Formal Methods
Module I: Automated Reasoning
Ch. 03: **Temporal Logics**

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Outline

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1. Transition Systems as Kripke Models
    - Kripke Models
    - Languages for Transition Systems (hints)
2. Properties and Temporal Logics
    - Properties
    - Temporal Logics
3. Linear Temporal Logic – LTL
    - LTL: Syntax and Semantics
    - Some LTL Model Checking Examples
4. Computation Tree Logic – CTL
    - CTL: Syntax and Semantics
    - Some CTL Model Checking Examples
5. LTL vs. CTL
6. Exercises

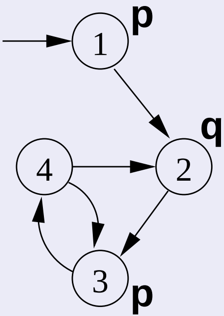
Outline

Kripke Models

- Theoretical role: the semantic framework for a variety of logics
  - Modal Logics
  - Description Logics
  - Temporal Logics
  - ...
- Practical role: used to describe reactive systems:
  - nonterminating systems with infinite behaviors (e.g. communication protocols, hardware circuits);
  - represent the dynamic evolution of modeled systems;
  - a state includes values to state variables, program counters, content of communication channels.
  - can be animated and validated before their actual implementation

Kripke Model: Formal Definition

- A Kripke model $\langle S, I, R, AP, L\rangle$ consists of
  - a finite set of states S;
  - a set of initial states $I \subseteq S$;
  - a set of transitions $R \subseteq S \times S$;
  - a set of atomic propositions AP;
  - a labeling function $L : S \longmapsto 2^{AP}$.
- We assume R total: for every state s, there exists (at least) one state s' s.t. $(s, s') \in R$
- Sometimes we use variables with discrete bounded values $v_i \in \{d_1, ..., d_k\}$ (can be encoded with $\lceil log(k) \rceil$ Boolean variables)

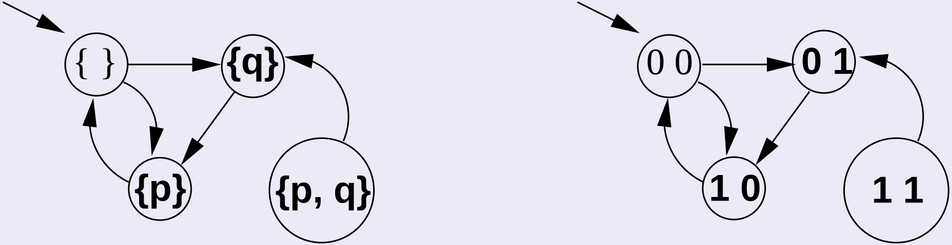


Remark

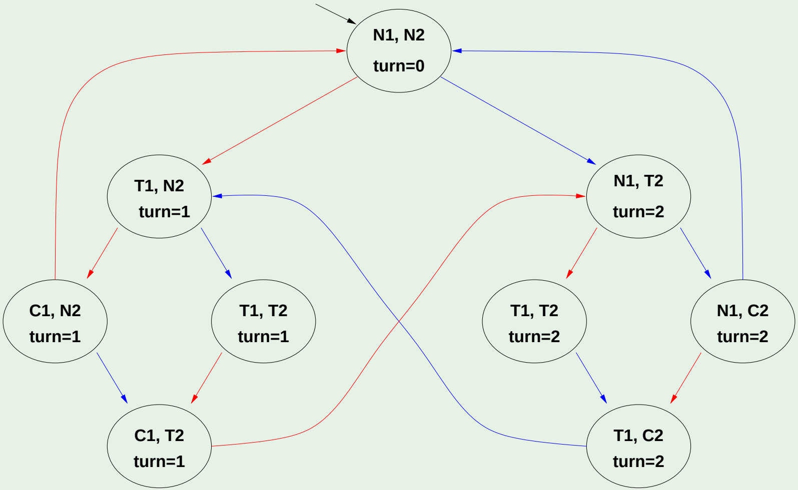
Unlike with other types of Automata (e.g., Buechi), in Kripke models the values of all variables are always assigned in each state.

Kripke Structures: Two Alternative Representations:

- each state identifies univocally the values of the atomic propositions which hold there
- each state is labeled by a bit vector



Example: a Kripke model for mutual exclusion



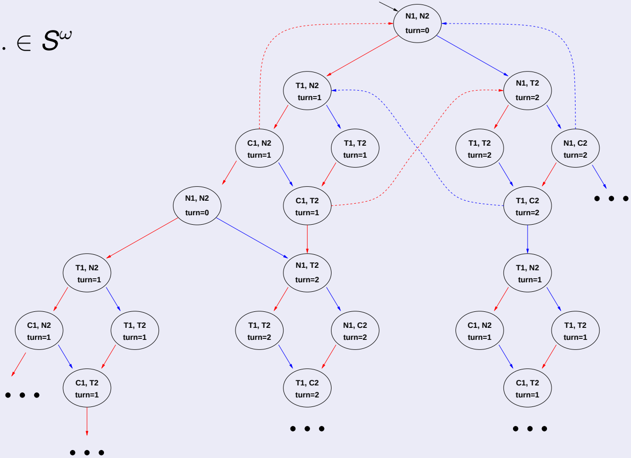
N = noncritical, T = trying, C = critical **User 1** **User 2**

Path in a Kripke Model

A path in a Kripke model M is an infinite sequence of states

$$\pi = s_0, s_1, s_2, \ldots \in S^\omega$$

such that $s_0 \in I$ and $(s_i, s_{i+1}) \in R$.



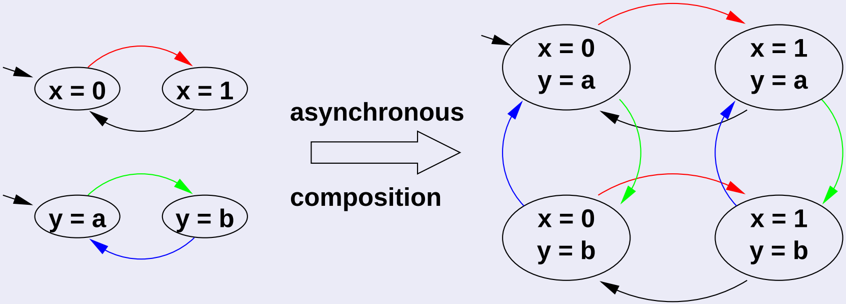
A state s is reachable in M if there is a path from the initial states to s.

Composing Kripke Models

- Complex Kripke Models are tipically obtained by composition of smaller ones
- Components can be combined via
  - asynchronous composition.
  - synchronous composition,

Asynchronous Composition

- Interleaving of evolution of components.
- At each time instant, one component is selected to perform a transition.



- Typical example: communication protocols.

Asynchronous Composition/Product: formal definition

Asynchronous product of Kripke models

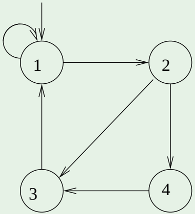
Let $M_1 \overset{\text{def}}{=} \langle S_1, I_1, R_1, AP_1, L_1 \rangle$, $M_2 \overset{\text{def}}{=} \langle S_2, I_2, R_2, AP_2, L_2 \rangle$. Then the asynchronous product $M \overset{\text{def}}{=} M_1 || M_2$ is $M \overset{\text{def}}{=} \langle S, I, R, AP, L \rangle$, where

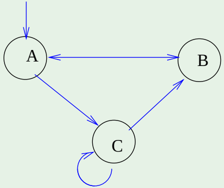
- $S \subseteq S_1 \times S_2$ s.t., $\forall \langle s_1, s_2 \rangle \in S,\ \forall l \in AP_1 \cap AP_2, l \in L_1(s_1)$ *iff* $l \in L_2(s_2)$
- $I \subseteq I_1 \times I_2$ s.t. $I \subseteq S$
- $R(\langle s_1, s_2 \rangle, \langle t_1, t_2 \rangle)$ *iff* $(R_1(s_1, t_1)$ **and** $s_2 = t_2)$ **or** $(s_1 = t_1$ **and** $R_2(s_2, t_2))$
- $AP = AP_1 \cup AP_2$
- $L : S \longmapsto 2^{AP}$ s.t. $L(\langle s_1, s_2 \rangle) \overset{\text{def}}{=} L_1(s_1) \cup L_2(s_2)$.

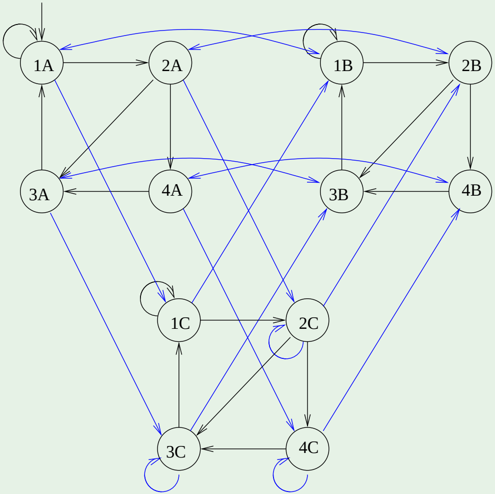
Note: combined states must agree on the values of Boolean variables.

