Formal Methods
Module II: Formal Verification
Ch. 10: **SMT-Based Model Checking**

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M.S. in Computer Science, Mathematics, & Artificial Intelligence Systems
Academic year 2022-2023

last update: Tuesday 30th May, 2023, 12:46

Outline

1. Motivations & Context
2. Background (from previous chapters)
3. SMT-Based Bounded Model Checking of Timed Systems
   - Basic Ideas
   - Basic Encoding
   - Improved & Extended Encoding
   - A Case-Study
4. SMT-Based Bounded Model Checking of Linear Hybrid Systems (hints)
5. Proposed Exercises

Outline

Motivations

- Model Checking for Timed Systems:
  - relevant improvements and results over the last decades
  - historically, “explicit-state” search style, based on DBMs
    - notable examples: Kronos, Uppaal
  - More recently, *symbolic* verification techniques:
    - extensions of decision diagrams
    - CDD, DDD, RED, ...
- Key problem: potential blow up in size
- A more recent and viable alternative to Binary Decision Diagrams: SAT-based MC
  - Bounded Model Checking (BMC), K-induction, IC3/PDR, ...

Context

First Idea: SMT-based BMC of Timed Systems

[Audemard et al. 2002], [Sorea, MTCS'02], [Niebert et al.,FTRTFT'02]

Leverage the SAT-based BMC approach to Timed Systems by means of SMT Solvers

Extensions

- SMT eventually applied to other SAT-based MC techniques
  - K-Induction
  - interpolant-based
  - IC3/PDR
- SMT applied to a variety of domains:
  - hybrid systems
  - verification of SW (loop invariants/proof obbligations, ...)
  - hardware verification
- Nowadays SMT leading backend technology for FV

We restrict to BMC for Timed/Hybrid Systems only

Outline

Bounded Model Checking [Biere et al., TACAS'99]

- Given a Kripke Structure M, an LTL property f and an integer bound k, is there an execution path of M of length (up to) k satisfying f? ($M \models_k \mathbf{E}f$)
- Problem converted into the satisfiability of the Boolean formula:

$$[[M]]_k^f := I(s^{(0)}) \wedge \bigwedge_{i=0}^{k-1} R(s^{(i)}, s^{(i+1)}) \wedge (\neg L_k \wedge [[f]]_k^0) \vee \bigvee_{l=0}^{k} ({}_lL_k \wedge {}_l[[f]]_k^0)$$

s.t. ${}_lL_k \stackrel{\text{def}}{=} R(s^{(k)}, s^{(l)})$, $L_k \stackrel{\text{def}}{=} \bigvee_{l=0}^{k} {}_lL_k$

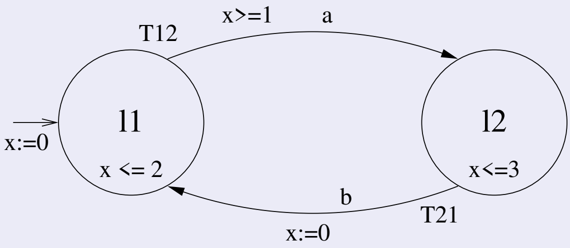
- A satisfying assignment represents a satisfying execution path.
- Test repeated for increasing values of k
- Incomplete
- Very effective for debugging, alternative to OBDDs
- Complemented with K-Induction [Sheeran et al. 2000]
- Further developments: IC3/PDR [Bradley, VMCAI 2011]

