseitin CNF-ization first

⇒  $\varphi$  is represented as a set of clauses

- **Search** for a refutation of  $\varphi$  (is  $\varphi$  unsatisfiable?)
  - recall:  $\alpha \models \beta$  iff  $\alpha \wedge \neg\beta$  unsatisfiable
- Basic idea: **apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either**
  - a false clause is generated, or
  - the resolution rule is no more applicable
- **Correct**: if returns an empty clause, then  $\varphi$  unsat ( $\alpha \models \beta$ )
- **Complete**: if  $\varphi$  unsat ( $\alpha \models \beta$ ), then it returns an empty clause
- **Time-inefficient**
- **Very Memory-inefficient (exponential in memory)**
- Many different strategies

## Resolution: basic strategy [10]

```
function  $DP(\Gamma)$ 
  if  $\perp \in \Gamma$                                 /* unsat */
    then return False;
  if (Resolve() is no more applicable to  $\Gamma$ ) /* sat   */
    then return True;
  if {a unit clause ( $l$ ) occurs in  $\Gamma$ }      /* unit   */
    then  $\Gamma := Unit\_Propagate(l, \Gamma)$ ;
    return  $DP(\Gamma)$ 
   $A := select\_variable(\Gamma)$ ;                /* resolve */
   $\Gamma = \Gamma \cup \bigcup_{A \in C', \neg A \in C''} \{Resolve(C', C'')\} \setminus \bigcup_{A \in C', \neg A \in C''} \{C', C''\}$ ;
  return  $DP(\Gamma)$ 
```

Hint: drops one variable  $A \in Atoms(\Gamma)$  at a time

## Resolution: Examples

$$\begin{array}{cccc} (A_1 \vee A_2) & (A_1 \vee \neg A_2) & (\neg A_1 \vee A_2) & (\neg A_1 \vee \neg A_2) \\ \Downarrow & & & \\ (A_2) & (A_2 \vee \neg A_2) & (\neg A_2 \vee A_2) & (\neg A_2) \\ \Downarrow & & & \\ \perp & & & \end{array}$$

$\Rightarrow$  UNSAT

## Resolution: Examples (cont.)

$$\begin{array}{c} (A \vee B \vee C) \quad (B \vee \neg C \vee \neg F) \quad (\neg B \vee E) \\ \Downarrow \\ (A \vee C \vee E) \quad (\neg C \vee \neg F \vee E) \\ \Downarrow \\ (A \vee E \vee \neg F) \end{array}$$

$\Rightarrow$  SAT

## Resolution: Examples

$$(A \vee B) \quad (A \vee \neg B) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(A) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(C) \quad (\neg C)$$
$$\Downarrow$$
$$\perp$$

$\Rightarrow$  UNSAT

## Resolution – summary

- Requires CNF
- $\Gamma$  may blow up
  - ⇒ May require **exponential space**
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

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## Semantic tableaux [39]

- **Search** for an assignment satisfying  $\varphi$
- applies recursively **elimination rules** to the connectives
- If a branch contains  $A_i$  and  $\neg A_i$ , ( $\psi_i$  and  $\neg\psi_i$ ) for some  $i$ , the branch is **closed**, otherwise it is **open**.
- if no rule can be applied to an open branch  $\mu$ , then  $\mu \models \varphi$ ;
- if all branches are **closed**, the formula is **not satisfiable**;



# Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \rightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2}$$

( $\wedge$ -elimination)

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi}$$

( $\neg\neg$ -elimination)

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \quad \Gamma, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \neg\varphi_2}$$

$$\frac{\Gamma, (\varphi_1 \rightarrow \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \varphi_2}$$

( $\vee$ -elimination)

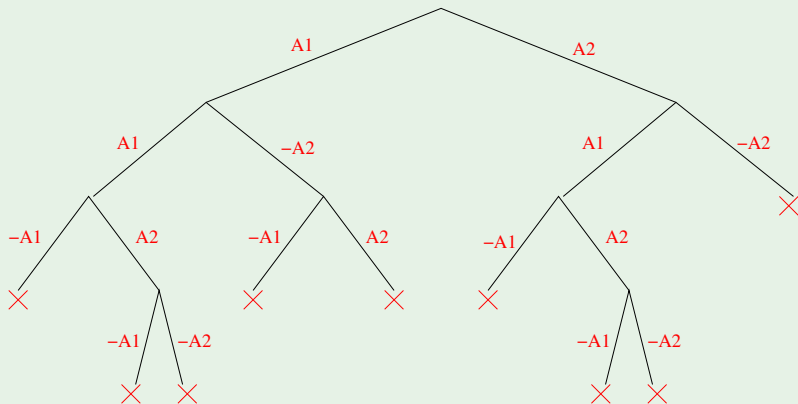
$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \quad \Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \quad \Gamma, \neg\varphi_1, \varphi_2}$$

( $\leftrightarrow$ -elimination).

# Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

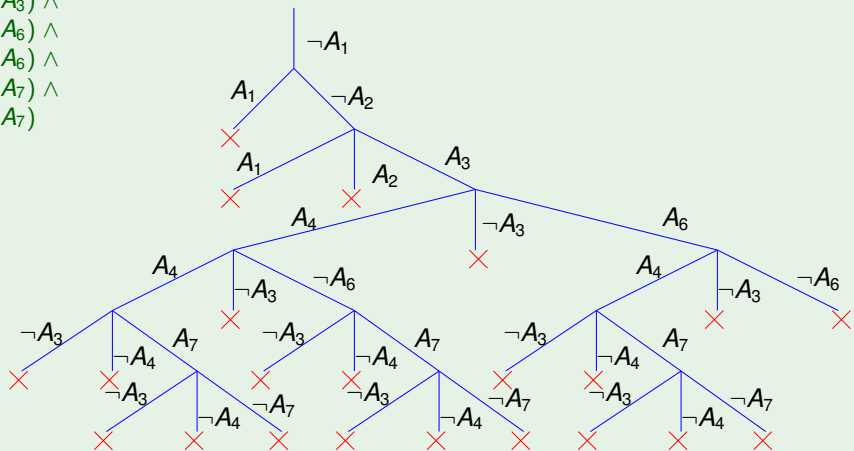


# Tableau algorithm

```
function Tableau( $\Gamma$ )  
  if  $A_i \in \Gamma$  and  $\neg A_i \in \Gamma$                                 /* branch closed */  
    then return False;  
  if  $(\varphi_1 \wedge \varphi_2) \in \Gamma$                                     /*  $\wedge$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1, \varphi_2\} \setminus \{(\varphi_1 \wedge \varphi_2)\}$ );  
  if  $(\neg\neg\varphi_1) \in \Gamma$                                         /*  $\neg\neg$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\neg\neg\varphi_1)\}$ );  
  if  $(\varphi_1 \vee \varphi_2) \in \Gamma$                                     /*  $\vee$ -elimination */  
    then return    Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ ) or  
                   Tableau( $\Gamma \cup \{\varphi_2\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ );  
  ...  
  return True;                                                /* branch expanded */
```

# Semantic Tableaux: Example

$(\neg A_1) \wedge$   
 $(A_1 \vee \neg A_2) \wedge$   
 $(A_1 \vee A_2 \vee A_3) \wedge$   
 $(A_4 \vee \neg A_3 \vee A_6) \wedge$   
 $(A_4 \vee \neg A_3 \vee \neg A_6) \wedge$   
 $(\neg A_3 \vee \neg A_4 \vee A_7) \wedge$   
 $(\neg A_3 \vee \neg A_4 \vee \neg A_7)$



$\Rightarrow$  unsat

## Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend  
⇒ loved by logicians.
- Rather inefficient  
⇒ avoided by computer scientists.
- Requires polynomial space

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- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment  $\mu$  satisfying  $\varphi$ ;
- At each step assigns a truth value to (all instances of) **one atom**.
- Performs **deterministic choices** first.

$$\frac{\varphi_1 \wedge (I)}{\varphi_1[I|\top]} \text{ (Unit)}$$

$$\frac{\varphi}{\varphi[I|\top]} \text{ (I Pure)}$$

$$\frac{\varphi}{\varphi[I|\top] \quad \varphi[I|\perp]} \text{ (split)}$$

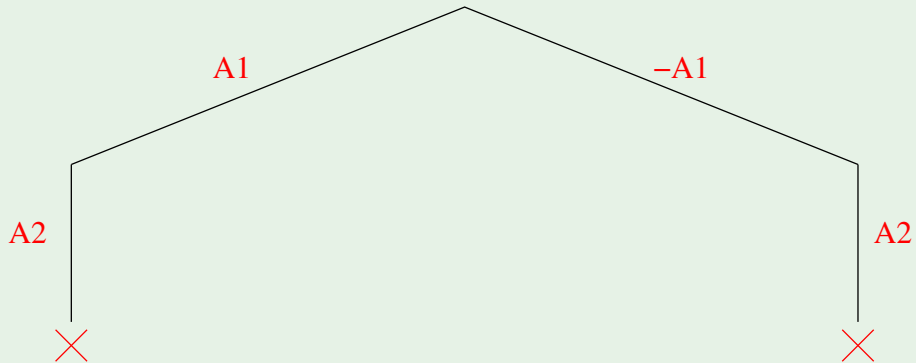
( $I$  is a **pure literal** in  $\varphi$  iff it occurs **only positively**).

- Split applied **if and only if the others cannot be applied**.
- Richer formalisms described in [40, 29, 30]



## DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



# DPLL Algorithm

```
function DPLL( $\varphi, \mu$ )  
  if  $\varphi = \top$                                 /* base */  
    then return True;  
  if  $\varphi = \perp$                                 /* backtrack */  
    then return False;  
  if {a unit clause (l) occurs in  $\varphi$ }        /* unit */  
    then return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ );  
  if {a literal l occurs pure in  $\varphi$ }        /* pure */  
    then return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ );  
  l := choose-literal( $\varphi$ );                    /* split */  
  return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ ) or  
        DPLL(assign( $\neg l$ ,  $\varphi$ ),  $\mu \wedge \neg l$ );
```

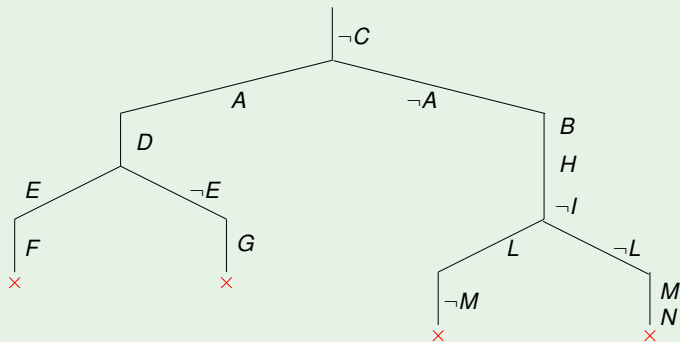
- The pure-literal rule is nowadays obsolete.
- *choose-literal*( $\varphi$ ) picks only variables still occurring in the formula

# DPLL – example

## DPLL (without pure-literal rule)

Here “choose-literal” selects variable in alphabetic, selecting true first.