Asynchronous composition is associative:
$(...(M_1 || M_2) || ...) || M_n) = (M1 || (M_2 || (... || M_n)...) = M_1 || M_2 || ... || M_n$

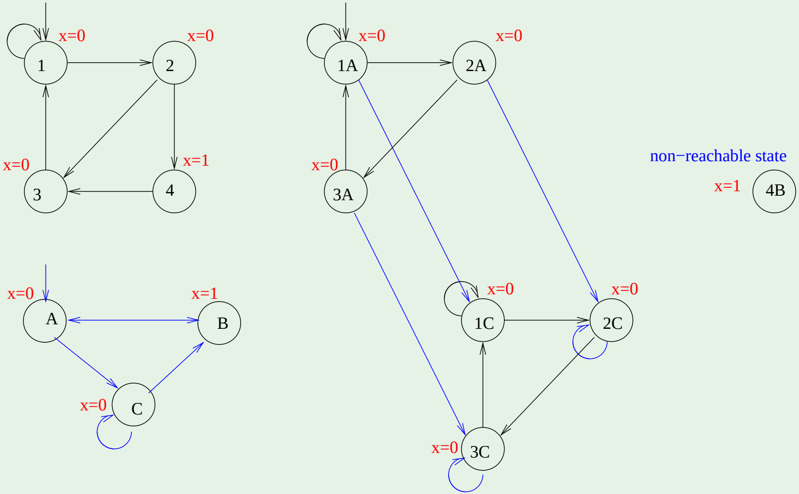
Asynchronous Composition: Example 1



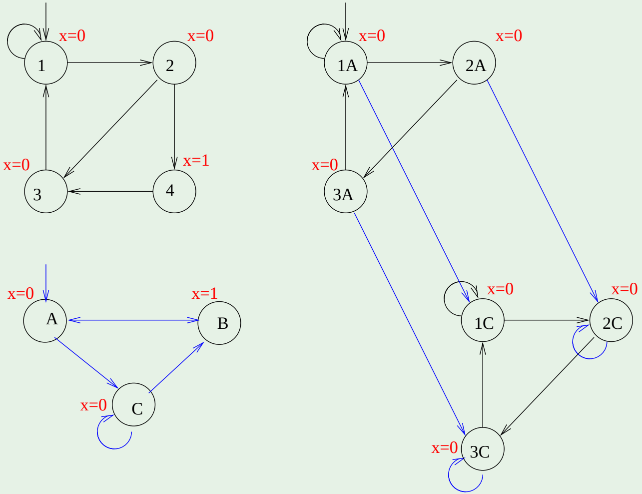




Asynchronous Composition: Example 2

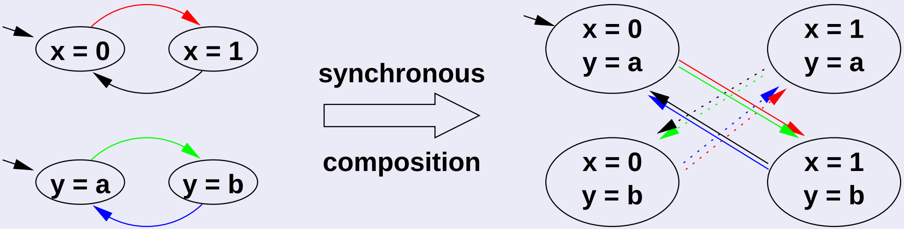


Asynchronous Composition: Example 2



Synchronous Composition

- Components evolve in parallel.
- At each time instant, every component performs a transition.



- Typical example: sequential hardware circuits.

Synchronous Composition/Product: formal definition

Synchronous product of Kripke models

Let $M_1 \stackrel{\text{def}}{=} \langle S_1, I_1, R_1, AP_1, L_1 \rangle$, $M_2 \stackrel{\text{def}}{=} \langle S_2, I_2, R_2, AP_2, L_2 \rangle$. Then the synchronous product $M \stackrel{\text{def}}{=} M_1 \times M_2$ is $M \stackrel{\text{def}}{=} \langle S, I, R, AP, L \rangle$, where

- $S \subseteq S_1 \times S_2$ s.t., $\forall \langle s_1, s_2 \rangle \in S,\ \forall l \in AP_1 \cap AP_2, l \in L_1(s_1)$ *iff* $l \in L_2(s_2)$
- $I \subseteq I_1 \times I_2$ s.t. $I \subseteq S$
- $R(\langle s_1, s_2 \rangle, \langle t_1, t_2 \rangle)$ *iff* $(R_1(s_1, t_1)$ **and** $R_2(s_2, t_2))$
- $AP = AP_1 \cup AP_2$
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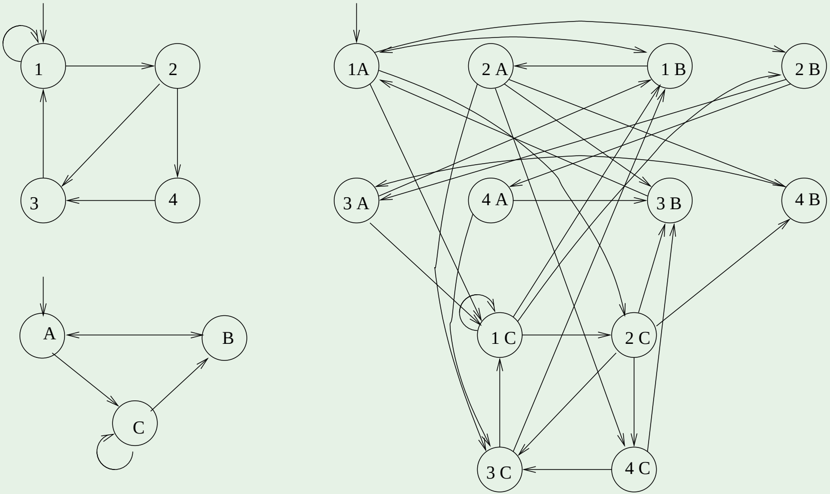
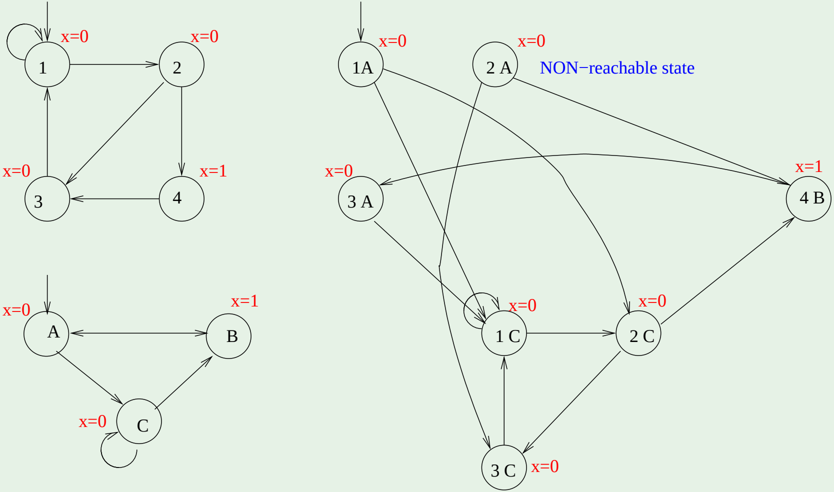
Note: combined states must agree on the values of Boolean variables.

Synchronous composition is associative:
$(...(M_1 \times M_2) \times ...) \times M_n) = (M1 \times (M_2 \times (... \times M_n)...) = M_1 \times M_2 \times ... \times M_n$

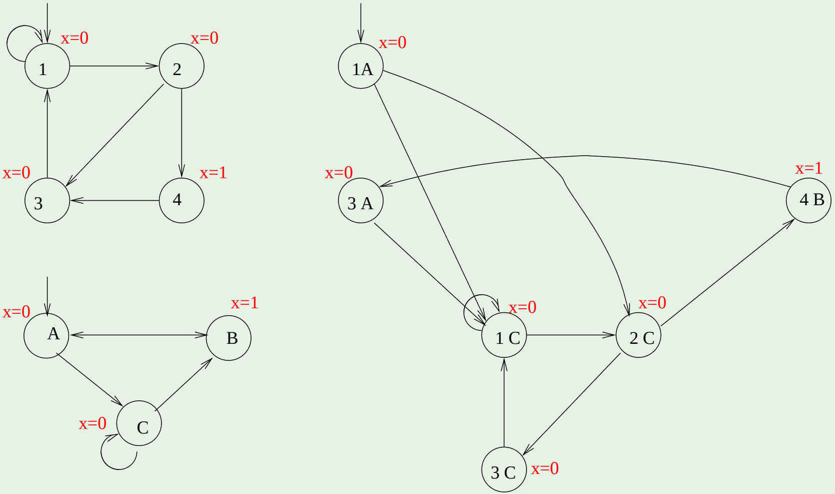
Synchronous Composition: Example 1



Synchronous Composition: Example 2



Synchronous Composition: Example 2 (cont.)



Outline

Description languages for Kripke Model

- Most often a Kripke model is not given explicitly (states, arcs),...
- ... rather it is usually presented in a structured language (e.g., SMV, PROMELA, StateCharts, VHDL, ...)
  - even a piece of SW can be seen as a Kripke model!
- Each component is presented by specifying
  - state variables: determine the set of atomic propositions AP, the state space S and the labeling L.
  - initial values of variables V: determine the set of initial states I.
    - described as a relation $I(V_0)$ in terms of state variables at step 0
  - instructions: determine the transition relation R.
    - described as a relation $R(V, V')$ in terms of current state variables V and next state variables V'
- Aka as symbolic representation of a Kripke model

Remark

Tipically symbolic description are much more compact (and intuitive) than the explicit representation of the Kripke model.

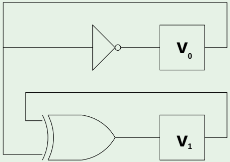
The SMV language

- The input language of the SMV M.C. (and NuSMV)
- Booleans, enumerative and bounded integers as data types
- now enriched with other constructs, e.g. in NuXMV language
- An SMV program consists of:
  - Declarations of the state variables (e.g., `b0`);
  - Assignments that define the initial states (e.g., `init(b0) := 0`).
  - Assignments that define the transition relation (e.g., `next(b0) := !b0`).
- Allows for both synchronous and asyncronous composition of modules (though synchronous interaction more natural)

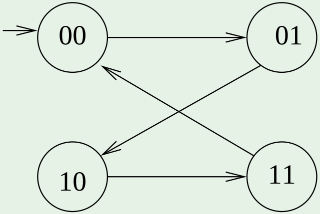
Example: a Simple Counter Circuit

```
MODULE main
 VAR
   v0     : boolean;
   v1     : boolean;
   out    : 0..3;

 ASSIGN
   init(v0)  := 0;
   next(v0)  := !v0;
   init(v1)  := 0;
   next(v1)  := (v0 xor v1);
   out := toint(v0) + 2*toint(v1);
```



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



$$
\begin{aligned}
I(V) &= (\neg v_0 \land \neg v_1) \\
R(V, V') &= (v_0' \leftrightarrow \neg v_0) \land (v_1' \leftrightarrow v_0 \bigoplus v_1)
\end{aligned}
$$

Standard Programming Languages

- Standard programming languages are typically sequential

$\Longrightarrow$ Transition relation defined in terms also of the program counter

- Numbers & values Booleanized

```
...
10. i = 0;
11. acc = 0.0;
12. while (i<dim) {
13.   acc += V[i];
14.   i++;
15. }
...
```

....

$$(pc = 10) \rightarrow ((i' = 0) \wedge (pc' = 11))$$
$$(pc = 11) \rightarrow ((acc' = 0.0) \wedge (pc' = 12))$$
$$(pc = 12) \rightarrow ((i < dim) \rightarrow (pc' = 13))$$
$$(pc = 12) \rightarrow (\neg(i < dim) \rightarrow (pc' = 16))$$
$$(pc = 13) \rightarrow ((acc' = acc + read(V, i)) \wedge (pc' = 14))$$
$$(pc = 14) \rightarrow (i' = i + 1) \wedge (pc' = 15))$$
$$(pc = 15) \rightarrow (pc' = 16))$$

...