General Encoding for LTL Formulae

f	$[[f]]_k^i$	${}_l[[f]]_k^i$
p	$p^{(i)}$	$p^{(i)}$
$\neg p$	$\neg p^{(i)}$	$\neg p^{(i)}$
$h \wedge g$	$[[h]]_k^i \wedge [[g]]_k^i$	${}_l[[h]]_k^i \wedge {}_l[[g]]_k^i$
$h \vee g$	$[[h]]_k^i \vee [[g]]_k^i$	${}_l[[h]]_k^i \vee {}_l[[g]]_k^i$
$\mathbf{X}g$	$\begin{cases} [[g]]_k^{i+1} & \textit{if } i < k \\ \bot & \textit{otherwise.} \end{cases}$	$\begin{cases} {}_l[[g]]_k^{i+1} & \textit{if } i < k \\ {}_l[[g]]_k^{l} & \textit{otherwise.} \end{cases}$
$\mathbf{G}g$	$\bot$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$\mathbf{F}g$	$\bigvee_{j=i}^{k} [[g]]_k^j$	$\bigvee_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$h\mathbf{U}g$	$\bigvee_{j=i}^{k} \left([[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} [[h]]_k^n \right)$	$\bigvee_{j=i}^{k} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_l[[h]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[h]]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_l[[h]]_k^n \right)$
$h\mathbf{R}g$	$\bigvee_{j=i}^{k} \left([[h]]_k^j \wedge \bigwedge_{n=i}^{j} [[g]]_k^n \right)$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j \vee$ $\bigvee_{j=i}^{k} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{j} {}_l[[g]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[g]]_k^n \wedge \bigwedge_{n=l}^{j} {}_l[[g]]_k^n \right)$

Timed Automata [Alur and Dill, TCS'94; Alur, CAV'99]

- Clocks: real variables (ex. x)
- Locations:
  - label: (ex. l_1),
  - invariants: (conjunctive) constraints on clocks values (ex. $x \leq 2$)
- Switches:
  - event labels (ex. a),
  - clock constraints (ex. $x \geq 1$),
  - reset statements (ex. $x := 0$)
- Time elapse: all clocks are increased by the same amount



$\mathcal{LRA}$-Formulae

[Audemard et al., CADE'02]; [Sorea, MTCS'02]; [Niebert et al.,FTRTFT'02]

- $\mathcal{LRA}$-formulae are Boolean combinations of
  - Boolean variables and
  - linear constraints over real variables (equalities and differences)
    - e.g., $(x - 2 \cdot y \geq 4) \wedge ((x = y) \vee \neg A)$
- An interpretation $\mathcal{I}$ for a $\mathcal{LRA}$ formula assigns
  - truth values to Boolean variables
  - real values to numerical variables and constants
    - e.g., $\mathcal{I}(x) = 3, \mathcal{I}(y) = -1, \mathcal{I}(A) = \bot$
- $\mathcal{I}$ satisfies a $\mathcal{LRA}$-formula ϕ, written "$\mathcal{I} \models \phi$", iff $\mathcal{I}(\phi)$ evaluates to true under the standard semantics of Boolean and mathematical operators.
  - E.g., $\mathcal{I}((x - 2 \cdot y \geq 4) \wedge ((x = y) \vee \neg A)) = \top$

The MathSAT Solver [Audemard et al., CADE'02]

- Bottom level: a $\mathcal{T}$-Solver for sets of $\mathcal{LRA}$ constraints
  - E.g. $\{..., z_1 - x_1 \leq 6, z_2 - x_2 \geq 8, x_1 = x_2, z_1 = z_2, ...\} \Longrightarrow$ unsat.
  - Combination of symbolic and numerical algorithms (equivalence class building, Belman-Ford, Simplex)
- Top level: a CDCL procedure for propositional satisfiability
  - mathematical predicates treated as propositional atoms
  - invokes $\mathcal{T}$-Solver on every assignment found
  - used as an enumerator of assignments
  - lots of enhancements

(see chapter on SMT)

Outline

Outline

SMT-Based BMC for Timed Systems

Independently developed approaches (2002):

- [Audemard et al. FORTE'02]: encoding into $\mathcal{LRA}$
  - all LTL properties
- [Sorea, MTCS'02]: encoding into $\mathcal{LRA}$
  - based on automata-theoretic approach for LTL
- [Niebert et al.,FTRTFT'02]: encoding into $\mathcal{DL}$
  - limited to reachability

Disclaimer

These slides are adapted from [Audemard et al. FORTE'02]:

> G. Audemard, A. Cimatti, A. Kornilowicz, R. Sebastiani
> *Bounded Model Checking for Timed Systems*,
> proc. FORTE 2002, Springer
> freely available as `https://disi.unitn.it/rseba/publist.html`

(with some simplification in the notation).