$(\neg C \vee A \vee C) \wedge$   
 $(B \vee A \vee C) \wedge$   
 $(\neg A \vee D \vee F) \wedge$   
 $(\neg E \vee \neg A \vee F) \wedge$   
 $(\neg E \vee \neg F \vee \neg A) \wedge$   
 $(G \vee \neg A \vee E) \wedge$   
 $(E \vee \neg G \vee \neg A) \wedge$   
 $(A \vee H \vee C) \wedge$   
 $(\neg H \vee \neg I \vee A) \wedge$   
 $(I \vee L \vee M) \wedge$   
 $(\neg L \vee C \vee \neg M) \wedge$   
 $(A \vee \neg L \vee M) \wedge$   
 $(L \vee N \vee \neg H) \wedge$   
 $(I \vee L \vee \neg N)$



⇒ UNSAT

## DPLL – summary

- Handles **CNF formulas** (non-CNF variant known [1, 15]).
- **Branches on truth values**  
⇒ all instances of an atom assigned simultaneously
- **Postpones branching as much as possible.**
- Mostly ignored by logicians.
- (The grandfather of) **the most efficient SAT algorithms**  
⇒ loved by computer scientists.
- Requires **polynomial space**
- **Choose\_literal()** critical!
- Many very efficient implementations [42, 38, 2, 28].

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# Ordered Binary Decision Diagrams (OBDDs) [8]

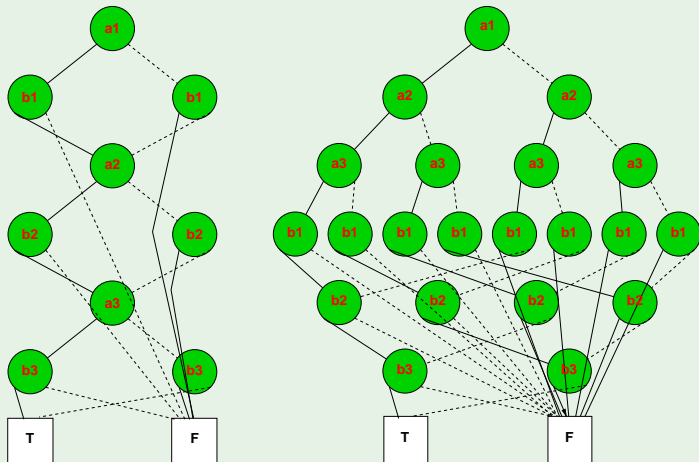
**Canonical** representation of Boolean formulas

- “If-then-else” binary direct acyclic graphs (DAGs) with one root and two leaves: **1**, **0** (or **T**, **⊥**; or **T**, **F**)
- **Variable ordering**  $A_1, A_2, \dots, A_n$  imposed a priori.
- Paths leading to **1** represent **models**  
Paths leading to **0** represent **counter-models**

## Note

Some authors call them **Reduced** Ordered Binary Decision Diagrams (**ROBDDs**)

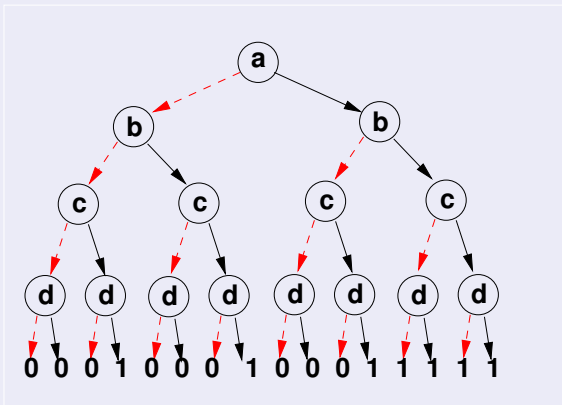
# OBDD - Examples



OBDDs of  $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$  with different variable orderings

# Ordered Decision Trees

- **Ordered Decision Tree:**  
from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for  $\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$



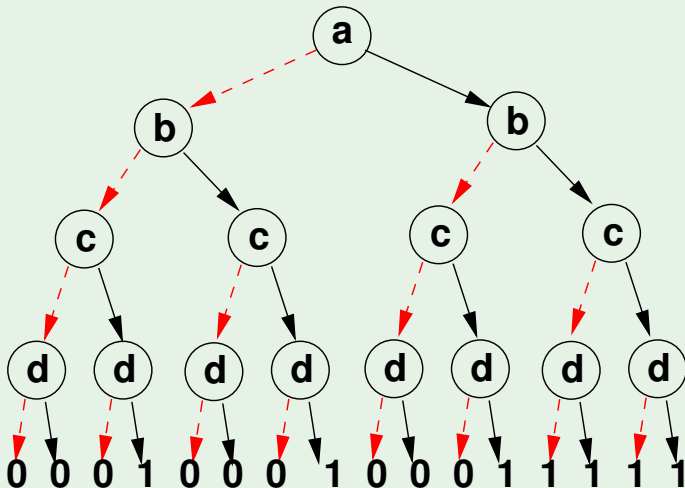


# From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following **reductions**:
  - **share subnodes**: point to the same occurrence of a subtree (via **hash consing**)
  - **remove redundancies**: nodes with same left and right children can be eliminated:  
"if  $A$  then  $B$  else  $B$ "  $\implies$  " $B$ "

# Reduction: example

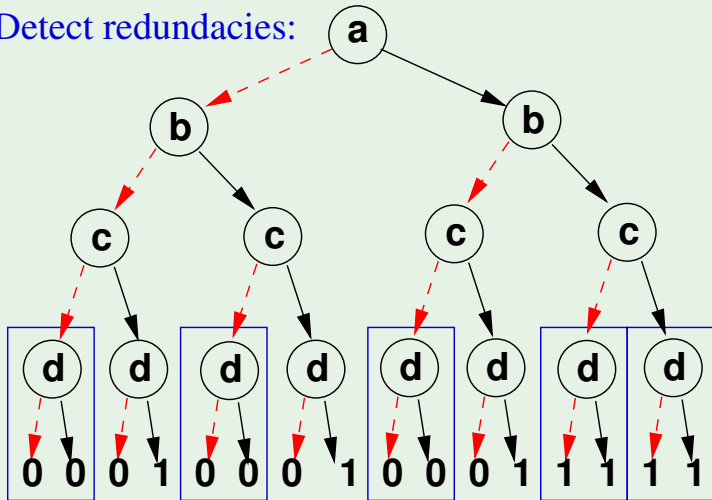
$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

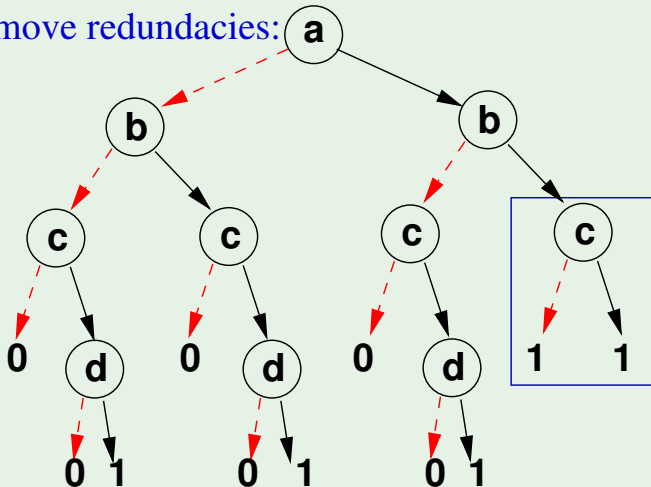
Detect redundancies:



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

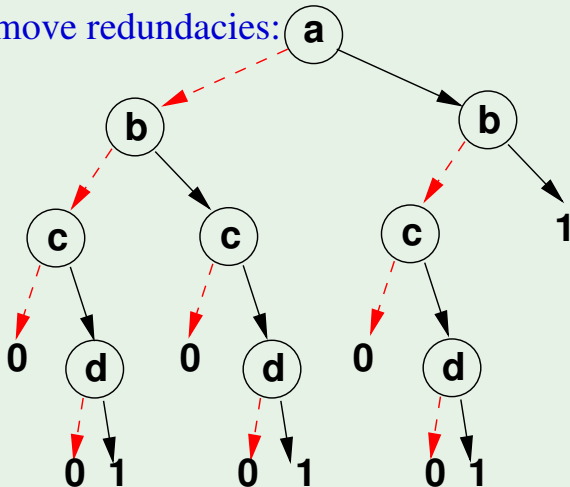
Remove redundancies:



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

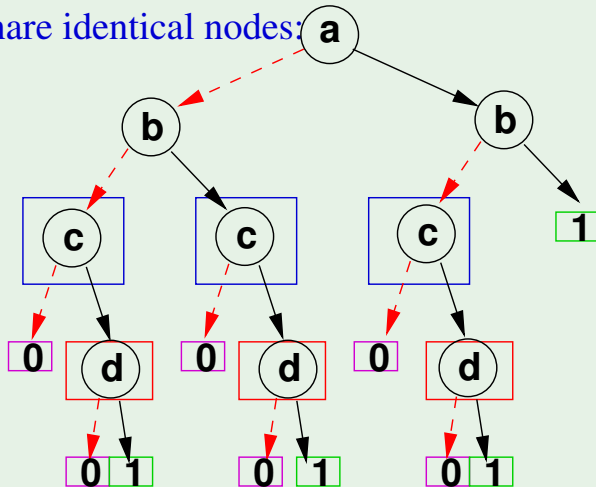
Remove redundancies:



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

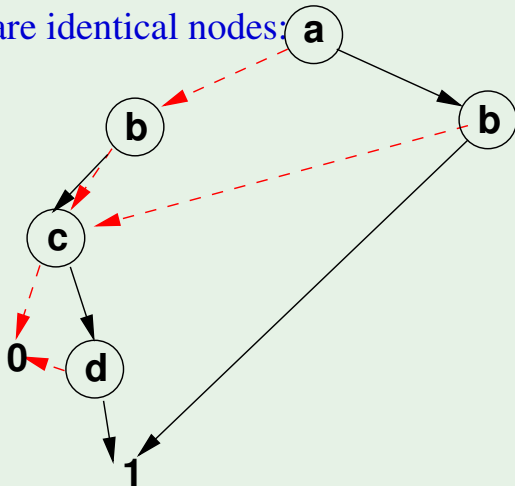
Share identical nodes:



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

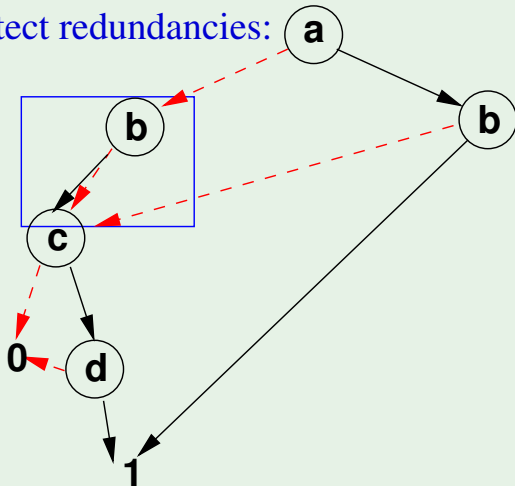
Share identical nodes:



# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

Detect redundancies:

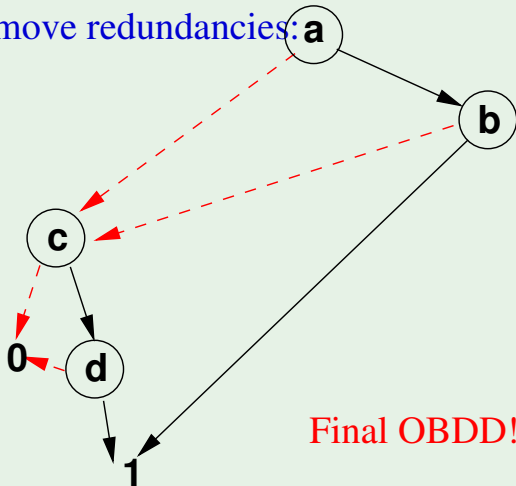




# Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

Remove redundancies:



# If-Then-Else Operators: “*ite*(...)”

## If-Then-Else Operators: “*ite*(...)”

- *ite*( $\phi, \varphi^{\top}, \varphi^{\perp}$ ): “If  $\phi$  Then  $\varphi^{\top}$  Else  $\varphi^{\perp}$ ”
- *ite*( $\phi, \varphi^{\top}, \varphi^{\perp}$ )  $\stackrel{\text{def}}{=} ((\neg\phi \vee \varphi^{\top}) \wedge (\phi \vee \varphi^{\perp})) \iff ((\phi \wedge \varphi^{\top}) \vee (\neg\phi \wedge \varphi^{\perp}))$

- properties:

$$\begin{aligned} \textit{ite}(\neg\phi, \varphi^{\top}, \varphi^{\perp}) &= \textit{ite}(\phi, \varphi^{\perp}, \varphi^{\top}) \\ \neg\textit{ite}(\phi, \varphi^{\top}, \varphi^{\perp}) &= \textit{ite}(\phi, \neg\varphi^{\top}, \neg\varphi^{\perp}) \\ \textit{ite}(\phi, \varphi_1^{\top}, \varphi_1^{\perp}) \textit{op} \textit{ite}(\phi, \varphi_2^{\top}, \varphi_2^{\perp}) &= \textit{ite}(\phi, (\varphi_1^{\top} \textit{op} \varphi_2^{\top}), (\varphi_1^{\perp} \textit{op} \varphi_2^{\perp})) \\ \textit{ite}(\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}) \textit{op} \textit{ite}(\phi_2, \varphi_2^{\top}, \varphi_2^{\perp}) &= \textit{ite}(\phi_1, (\varphi_1^{\top} \textit{op} \textit{ite}(\phi_2, \varphi_2^{\top}, \varphi_2^{\perp})), \\ &\quad (\varphi_1^{\perp} \textit{op} \textit{ite}(\phi_2, \varphi_2^{\top}, \varphi_2^{\perp}))) \\ &= \textit{ite}(\phi_2, (\textit{ite}(\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}) \textit{op} \varphi_2^{\top}), \\ &\quad (\textit{ite}(\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}) \textit{op} \varphi_2^{\perp})) \end{aligned} \quad \textit{op} \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$$

# Recursive structure of an OBDD

Assume the variable ordering  $A_1, A_2, \dots, A_n$ :

$$\begin{aligned} \text{OBDD}(\top, \{A_1, A_2, \dots, A_n\}) &= 1 \\ \text{OBDD}(\perp, \{A_1, A_2, \dots, A_n\}) &= 0 \\ \text{OBDD}(\varphi, \{A_1, A_2, \dots, A_n\}) &= \begin{aligned} &\text{if } A_1 \\ &\text{then } \text{OBDD}(\varphi[A_1|\top], \{A_2, \dots, A_n\}) \\ &\text{else } \text{OBDD}(\varphi[A_1|\perp], \{A_2, \dots, A_n\}) \end{aligned} \end{aligned}$$

# Incrementally building an OBDD

- $obdd\_build(\top, \{\dots\}) := \top$ ,
- $obdd\_build(\perp, \{\dots\}) := \perp$ ,
- $obdd\_build(A_i, \{\dots\}) := ite(A_i, \top, \perp)$ ,
- $obdd\_build((\neg\varphi), \{A_1, \dots, A_n\}) := apply(\neg, obdd\_build(\varphi, \{A_1, \dots, A_n\}))$
- $obdd\_build((\varphi_1 \text{ op } \varphi_2), \{A_1, \dots, A_n\}) :=$   
     $reduce($   
         $apply($     $op,$   
             $obdd\_build(\varphi_1, \{A_1, \dots, A_n\}),$     $op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$   
             $obdd\_build(\varphi_2, \{A_1, \dots, A_n\})$   
         $)$   
     $)$

## Incrementally building an OBDD (cont.)

- $apply(op, O_i, O_j) := (O_i op O_j)$  **if**  $(O_i \in \{\top, \perp\}$  or  $O_j \in \{\top, \perp\})$
- $apply(\neg, ite(A_i, \varphi_i^\top, \varphi_i^\perp)) :=$   
 $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\perp))$
- $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), ite(A_j, \varphi_j^\top, \varphi_j^\perp)) :=$   
**if**  $(A_i = A_j)$  **then**  $ite(A_i, apply(op, \varphi_i^\top, \varphi_j^\top),$   
 $apply(op, \varphi_i^\perp, \varphi_j^\perp))$   
**if**  $(A_i < A_j)$  **then**  $ite(A_i, apply(op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\perp)),$   
 $apply(op, \varphi_i^\perp, ite(A_j, \varphi_j^\top, \varphi_j^\perp)))$   
**if**  $(A_i > A_j)$  **then**  $ite(A_j, apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\top),$   
 $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\perp))$

$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

## Incrementally building an OBDD: Examples

- Ex: build the obdd for  $A_1 \vee A_2$  from those of  $A_1, A_2$  (order:  $A_1, A_2$ ):

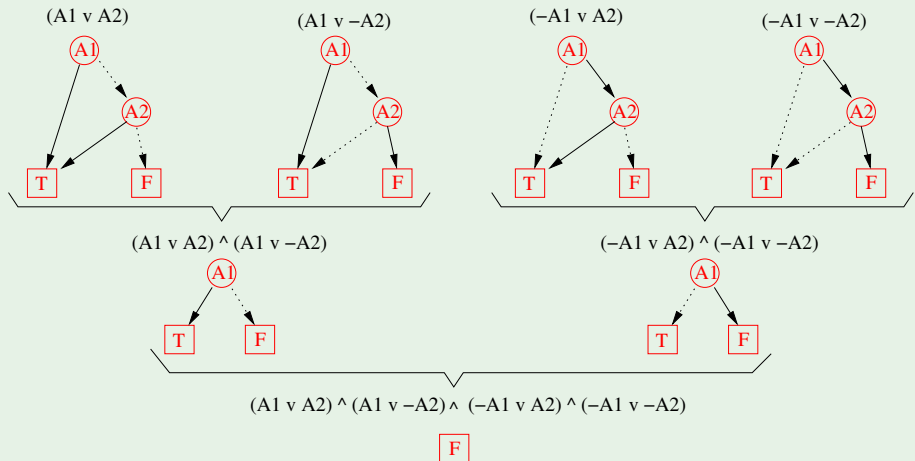
$$\begin{aligned} & \text{apply}(\vee, \overbrace{\text{ite}(A_1, \top, \perp)}^{A_1}, \overbrace{\text{ite}(A_2, \top, \perp)}^{A_2}) \\ &= \text{ite}(A_1, \text{apply}(\vee, \top, \text{ite}(A_2, \top, \perp)), \text{apply}(\vee, \perp, \text{ite}(A_2, \top, \perp))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp)) \end{aligned}$$

- Ex: build the obdd for  $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$  from those of  $(A_1 \vee A_2), (A_1 \vee \neg A_2)$  (order:  $A_1, A_2$ ):

$$\begin{aligned} & \text{apply}(\wedge, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp))}^{(A_1 \vee A_2)}, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \perp, \top))}^{(A_1 \vee \neg A_2)}), \\ &= \text{ite}(A_1, \text{apply}(\wedge, \top, \top), \text{apply}(\wedge, \text{ite}(A_2, \top, \perp), \text{ite}(A_2, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \text{apply}(\wedge, \top, \perp), \text{apply}(\wedge, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \perp, \perp)) \\ &= \text{ite}(A_1, \top, \perp) \end{aligned}$$

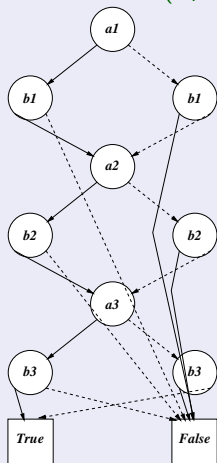
# OBDD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

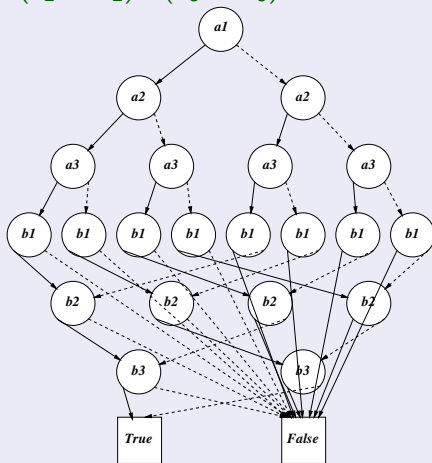


# Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$



Linear size



Exponential size



# OBDD's as canonical representation of Boolean formulas

- An OBDD is a **canonical representation** of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff \text{OBDD}(\varphi_1) = \text{OBDD}(\varphi_2)$$

- equivalence check requires **constant time!**
  - ⇒ validity check requires constant time! ( $\varphi \leftrightarrow \top$ )
  - ⇒ (un)satisfiability check requires constant time! ( $\varphi \leftrightarrow \perp$ )
- the set of the paths from the root to 1 represent all the **models** of the formula
- the set of the paths from the root to 0 represent all the **counter-models** of the formula

# Exponentiality of OBDD's

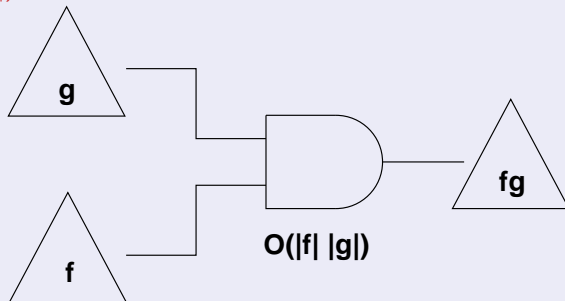
- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless  $P = co-NP$ )
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

## Note

The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

# Useful Operations over OBDDs

- the **equivalence check** between two OBDDs is simple
  - are they the same OBDD? ( $\implies$  constant time)
- the size of a **Boolean composition** is up to the product of the size of the operands:  
 $|f \text{ op } g| = O(|f| \cdot |g|)$



(but typically much smaller on average).