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   - Languages for Transition Systems (hints)
2. **Properties and Temporal Logics**
   - Properties
   - Temporal Logics
3. Linear Temporal Logic – LTL
   - LTL: Syntax and Semantics
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4. Computation Tree Logic – CTL
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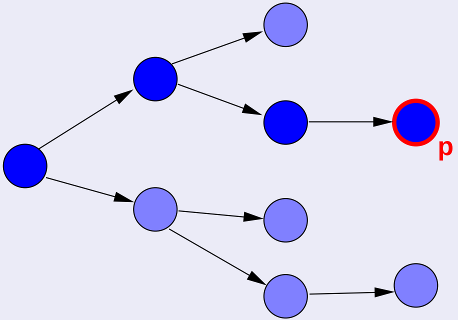
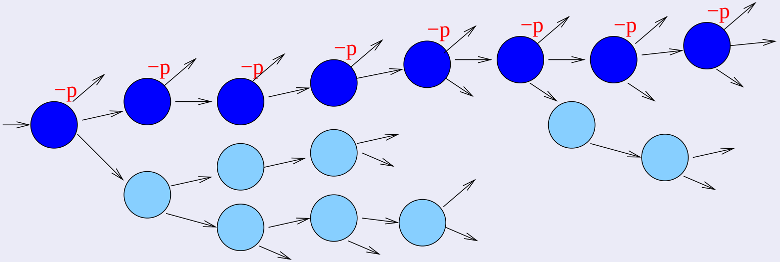
Safety Properties

- Bad events never happen
  - deadlock: two processes waiting for input from each other, the system is unable to perform a transition.
  - no reachable state satisfies a “bad” condition, e.g. never two processes in critical section at the same time
- Can be refuted by a finite behaviour
- Ex.: it is never the case that *p*.



Liveness Properties

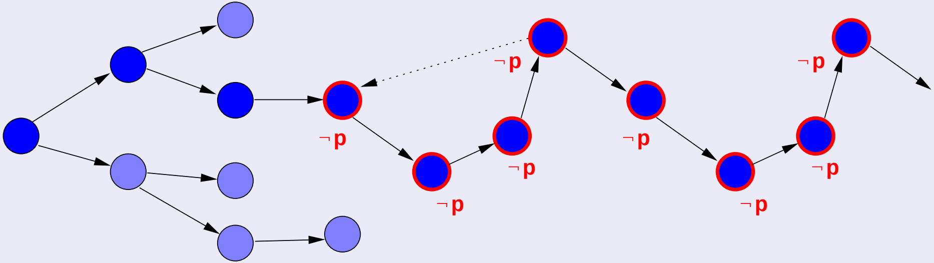
- Something desirable will eventually happen
  - sooner or later this will happen
- Can be refuted by infinite behaviour



  - an infinite behaviour can be typically presented as a loop

Fairness Properties

- Something desirable will happen infinitely often
  - important subcase of liveness
  - whenever a subroutine takes control, it will always return it (sooner or later)
- Can be refuted by infinite behaviour
  - a subroutine takes control and never returns it



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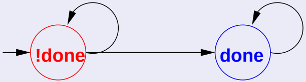
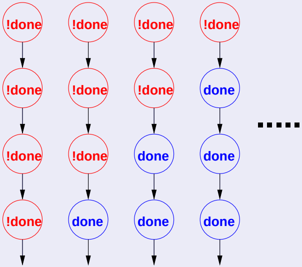
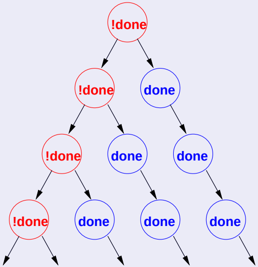
Computation tree vs. computation paths

- Consider the following Kripke structure:



- Its execution can be seen as:
    - an infinite set of computation paths
    - an infinite computation tree





Temporal Logics

- Express properties of “Reactive Systems”
    - nonterminating behaviours,
    - without explicit reference to time.
- Linear Temporal Logic (LTL)
    - interpreted over each path of the Kripke structure
    - linear model of time
    - temporal operators
    - “Medieval”: “since birth, one’s destiny is set”.
- Computation Tree Logic (CTL)
    - interpreted over computation tree of Kripke model
    - branching model of time
    - temporal operators plus path quantifiers
    - “Humanistic”: “one makes his/her own destiny step-by-step”.

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Linear Temporal Logic (LTL): Syntax

- An atomic proposition is a LTL formula;
- if φ_1 and φ_2 are LTL formulae, then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$, $\varphi_1 \oplus \varphi_2$ are LTL formulae;
- if φ_1 and φ_2 are LTL formulae, then $\mathbf{X}\varphi_1$, $\mathbf{G}\varphi_1$, $\mathbf{F}\varphi_1$, $\varphi_1\mathbf{U}\varphi_2$ are LTL formulae, where **X**, **G**, **F**, **U** are the “next”, “globally”, “eventually”,“until” temporal operators respectively.
- Another operator **R** “releases” (the dual of **U**) is used sometimes.

LTL semantics: intuitions

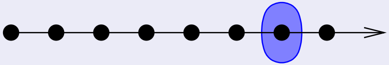
LTL is given by the standard boolean logic enhanced with the following temporal operators, which operate through paths $\langle s_0, s_1, ..., s_k, ...\rangle$:

- "Next" **X**: $\mathbf{X}\varphi$ is true in s_t iff φ is true in s_{t+1}
- "Finally" (or "eventually") **F**: $\mathbf{F}\varphi$ is true in s_t iff φ is true in **some** $s_{t'}$ with $t' \geq t$
- "Globally" (or "henceforth") **G**: $\mathbf{G}\varphi$ is true in s_t iff φ is true in **all** $s_{t'}$ with $t' \geq t$
- "Until" **U**: $\varphi\mathbf{U}\psi$ is true in s_t iff, for some state $s_{t'}$ s.t $t' \geq t$:
  - ψ is true in $s_{t'}$ **and**
  - φ is true in all states $s_{t''}$ s.t. $t \leq t'' < t'$
- "Releases" **R**: $\varphi\mathbf{R}\psi$ is true in s_t iff, for all states $s_{t'}$ s.t. $t' \geq t$:
  - ψ is true **or**
  - φ is true in some states $s_{t''}$ with $t \leq t'' < t'$

  "ψ can become false only if φ becomes true first"

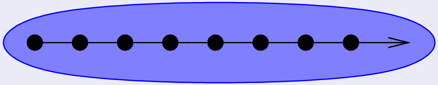
LTL semantics: intuitions

finally p



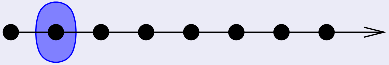
F p

globally p



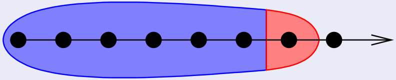
G p

next p



X p

p until q



p U q

LTL: Some Noteworthy Examples

- Safety: “it never happens that a train is arriving and the bar is up”

$$\mathbf{G}(\neg(\text{train\_arriving} \wedge \text{bar\_up}))$$

- Liveness: “if input, then eventually output”

$$\mathbf{G}(\text{input} \rightarrow \mathbf{F}\text{output})$$

- Releases: “the device is not working if you don’t first repair it”

$$(\text{repair\_device}\ \mathbf{R}\ \neg\text{working\_device})$$

- Fairness: “infinitely often send ”

$$\mathbf{GF}\text{send}$$

- Strong fairness: “infinitely often send implies infinitely often recv.”

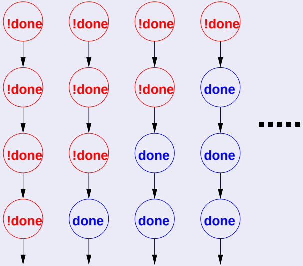
$$\mathbf{GF}\text{send} \rightarrow \mathbf{GF}\text{recv}$$

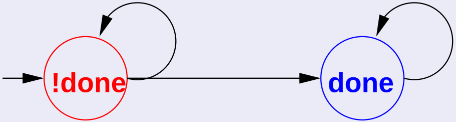
LTL Formal Semantics

$$
\begin{array}{lllrll}
\pi, s_i & \models & a & \text{iff} & a \in L(s_i) & \\
\pi, s_i & \models & \neg\varphi & \text{iff} & \pi, s_i & \not\models \varphi \\
\pi, s_i & \models & \varphi \land \psi & \text{iff} & \pi, s_i & \models \varphi \textit{ and} \\
 & & & & \pi, s_i & \models \psi \\
\pi, s_i & \models & \mathbf{X}\varphi & \text{iff} & \pi, s_{i+1} & \models \varphi \\
\pi, s_i & \models & \mathbf{F}\varphi & \text{iff} & \textit{for some } j \geq i \ : \pi, s_j & \models \varphi \\
\pi, s_i & \models & \mathbf{G}\varphi & \text{iff} & \textit{for all } j \geq i \ : \pi, s_j & \models \varphi \\
\pi, s_i & \models & \varphi\mathbf{U}\psi & \text{iff} & \textit{for some } j \geq i \ : (\pi, s_j & \models \psi \ \textit{ and} \\
 & & & & \textit{for all } k \textit{ s.t. } i \leq k < j \ : \pi, s_k & \models \varphi) \\
\pi, s_i & \models & \varphi\mathbf{R}\psi & \text{iff} & \textit{for all } j \geq i \ : (\pi, s_j & \models \psi \ \textit{ or} \\
 & & & & \textit{for some } k \textit{ s.t. } i \leq k < j \ : \pi, s_k & \models \varphi)
\end{array}
$$

LTL Formal Semantics (cont.)