BMC for Timed Systems

Basic ingredients:

- An extension of propositional logic expressive enough to represent timed information: "$\mathcal{LRA}$-formulae"
- A SMT($\mathcal{LRA}$) solver for deciding $\mathcal{LRA}$-formulae
  $\Longrightarrow$ e.g., the MATHSAT solver
- An encoding from timed BMC problems into $\mathcal{LRA}$-formulae
  - $\mathcal{LRA}$-satisfiable iff an execution path within the bound exists

Outline

The encoding

Given a timed automaton A and a LTL formula f:

- The encoding $[[A, f]]_k$ is obtained following the same schema as in propositional BMC:

$$[[A, f]]_k := I(s^{(0)}) \wedge \bigwedge_{i=0}^{k-1} R(s^{(i)}, s^{(i+1)}) \wedge (\neg L_k \wedge \; [[f]]_k^0) \vee \bigvee_{l=0}^{k} (\, _lL_k \wedge \; _l[[f]]_k^0)$$

- $[[M, f]]_k$ is a $\mathcal{LRA}$-formula, where
  - Boolean variables encode the discrete part of the state of the automaton
  - constraints on real variables represent the temporal part of the state

Encoding: Boolean Variables

- Locations: an array $\underline{l}$ of $n \stackrel{\text{def}}{=} \lceil log_2(|L|) \rceil$ Boolean variables
  - $\underline{l_i}$ holds iff the system is in the location l_i
  - ex: "$\neg \underline{l_i}[3] \wedge \underline{l_i}[2] \wedge \neg \underline{l_i}[1] \wedge \underline{l_i}[0]$" means "the system is in location $\underline{l_5}$"
  - "$(\underline{l_i} = \underline{l_j})$" stands for "$\bigwedge_n (\underline{l_i}[n] \leftrightarrow \underline{l_j}[n])$",
  - "primed" variables $\underline{l_i}'$ to represent location after transition
- Events: for each event $a \in \Sigma$, a Boolean variable $\underline{a}$
  - $\underline{a}$ holds iff the system executes a switch with event a.
- Switches: for each switch $\langle l_i, a, \varphi, \lambda, l_j \rangle \in E$, a Boolean variable T,
  - T holds iff the system executes the corresponding switch
- Time elapse and null transitions: two variables T_δ and T^j_{null}
  - T_δ holds iff time elapses by some $\delta > 0$
  - T^j_{null} holds if and only A_j does nothing (specific for automaton A_j)

Note: also for events, switches&transitions it is possible to use arrays of Boolean variables of size $\lceil log_2(|\Sigma|) \rceil$, $\lceil log_2(|E| + 2) \rceil$ respectively

Encoding: Clock Values and Constraints

- Clocks values x are "normalized" wrt absolute time $(t - x)$:
  - a clock value x is written as difference $t - x$
  - t represents the absolute time
  - "offset" variable x represents the absolute time when the clock was reset last time
- Clock constraints reduce to $(t - x \bowtie c)$, $\bowtie \in \{\leq, \geq, <, >\}$, $c \in \mathbb{Z}$
- Clock reset conditions reduce to $(x := t)$
- Clock equalities like $(x_k = x_l)$ reduce to $(t_k - x_k = t_l - x_l)$
  - appear only in loops
  - only place where full $\mathcal{LRA}$ is needed (rather than $\mathcal{DL}$)

  $\Longrightarrow$ for invariant checking (no loops) $\mathcal{DL}$ suffices
- Encoding the effect of transitions:
  - with a time-elapse transition:
    - $t' > t$, and $x' = x$
  - otherwise:
    - $t' = t$, absolute time does not elapse
    - $x' = t'$, if the clock is reset
    - $x' = x$, if the clock is not reset

Encoding: Initial Conditions

Initial condition $I(s)$:

- Initially, the automaton is in an initial location:

$$\bigvee_{l_i \in L^0} \underline{l_i}$$

- Initially, clocks have a null value:

$$\bigwedge_{x \in X} (x = t)$$

Remark

Here and hereafter: in the encoding, when we write a formula φ, we implicitly mean "any formula logically equivalent to φ"

