Formal Methods
Module II: Formal Verification
Ch. 05: **Explicit-State CTL Model Checking**

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Outline

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CTL Model Checking

CTL Model Checking is a formal verification technique where...

- ...the system is represented as a Finite State Machine M:
- ...the property is expressed a CTL formula φ:

$$\mathbf{AG}(p \rightarrow \mathbf{AF}q)$$

- ...the model checking algorithm checks whether in all initial states of M all the executions of the model satisfy the formula ($M \models \varphi$).

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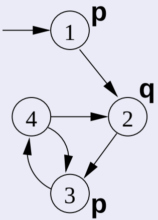

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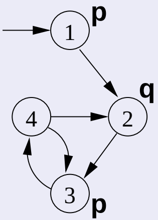

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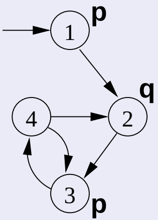

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CTL Model Checking: General Idea

Two macro-steps:

1. construct the set of states where the formula holds:
   $[\varphi] := \{s \in S : M, s \models \varphi\}$
   ($[\varphi]$ is called the denotation of φ)
2. then compare with the set of initial states:
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CTL Model Checking: General Idea [cont.]

In order to compute $[\varphi]$:

- proceed "bottom-up" on the structure of the formula, computing $[\varphi_i]$ for each subformula φ_i of $\mathbf{AG}(p \rightarrow \mathbf{AF}q)$:
  - $[q]$,
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- assign Propositional atoms by labeling function
- handle Boolean operators by standard set operations
- handle temporal operators **AX**, **EX** by computing pre-images
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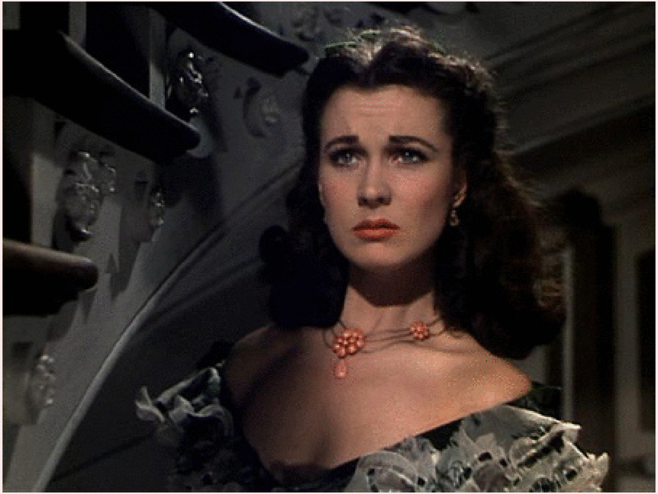
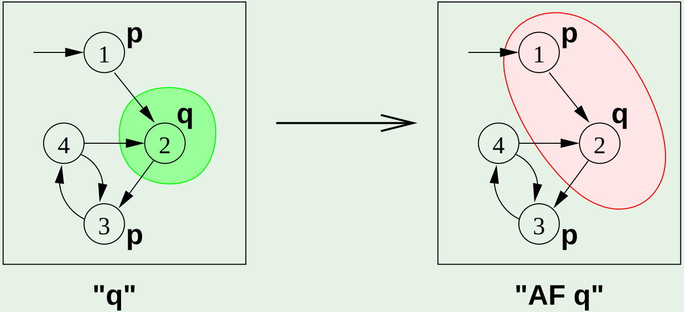
Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

CTL Model Checking: Example: $\mathbf{AG}(p \rightarrow \mathbf{AF}q)$



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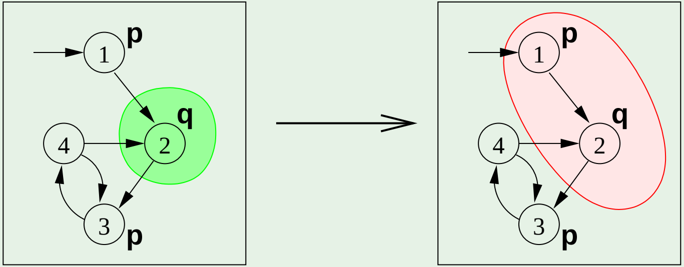

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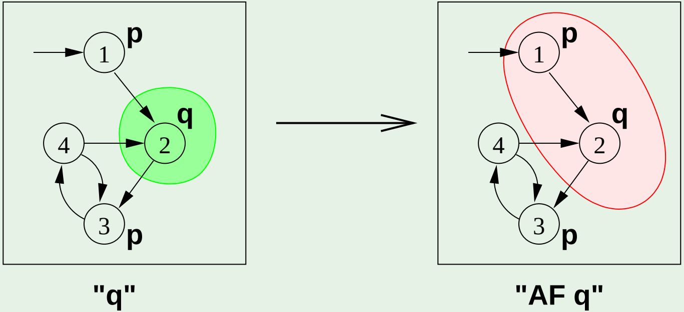

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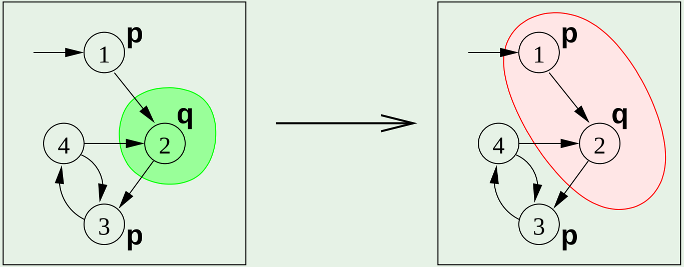

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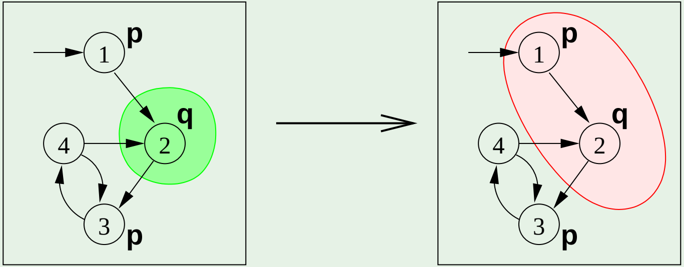

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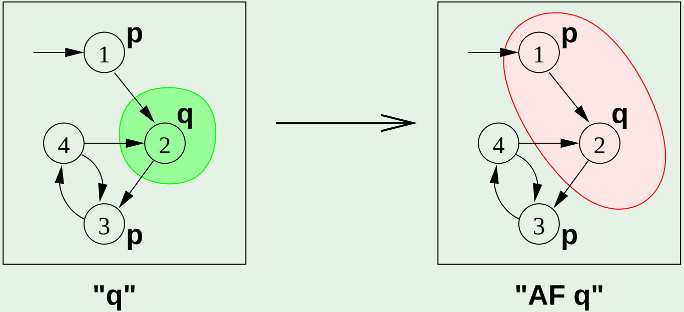

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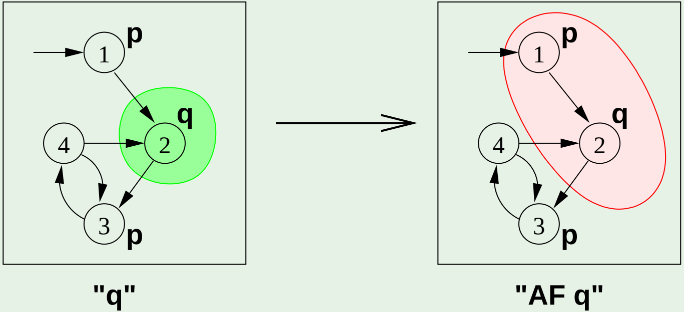