- LTL properties are evaluated over paths, i.e., over infinite, linear sequences of states:
  $\pi = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_t \rightarrow s_{t+1} \rightarrow \cdots$
- Given an infinite sequence $\pi = s_0, s_1, s_2, \ldots$
  - $\pi, s_i \models \phi$ if ϕ is true in state s_i of π.
  - $\pi \models \phi$ if ϕ is true in the initial state s_0 of π.
- The LTL model checking problem $\mathcal{M} \models \phi$
  - check if $\pi \models \phi$ for every path π of the Kripke structure $\mathcal{M}$ (e.g., $\phi = \mathbf{F} \mathit{done}$)





The LTL model checking problem $\mathcal{M} \models \phi$: remark

The LTL model checking problem $\mathcal{M} \models \phi$

$\pi \models \phi$ for every path π of the Kripke structure $\mathcal{M}$

Important Remark

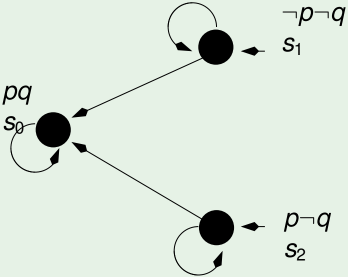
$\mathcal{M} \not\models \phi \not\Longrightarrow \mathcal{M} \models \neg\phi$ (!!)

- E.g. if ϕ is a LTL formula and two paths π_1 and π_2 are s.t. $\pi_1 \models \phi$ and $\pi_2 \models \neg\phi$.

Example: $\mathcal{M} \not\models \phi \not\Longrightarrow \mathcal{M} \models \neg\phi$

Let $\pi_1 \overset{\text{def}}{=} \{s_1\}^\omega$, $\pi_2 \overset{\text{def}}{=} \{s_2\}^\omega$.

- $\mathcal{M} \not\models \mathbf{G}p$, in fact:
  - $\pi_1 \not\models \mathbf{G}p$
  - $\pi_2 \models \mathbf{G}p$
- $\mathcal{M} \not\models \neg\mathbf{G}p$, in fact:
  - $\pi_1 \models \neg\mathbf{G}p$
  - $\pi_2 \not\models \neg\mathbf{G}p$



Syntactic properties of LTL operators

$$\begin{array}{rcl}
\varphi_1 \vee \varphi_2 & \iff & \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\
\dots & & \\
\mathbf{F}\ \varphi_1 & \iff & \top\mathbf{U}\varphi_1 \\
\mathbf{G}\ \varphi_1 & \iff & \bot\mathbf{R}\varphi_1 \\
\mathbf{F}\varphi_1 & \iff & \neg\mathbf{G}\neg\varphi_1 \\
\mathbf{G}\varphi_1 & \iff & \neg\mathbf{F}\neg\varphi_1 \\
\neg\mathbf{X}\varphi_1 & \iff & \mathbf{X}\neg\varphi_1 \\
\varphi_1\mathbf{R}\varphi_2 & \iff & \neg(\neg\varphi_1\mathbf{U}\neg\varphi_2) \\
\varphi_1\mathbf{U}\varphi_2 & \iff & \neg(\neg\varphi_1\mathbf{R}\neg\varphi_2)
\end{array}$$

Note

LTL can be defined in terms of $\wedge$, $\neg$, $\mathbf{X}$, $\mathbf{U}$ only

Exercise

Prove that $\varphi_1\mathbf{R}\varphi_2 \iff \mathbf{G}\varphi_2 \vee \varphi_2\mathbf{U}(\varphi_1 \wedge \varphi_2)$

Proof of $\varphi \mathbf{R} \psi \Leftrightarrow (\mathbf{G}\psi \vee \psi \mathbf{U}(\varphi \wedge \psi))$

[Solution proposed by the student Samuel Valentini, 2016]

(All state indexes below are implicitly assumed to be ≥ 0.)

$\Rightarrow$: Let π be s.t. $\pi, s_0 \models \varphi \mathbf{R} \psi$

- If $\forall j, \pi, s_j \models \psi$, then $\pi, s_0 \models \mathbf{G}\psi$.
- Otherwise, let s_k be the first state s.t. $\pi, s_k \not\models \psi$.
- Since $\pi, s_0 \models \varphi \mathbf{R} \psi$, then $k > 0$ and exists $k' < k$ s.t. $\pi, S_{k'} \models \varphi$
- By construction, $\pi, s_{k'} \models \varphi \wedge \psi$ and, for every $w < k'$, $\pi, s_w \models \psi$, so that $\pi, s_0 \models \psi \mathbf{U}(\varphi \wedge \psi)$.
- Thus, $\pi, s_0 \models \mathbf{G}\psi \vee \psi \mathbf{U}(\varphi \wedge \psi)$

$\Leftarrow$: Let π be s.t. $\pi, s_0 \models \mathbf{G}\psi \vee \psi \mathbf{U}(\varphi \wedge \psi)$

- If $\pi, s_0 \models \mathbf{G}\psi$, then $\forall j, \pi, s_j \models \psi$, so that $\pi, s_0 \models \varphi \mathbf{R} \psi$.
- Otherwise, $\pi, s_0 \models \psi \mathbf{U}(\varphi \wedge \psi)$.
- Let s_k be the first state s.t. $\pi, s_k \not\models \psi$.
- by construction, $\exists k'$ such that $\pi, S_{k'} \models \varphi \wedge \psi$
- by the definition of k, we have that $k' < k$ and $\forall w < k, \pi, S_w \models \psi$.
- Thus $\pi, s_0 \models \varphi \mathbf{R} \psi$

Strength of LTL operators

- $\mathbf{G}\varphi \models \varphi \models \mathbf{F}\varphi$
- $\mathbf{G}\varphi \models \mathbf{X}\varphi \models \mathbf{F}\varphi$
- $\mathbf{G}\varphi \models \mathbf{XX}...\mathbf{X}\varphi \models \mathbf{F}\varphi$
- $\varphi\mathbf{U}\psi \models \mathbf{F}\psi$
- $\mathbf{G}\psi \models \varphi\mathbf{R}\psi$

LTL tableaux rules

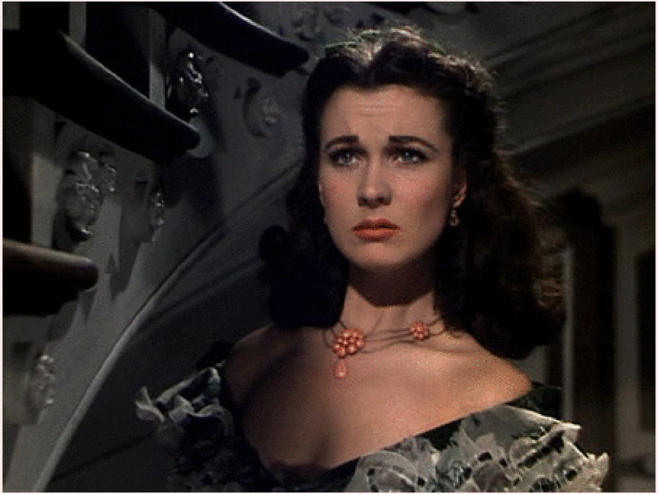
- Let φ_1 and φ_2 be LTL formulae:

$$\begin{aligned}
\mathbf{F}\varphi_1 &\iff (\varphi_1 \vee \mathbf{XF}\varphi_1)\\
\mathbf{G}\varphi_1 &\iff (\varphi_1 \wedge \mathbf{XG}\varphi_1)\\
\varphi_1\mathbf{U}\varphi_2 &\iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X}(\varphi_1\mathbf{U}\varphi_2)))\\
\varphi_1\mathbf{R}\varphi_2 &\iff (\varphi_2 \wedge (\varphi_1 \vee \mathbf{X}(\varphi_1\mathbf{R}\varphi_2)))
\end{aligned}$$

- If applied recursively, rewrite an LTL formula in terms of atomic and **X**-formulas:

$$(p\mathbf{U}q) \wedge (\mathbf{G}\neg p) \Longrightarrow (q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))) \wedge (\neg p \wedge \mathbf{XG}\neg p)$$

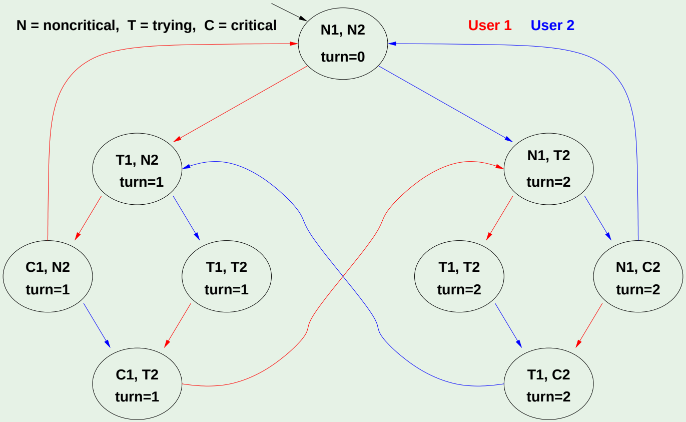
Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

Outline

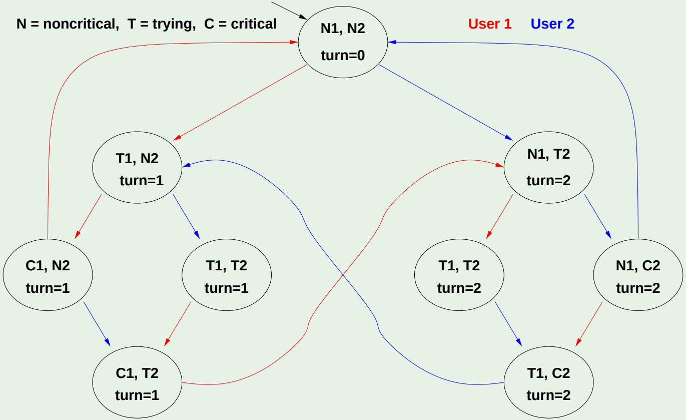
Example 1: mutual exclusion (safety)



$$M \models \mathbf{G}\neg(C_1 \wedge C_2)\ ?$$

YES: There is no reachable state in which $(C_1 \wedge C_2)$ holds!

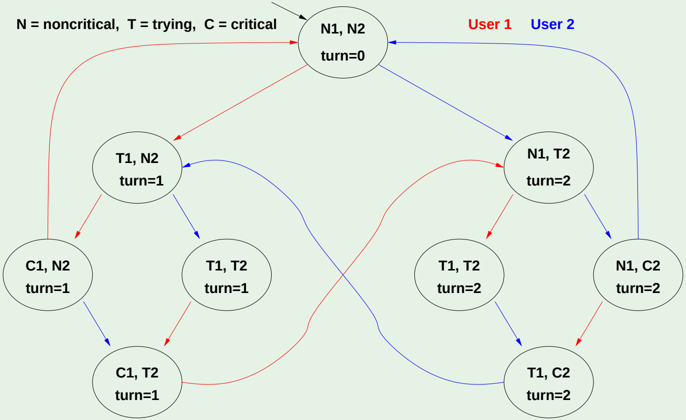
Example 2: liveness



$$M \models \mathbf{F} C_1 \ ?$$

NO: there is an infinite cyclic solution in which C_1 never holds!