- in particular when encoding symbolically the discrete part of the system
- e.g., there is probably a much more compact formula equivalent to $\bigvee_{l_i \in L^0} \underline{l_i}$

Encoding: Invariants

Transition relation $R(s, s')$: Invariants

- Always, being in a location implies the corresponding invariant constraints:

$$\bigwedge_{l_i \in L} (\underline{l_i} \rightarrow \bigwedge_{\psi \in I(l_i)} \psi),$$

Encoding: Transitions

Transition relation $T(s, s')$:

- Switches:

$$\bigwedge_{T \stackrel{\text{def}}{=} \langle l_i, a, \varphi, \lambda, l_j \rangle \in E} T \rightarrow \left(\underline{l_i} \wedge \underline{a} \wedge \varphi \wedge \underline{l_j}' \wedge (t' = t) \wedge \bigwedge_{x \in \lambda} (x' = t') \wedge \bigwedge_{x \notin \lambda} (x' = x) \right)$$

- Time elapse:

$$T_\delta \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' - t > 0) \wedge \bigwedge_{x \in X} (x' = x) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

- Null transition:

$$T^j_{null} \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' = t) \wedge \bigwedge_{x \in X} (x' = x) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

Encoding: Relations between Transitions

- Mutual exclusion between events:

$$\bigwedge_{a_k, a_r \in \Sigma, a_k \neq a_r} (\neg \underline{a}_k \vee \neg \underline{a}_r)$$

- At least one transition takes place:

$$T^j_{null} \vee T_\delta \vee \bigvee_{T \in E} T$$

- Mutual exclusion between transitions:

$$\bigwedge_{T_k, T_r \in E \cup \{T^j_{null}\} \cup \{T_\delta\}, T_k \neq T_r} (\neg T_k \vee \neg T_r)$$

If events and transitions are encoded via arrays of Booleans, mutual exclusion constraints are not needed

Automata Product Construction

- The encoding is compositional wrt. product of automata
- The encoding of $A = A_1 || A_2$ is given by the conjunction of the encodings of A_1 and A_2, plus a few extra axioms
- Mutual exclusion between events that are local

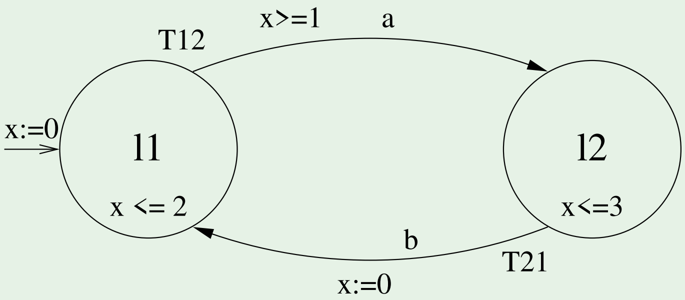
$$\bigwedge_{\substack{a_1 \in \Sigma_1 \setminus \Sigma_2 \\ a_2 \in \Sigma_2 \setminus \Sigma_1}} (\neg \underline{a}_1 \vee \neg \underline{a}_2)$$

- Forcing system activity:

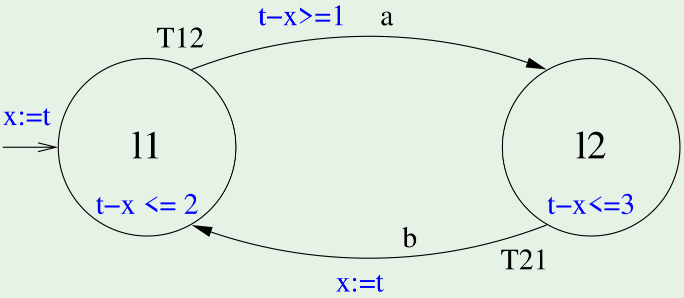
$$\bigvee_{j=0}^{N-1} \neg T^j_{null}$$

  - one distinct T^j_{null} for each automaton A_j
  - T_δ is common to all automata A_j