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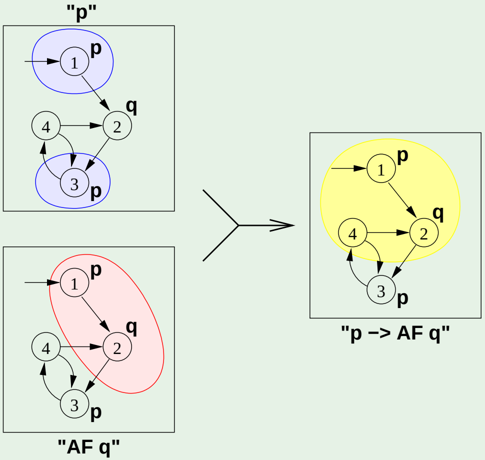

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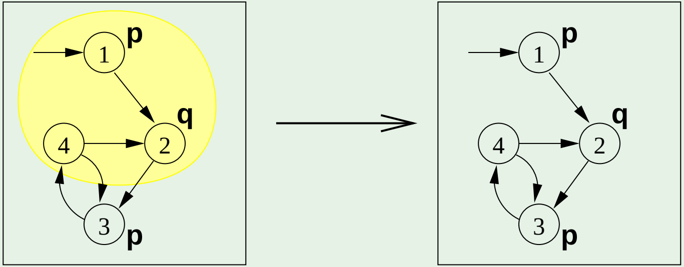

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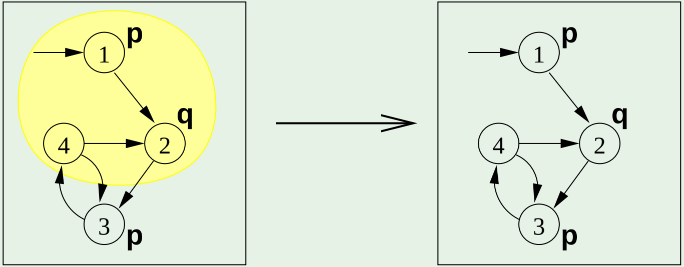

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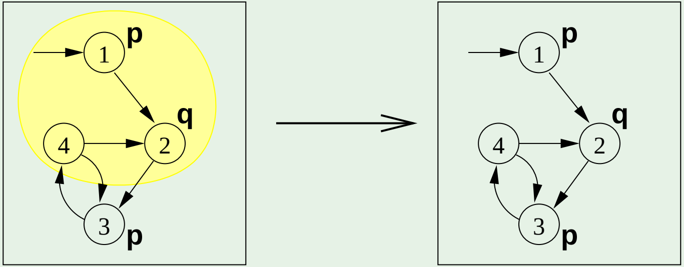

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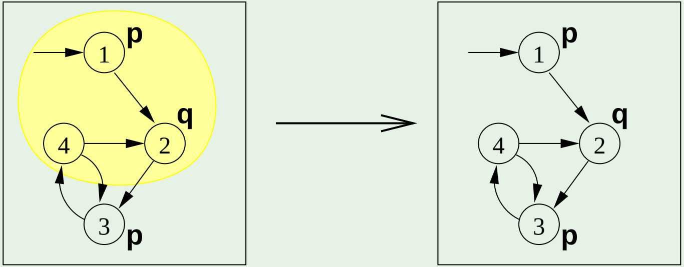

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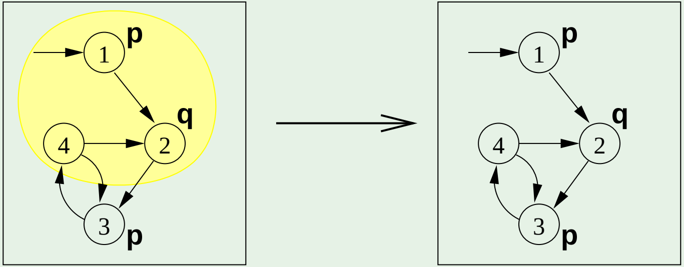

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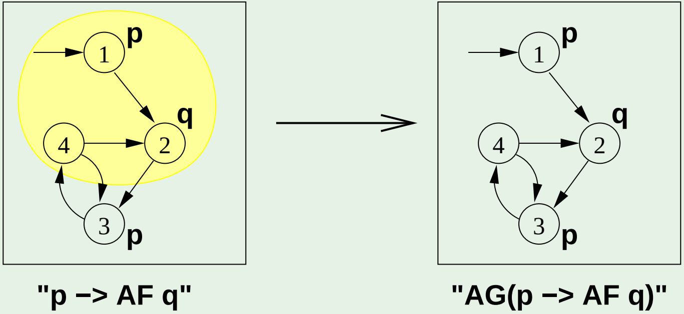

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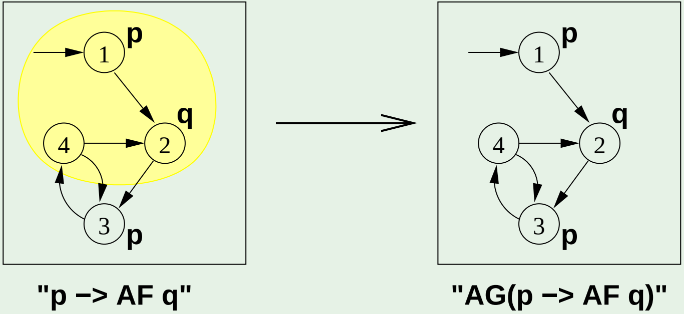

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- The set of states where the formula holds is empty
  $\Longrightarrow$ the initial state does not satisfy the property
  $\Longrightarrow M \not\models \mathbf{AG}(p \rightarrow \mathbf{AF}q)$
- Counterexample: a lazo-shaped path: $1, 2, \{3, 4\}^{\omega}$ (satisfying $\mathbf{EF}(p \wedge \mathbf{EG}\neg q)$)

Note

Counter-example reconstruction in general is not trivial, based on intermediate sets.

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Outline

The fixed-point theory of lattice of sets

Definition

Let 2^S denote the power set of S, i.e., the set of all subsets of S.

- For any finite set S, the structure $\langle 2^S, \subseteq \rangle$ forms a complete lattice with $\cup$ as join and $\cap$ as meet operations.
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Iteratively computing fixed points

Tarski's Theorem

A monotonic function over a complete finite lattice has a least and a greatest fixed point.

(A corollary of) Kleene's Theorem

A monotonic function F over a complete finite lattice has a least and a greatest fixed point, which can be computed as follows:

- the least fixed point of F is the limit of the chain $\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \ldots$,
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CTL Model Checking and Lattices

- If $M = \langle S, I, R, L, AP \rangle$ is a Kripke structure, then $\langle 2^S, \subseteq \rangle$ is a complete lattice
- We identify φ with its denotation $[\varphi]$

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Denotation of a CTL formula φ: $[\varphi]$

Definition of $[\varphi]$

$[\varphi] := \{s \in S : M, s \models \varphi\}$

Recursive definition of $[\varphi]$

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\begin{array}{lcl}
[\top] & = & S \\
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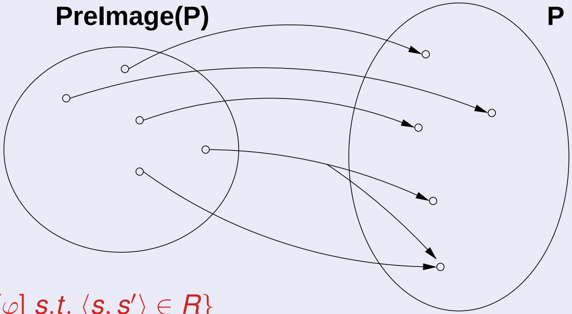