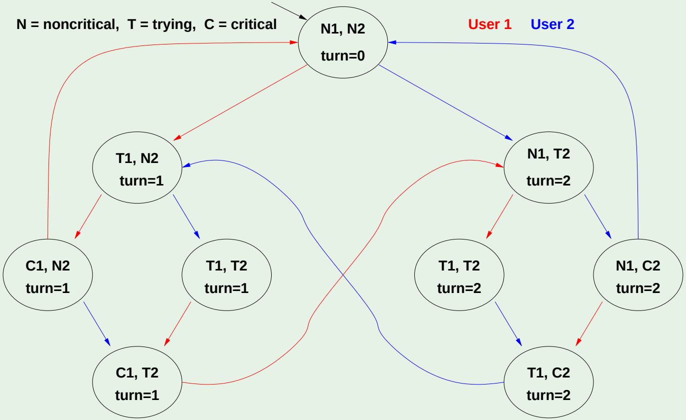
Example 3: liveness



$$M \models \mathbf{G}(T_1 \rightarrow \mathbf{F} C_1)\ ?$$

YES: every path starting from each state where T_1 holds passes through a state where C_1 holds.

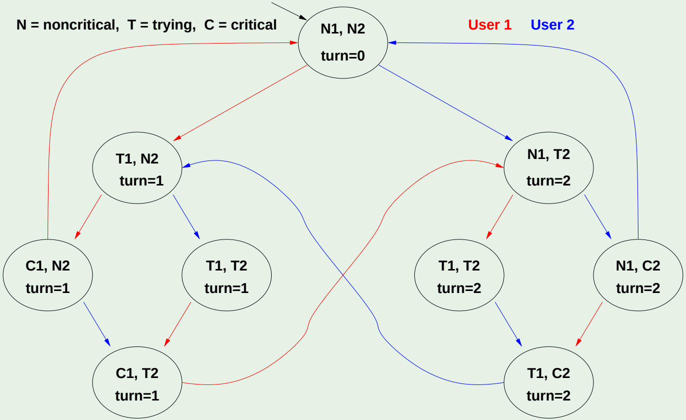
Example 4: fairness



$M \models \mathbf{GF}C_1$?

NO: e.g., in the initial state, there is an infinite cyclic solution in which C_1 never holds!

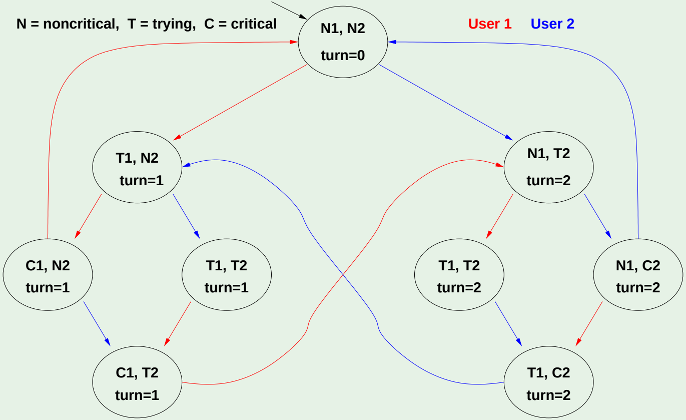
Example 5: strong fairness



$$M \models \mathbf{GF}\,T_1 \rightarrow \mathbf{GF}\,C_1 \ ?$$

YES: every path which visits T_1 infinitely often also visits C_1 infinitely often (see liveness property of previous example).

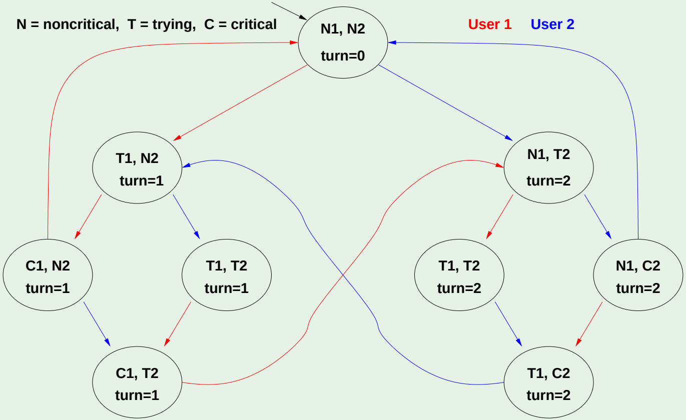
Example 6: blocking



$$M \models \mathbf{G}(N_1 \rightarrow \mathbf{F}\, T_1)\ ?$$

NO: e.g., in the initial state, there is an infinite cyclic solution in which N_1 holds and T_1 never holds!

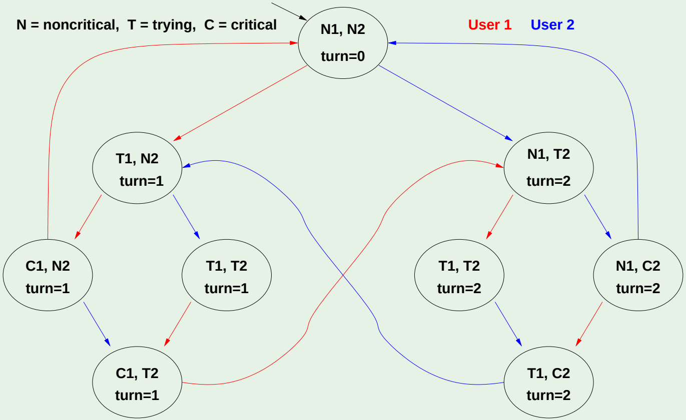
Example 7: Releases



$$M \models T_1 \mathbf{R} \neg C_1 \ ?$$

YES: C_1 in paths only strictly after T_1 has occured.

Example 8: **XF**



$M \models \mathbf{XF}(turn = 0)$?

NO: a counter-example is the ∞-shaped loop:
$(N1, N2), \{(T1, N2), (C1, N2), (C1, T2), (N1, T2), (N1, C2), (T1, C2)\}^{\omega}$

Exercise: $\mathbf{G}(T \to \mathbf{F}C)$ vs. $\mathbf{GF}T \to \mathbf{GF}C$

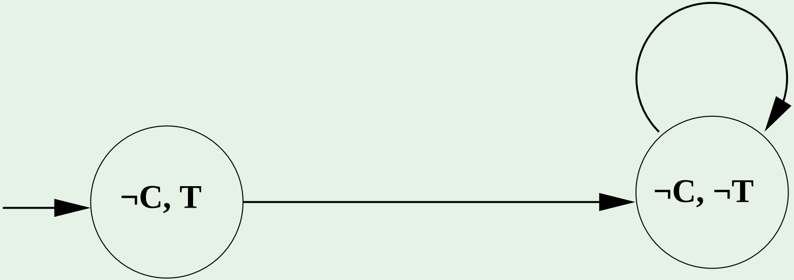
- Prove that $\mathbf{G}(T \to \mathbf{F}C) \implies \mathbf{GF}T \to \mathbf{GF}C$, or produce a counterexample
- Prove that $\mathbf{GF}T \to \mathbf{GF}C \implies \mathbf{G}(T \to \mathbf{F}C)$, or produce a counterexample

Example: $\mathbf{G}(T \rightarrow \mathbf{F}C)$ vs. $\mathbf{GF}T \rightarrow \mathbf{GF}C$

- $\mathbf{G}(T \rightarrow \mathbf{F}C) \implies \mathbf{GF}T \rightarrow \mathbf{GF}C$?
- YES: if $M \models \mathbf{G}(T \rightarrow \mathbf{F}C)$, then $M \models \mathbf{GF}T \rightarrow \mathbf{GF}C$!
- let $M \models \mathbf{G}(T \rightarrow \mathbf{F}C)$.
  let $\pi \in M$ s.t. $\pi \models \mathbf{GF}T$
  $\implies \pi, s_i \models \mathbf{F}T$ for each $s_i \in \pi$
  $\implies \pi, s_j \models T$ for each $s_i \in \pi$ and for some $s_j \in \pi$ $s.t.j \geq i$
  $\implies \pi, s_j \models FC$ for each $s_i \in \pi$ and for some $s_j \in \pi$ $s.t.j \geq i$
  $\implies \pi, s_k \models C$ for each $s_i \in \pi$, for some $s_j \in \pi$ $s.t.j \geq i$ and for some $k \geq j$
  $\implies \pi, s_k \models C$ for each $s_i \in \pi$ and for some $k \geq i$
  $\implies \pi \models \mathbf{GF}C$
  $\implies M \models \mathbf{GF}T \rightarrow \mathbf{GF}C$.

Example: $\mathbf{G}(T \rightarrow \mathbf{F}C)$ vs. $\mathbf{GF}T \rightarrow \mathbf{GF}C$

- $\mathbf{G}(T \rightarrow \mathbf{F}C) \Longleftarrow \mathbf{GF}T \rightarrow \mathbf{GF}C$?
- NO!.
- Counter example:



  - $\mathbf{GF}T \rightarrow \mathbf{GF}C$ is satisfied
  - $\mathbf{G}(T \rightarrow \mathbf{F}C)$ is not satisfied

(Counter-example proposed by the student Vaishak Belle, 2008)

Outline

Outline

Computational Tree Logic (CTL): Syntax

- An atomic proposition is a CTL formula;
- if φ_1 and φ_2 are CTL formulae, then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$ are CTL formulae;
- if φ_1 and φ_2 are CTL formulae, then $\mathbf{AX}\varphi_1$, $\mathbf{A}(\varphi_1\mathbf{U}\varphi_2)$, $\mathbf{AG}\varphi_1$, $\mathbf{AF}\varphi_1$, $\mathbf{EX}\varphi_1$, $\mathbf{E}(\varphi_1\mathbf{U}\varphi_2)$, $\mathbf{EG}\varphi_1$, $\mathbf{EF}\varphi_1$,, are CTL formulae.
  ($\mathbf{E}(\varphi_1\mathbf{R}\varphi_2)$ and $\mathbf{A}(\varphi_1\mathbf{R}\varphi_2)$ never used in practice.)