# [Recall] Boolean Quantification

## Shannon's expansion:

- If  $v$  is a Boolean variable and  $f$  is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- $v$  does no more occur in  $\exists v.\varphi$  and  $\forall v.\varphi$  !!
- Multi-variable quantification:  $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

- Intuition:

- $\mu \models \exists v.\varphi$  iff exists *truthvalue*  $\in \{\top, \perp\}$  s.t.  $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$  iff forall *truthvalue*  $\in \{\top, \perp\}$ ,  $\mu \cup \{v := \text{truthvalue}\} \models \varphi$

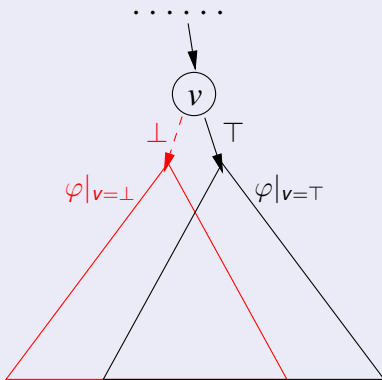
- Example:  $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

## Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

# OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if  $f$  is a sub-OBDD labeled by variable  $v$ , then  $\varphi|_{v=T}$  and  $\varphi|_{v=\perp}$  are the “then” and “else” branches of  $f$

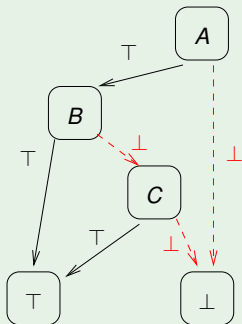


⇒ lots of sharing of subformulae!

# Example

Let  $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$  and  $\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. \varphi$ . Using the variable ordering “A, B, C”, draw the OBDD corresponding to the formulas  $\varphi$  and  $\varphi'$ .

$$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$$

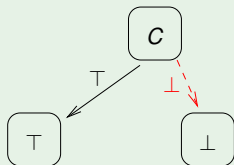


## Example (cont.)

$$\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. (A \wedge (B \vee C))$$

$$\begin{aligned} \varphi' &\stackrel{\text{def}}{=} \exists A. \forall B. \varphi \\ &= \forall B. (A \wedge (B \vee C)) [A := \top] && \vee (\forall B. (A \wedge (B \vee C))) [A := \perp] \\ &= \forall B. (B \vee C) && \vee \forall B. \perp \\ &= ((B \vee C) [B := \top] \quad \wedge \quad (B \vee C) [B := \perp]) && \vee \perp \\ &= (\top \quad \wedge \quad C) \\ &= C \end{aligned}$$

which corresponds to the following OBDD:



## OBDD – summary

- **Factorize** common parts of the search tree (DAG)
- Require setting a **variable ordering** a priori (**critical!**)
- **Canonical representation** of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents **all** models and counter-models of the formula.
- Require **exponential space** in worst-case
- **Very efficient** for some practical problems (circuits, symbolic model checking).



# Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
  - Generalities
  - Resolution
  - Tableaux
  - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers**
  - Limitations of Chronological Backtracking
  - Conflict-Driven Clause-Learning SAT solvers
  - Further Improvements
  - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

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## DPLL: “Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment /
- / is toggled, is labeled as “closed”, and the search proceeds.

## DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible

⇒ lots of useless search!

# DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

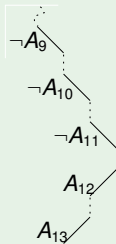
$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

...

# DPLL Chronological Backtracking: Example

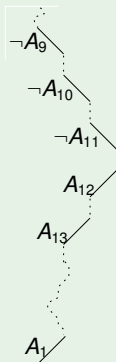
$C_1 : \neg A_1 \vee A_2$   
 $C_2 : \neg A_1 \vee A_3 \vee A_9$   
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$   
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$   
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$   
 $C_6 : \neg A_5 \vee \neg A_6$   
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$   
 $C_8 : A_1 \vee A_8$   
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$   
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$   
(initial assignment)

# DPLL Chronological Backtracking: Example

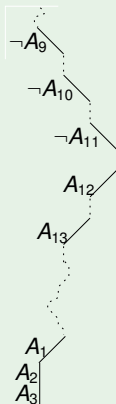
- $C_1 : \neg A_1 \vee A_2$
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- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$   
... (branch on  $A_1$ )

# DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
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- ...

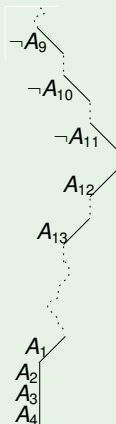


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$   
(unit  $A_2, A_3$ )



# DPLL Chronological Backtracking: Example

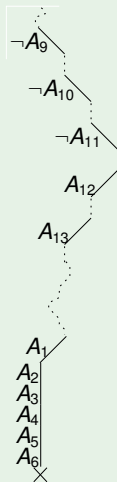
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4 \quad \checkmark$
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- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$   
(unit  $A_4$ )

# DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$   
 $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$   
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4 \quad \checkmark$   
 $C_4 : \neg A_4 \vee A_5 \vee A_{10} \quad \checkmark$   
 $C_5 : \neg A_4 \vee A_6 \vee A_{11} \quad \checkmark$   
 $C_6 : \neg A_5 \vee \neg A_6 \quad \times$   
 $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$   
 $C_8 : A_1 \vee A_8 \quad \checkmark$   
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$   
 ...

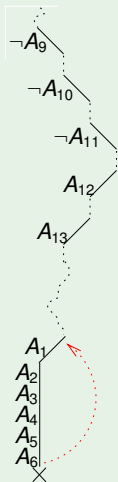


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, \neg A_{12}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4, A_5, A_6\}$   
 (unit  $A_5, A_6$ )  $\implies$  conflict

# DPLL Chronological Backtracking: Example

$C_1 : \neg A_1 \vee A_2$   
 $C_2 : \neg A_1 \vee A_3 \vee A_9$   
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$   
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$   
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$   
 $C_6 : \neg A_5 \vee \neg A_6$   
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$   
 $C_8 : A_1 \vee A_8$   
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$   
...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$   
 $\implies$  backtrack up to  $A_1$



# DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2 \quad \checkmark$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

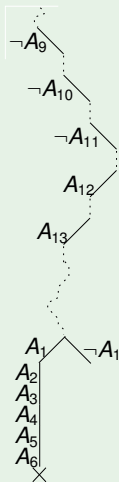
$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

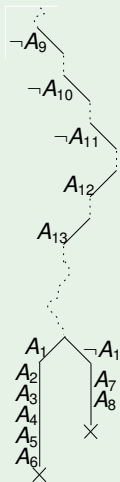
...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1\}$   
(unit  $\neg A_1$ )



# DPLL Chronological Backtracking: Example

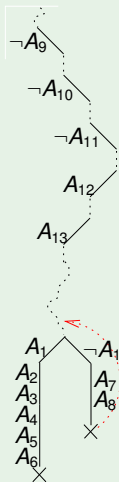
- $C_1 : \neg A_1 \vee A_2$  ✓  
 $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓  
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$   
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$   
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$   
 $C_6 : \neg A_5 \vee \neg A_6$   
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓  
 $C_8 : A_1 \vee A_8$  ✓  
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✗  
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8\}$   
(unit  $A_7, A_8$ )  $\implies$  conflict

# DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

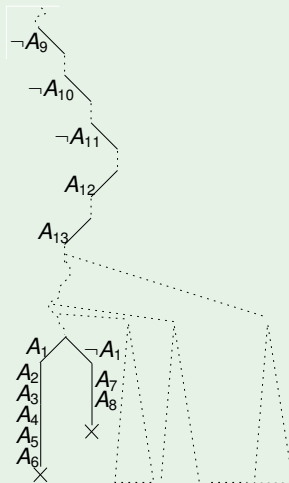


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

$\implies$  backtrack to the most recent open branching point

# DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

$\Rightarrow$  lots of useless search before backtracking up to  $A_{13}$ !

# Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
  - Generalities
  - Resolution
  - Tableaux
  - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 **Modern CDCL SAT Solvers**
  - Limitations of Chronological Backtracking
  - **Conflict-Driven Clause-Learning SAT solvers**
  - Further Improvements
  - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization



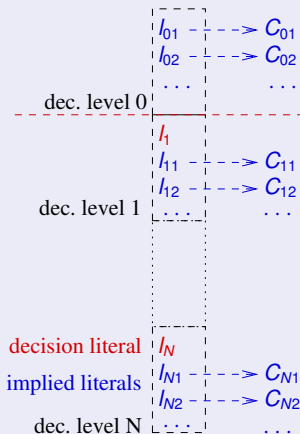
# Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
  - inspired to conflict-driven backjumping and learning in CSPs
  - learns implied clauses as nogoods
- Random restarts
  - abandon the current search tree and restart on top level
  - previously-learned clauses maintained
- Smart literal selection heuristics (ex: VSIDS)
  - “static”: scores updated only at the end of a branch
  - “local”: privileges variable in recently learned clauses
- Smart preprocessing/inprocessing technique to simplify formulas
- Smart indexing techniques (e.g. 2-watched literals)
  - efficiently do/undo assignments and reveal unit clauses
- Allow Incremental Calls (stack-based interface)
  - allow for reusing previous search on “similar” problems

Can handle industrial problems with  $10^6$  –  $10^7$  variables and clauses!