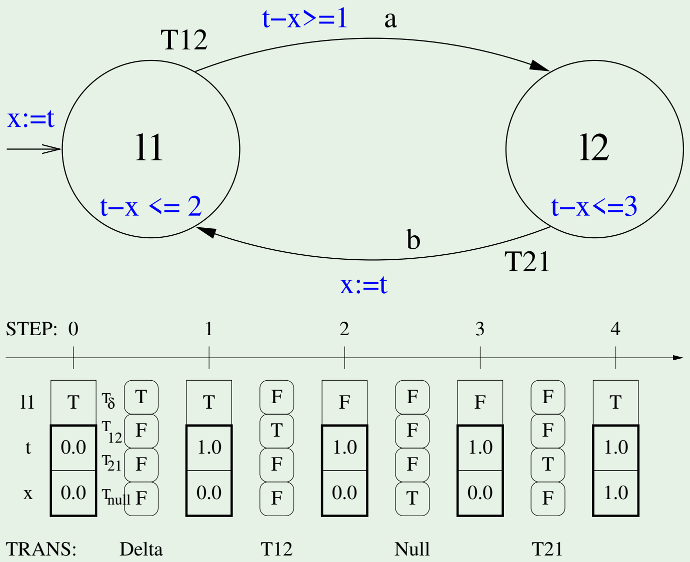
A Simple Example



A Simple Example



A Simple Example



Outline

Encoding: Extension

Adding Global Variables

Dealing with some global variable v on discrete domain:

- A switch $T \stackrel{\text{def}}{=} \langle l_i, a, \varphi, \lambda, l_j \rangle$ can
  - be subject to a condition $\psi(v)$

  $\Longrightarrow$ add $T \rightarrow \psi(v)$
  - assign v to some value n or keep its value

  $\Longrightarrow$ add $T \rightarrow (v' = n)$ or add $T \rightarrow (v' = v)$
- T_δ mantains the value of v:

  $\Longrightarrow$ add $T_\delta \rightarrow (v' = v)$
- T^j_{null} imposes no constraint on v:

  $\Longrightarrow$ add nothing (for A_j)

MATHSAT: Optimizations

Customization of MATHSAT

- Limit Boolean variable-selection heuristic to pick transition variables, in forward order

Encoding: Optimizations

Boolean Propagation of Math Constraints:

Idea: add small and mathematically-obvious lemmas

$$
\begin{array}{rcl}
\neg(t' = t) & \leftrightarrow & (t' - t > 0) \\
\bigwedge_{x \in X}(\neg(x = t) & \leftrightarrow & (t - x > 0)) \\
\bigwedge_{x \in X} \neg(x' = x) & \leftrightarrow & (x' - x > 0)
\end{array}
$$

$$
\begin{array}{lllllllll}
\bigwedge_{x \in X}(& (x = t) & \wedge & (x' = x) & \wedge & (t' = t)) & \rightarrow & (x' = t') \\
\bigwedge_{x \in X}(& \neg(x = t) & \wedge & (x' = x) & \wedge & (t' = t)) & \rightarrow & \neg(x' = t') \\
\bigwedge_{x \in X}(& (x = t) & \wedge & \neg(x' = x) & \wedge & (t' = t)) & \rightarrow & \neg(x' = t') \\
\bigwedge_{x \in X}(& (x = t) & \wedge & (x' = x) & \wedge & \neg(t' = t)) & \rightarrow & \neg(x' = t') \\
\bigwedge_{x \in X}(& (x' = x) & \wedge & (t' - t > 0) & \wedge & (t - x > 0)) & \rightarrow & (t' - x' > 0) \\
\bigwedge_{x \in X}(& (t' = t) & \wedge & \neg(t - x > 0) & \wedge & (x' - x > 0)) & \rightarrow & \neg(t' - x' > 0) \\
\bigwedge_{x \in X}(& (t - x \bowtie c) & \wedge & (x' = x) & \wedge & (t' = t)) & \rightarrow & (t' - x' \bowtie c) \\
\bigwedge_{x \in X}(& \neg(t - x \bowtie c) & \wedge & (x' = x) & \wedge & (t' = t)) & \rightarrow & \neg(t' - x' \bowtie c)
\end{array}
$$