CTL semantics: intuitions

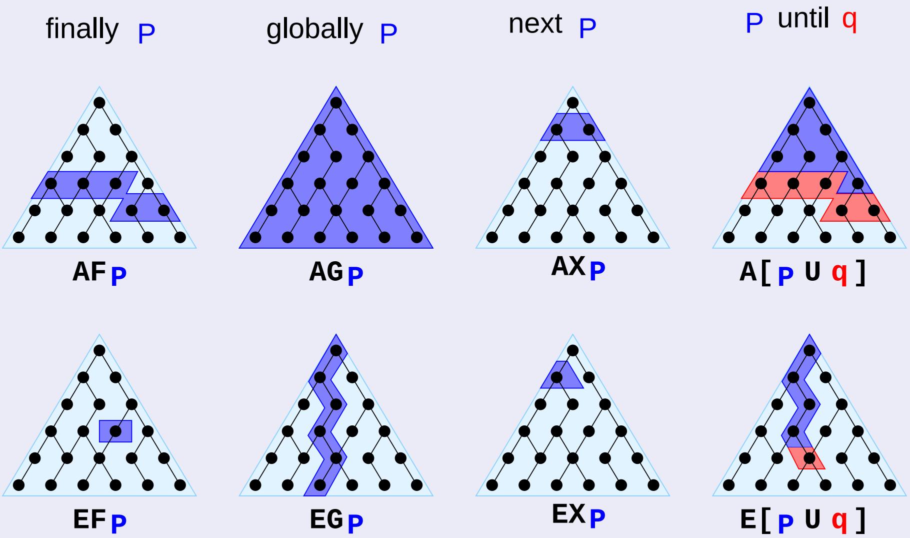
CTL is given by the standard boolean logic enhanced with the operators **AX**, **AG**, **AF**, **AU**, **EX**, **EG**, **EF**, **EU**:

- "Necessarily Next" **AX**: **AX**φ is true in s_t iff φ is true in every successor state s_{t+1}
- "Possibly Next" **EX**: **EX**φ is true in s_t iff φ is true in one successor state s_{t+1}
- "Necessarily in the future" (or "Inevitably") **AF**: **AF**φ is true in s_t iff φ is inevitably true in **some** $s_{t'}$ with $t' \geq t$
- "Possibly in the future" (or "Possibly") **EF**: **EF**φ is true in s_t iff φ may be true in **some** $s_{t'}$ with $t' \geq t$

CTL semantics: intuitions [cont.]

- "Globally" (or "always") **AG**: **AG**φ is true in s_t iff φ is true in **all** $s_{t'}$ with $t' \geq t$
- "Possibly henceforth" **EG**: **EG**φ is true in s_t iff φ is possibly true henceforth
- "Necessarily Until" **AU**: **A**(φ**U**ψ) is true in s_t iff necessarily φ holds until ψ holds.
- "Possibly Until" **EU**: **E**(φ**U**ψ) is true in s_t iff possibly φ holds until ψ holds.

CTL semantics: intuitions [cont.]



finally p | globally p | next p | p until q

AF p

AG p

AX p

A[p U q]

EF p

EG p

EX p

E[p U q]

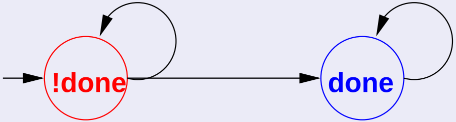
CTL Formal Semantics

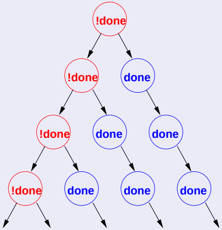
Let $(s_i, s_{i+1}, \ldots)$ be a path outgoing from state s_i in M

$$\begin{array}{llllll}
M, s_i & \models & a & \text{iff} & a \in L(s_i) & \\
M, s_i & \models & \neg\varphi & \text{iff} & M, s_i \not\models \varphi & \\
M, s_i & \models & \varphi \vee \psi & \text{iff} & M, s_i \models \varphi \ \mathit{or} & \\
 & & & & M, s_i \models \psi & \\
M, s_i & \models & AX\varphi & \text{iff} & \mathit{for\ all}\ (s_i, s_{i+1}, \ldots), & M, s_{i+1} \models \varphi \\
M, s_i & \models & EX\varphi & \text{iff} & \mathit{for\ some}\ (s_i, s_{i+1}, \ldots), & M, s_{i+1} \models \varphi \\
M, s_i & \models & AG\varphi & \text{iff} & \mathit{for\ all}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ all}\ j \geq i.M, s_j \models \varphi \\
M, s_i & \models & EG\varphi & \text{iff} & \mathit{for\ some}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ all}\ j \geq i.M, s_j \models \varphi \\
M, s_i & \models & AF\varphi & \text{iff} & \mathit{for\ all}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ some}\ j \geq i.M, s_j \models \varphi \\
M, s_i & \models & EF\varphi & \text{iff} & \mathit{for\ some}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ some}\ j \geq i.M, s_j \models \varphi \\
M, s_i & \models & A(\varphi U \psi) & \text{iff} & \mathit{for\ all}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ some}\ j \geq i. \\
 & & & & & (M, s_j \models \psi \ \ \mathit{and} \\
 & & & & & \mathit{forall}\ k\ s.t.\ i \leq k < j.M, s_k \models \varphi) \\
M, s_i & \models & E(\varphi U \psi) & \text{iff} & \mathit{for\ some}\ (s_i, s_{i+1}, \ldots), & \mathit{for\ some}\ j \geq i. \\
 & & & & & (M, s_j \models \psi \ \ \mathit{and} \\
 & & & & & \mathit{forall}\ k\ s.t.\ i \leq k < j.M, s_k \models \varphi)
\end{array}$$

Formal Semantics (cont.)

- CTL properties (e.g. **AF***done*) are evaluated over trees.





- Every temporal operator (**F**, **G**, **X**, **U**) is preceded by a path quantifier (**A** or **E**).
- Universal modalities (**AF**, **AG**, **AX**, **AU**): the temporal formula is true in **all** the paths starting in the current state.
- Existential modalities (**EF**, **EG**, **EX**, **EU**): the temporal formula is true in **some** path starting in the current state.

The CTL model checking problem $\mathcal{M} \models \phi$

The CTL model checking problem $\mathcal{M} \models \phi$

$\mathcal{M}, s \models \phi$ for every initial state $s \in I$ of the Kripke structure

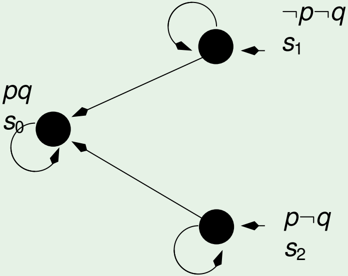
Important Remark

$\mathcal{M} \not\models \phi \not\Longrightarrow \mathcal{M} \models \neg\phi$ (!!)

- E.g. if ϕ is a universal formula **A**... and two initial states s_0, s_1 are s.t. $\mathcal{M}, s_0 \models \phi$ and $\mathcal{M}, s_1 \not\models \phi$
- $\mathcal{M} \not\models \phi \Longrightarrow \mathcal{M} \models \neg\phi$ if $\mathcal{M}$ has only one initial state

Example: $\mathcal{M} \not\models \phi \not\Longrightarrow \mathcal{M} \models \neg\phi$

- $\mathcal{M} \not\models \mathbf{AG}p$, in fact:
  - $\mathcal{M}, s_1 \not\models \mathbf{AG}p$
    (e.g., $\{s_1, ...\}$ is a counter-example)
  - $\mathcal{M}, s_2 \models \mathbf{AG}p$
- $\mathcal{M} \not\models \neg\mathbf{AG}p$, in fact:
  - $\mathcal{M}, s_1 \models \neg\mathbf{AG}p$
    (i.e., $\mathcal{M}, s_1 \models \mathbf{EF}\neg p$)
  - $\mathcal{M}, s_2 \not\models \neg\mathbf{AG}p$
    (i.e., $\mathcal{M}, s_2 \not\models \mathbf{EF}\neg p$)



Syntactic properties of CTL operators

$$\begin{array}{lcl}
\varphi_1 \vee \varphi_2 & \Longleftrightarrow & \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\
\dots & & \\
\mathbf{A}(\varphi_1 \mathbf{U} \varphi_2) & \Longleftrightarrow & \neg\mathbf{E}(\neg\varphi_2 \mathbf{U}(\neg\varphi_1 \wedge \neg\varphi_2)) \wedge \neg\mathbf{EG}\neg\varphi_2 \\
\mathbf{EF}\,\varphi_1 & \Longleftrightarrow & \mathbf{E}(\top \mathbf{U} \varphi_1) \\
\mathbf{AG}\varphi_1 & \Longleftrightarrow & \neg\mathbf{EF}\neg\varphi_1 \\
\mathbf{AF}\,\varphi_1 & \Longleftrightarrow & \neg\mathbf{EG}\neg\varphi_1 \\
\mathbf{AX}\varphi_1 & \Longleftrightarrow & \neg\mathbf{EX}\neg\varphi_1
\end{array}$$

Note

CTL can be defined in terms of $\wedge$, $\neg$, **EX**, **EG**, **EU** only

Exercise:

prove that $\mathbf{A}(\varphi_1 \mathbf{U} \varphi_2) \Longleftrightarrow \neg\mathbf{EG}\neg\varphi_2 \wedge \neg\mathbf{E}(\neg\varphi_2 \mathbf{U}(\neg\varphi_1 \wedge \neg\varphi_2))$

Strength of CTL operators

- $\mathbf{A[OP]}\varphi \models \mathbf{E[OP]}\varphi$, s.t. $\mathbf{[OP]} \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U}\}$
- $\mathbf{AG}\varphi \models \varphi \models \mathbf{AF}\varphi$, $\mathbf{EG}\varphi \models \varphi \models \mathbf{EF}\varphi$
- $\mathbf{AG}\varphi \models \mathbf{AX}\varphi \models \mathbf{AF}\varphi$, $\mathbf{EG}\varphi \models \mathbf{EX}\varphi \models \mathbf{EF}\varphi$
- $\mathbf{AG}\varphi \models \mathbf{AX}...\mathbf{AX}\varphi \models \mathbf{AF}\varphi$, $\mathbf{EG}\varphi \models \mathbf{EX}...\mathbf{EX}\varphi \models \mathbf{EF}\varphi$
- $\mathbf{A}(\varphi\mathbf{U}\psi) \models \mathbf{AF}\psi$, $\mathbf{E}(\varphi\mathbf{U}\psi) \models \mathbf{EF}\psi$