# Stack-based representation of a truth assignment $\mu$

- assign one truth-value at a time (add one literal to a stack representing  $\mu$ )
- stack partitioned into **decision levels**:
  - one **decision literal**
  - its **implied literals**
  - each implied literal tagged with the clause causing its unit-propagation (**antecedent clause**)
- equivalent to an **implication graph**

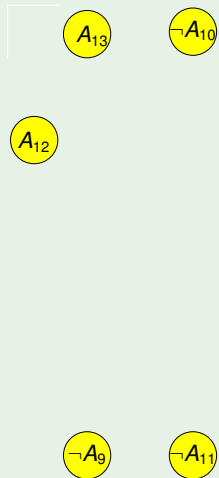
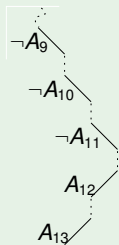


# Implication graph

- An **implication graph** is a DAG s.t.:
  - each node represents a variable assignment (literal)
  - each edge  $l_i \xrightarrow{c} l$  is labeled with a clause
  - the node of a decision literal has no incoming edges
  - all edges incoming into a node  $l$  are labeled with the same clause  $c$ , s.t.  $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$  iff  $c = \neg l_1 \vee \dots \vee \neg l_n \vee l$   
( $c$  is said to be the **antecedent clause** of  $l$ )
  - when both  $l$  and  $\neg l$  occur in the graph, we have a **conflict**.
- Intuition:
  - representation of the dependencies between literals in  $\mu$
  - the graph contains  $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$  iff  $l$  has been obtained from  $l_1, \dots, l_n$  by unit propagation on  $c$
  - a partition of the graph with all decision literals on one side and the conflict on the other represents a **conflict set**

# Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

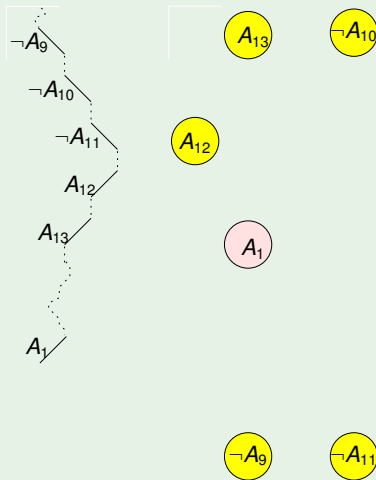


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

(Initial assignment. Note:  $c_1, \dots, c_9$  inconsistent.)

# Example

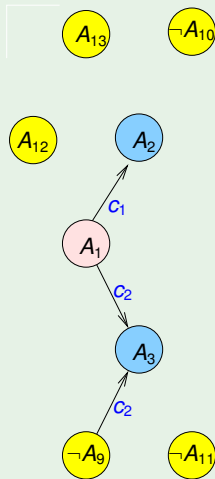
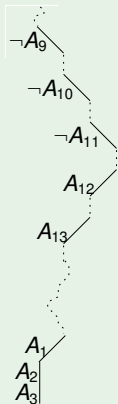
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$   
... (decide  $A_1$ )

# Example

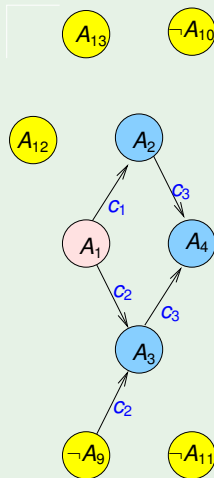
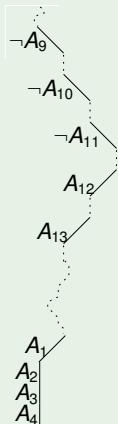
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$   
(unit  $A_2, A_3$ )

# Example

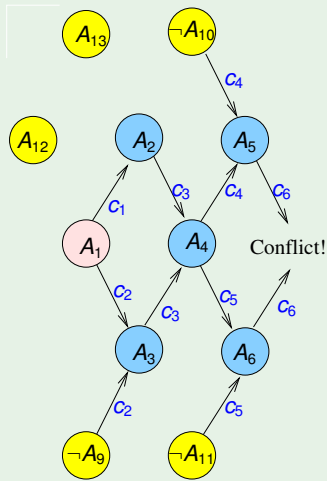
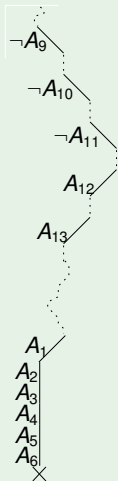
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4 \quad \checkmark$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$   
(unit  $A_4$ )

# Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_1, \neg A_4, A_1, A_2, A_3, A_4, A_5, A_6\}$   
 (unit  $A_5, A_6$ )  $\implies$  conflict



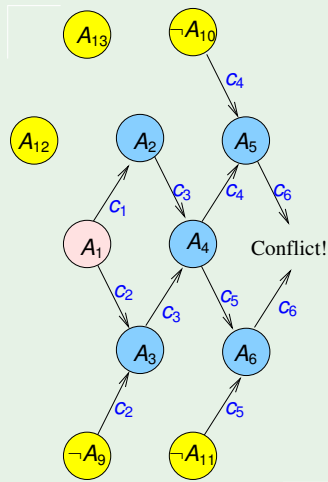
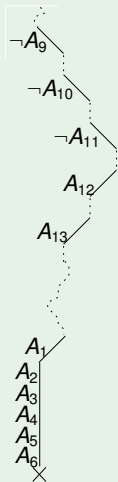
## Unique implication point - UIP [44]

- A node  $l$  in an implication graph is an **unique implication point** (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through  $l$ .
  - the most recent decision node is an UIP (**last UIP**)
  - all other UIP's have been assigned after the most recent decision

# Unique implication point - UIP - example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...

- $A_1$  is the last UIP
- $A_4$  is the 1<sup>st</sup> UIP



## Schema of a CDCL DPLL solver [38, 45]

```
Function CDCL-SAT (formula:  $\varphi$ , assignment &  $\mu$ ) {
  status := preprocess( $\varphi, \mu$ );
  while (1) {
    while (1) {
      status := deduce( $\varphi, \mu$ );
      if (status == Sat)
        return Sat;
      if (status == Conflict) {
         $\langle \text{blevel}, \eta \rangle := \text{analyze\_conflict}(\varphi, \mu)$ ;
        //  $\eta$  is a conflict set
        if (blevel == 0)
          return Unsat;
        else backtrack(blevel,  $\varphi, \mu$ );
      }
      else break;
    }
    decide_next_branch( $\varphi, \mu$ );
  }
}
```

## Schema of a CDCL DPLL solver [38, 45] (cont.)

- `preprocess( $\varphi, \mu$ )` simplifies  $\varphi$  into an easier equisatisfiable formula, updating  $\mu$ .
- `decide_next_branch( $\varphi, \mu$ )` chooses a new decision literal from  $\varphi$  according to some heuristic, and adds it to  $\mu$
- `deduce( $\varphi, \mu$ )` performs all deterministic assignments (unit-propagations plus others), and updates  $\varphi, \mu$  accordingly.
- `analyze_conflict( $\varphi, \mu$ )` Computes the subset  $\eta$  of  $\mu$  causing the conflict (conflict set), and returns the “wrong-decision” level suggested by  $\eta$  (“0” means that  $\eta$  is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack( $blevel, \varphi, \mu$ )` undoes the branches up to  $blevel$ , and updates  $\varphi, \mu$  accordingly

## Backjumping and learning: general ideas [2, 38]

- When a branch  $\mu$  fails:
  - (i) **conflict analysis**: reveal the sub-assignment  $\eta \subseteq \mu$  causing the failure (**conflict set  $\eta$** )
  - (ii) **learning**: add the **conflict clause**  $C \stackrel{\text{def}}{=} \neg\eta$  to the clause set
  - (iii) **backjumping**: use  $\eta$  to decide the point where to backtrack
- Jump back up much more than one decision level in the stack  
 $\implies$  **may avoid lots of redundant search!!**
- We illustrate two main backjumping & learning strategies:
  - the original strategy presented in [38]
  - the state-of-the-art 1<sup>st</sup>UIP strategy of [44]

# Conflict analysis

1.  $C :=$  falsified clause (**conflicting clause**)
2. repeat
  - (i) resolve the current clause  $C$  with the antecedent clause of the last unit-propagated literal  $l$  in  $C$until  $C$  verifies some given termination criteria

# Conflict analysis

1.  $C :=$  falsified clause (**conflicting clause**)
2. repeat
  - (i) resolve the current clause  $C$  with the antecedent clause of the last unit-propagated literal  $l$  in  $C$until  $C$  verifies some given termination criteria

criterion: **decision**

...until  $C$  contains only decision literals

$$\begin{array}{r} \frac{\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_5 \vee \neg A_6}}{\text{Conflicting cl.}} \quad (A_6) \end{array}$$

(A<sub>6</sub>)

(A<sub>5</sub>)

(A<sub>4</sub>)

(A<sub>3</sub>)

(A<sub>2</sub>)

# Conflict analysis

1.  $C :=$  falsified clause (**conflicting clause**)
2. repeat
  - (i) resolve the current clause  $C$  with the antecedent clause of the last unit-propagated literal  $l$  in  $C$until  $C$  verifies some given termination criteria

## criterion: last UIP

... until  $C$  contains only one literal assigned at current decision level, and it is the decision literal (**last UIP**)

$$\begin{array}{r} \frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_6) \end{array}$$

*Conflicting cl.*

$$\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_5)$$
$$\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_4)$$
$$\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_3)$$
$$\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_2)$$
$$\frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}} \quad (A_1)$$



# Conflict analysis

1.  $C :=$  falsified clause (**conflicting clause**)
2. repeat
  - (i) resolve the current clause  $C$  with the antecedent clause of the last unit-propagated literal  $l$  in  $C$
 until  $C$  verifies some given termination criteria

criterion: **1st UIP**

... until  $C$  contains only one literal assigned at current decision level (**1st UIP**)

$$\begin{array}{c}
 \text{Conflicting cl.} \\
 \overline{\neg A_4 \vee A_6 \vee A_{11}} \quad \overline{\neg A_5 \vee \neg A_6} \quad (A_6) \\
 \hline
 \neg A_4 \vee A_5 \vee A_{10} \quad \neg A_4 \vee \neg A_5 \vee A_{11} \quad (A_5) \\
 \hline
 \underbrace{\neg A_4}_{\text{1st UIP}} \vee A_{10} \vee A_{11}
 \end{array}$$

# Conflict analysis

1.  $C :=$  falsified clause (**conflicting clause**)
2. repeat
  - (i) resolve the current clause  $C$  with the antecedent clause of the last unit-propagated literal  $l$  in  $C$until  $C$  verifies some given termination criteria

## Note:

$\varphi \models C$ , so that  $C$  can be safely added to  $\varphi$ .

## Note:

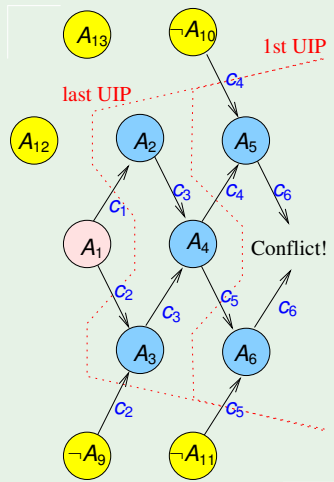
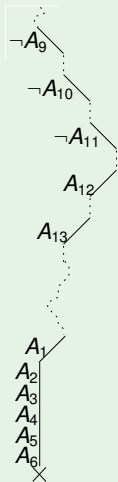
Equivalent to finding a partition in the implication graph of  $\mu$  with all decision literals on one side and the conflict on the other.