$\Longrightarrow$ force assignments by unit-propagation,

$\Longrightarrow$ saves calls to the $\mathcal{T}$-Solvers

Encoding Variants

Shortening counter-examples:

- Collapsing consequent time elapsing transitions:
  - $s \stackrel{\delta}{\longmapsto} s$, $s \stackrel{\delta'}{\longmapsto} s$ reduced to $s \stackrel{\delta+\delta'}{\longmapsto} s$
  - add $\neg T_\delta \vee \neg T'_\delta$ to transition relation $R(s, s')$

  $\Longrightarrow$ implements the notion of “non-Zeno-ness” (see previous chapter)
- Allow multiple parallel transitions
  - remove mutex between labels local to processes

  $\Longrightarrow$ allows a form of parallel progression

Remark: may change the notion of “next step”
$\Longrightarrow$ only if no “**X**” operators occurs in property!

Encoding Variants (cont.)

A limited form of symmetry reduction

If N automata are symmetric (frequent with protocol verification):

- Intuition: restrict executions s.t.
  - At step 0 only A_0 can move
  - At step 1 only A_0, A_1 can move
  - At step 2 only A_0, A_1, A_2 can move
  - ...

  $\Longrightarrow$ we name ”0” the first automata who acts, “1” the second one, etc.
- for step $i < N - 1$, we drop the disjunct $\neg T^{i+1\ (i)}_{null} \vee \ldots \vee \neg T^{N-1\ (i)}_{null}$:

$$\textit{set} \quad \bigvee_{j=0}^{min(i,N-1)} \neg T^{j\ (i)}_{null} \quad \textit{rather than} \quad \bigvee_{j=0}^{N-1} \neg T^{j\ (i)}_{null}$$

$\Longrightarrow$ drops “symmetric” executions

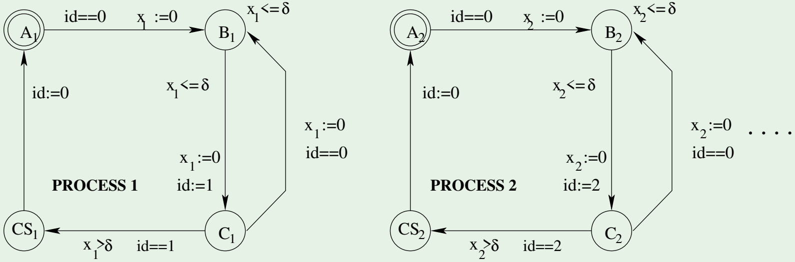
$\Longrightarrow$ reduces the search space of a up to $2^{N(N-1)/2}$ factor!

Outline

A Case-study: Fischer's Protocol

A Mutual-Exclusion Real-Time Protocol

- N identical processes accessing one critical section
- shared variable $id \in \{0, 1, 2, ..., N\}$: process identifier (0: none)
  - when entering wait state C_j, agent A_j writes its code on id
  - if $id = j$ after δ, then A_j can enter the critical session
- Two properties under test
  - Reachability: $\mathbf{EF} \bigwedge_i P_i.C$ (reached in N+1 steps)
  - Fairness: $\mathbf{E}\neg(\mathbf{GF}P_i.B \rightarrow \mathbf{GF}P_i.CS)$ (reached in N+5 steps)



Fischer's protocol: (cont.)

Exercise:

- Why is $\mathbf{EF} \bigwedge_i P_i.C$ reached in N+1 steps?
- Why is $\mathbf{E}\neg(\mathbf{GF}P_i.B \rightarrow \mathbf{GF}P_i.CS)$ reached in N+5 steps?

(See [Audemard et al, FORTE'02] for the solution.)