CTL tableaux rules

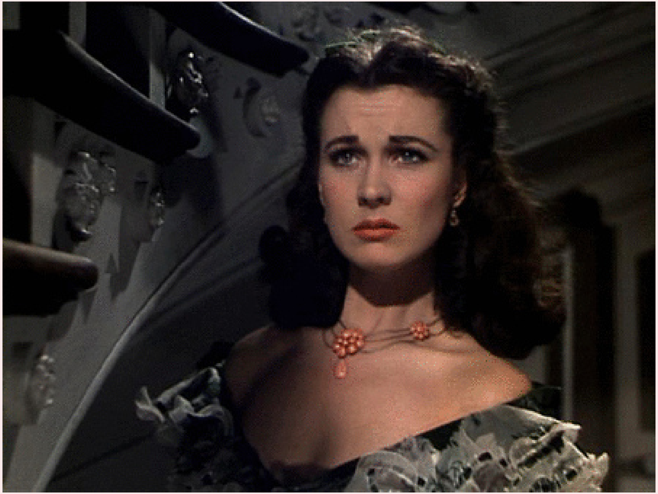
- Let φ_1 and φ_2 be CTL formulae:

$$\begin{array}{rcl}
\mathbf{AF}\varphi_1 & \Longleftrightarrow & (\varphi_1 \vee \mathbf{AXAF}\varphi_1) \\
\mathbf{AG}\varphi_1 & \Longleftrightarrow & (\varphi_1 \wedge \mathbf{AXAG}\varphi_1) \\
\mathbf{A}(\varphi_1 \mathbf{U} \varphi_2) & \Longleftrightarrow & (\varphi_2 \vee (\varphi_1 \wedge \mathbf{AXA}(\varphi_1 \mathbf{U} \varphi_2))) \\
\mathbf{EF}\varphi_1 & \Longleftrightarrow & (\varphi_1 \vee \mathbf{EXEF}\varphi_1) \\
\mathbf{EG}\varphi_1 & \Longleftrightarrow & (\varphi_1 \wedge \mathbf{EXEG}\varphi_1) \\
\mathbf{E}(\varphi_1 \mathbf{U} \varphi_2) & \Longleftrightarrow & (\varphi_2 \vee (\varphi_1 \wedge \mathbf{EXE}(\varphi_1 \mathbf{U} \varphi_2)))
\end{array}$$

- Recursive definitions of **AF**, **AG**, **AU**, **EF**, **EG**, **EU**.
- If applied recursively, rewrite a CTL formula in terms of atomic, **AX**- and **EX**-formulas:

$$\mathbf{A}(p\mathbf{U}q) \wedge (\mathbf{EG}\neg p) \Longrightarrow (q \vee (p \wedge \mathbf{AXA}(p\mathbf{U}q))) \wedge (\neg p \wedge \mathbf{EXEG}\neg p)$$

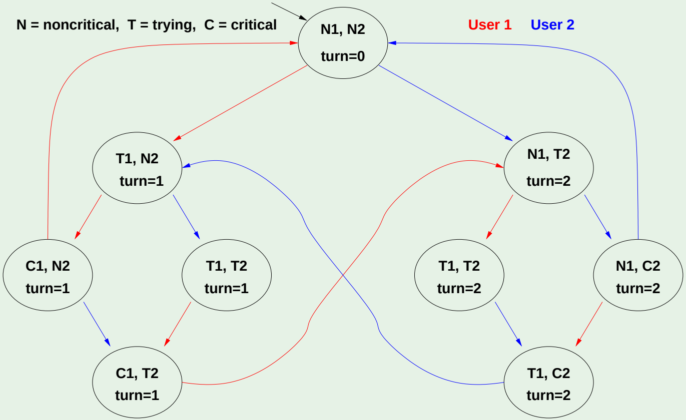
Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

Outline

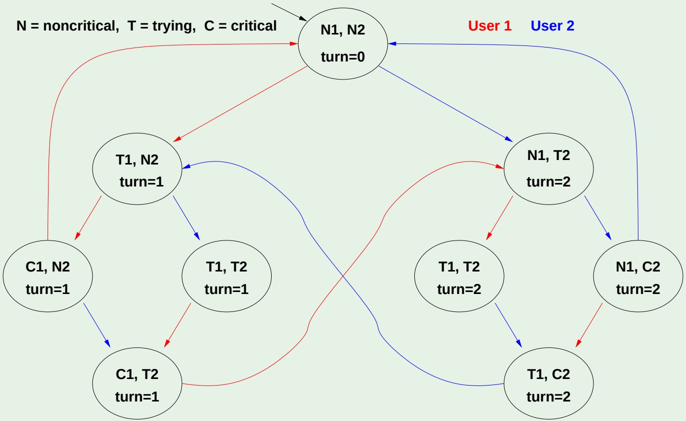
Example 1: mutual exclusion (safety)



$M \models \mathbf{AG}\neg(C_1 \wedge C_2)$?

YES: There is no reachable state in which $(C_1 \wedge C_2)$ holds!
(Same as the $\mathbf{G}\neg(C_1 \wedge C_2)$ in LTL.)

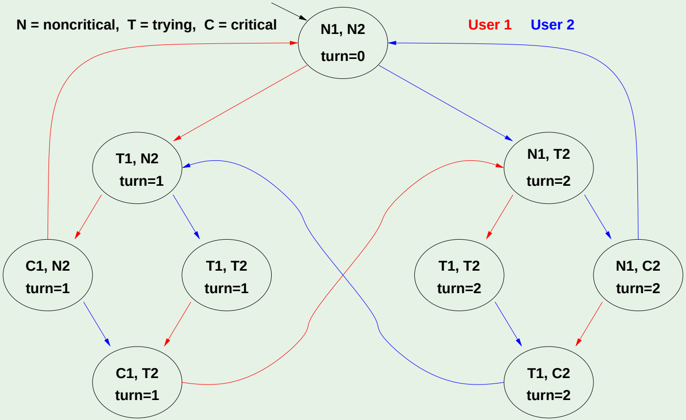
Example 2: liveness



$M \models \mathbf{AF}\, C_1$?

No: there is an infinite cyclic solution in which C_1 never holds!
(Same as $\mathbf{F} C_1$ in LTL.)

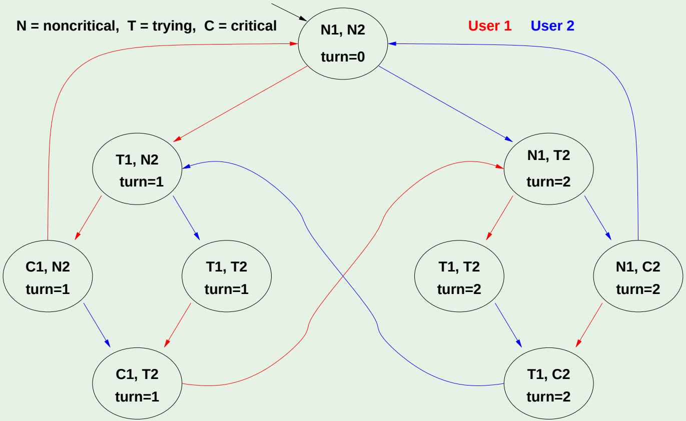
Example 3: liveness



$$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF}\, C_1)\ ?$$

YES: every path starting from each state where T_1 holds passes through a state where C_1 holds (Same as $\mathbf{G}(T_1 \rightarrow \mathbf{F} C_1)$ in LTL.)

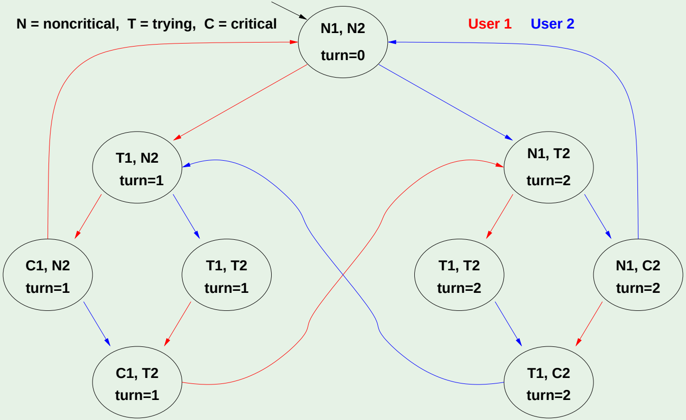
Example 4: fairness



$M \models \mathbf{AGAF} C_1$?

NO: e.g., in the initial state, there is an infinite cyclic solution in which C_1 never holds!
(Same as $\mathbf{GF} C_1$ in LTL.)

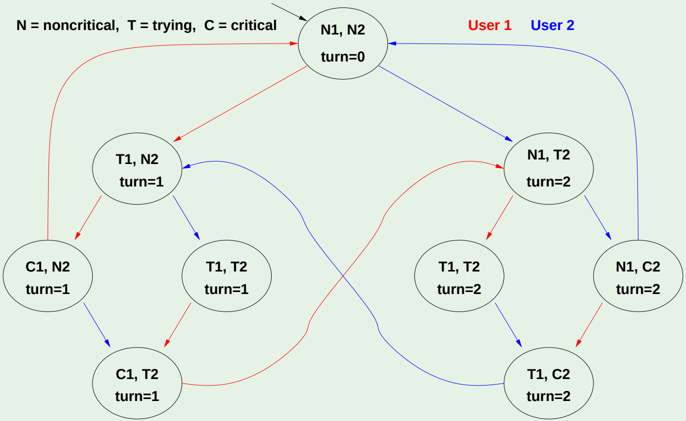
Example 5: fairness (2)



$M \models \mathbf{AGAF}(turn = 0)$?

NO: there is an infinite 8-shaped cyclic solution in which $(turn = 0)$ never holds!