# Conflict analysis and implication graph - example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...

Note: in

this case decision and last-UIP criteria produce the same partition

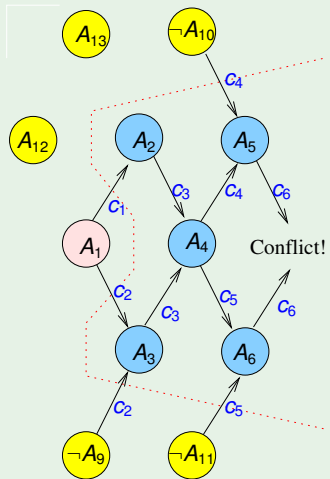
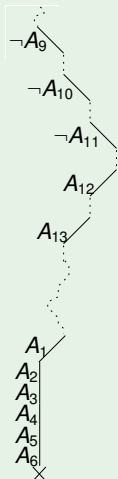


## The original backjumping and learning strategy of [38]

- Idea: when a branch  $\mu$  fails,
  - (i) **conflict analysis**: find the conflict set  $\eta \subseteq \mu$  by generating the conflict clause  $C \stackrel{\text{def}}{=} \neg\eta$  via resolution from the falsified clause (conflicting clause) using the “Decision” criterion;
  - (ii) **learning**: add the conflict clause  $C$  to the clause set
  - (iii) **backjumping**: backtrack to the most recent branching point s.t. the stack does not fully contain  $\eta$ , and then unit-propagate the unassigned literal on  $C$

# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...

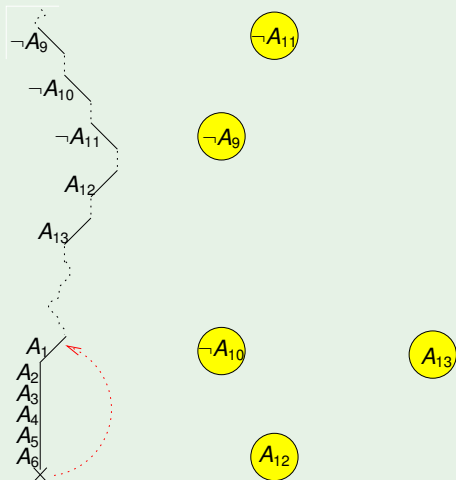


- ⇒ Conflict set:  $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$  ("decision" schema)
- ⇒ learn the conflict clause  $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
- ...

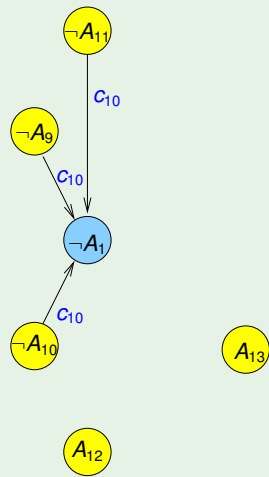
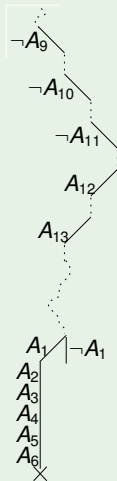
$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$   
 $\implies$  backtrack up to  $A_1$



# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$  ✓
- ...

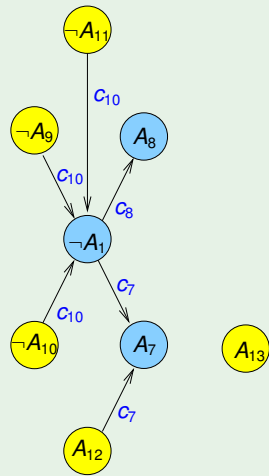
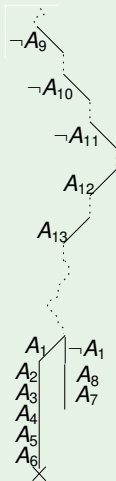
$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1\}$   
 (unit  $\neg A_1$ )



# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$  ✓
- ...

{ ...,  $\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8$  }  
 (unit  $A_7, A_8$ )

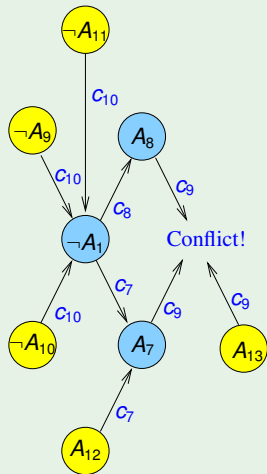
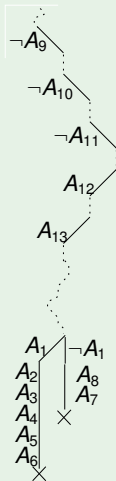




# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✗
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$  ✓
- ...

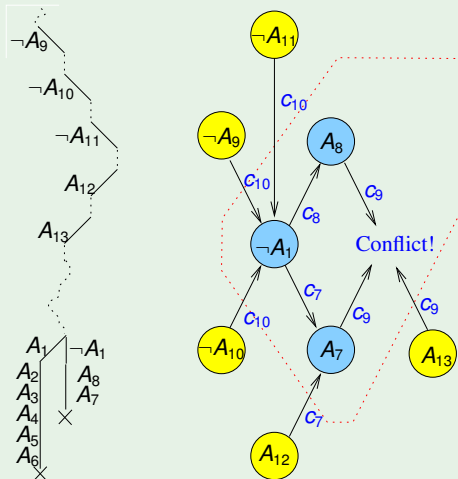
{...,  $\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8$ }  
**Conflict!**



# The Original Backjumping Strategy: Example

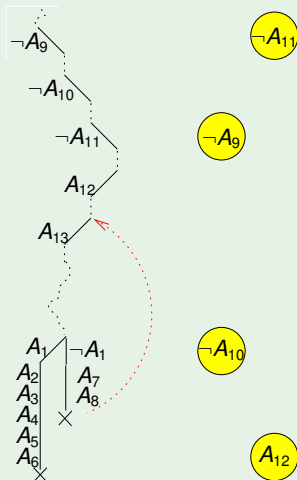
- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✗
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$  ✓
- ...

⇒ conflict set:  $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$  .  
 ⇒ learn  $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$



# The Original Backjumping Strategy: Example

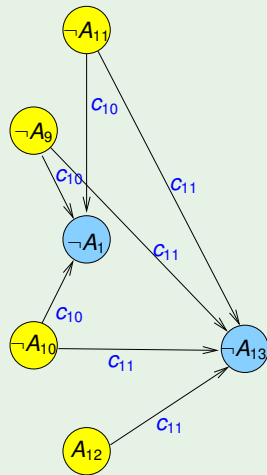
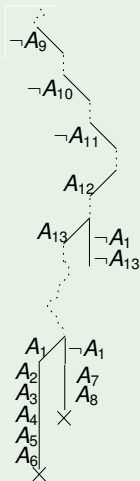
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$
- ...



⇒ backtrack to  $A_{13}$  ⇒ Lots of search saved!

# The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$  ✓
- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$  ✓
- ...



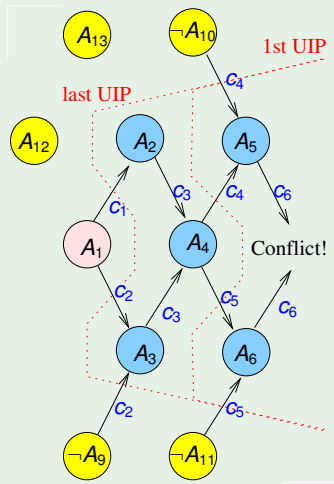
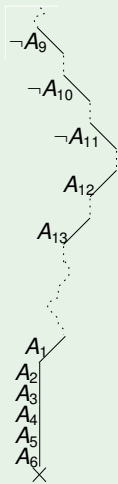
⇒ backtrack to  $A_{13}$ , then set  $A_{13}$  and  $A_1$  to  $\perp, \dots$

## State-of-the-art backjumping and learning [44]

- Idea: when a branch  $\mu$  fails,
  - (i) **conflict analysis**: find the conflict set  $\eta \subseteq \mu$  by generating the conflict clause  $C \stackrel{\text{def}}{=} \neg\eta$  via resolution from the falsified clause, according to the **1<sup>st</sup>UIP strategy**
  - (ii) **learning**: add the conflict clause  $C$  to the clause set
  - (iii) **backjumping**: **backtrack to the highest branching point s.t. the stack contains all-but-one literals in  $\eta$ , and then unit-propagate the unassigned literal on  $C$**

# 1st UIP strategy – example (7)

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...



⇒ Conflict set:  $\{\neg A_{10}, \neg A_{11}, A_4\}$ , learn  $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

## 1st UIP strategy and backjumping [44]

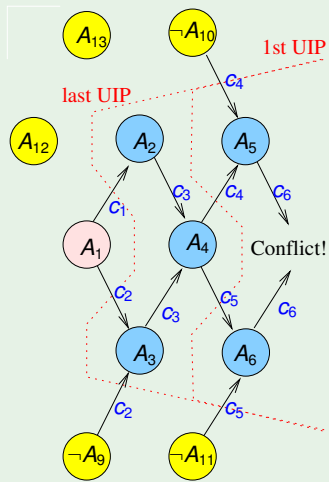
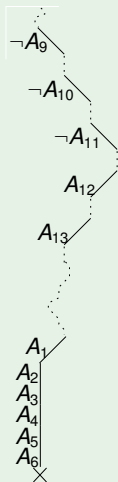
- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.:  $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

$\implies$  backtrack to  $A_{11}$ , then assign  $\neg A_4$

# 1st UIP strategy – example (7)

- $C_1 : \neg A_1 \vee A_2$  ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$  ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$  ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$  ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$  ✓
- $C_6 : \neg A_5 \vee \neg A_6$  ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$  ✓
- $C_8 : A_1 \vee A_8$  ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$  ✓
- ...