Fischer's protocol: (reachability)

$$M \models_k \mathbf{EF} \bigwedge_i P_i.C$$

	MATHSAT		MATHSAT,Sym		DDD		UPPAL		KRONOS		RED		RED,Sym	
N	Time	Size	Time	Size	Time	Size	Time	Size	Time	Size	Time	Size	Time	Size
3	0.05	2.9	0.04	2.9	0.11	106	0.01	1.7	0.01	0.8	0.23	2.0	0.19	2.0
4	0.09	3.0	0.08	3.0	0.14	106	0.02	1.9	0.02	2.2	1.00	2.1	0.70	2.1
5	0.20	3.2	0.16	3.2	0.24	106	0.21	1.9	0.09	19	3.70	2.2	2.00	2.4
6	0.60	3.7	0.23	3.7	0.47	106	3.44	6.7	0.39	236	12.00	2.7	5.20	3.1
7	3.20	4.2	0.36	4.2	1.30	106	153	54		MEM	38	4.0	12	4.7
8	29	4.9	0.52	4.9	3.96	106	TIME				121	7.6	26	7.8
9	343	5.9	0.75	5.9	14	106					416	16.6	49	13.3
10	3331	6.5	1.01	6.5	62	106					1382	39	90	23
11	TIME		1.39	7.0	691	106					TIME		157	38
12			1.89	7.5		MEM							266	63
13			2.44	8.2									439	100
14			3.24	8.9									709	155
15			4.11	9.7									1118	225
16			5.10	10.7									1717	342
17			6.30	11.7									2582	492
18			8.00	12.9									TIME	
19			9.50	14.2										

(MATHSAT times are sum of all instances up to k)

Fischer's protocol (liveness violation)

$$M \models_k \mathbf{E}\neg(\mathbf{GF}P_i.B \rightarrow \mathbf{GF}P_i.CS)$$

	MATHSAT					MATHSAT with Boehm heuristic				
$k \backslash N$	2	3	4	5	6	2	3	4	5	6
2	0.01	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.02
3	0.01	0.02	0.01	0.01	0.03	0.01	0.01	0.02	0.03	0.04
4	0.01	0.02	0.02	0.02	0.04	0.01	0.02	0.04	0.07	0.17
5	0.02	0.03	0.05	0.09	0.18	0.01	0.03	0.09	0.30	1.16
6	0.03	0.10	0.21	0.54	1.35	0.02	0.07	0.31	1.52	7.74
7	0.04	0.26	0.97	3.20	9.83	0.02	0.18	1.19	7.14	45.00
8		0.65	4.80	19.72	70.70		0.06	4.70	33.50	242.00
9			5.55	112.17	478.00			0.61	165.90	1348.00
10				303.17	3086.00				9.92	7824.00
11					5002.00					252.00
Σ	0.12	1.08	11.62	438.93	8648.15	0.07	0.37	6.98	218.40	9720.13

Outline

The encoding

Given a Linear hybrid automaton A and a LTL formula f:

- The encoding $[[A, f]]_k$ is obtained following the same schema as in propositional BMC:

$$[[A, f]]_k := I(s^{(0)}) \wedge \bigwedge_{i=0}^{k-1} R(s^{(i)}, s^{(i+1)}) \wedge (\neg L_k \wedge [[f]]_k^0) \vee \bigvee_{l=0}^{k} ({}_l L_k \wedge {}_l[[f]]_k^0)$$

- $[[M, f]]_k$ is a $\mathcal{LRA}$-formula, where
  - Boolean variables encode the discrete part of the state of the automaton
  - a real variable t (rational for rectangular automata) encodes absolute time elapse
  - real (rational) variables $x \in X$ encode continuous variables
  - constraints on real (rational) variables represent the continuous flow part of the state

Encoding: Boolean Variables

- Locations: $\underline{l}$, as with timed systems
- Events: $a \in \Sigma$, as with timed systems
- Switches: T, as with timed systems
- Time elapse and null transitions: T_δ and T^j_{null}, as with timed systems