- $[\mathbf{EX}\varphi] = \{s \mid \exists s' \in [\varphi] \ s.t. \ \langle s, s' \rangle \in R\}$
- $[\mathbf{EX}\varphi]$ is said to be the Pre-image of $[\varphi]$ ($Preimage([\varphi])$)
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Preimage() is monotonic: $X \subseteq X' \Longrightarrow Preimage(X) \subseteq Preimage(X')$

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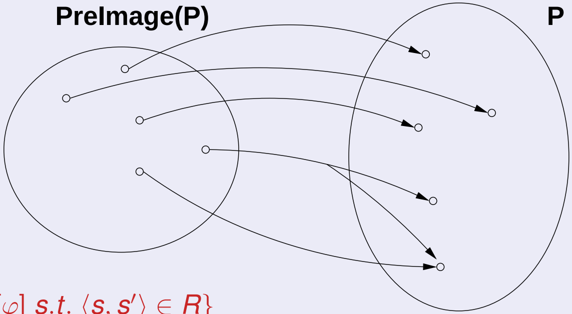

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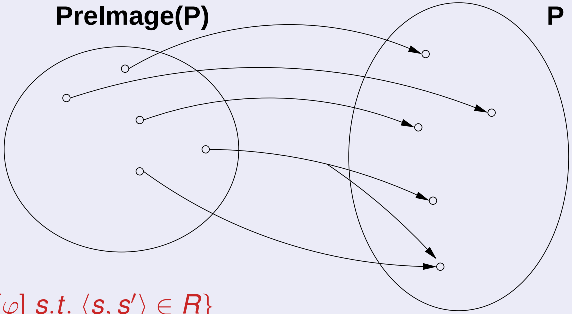

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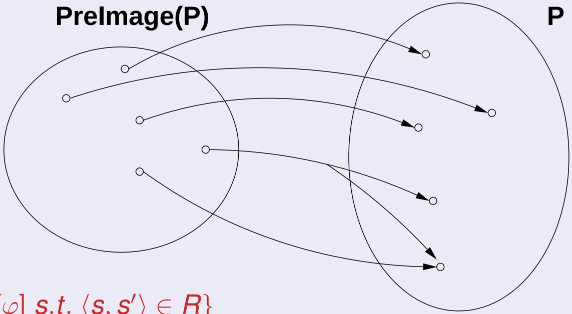

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- $\nu Z.(\ [\beta] \cap [\mathbf{EX}Z]\ )$: greatest fixed point of the function $F_\beta : 2^S \longmapsto 2^S$, s.t.

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- We can compute $X := [\mathbf{EG}\beta]$ inductively as follows:

$$\begin{array}{lcllcl} X_0 & := & S & & & \\ X_1 & := & F_\beta(S) & = & [\beta] & \\ X_2 & := & F_\beta(F_\beta(S)) & = & [\beta] \cap Preimage(X_1) \\ \ldots & & & & & \\ X_{j+1} & := & F_\beta^{j+1}(S) & = & [\beta] \cap Preimage(X_j) \end{array}$$

- Noticing that $X_1 = [\beta]$ and $X_{j+1} \subseteq X_j$ for every $j \geq 0$, and that $([\beta] \cap Y) \subseteq X_j \subseteq [\beta] \Longrightarrow ([\beta] \cap Y) = (X_j \cap Y)$, we can use instead the following inductive schema:
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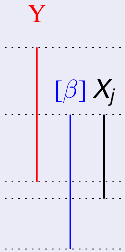

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Consider $\mathbf{E}(\beta_1 \mathbf{U} \beta_2)$:

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  - (Kleene's theorem): $\mu x.F_{\beta_1,\beta_2}(x)$ can be computed as the limit $\emptyset \subseteq F_{\beta_1,\beta_2}(\emptyset) \subseteq F_{\beta_1,\beta_2}(F_{\beta_1,\beta_2}(\emptyset)) \subseteq \ldots$, in a finite number of steps.

Theorem (Clarke & Emerson)

$[\mathbf{E}(\beta_1 \mathbf{U} \beta_2)] = \mu Z.(\ [\beta_2] \cup ([\beta_1] \cap [\mathbf{EX}Z])\ )$

Case **EU** [cont.]

- We can compute $X := [\mathbf{E}(\beta_1 \mathbf{U} \beta_2)]$ inductively as follows:

$$
\begin{array}{lcllcl}
X_0 & := & \emptyset & & & \\
X_1 & := & F_{\beta_1,\beta_2}(\emptyset) & = & [\beta_2] \\
X_2 & := & F_{\beta_1,\beta_2}(F_{\beta_1,\beta_2}(\emptyset)) & = & [\beta_2] \cup ([\beta_1] \cap \mathit{Preimage}(X_1)) \\
\ldots & & & & \\
X_{j+1} & := & F^{j+1}_{\beta_1,\beta_2}(\emptyset)) & = & [\beta_2] \cup ([\beta_1] \cap \mathit{Preimage}(X_j))
\end{array}
$$

- Noticing that $X_1 = [\beta_2]$ and $X_{j+1} \supseteq X_j$ for every $j \geq 0$, and that $([\beta_2] \cup Y) \supseteq X_j \supseteq [\beta_2] \Longrightarrow ([\beta_2] \cup Y) = (X_j \cup Y)$, we can use instead the following inductive schema:
  - $X_1 := [\beta_2]$
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Case EU [cont.]

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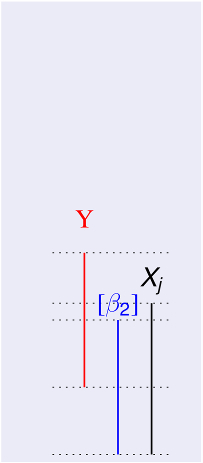

A relevant subcase: **EF**

- $\mathbf{EF}\beta = \mathbf{E}(\top \mathbf{U} \beta)$
- $[\top] = S \implies [\top] \cap \mathit{Preimage}(X_j) = \mathit{Preimage}(X_j)$
- We can compute $X := [\mathbf{EF}\beta]$ inductively as follows:
  - $X_1 := [\beta]$
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Outline

General Schema

- Assume φ written in terms of $\neg$, $\wedge$, **EX**, **EU**, **EG**
- A general M.C. algorithm (fix-point):
  1. for every $\varphi_i \in Sub(\varphi)$, find $[\varphi_i]$
  2. Check if $I \subseteq [\varphi]$
- Subformulas $Sub(\varphi)$ of φ are checked bottom-up
- To compute each $[\varphi_i]$: if the main operator of φ_i is a
  - Propositional atoms: apply labeling function
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General M.C. Procedure

```
state_set Check(CTL_formula β) {
    case β of
    ⊤:          return S;
    ⊥:          return {};
    p:          return {s | p ∈ L(s)};
    ¬β1:        return S / Check(β1);
    β1 ∧ β2:    return Check(β1) ∩ Check(β2);
    EXβ1:       return PreImage(Check(β1));
    EGβ1:       return Check_EG(Check(β1));
    E(β1Uβ2):   return Check_EU(Check(β1),Check(β2));
}
```

PreImage

Compute [EXβ]

state_set PreImage(**state_set** [β]) {
    $X := \{\}$;
    for each $s \in S$ **do**
        for each s' $s.t.$ $s' \in [\beta]$ and $\langle s, s' \rangle \in R$ **do**
            $X := X \cup \{s\}$;
return X;
}