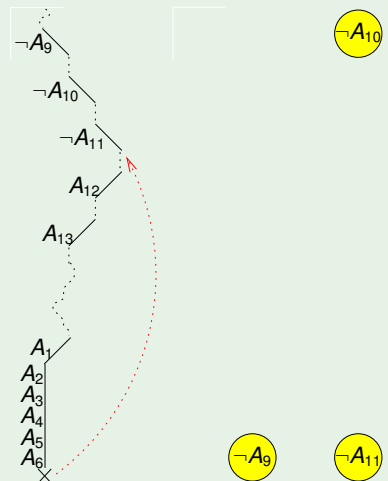


⇒ Conflict set:  $\{\neg A_{10}, \neg A_{11}, A_4\}$ , learn  $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$



# 1st UIP strategy – example (8)

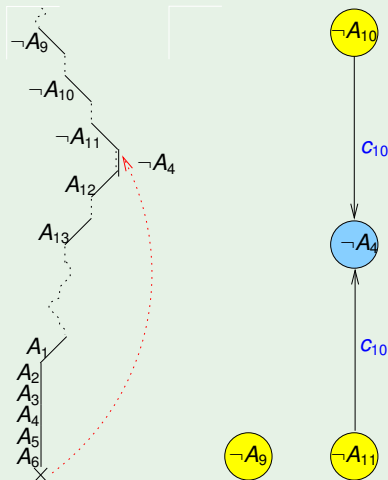
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4$
- ...



$\Rightarrow$  backtrack up to  $A_{11} \Rightarrow \{ \dots, \neg A_9, \neg A_{10}, \neg A_{11} \}$

# 1st UIP strategy – example (9)

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10} \quad \checkmark$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11} \quad \checkmark$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4 \quad \checkmark$
- ...



$\Rightarrow$  unit propagate  $\neg A_4 \Rightarrow \{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_4\} \dots$

# 1st UIP strategy and backjumping – intuition

- An UIP is a **single** reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
  - does not enlarge the conflict
  - requires less resolution steps to compute  $C$
  - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
  - jump higher
  - allows for assigning (the negation of) the UIP as high as possible in the search tree.

## Learning [2, 38]

Idea: When a conflict set  $\eta$  is revealed, then  $C \stackrel{\text{def}}{=} \neg\eta$  is added to  $\varphi$

$\implies$  the solver will no more generate an assignment containing  $\eta$ :

when  $|\eta| - 1$  literals in  $\eta$  are assigned, the other is set  $\perp$  by unit-propagation on  $C$

$\implies$  **Drastic pruning of the search!**

# Learning – example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

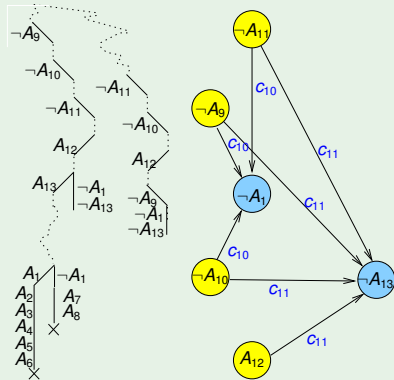
$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13} \quad \checkmark$$

$$C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1 \quad \checkmark$$

$$C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13} \quad \checkmark$$

...

⇒ Unit:  $\{\neg A_1, \neg A_{13}\}$



# Drawbacks of Learning & Clause discharging

## Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

## A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their **activity**.

A clause is currently **active** if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

# Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

## Property (see, e.g., [30])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.

⇒ CDCL solvers require polynomial space

## “Lazy” Strategy

- when a clause is involved in conflict analysis, increase its activity
- when needed, drop the least-active clauses

# State-of-the-art backjumping and learning: intuitions

- **Backjumping:** allows for climbing up to many decision levels in the stack
  - intuition: “go back to the oldest decision where you’d have done something different if only you had known  $C$ ”  
⇒ may avoid lots of redundant search
- **Learning:** in future branches, when all-but-one literals in  $\eta$  are assigned, the remaining literal is assigned to false by unit-propagation:
  - intuition: “when you’re about to repeat the mistake, do the opposite of the last step”  
⇒ avoid finding the same conflict again



## Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
  - Backjumping: if causes the highest backjump (“local” role)
  - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented (see, e.g., [2, 38, 44])

# Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
  - Generalities
  - Resolution
  - Tableaux
  - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers**
  - Limitations of Chronological Backtracking
  - Conflict-Driven Clause-Learning SAT solvers
  - Further Improvements**
  - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

# Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
  - detect and remove subsumed clauses
  - detect & collapse equivalent literals
  - apply basic resolution steps
  - ...

## Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\begin{array}{c} \varphi_1 \wedge (b_2 \vee h_1) \wedge \varphi_2 \wedge (b_2 \vee b_3 \vee h_1) \wedge \varphi_3 \\ \Downarrow \\ \varphi_1 \wedge (h_1 \vee b_2) \wedge \varphi_2 \wedge \varphi_3 \end{array}$$

# Preprocessing/Inprocessing (cont.)

## Detect & collapse equivalent literals [7]

### Repeat:

- (i) build the implication graph induced by binary clauses
- (ii) detect **strongly connected cycles**  $\implies$  **equivalence classes of literals**
- (iii) perform substitutions
- (iv) perform unit and pure literal.

**Until** (no more simplification is possible).

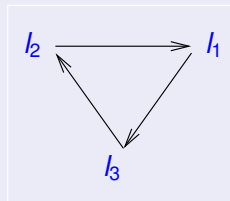
- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$

$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.



## Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$

$\Downarrow$  *resolve*

$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$

$\Downarrow$  *unit-propagate*

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

## Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- **Branch** is the source of non-determinism for DPLL
  - ⇒ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

# Some Heuristics

- **MOMS** heuristics (DPLL): pick the literal occurring **most** often in the **minimal size** clauses  
⇒ fast and simple, many variants
- **Jeroslow-Wang** (DPLL): choose the literal with maximum

$$\text{score}(l) := \sum_{I \in c \ \& \ c \in \varphi} 2^{-|c|}$$

⇒ estimates  $l$ 's contribution to the satisfiability of  $\varphi$

- **Satz** [21] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set  
⇒ maximizes the effects of unit propagation
- **VSIDS** [28] (CDCL+): **v**ariable **s**tate **i**ndependent **d**ecaying **s**um
  - “static”: scores updated only at the end of a branch
  - “local”: privileges variable in recently learned clauses



## Restarts [16]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
  - The clauses learned prior to the restart are still there after the restart and can help pruning the search space
  - avoid getting stuck in certain areas of the search space
- ⇒ may significantly reduce the overall search space

# Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
  - Generalities
  - Resolution
  - Tableaux
  - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 **Modern CDCL SAT Solvers**
  - Limitations of Chronological Backtracking
  - Conflict-Driven Clause-Learning SAT solvers
  - Further Improvements
  - **SAT Under Assumptions & Incremental SAT**
- 5 SAT Functionalities: proofs, unsat cores, optimization

## SAT under assumptions: $SAT(\varphi, \{l_1, \dots, l_n\})$ [12]

- Many SAT solvers allow for solving a CNF formula  $\varphi$  under a set of assumption literals  $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, \dots, l_n\}$ :  $SAT(\varphi, \{l_1, \dots, l_n\})$ 
  - $SAT(\varphi, \{l_1, \dots, l_n\})$ : same result as  $SAT(\varphi \wedge \bigwedge_{i=1}^n l_i)$
  - often useful to call the same formula with different assumption lists:  $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2), \dots$
- Idea:
  - $l_1, \dots, l_n$  “decided” at decision level 0 before starting the search
  - if backtrack to level 0 on  $C \stackrel{\text{def}}{=} \neg\eta$  s.t.  $\eta \subseteq \mathcal{A}$ , then return UNSAT

### Property

If the “decision” strategy for conflict analysis is used,  
then  $\eta$  is the subset of assumptions causing the inconsistency

# Selection of sub-formulas

Idea: select clauses [12, 23]

Let  $\varphi$  be  $\bigwedge_{i=1}^n C_i$ .

- let  $S_1 \dots S_n$  be fresh Boolean atoms (selection variables).

- let  $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$

$\Rightarrow$   $\text{SAT}(\bigwedge_{i=1}^n (\neg S_i \vee C_i), \mathcal{A})$ : same as  $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (C_i))$

- if  $S_i$  is not assumed, then  $\neg S_i \vee C_i$  does not contribute to search

$\Rightarrow$  “Select” (activate) only a subset of the clauses in  $\varphi$  at each call.

Generalised Idea: select blocks of clauses

Let  $\varphi$  be  $\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} C_{ij})$ .

- let  $S_1 \dots S_n$  be fresh Boolean atoms (selection variables).

- let  $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$

- $\text{SAT}(\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} (\neg S_i \vee C_{ij})), \mathcal{A})$ : same as  $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (\bigwedge_{j=1}^{n_i} C_{ij}))$

$\Rightarrow$  Allows for “selecting” block of clauses at each call.

# Example

- Initial formula  $\varphi$ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$  // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$  // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$  // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

- Augmented formula  $\varphi'$ :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$  // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$  // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$  // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

- $SAT(\varphi', \{S_2, S_3\})$ : activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$ : activates group 1,3

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$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$  // group 1

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$(\neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$  // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

- Augmented formula  $\varphi'$ :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$  // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$  // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$  // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

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$(A_3 \vee A_5 \vee \neg A_8) \wedge$  // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

- Augmented formula  $\varphi'$ :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$  // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$  // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$  // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

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$(\neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$  // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

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$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$  // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$  // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$  // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$  // group 3

- $SAT(\varphi', \{S_2, S_3\})$ : activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$ : activates group 1,3



# Incremental SAT solving [12, 11]

- Many CDCL solvers provide a **stack-based incremental interface**
    - it is possible to push/pop  $\phi_i$  into a stack of subformulas  $\{\phi_1, \dots, \phi_k\}$
    - check incrementally the satisfiability of  $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^k \phi_i$ .
  - Maintains the **status** of the search from one call to the other
    - in particular it records the **learned clauses** (plus other information)
    - $\Rightarrow$  **reuses search from one call to another**
  - Very useful in many applications (in particular in FV)
- 
- Idea: **incremental** calls  $SAT(\varphi', \mathcal{A}_1), SAT(\varphi', \mathcal{A}_2), \dots$ 
    - $\varphi' \stackrel{\text{def}}{=} \bigwedge_i (\neg S_i \vee \phi_i)$ ,  $\mathcal{A}_j \subseteq \{S_1, \dots, S_k\}$ ,  $(\neg S_i \vee \bigwedge_j C_{ij}) \stackrel{\text{def}}{=} \bigwedge_j (\neg S_i \vee C_{ij})$
    - push/pop selection variables  $S_i$
    - in practice, also subformulas  $\phi_i$  can be pushed/popped
  - Key efficiency issue: **learned clauses safely reused from call to call (even if assumptions have been popped)**
    - a learned clause  $C \stackrel{\text{def}}{=} \bigvee_j \neg S_j \vee C'$  is s.t.  $\bigwedge_j (\neg S_j \vee \phi_j) \models C$
    - $\Rightarrow$   $C$  contains the vars selecting the subformulas it is derived from
    - $\Rightarrow$  in  $SAT(\varphi', \mathcal{A})$ , if some  $S_j \notin \mathcal{A}$ , then  $C$  is not active

# Example

- Initial formula  $\varphi$ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

- Augmented formula  $\varphi'$ :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