Encoding: Continuous variables and constraints

- Continuous variables:
  - t represents the absolute time
  - real (rational) variables x represent continuous values
- Continuous constraints (initial, guards, invariants) reduce to linear constraints on X:
  $\sum_{x_i \in X} a_i x_i \bowtie c$ s.t. $\bowtie \in \{\leq, \geq, <, >\}$, $c \in \mathbb{Q}$
  - $x_i \bowtie c_i$ with rectangular automata
- Encoding the effect of discrete transitions:
  - $t' = t$, absolute time does not elapse
  - Jump relations reduce to linear transformations: $\bigwedge_j x_j' := \sum_i a_{ij} x_i + c_j$
    - $\bigwedge_{x_i \in X} (x_i' := c_i)$ with rectangular automata
- Encoding the effect of time-elapse transitions:
  - $t' > t$
  - $\bigwedge_j \Psi_j(X, t, X', t) \geq 0$
    where $\Psi_j(X, t, X', t) \stackrel{\text{def}}{=} \sum_i a_{ij}(x_i' - x_i) + c(t' - t) \geq 0$, given $\bigwedge_j \sum_i a_{ij} \frac{dx_i}{dt} + c \geq 0$
    - with rectangular automata:
      $(x_i' - x_i \leq c_i^M (t' - t) + b_i^M)$, $(x_i' - x_i \geq c_i^m (t' - t) + b_i^m)$ s.t. $c_i^M \stackrel{\text{def}}{=} max\{\frac{dx_i}{dt}\}$, $c_i^m \stackrel{\text{def}}{=} min\{\frac{dx_i}{dt}\}$,

Encoding: Initial Conditions and Invariants

Initial condition $I(s)$:

- Initially, the automaton is in an initial location:

$$t = 0 \to \bigvee_{l_i \in L^0} \underline{l_i}$$

- Initially, clocks comply with initial conditions:

$$t = 0 \to \bigwedge_{l_i \in L^0} (\underline{l_i} \to Init_l(X))$$

Transition relation $R(s, s')$: Invariants

- Always, being in a location implies the corresponding invariant constraints:

$$\bigwedge_{l_i \in L} (\underline{l_i} \to \bigwedge_{\psi \in I(l_i)} \psi),$$

Encoding (linear automata): Transitions

Transition relation $T(s, s')$:

- Switches:

$$\bigwedge_{T \stackrel{\text{def}}{=} \langle l_i, a, \varphi, J, l_j \rangle \in E} T \rightarrow \left(\underline{l_i} \wedge \underline{a} \wedge \varphi \wedge \underline{l_j}' \wedge (t' = t) \wedge \bigwedge_{x_j \in X} (x_j' := \sum_i a_{ij} x_i + c_j) \right)$$

- Time elapse:

$$T_\delta \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' - t > 0) \wedge (\bigwedge_j \Psi_j(X, t, X', t) \geq 0) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

- Null transition:

$$T^j_{null} \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' = t) \wedge \bigwedge_{x_i \in X} (x_i' = x_i) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

Encoding (rectangular automata): Transitions

Transition relation $T(s, s')$:

- Switches:

$$\bigwedge_{T \stackrel{\text{def}}{=} \langle l_i, a, \varphi, \lambda, l_j \rangle \in E} T \rightarrow \left(\underline{l_i} \wedge \underline{a} \wedge \varphi \wedge \underline{l_j}' \wedge (t' = t) \wedge \bigwedge_{x_i \in X} (x_i' := c_i) \right)$$

- Time elapse:

$$T_\delta \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' - t > 0) \wedge \bigwedge_{x_i \in X} (x_i' - x_i \leq c_i^M (t' - t) + b_i^M) \wedge (x_i' - x_i \geq c_i^m (t' - t) + b_i^m) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

- Null transition:

$$T_{null}^j \rightarrow \left((\underline{l}' = \underline{l}) \wedge (t' = t) \wedge \bigwedge_{x_i \in X} (x_i' = x_i) \wedge \bigwedge_{a \in \Sigma} \neg \underline{a} \right)$$

Outline

Proposed Exercise

Proposed Exercise

- Consider the Train-gate-controller example from [Alur CAV'99] (see previous chapter)
  - Encode the Initial state formula
  - Encode the transition relation
  - Encode the BMC problem for the formula $\mathbf{G}(s_2 \rightarrow t_2)$
- As above, reducing the delay time for the controller from 1 to 0.5
  - what happens?
  - in how many steps?
- Encode the above into MathSAT

Proposed Exercise

Proposed Exercise

- Consider the rectangular automaton of the Train-gate example (see previous chapter)
  - Encode the Initial state formula $I(s^{(0)})$
  - Encode the transition relation $R(s^{(i)}, s^{(i+1)})$

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