Check_EG

Compute [**EG**β]

state_set Check_EG(**state_set** [β]) {
  $X' := [\beta]$; $j := 1$;
  repeat
    $X := X'$; $j := j + 1$;
    $X' := X \cap PreImage(X)$;
  until $(X' = X)$;
return X;
}

Check_EU

Compute $[\mathbf{E}(\beta_1 \mathbf{U} \beta_2)]$

state_set Check_EU(**state_set** $[\beta_1]$,$[\beta_2]$) {
$\quad X' := [\beta_2];\ j := 1;$
$\quad$ **repeat**
$\quad\quad X := X';\ j := j + 1;$
$\quad\quad X' := X \cup ([\beta_1] \cap \mathit{PreImage}(X));$
$\quad$ **until** $(X' = X)$;
return X;
}

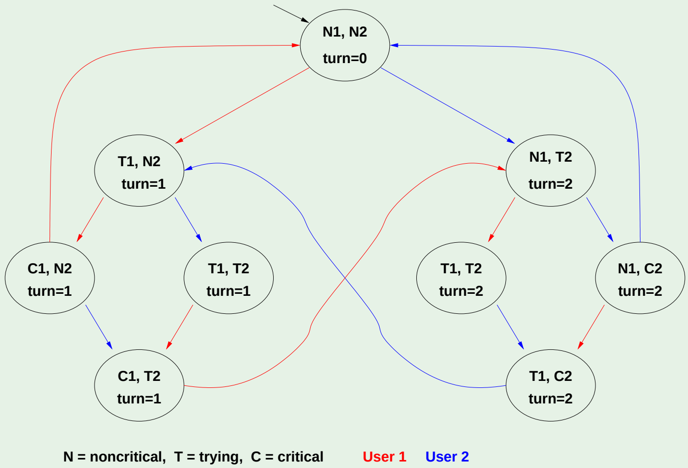
A relevant subcase: Check_EF

Compute [EFβ]

state_set Check_EF(**state_set** [β]) {
  $X' := [\beta]$; $j := 1$;
  repeat
    $X := X'$; $j := j + 1$;
    $X' := X \cup PreImage(X)$;
  until $(X' = X)$;
return X;
}

Outline

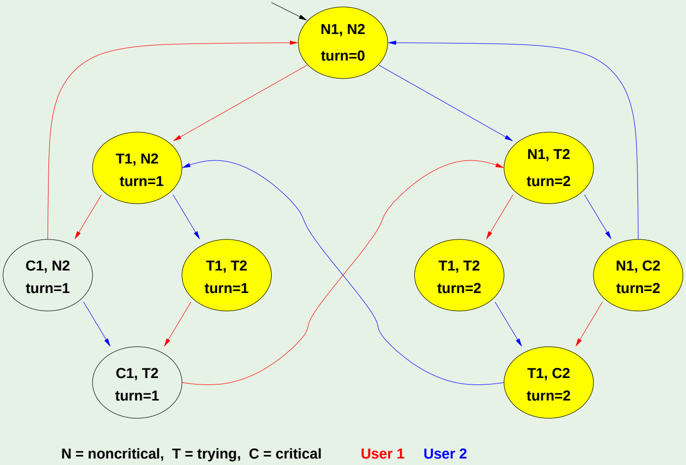
Example 1: fairness



$M \models \mathbf{AGAF}C_1$? $\Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

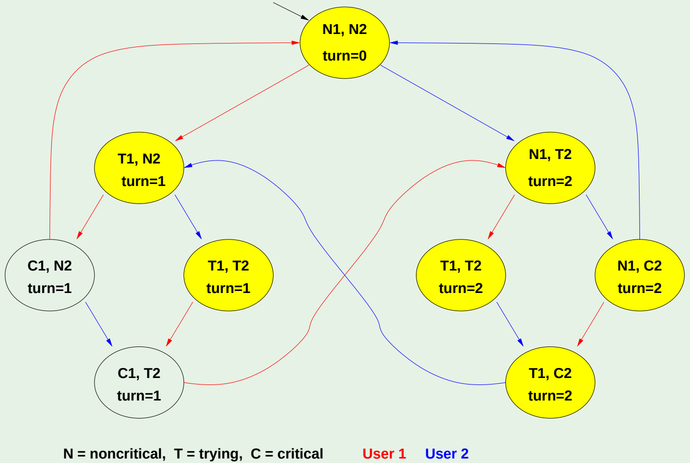
$[\neg C_1]$



$M \models \textbf{AGAF} C_1$? $\Longrightarrow M \models \neg \textbf{EFEG} \neg C_1$?

Example 1: fairness

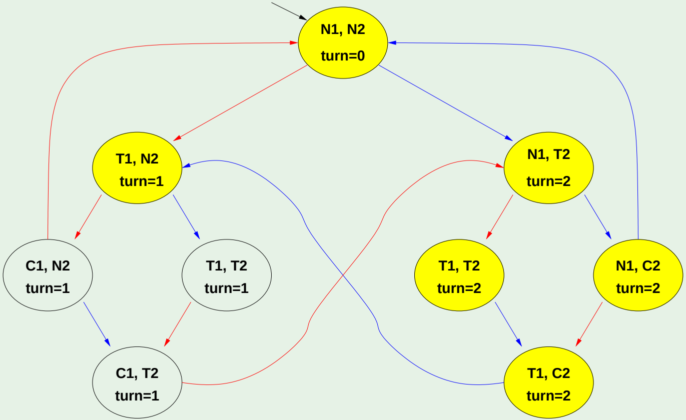
[**EG**$\neg C_1$], step 0:



$M \models$ **AGAF**C_1 ? $\Longrightarrow$ $M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

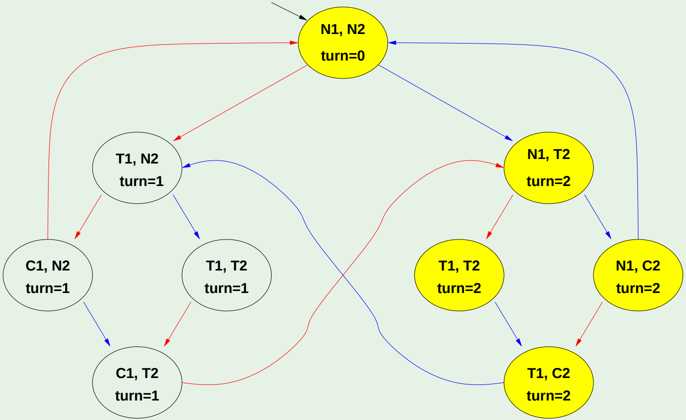
[**EG**$\neg C_1$], step 1:



$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

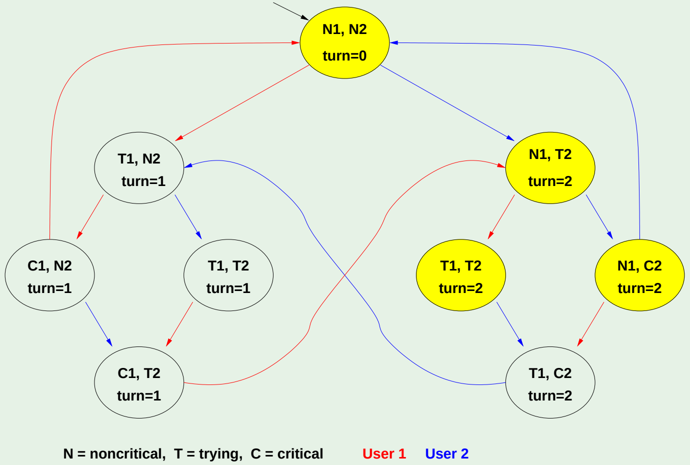
[**EG**$\neg C_1$], step 2:



$M \models$ **AGAF**C_1 ? $\Longrightarrow$ $M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

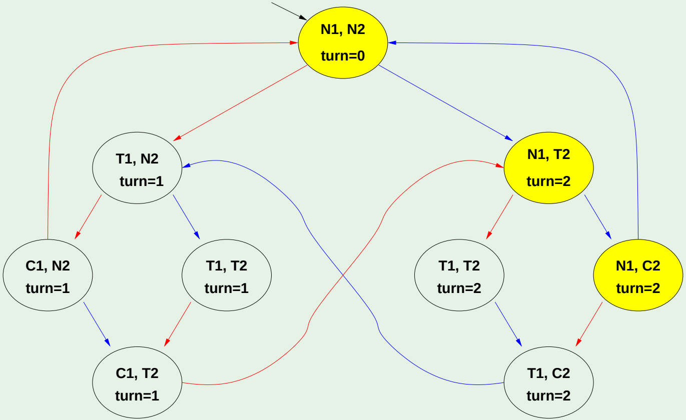
[**EG**$\neg C_1$], step 3:



$M \models \mathbf{AGAF}C_1$? $\Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

[**EG**$\neg C_1$], step 4:

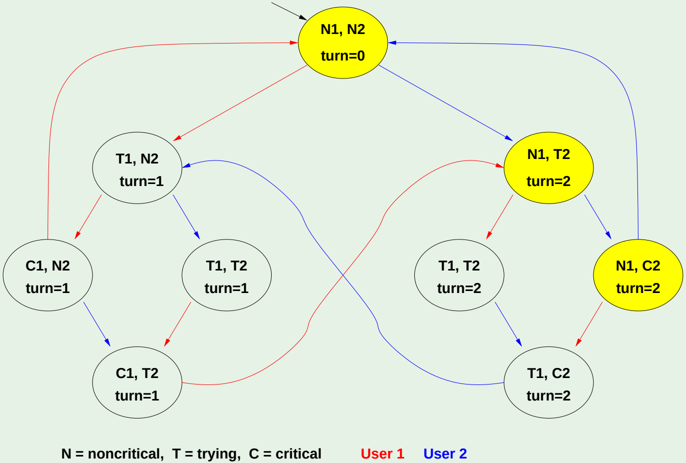


N = noncritical, T = trying, C = critical **User 1** **User 2**

$M \models \mathbf{AGAF} C_1$? $\Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

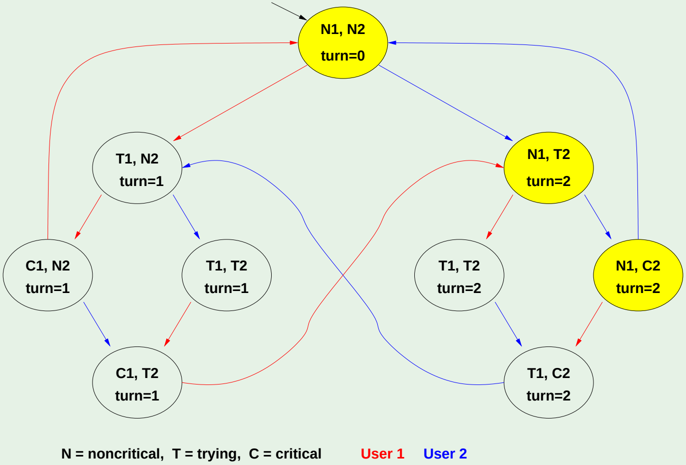
[**EG**$\neg C_1$], FIXPOINT!



$M \models \mathbf{AGAF}C_1$? $\Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

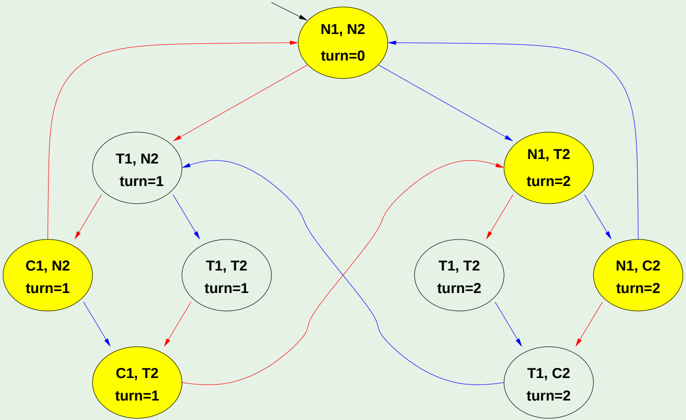
$[\mathbf{EFEG}\neg C_1]$, STEP 0



$M \models \mathbf{AGAF} C_1 \; ? \Longrightarrow M \models \neg \mathbf{EFEG} \neg C_1 \; ?$

Example 1: fairness

$[\mathbf{EFEG}\neg C_1]$, STEP 1

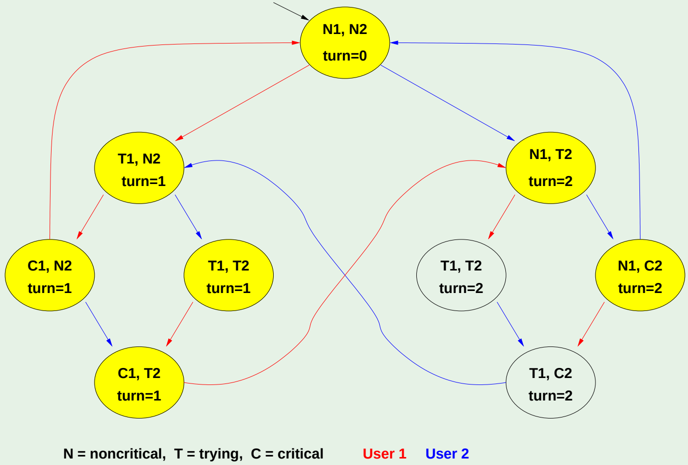


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF}C_1 \ ? \Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1 \ ?$

Example 1: fairness

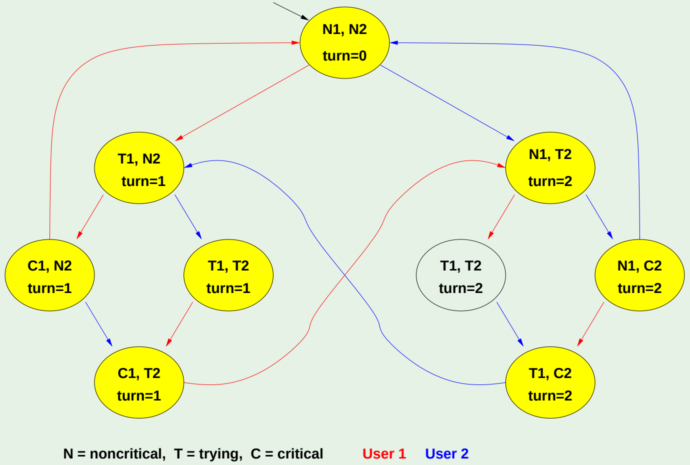
[**EFEG**$\neg C_1$], STEP 2



$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

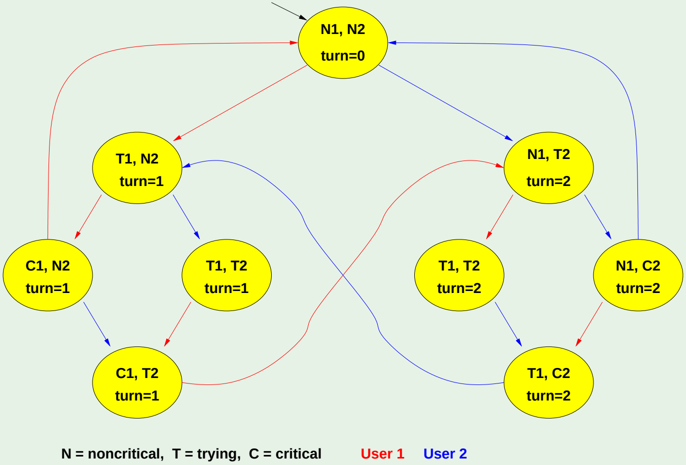
$[\mathbf{EFEG}\neg C_1]$, STEP 3



$M \models \mathbf{AGAF}C_1 \ ? \Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1 \ ?$