[push( $S_1$ )]:  $SAT(\varphi', \{\dots, S_1\})$ :  $\phi_1$  active  $\implies$  learn  $C_1$  from  $\phi_1$

- $C_1$  derived from  $\phi_1 \implies C_1$  active only when  $\phi_1$  is active
- $C_2$  derived from  $\phi_1, \phi_2 \implies C_2$  active only when both  $\phi_1, \phi_2$  are active

# Example

- Initial formula  $\varphi$ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

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$$(\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1$$

[push( $S_1$ )]: SAT( $\varphi', \{\dots, S_1\}$ ):  $\phi_1$  active  $\implies$  learn  $C_1$  from  $\phi_1$

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# Example

- Initial formula  $\varphi$ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \\ (\neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \end{array}$$

- Augmented formula  $\varphi'$ :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \end{array}$$

[push( $S_2$ ): SAT( $\varphi'$ ,  $\{\dots, S_1, S_2\}$ ):  $\phi_1, \phi_2$  active  $\implies$  learn  $C_2$  from  $\phi_1, \phi_2$

- $C_1$  derived from  $\phi_1 \implies C_1$  active only when  $\phi_1$  is active
- $C_2$  derived from  $\phi_1, \phi_2 \implies C_2$  active only when both  $\phi_1, \phi_2$  are active

# Example

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$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \\ (\neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \end{array}$$

- Augmented formula  $\varphi'$ :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_2, \text{ inactive} \end{array}$$

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# Example

- Initial formula  $\varphi$ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

$$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3$$

- Augmented formula  $\varphi'$ :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2, \textit{inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2, \textit{inactive} \\ (\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3 \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_2, \textit{inactive} \end{array}$$

[pop( $S_2$ );push( $S_3$ ): SAT( $\varphi'$ , { $\dots$ ,  $S_1$ ,  $S_3$ })]:  $\phi_1, \phi_3$  active  $\implies \dots$

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# Example

- Initial formula  $\varphi$ :

$$\begin{array}{l} \dots \\ (\neg A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

$$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3$$

- Augmented formula  $\varphi'$ :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2, \textit{inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2, \textit{inactive} \\ (\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3 \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_2, \textit{inactive} \end{array}$$

[pop( $S_2$ );push( $S_3$ ): SAT( $\varphi'$ , { $\dots$ ,  $S_1$ ,  $S_3$ })]:  $\phi_1, \phi_3$  active  $\implies \dots$

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# Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

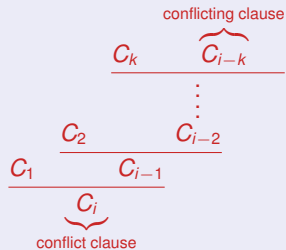
- Building **proofs of unsatisfiability**
- Extracting **unsatisfiable Cores**
- Enumeration in SAT: **AlISAT** (hints)
- Optimization in SAT: **MaxSAT** (hints)

# Building Proofs of Unsatisfiability

- When  $\varphi$  is unsat, it is very important to build a (resolution) proof of unsatisfiability:
  - to verify the result of the solver
  - to understand a “reason” for unsatisfiability
  - to build unsatisfiable cores and interpolants
- Can be built by **keeping track of the resolution steps performed when constructing the conflict clauses.**

# Building Proofs of Unsatisfiability

- Recall: each conflict clause  $C_i$  learned is computed from the conflicting clause  $C_{i-k}$  by backward resolving with the antecedent clause of one literal



- $C_1, \dots, C_k$ , and  $C_{i-k}$  can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

$k \quad i-k \quad \rightarrow \quad i-k-1$

$\dots$

$2 \quad i-2 \quad \rightarrow \quad i-1$

$1 \quad i-1 \quad \rightarrow \quad i$

# Building Proofs of Unsatisfiability

- ... in particular, if  $\varphi$  is unsatisfiable, the last step produces “false” as conflict clause:

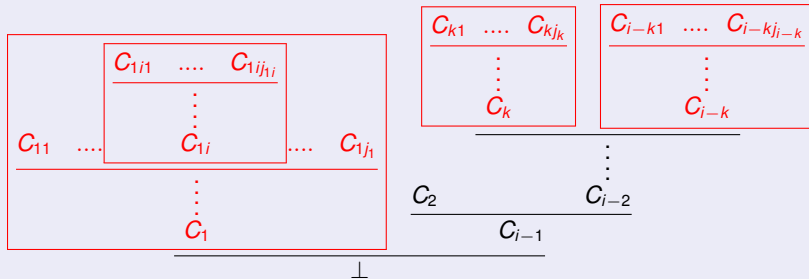
$$\begin{array}{c} \text{conflicting clause} \\ C_k \quad \overbrace{C_{i-k}} \\ \hline \vdots \\ C_2 \quad C_{i-2} \\ \hline C_1 \quad C_{i-1} \\ \hline \perp \end{array}$$

- note:  $C_1 = l$ ,  $C_{i-1} = \neg l$  for some literal  $l$
- $C_1, \dots, C_k$ , and  $C_{i-k}$  can be original or learned clauses...

# Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

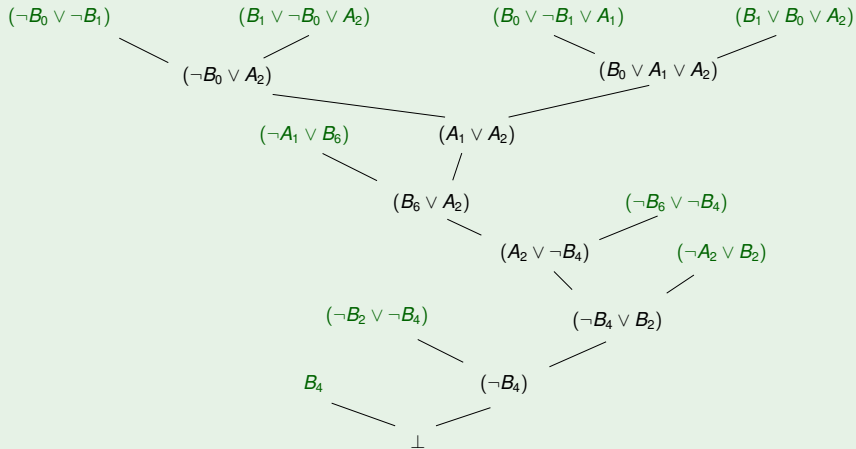
- for every **learned** leaf clause  $C_i$ , substitute  $C_i$  with the resolution proof generating it until all leaf clauses are original clauses



$\Rightarrow$  We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in  $\varphi$

# Building Proofs of Unsatisfiability: example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$   
 $(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$



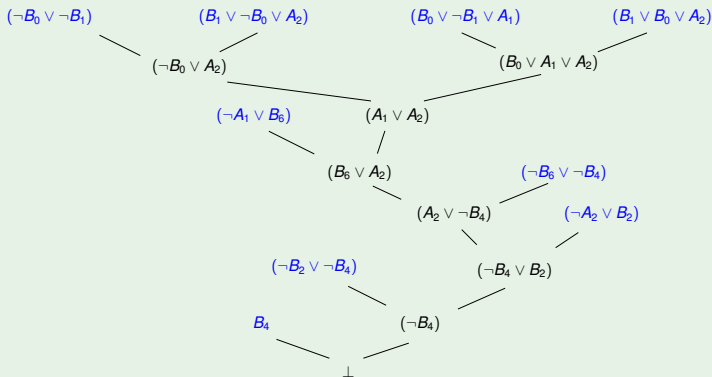
# Extraction of unsatisfiable cores

- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset
  - ⇒ **unsatisfiable cores** (aka **(Minimal) Unsatisfiable Subsets, (M)US**)
- Lots of literature on the topic [46, 24, 26, 31, 43, 19, 13, 6]
- We recognize two main approaches:
  - **Proof-based** approach [46]: byproduct of finding a resolution proof
  - **Assumption-based** approach [24]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
  - repeat the process up to fixpoint
  - remove clauses one-by one, until satisfiability is obtained
  - combinations of the two processed above
  - ...

# The proof-based approach to core extraction [46]

Unsat core: the set of leaf clauses of a resolution proof

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge \\ (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$





## The assumption-based approach to core extraction [24]

Based on the following process:

- (i) each clause  $C_i$  is substituted by  $\neg S_i \vee C_i$ , s.t.  $S_i$  fresh “selector” variable
- (ii) before starting the search each  $S_i$  is forced to true.
- (iii) final conflict clause at dec. level 0:  $\bigvee_j \neg S_j$   
 $\implies \{C_j\}_j$  is the unsat core!

# The assumption-based approach to core extraction

## Example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge \\ B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$

(i) add selector variables:

$$\begin{aligned} & (\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge \\ & (\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge \\ & (\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge \\ & (\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7) \end{aligned}$$

(ii) The conflict analysis returns:  $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$ ,

(iii) corresponding to the unsat core:

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge \\ B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$$

# All-SAT (hints)

## All-SAT & Projected All-SAT

- **All-SAT**: enumerate all truth assignments satisfying  $\varphi$
- **Projected All-SAT**: given an “important” subset of atoms  $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$ , enumerate all assignments over  $\mathbf{P}$  which can be extended to truth assignments satisfying  $\varphi$
- Algorithms
  - **BCLT** [Lahiri et al, CAV'06]:  
each time a satisfiable assignment  $\{l_1, \dots, l_n\}$  is found, perform conflict-driven backjumping as if the restricted clause  $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$  belonged to the clause set
  - **MathSAT/NuSMV** [Cavada et al, FMCAD'07]:  
As above, plus the Boolean search of the SAT solver is driven by an OBDD.

# MaxSAT (hints)

- **MaxSAT**: given a pair of CNF formulas  $\langle \varphi_h, \varphi_s \rangle$  s.t.  $\varphi_h \wedge \varphi_s \models \perp$ ,  $\varphi_s \stackrel{\text{def}}{=} \{C_1, \dots, C_k\}$ , find a truth assignment  $\mu$  satisfying  $\varphi_h$  and maximizing the amount of the satisfied clauses in  $\varphi_s$ .
- **Weighted MaxSAT**: given also the positive integer **penalties**  $\{w_1, \dots, w_k\}$ ,  $\mu$  must satisfy  $\varphi_h$  and maximize the sum of penalties of the satisfied clauses in  $\varphi_s$
- Generalization of SAT to **optimization**  
 $\implies$  much harder than SAT
- Many different approaches (see e.g. [22])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \quad \varphi_s \stackrel{\text{def}}{=} \left( \begin{array}{l} (A_1 \vee \neg A_2) \wedge [4] \\ (\neg A_1 \vee A_2) \wedge [3] \\ (\neg A_1 \vee \neg A_2) \wedge [2] \end{array} \right)$$

$$\implies \mu = \{A_1, A_2\} \text{ (penalty} = 2)$$

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