Example 1: fairness

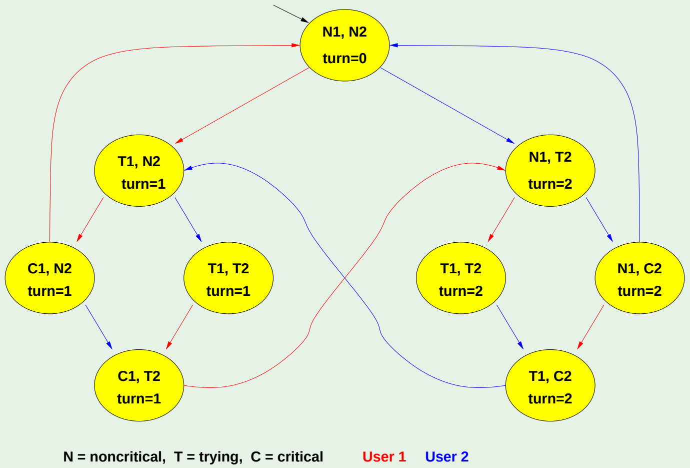
[**EFEG**$\neg C_1$], STEP 4



$M \models \mathbf{AGAF}C_1$? $\Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

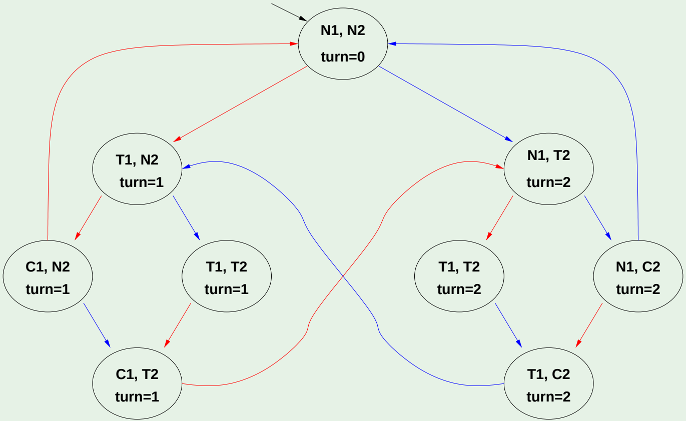
[**EFEG**$\neg C_1$], FIXPOINT!



$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

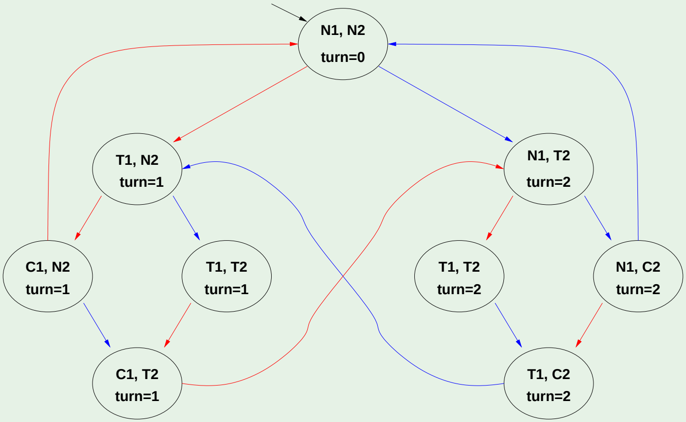
$[\neg \mathbf{EFEG} \neg C_1]$



N = noncritical, T = trying, C = critical **User 1** **User 2**

$M \models \mathbf{AGAF} C_1 \ ? \Longrightarrow M \models \neg \mathbf{EFEG} \neg C_1 \ ? \Longrightarrow$ NO!

Example 2: liveness

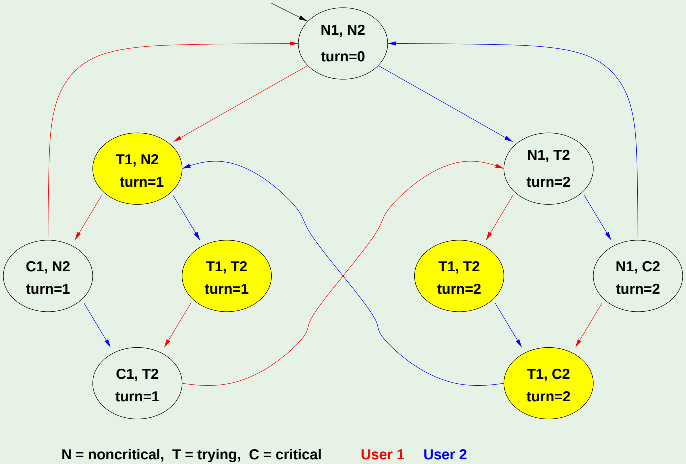


N = noncritical, T = trying, C = critical   User 1   User 2

$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF} C_1)$? $\Longrightarrow M \models \neg \mathbf{EF}(T_1 \wedge \mathbf{EG} \neg C_1)$?

Example 2: liveness

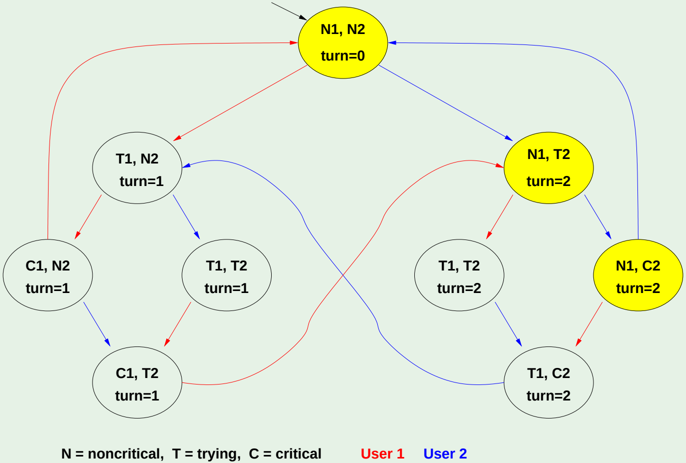
$[T_1]$:



$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF}C_1)$? $\Longrightarrow M \models \neg\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)$?

Example 2: liveness

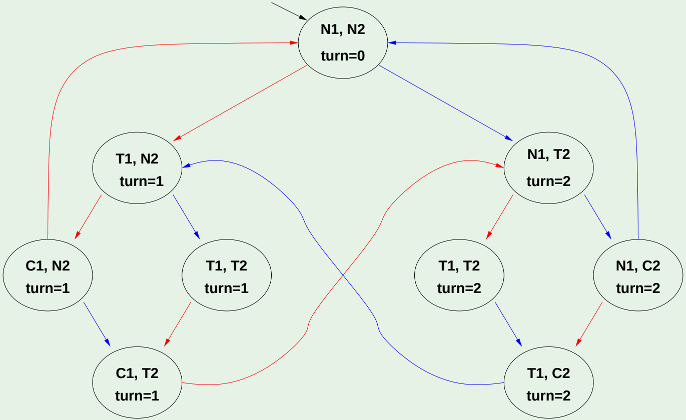
$[\mathbf{EG}\neg C_1]$, STEPS 0-4: (see previous example)



$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF}C_1)\ ? \Longrightarrow M \models \neg\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)\ ?$

Example 2: liveness

$[T_1 \wedge \mathbf{EG}\neg C_1]$:



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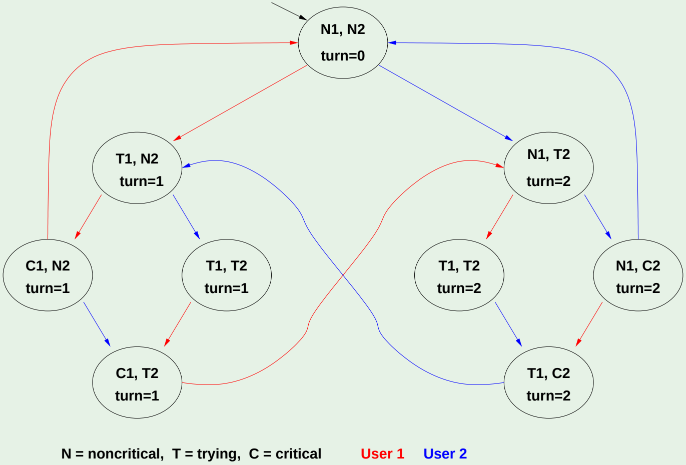
Example 2: liveness

$[\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)]$:



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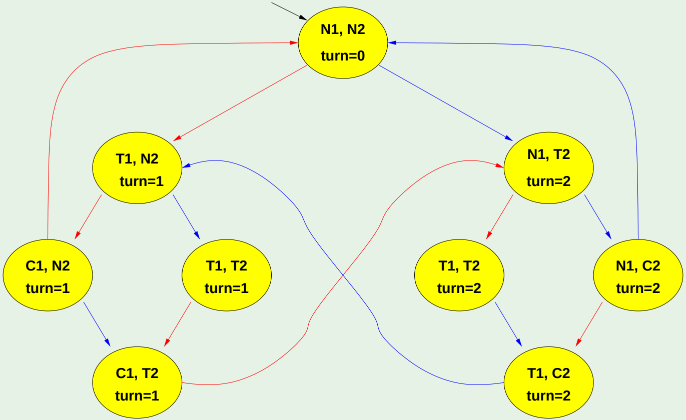

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The property verified is...

Homework

Apply the same process to all the CTL examples of Chapter 3.

Complexity of CTL Model Checking: $M \models \varphi$

- Step 1: compute $[\varphi]$
  - Compute $[\varphi]$ bottom-up on the $O(|\varphi|)$ sub-formulas of φ: $O(|\varphi|)$ steps...
  - ... each requiring at most exploring $O(|M|)$ states

  $\Longrightarrow O(|M| \cdot |\varphi|)$ steps
- Step 2: check $I \subseteq [\varphi]$: $O(|M|)$

$\Longrightarrow O(|M| \cdot |\varphi|)$

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  - ... each requiring at most exploring $O(|M|)$ states

  $\Longrightarrow O(|M| \cdot |\varphi|)$ steps
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$\Longrightarrow O(|M| \cdot |\varphi|)$

Outline

Model Checking of Invariants

- Invariant properties have the form **AG p** (e.g., **AG**$\neg$*bad*)
- Checking invariants is the negation of a reachability problem:
  - is there a reachable state that is also a bad state? ($\mathbf{AG}\neg bad = \neg\mathbf{EF} bad$)
- Standard M.C. algorithm reasons backward from the *bad* by iteratively applying PreImage:

$$Y' := Y \cup PreImage(Y)$$

  until a fixed point is reached.
  Then the complement is computed and I is checked for inclusion in the resulting set.
- Better algorithm: reasons backward from the *bad* by iteratively applying PreImage:

$$Y' := Y \cup PreImage(Y)$$

  until (i) it intersect $[I]$ or (ii) a fixed point is reached

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Model Checking of Invariants

- Invariant properties have the form **AG p** (e.g., $\mathbf{AG}\neg bad$)
- Checking invariants is the negation of a reachability problem:
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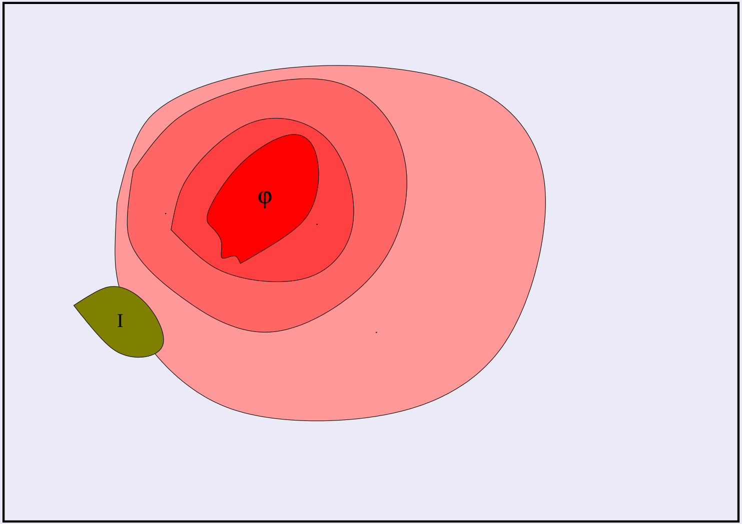
Model Checking of Invariants [cont.]



Forward Model Checking of Invariants

Alternative algorithm (often more efficient): forward checking

- Compute the set of bad states $[bad]$
- Compute the set of initial states I
- Compute incrementally the set of reachable states from I until (i) it intersect $[bad]$ or (ii) a fixed point is reached
- Basic step is the (Forward) Image:

$$Image(Y) \stackrel{\text{def}}{=} \{s' \mid s \in Y \text{ and } R(s, s') \text{ holds}\}$$

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- Simplest form: compute the set of reachable states.

Computing Reachable states: basic

State_Set Compute_reachable() {
$Y' := I; Y := \emptyset; j := 1;$
while $(Y' \neq Y)$
$j := j + 1;$
$Y := Y';$
$Y' := Y \cup Image(Y);$
}
return Y;
}

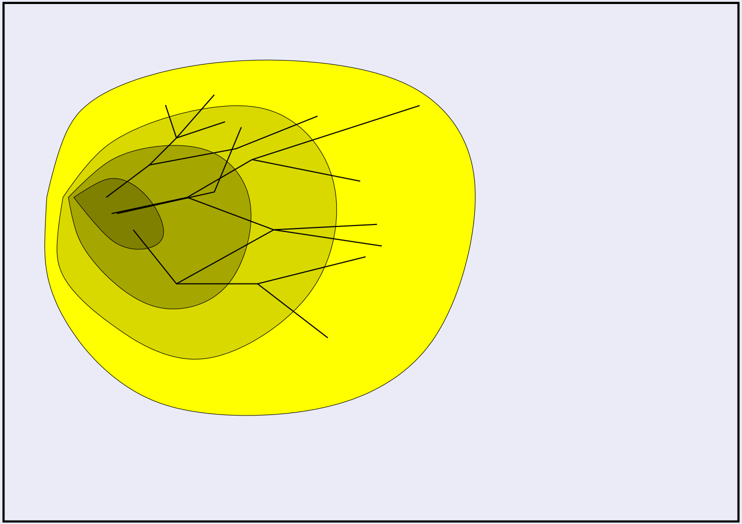
Y=reachable

Computing Reachable states: advanced

State_Set Compute_reachable() {
    $Y := F := I$; $j := 1$;
    while $(F \neq \emptyset)$
        $j := j + 1$;
        $F := Image(F) \setminus Y$;
        $Y := Y \cup F$;
    }
return Y;
}

Y=reachable;F=frontier (new)

Computing Reachable states [cont.]



Checking of Invariant Properties: basic

bool Forward_Check_EF(**State_Set** *BAD*) {
    $Y := I;\ Y' := \emptyset;\ j := 1;$
    while $(Y' \neq Y)$ **and** $(Y' \cap BAD) = \emptyset$
        $j := j + 1;$
        $Y := Y';$
        $Y' := Y \cup Image(Y);$
    }
    if $(Y' \cap BAD) \neq \emptyset$ // counter-example
        return *true*
    else   // fixpoint reached
        return *false*
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Checking of Invariant Properties: advanced

bool Forward_Check_EF(**State_Set** *BAD*) {
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        $F := Image(F) \setminus Y;$
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    }
    if $(F \cap BAD) \neq \emptyset$ // counter-example
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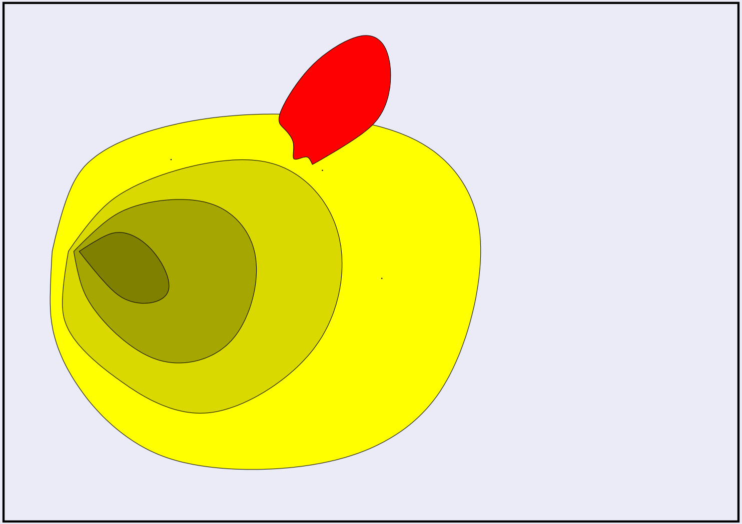
Checking of Invariant Properties [cont.]



Checking of Invariants: Counterexamples

- if layer n intersects with the bad states, then the property is violated
- a counterexample can be reconstructed proceeding backwards
  (i) select any state of $BAD \cap F[n]$ (we know it is satisfiable), call it $t[n]$
  (ii) compute $Preimage(t[n])$, i.e. the states that can result in $t[n]$ in one step
  (iii) compute $Preimage(t[n]) \cap F[n-1]$, and select one state $t[n-1]$
- iterate (i)-(iii) until the initial states are reached
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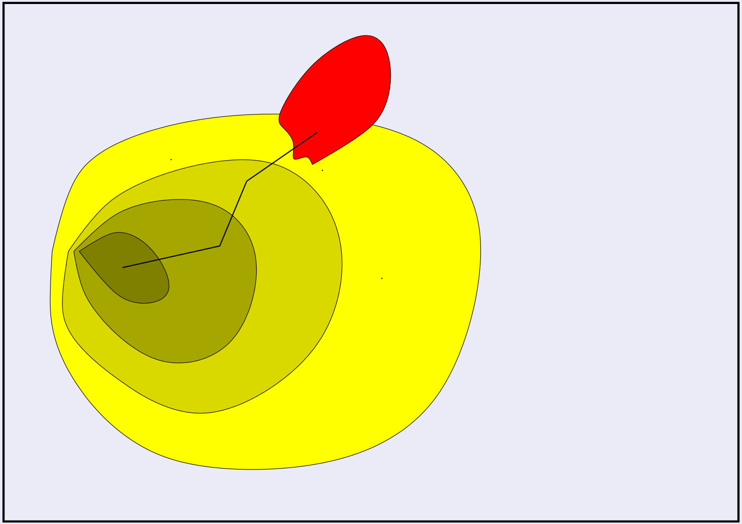
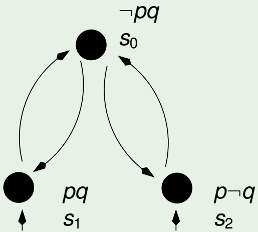


Outline

Ex: CTL Model Checking

Consider the Kripke Model M below, and the CTL property $\varphi \stackrel{\text{def}}{=} \mathbf{AG}((p \wedge q) \rightarrow \mathbf{EG}q)$.



(a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only.

(b) Compute bottom-up the denotations of all subformulas of φ'. (Ex: $[p] = \{s_1, s_2\}$)

(c) As a consequence of point (b), say whether $M \models \varphi$ or not.

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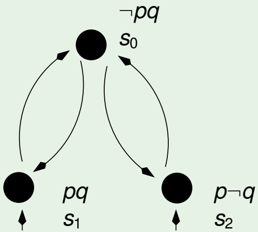

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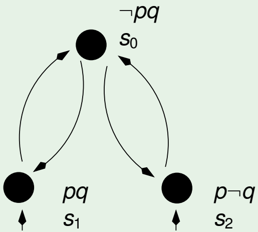

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[Solution:

$[p]$	$=$	$\{s_1, s_2\}$	$[\neg\mathbf{EG}q]$	$=$	$\{s_2\}$
$[q]$	$=$	$\{s_0, s_1\}$	$[((p \land q) \land \neg\mathbf{EG}q)]$	$=$	$\{\}$
$[(p \land q)]$	$=$	$\{s_1\}$	$[\mathbf{EF}((p \land q) \land \neg\mathbf{EG}q)]$	$=$	$\{\}$
$[\mathbf{EG}q]$	$=$	$\{s_0, s_1\}$	$[\neg\mathbf{EF}((p \land q) \land \neg\mathbf{EG}q)]$	$=$	$\{s_0, s_1, s_2\}$

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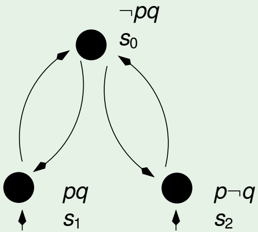

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[Solution: Yes, $\{s_1, s_2\} \subseteq [\varphi']$.]

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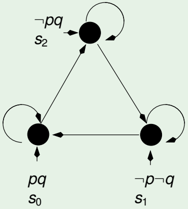

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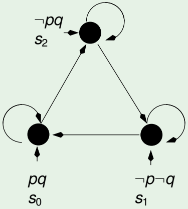

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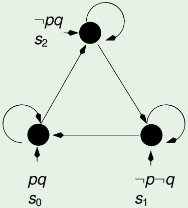

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[Solution:

$$
\begin{array}{lcl lcl}
[p] & = & \{s_0\} & [\neg q] & = & \{s_1\} \\
[\neg p] & = & \{s_1, s_2\} & [\mathbf{EG}\neg q] & = & \{s_1\} \\
[\mathbf{EG}\neg p] & = & \{s_1, s_2\} & [\neg\mathbf{EG}\neg p \wedge \mathbf{EG}\neg q] & = & \{\} \\
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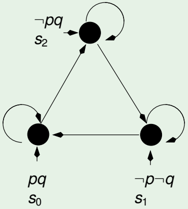

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