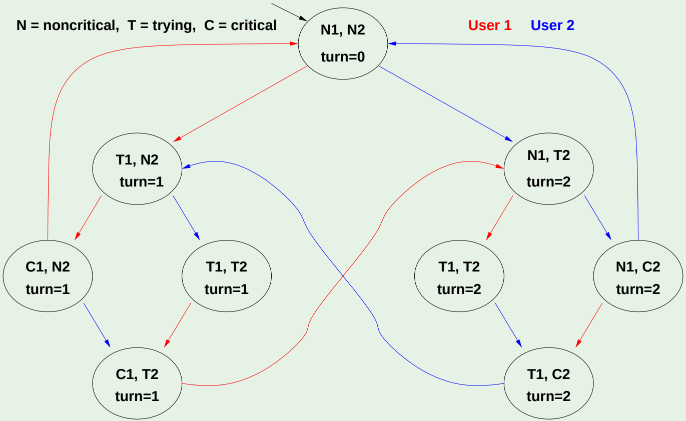
Example 6: blocking



$$M \models \mathbf{AG}(N_1 \to \mathbf{EF}\, T_1) \text{ ?}$$

YES: from each state where N_1 holds there is a path leading to a state where T_1 holds
(No corresponding LTL formula.)

Example 7: blocking (2)

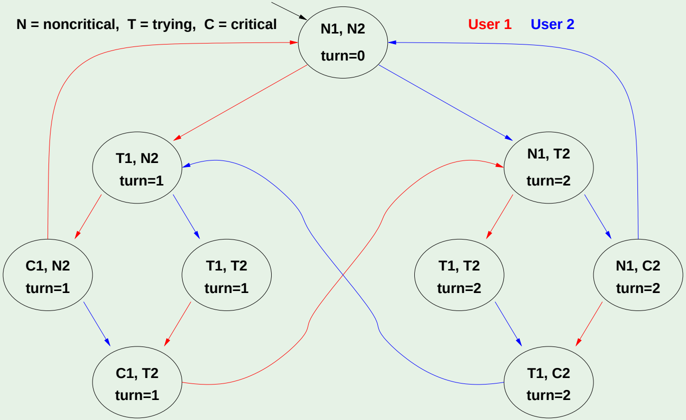


$$M \models \mathbf{AG}(N_1 \rightarrow \mathbf{AF}\ T_1)\ ?$$

NO: e.g., in the initial state, there is an infinite cyclic solution in which N_1 holds and T_1 never holds!

(Same as LTL formula $\mathbf{G}(N_1 \rightarrow \mathbf{F}\, T_1)$.)

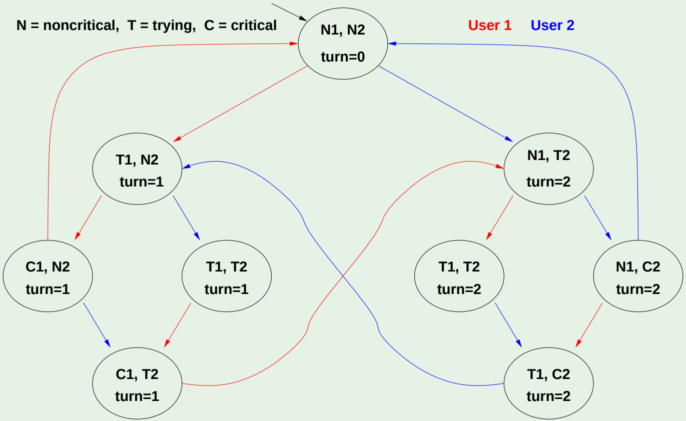
Example 8:



$$M \models \mathbf{EG} N_1 \ ?$$

YES: there is an infinite cyclic solution where N_1 always holds
(No corresponding LTL formula.)

Example 9:



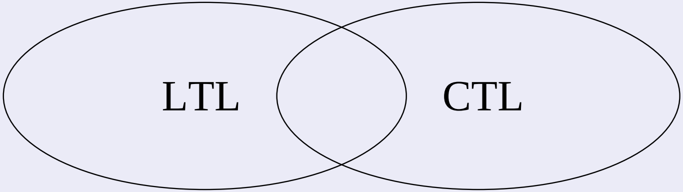
$M \models \mathbf{AFEG} N_1$?

YES: there is an infinite cyclic solution where N_1 always holds, and from every state you necessarily reach one state of such cycle
(No corresponding LTL formula.)

Outline

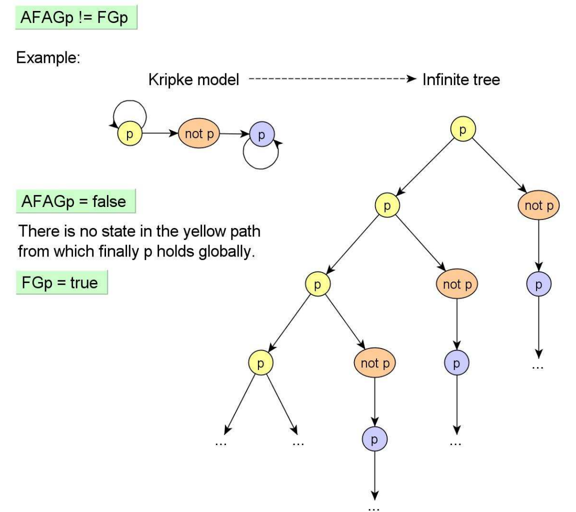
LTL vs. CTL: expressiveness

- Many CTL formulas cannot be expressed in LTL (e.g., those containing existentially quantified subformulas) E.g., $\mathbf{AG}(N_1 \rightarrow \mathbf{EF}T_1)$, $\mathbf{AFAG}\varphi$
- Many LTL formulas cannot be expressed in CTL (e.g. fairness LTL formulas) E.g., $\mathbf{GF}T_1 \rightarrow \mathbf{GF}C_1$, $\mathbf{FG}\varphi$
- Some formulas can be expressed both in LTL and in CTL (typically LTL formulas with operators of nesting depth 1, and/or with operators occurring positively) E.g., $\mathbf{G}\neg(C_1 \wedge C_2)$, $\mathbf{F}C_1$, $\mathbf{G}(T_1 \rightarrow \mathbf{F}C_1)$, $\mathbf{GF}C_1$



Example: *AFAGp* vs. *FGp*

(Example developed by the students Andrea Mattioli and Mirko Boniatti, 2005.)



AFAGp != FGp

Example:

AFAGp = false

There is no state in the yellow path from which finally p holds globally.

FGp = true

LTL vs. CTL: M.C. Algorithms

- LTL M.C. problems are typically handled with automata- based M.C. approaches (Wolper & Vardi)
- CTL M.C. problems are typically handled with symbolic M.C. approaches (Clarke & McMillan)
- LTL M.C. problems can be reduced to CTL M.C. problems under fairness constraints (Clarke et al.)

CTL*

- Syntax: let p's, φ's, ψ's being propositions, state formulae and path formulae respectively:
  - p, $\neg\varphi$, $\varphi_1 \land \varphi_2$, $\mathbf{A}\psi$, $\mathbf{E}\psi$ are state formulae
    (properties of the set of paths starting from a state)
  - φ, $\neg\psi$, $\psi_1 \land \psi_2$, $\mathbf{X}\psi$, $\mathbf{G}\psi$, $\mathbf{F}\psi$, $\psi_1\mathbf{U}\psi_2$ are path formulae
    (properties of a path)
- Semantics: **A**, **E**, **X**, **G**, **F**, **U** as in CTL
  - **A**, **E**: quantify on paths (as in CTL)
  - **X**, **G**, **F**, **U**: (as in LTL)
  - as in CTL, but **X**, **G**, **F**, **U** not necessarily preceded by **A**,**E**

Remark

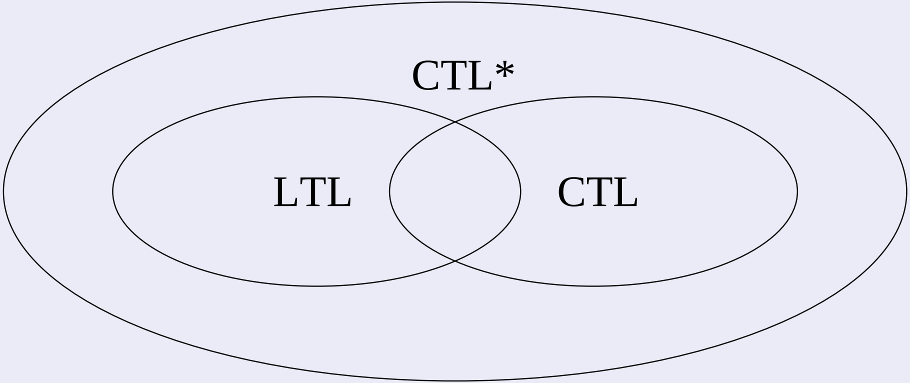
In principle in CTL* one may have sequences of nested path quantifiers.
In such case, the most internal one dominates:

$$M, s \models \mathbf{AE}\psi \textit{ iff } M, s \models \mathbf{E}\psi, \qquad M, s \models \mathbf{EA}\psi \textit{ iff } M, s \models \mathbf{A}\psi.$$

CTL* vs LTL & CTL

CTL* subsumes both CTL and LTL

- φ in CTL $\Longrightarrow$ φ in CTL* (e.g., $\mathbf{AG}(N_1 \rightarrow \mathbf{EF}\, T_1)$)
- φ in LTL $\Longrightarrow$ $\mathbf{A}\varphi$ in CTL* (e.g., $\mathbf{A}(\mathbf{GF}\, T_1 \rightarrow \mathbf{GF}\, C_1)$)
- LTL $\cup$ CTL $\subset$ CTL* (e.g., $\mathbf{E}(\mathbf{GF}p \rightarrow \mathbf{GF}q)$)



“You have no respect for logic. (...)
I have no respect for those who have no respect for logic.”

`https://www.youtube.com/watch?v=uGstM8QMCjQ`

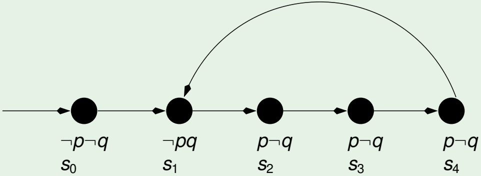
(Arnold Schwarzenegger in “Twins”)

Outline

1. Transition Systems as Kripke Models
   - Kripke Models
   - Languages for Transition Systems (hints)
2. Properties and Temporal Logics
   - Properties
   - Temporal Logics
3. Linear Temporal Logic – LTL
   - LTL: Syntax and Semantics
   - Some LTL Model Checking Examples
4. Computation Tree Logic – CTL
   - CTL: Syntax and Semantics
   - Some CTL Model Checking Examples
5. LTL vs. CTL
6. **Exercises**

Exercise: LTL Model Checking (path)

Consider the following path π:



For each of the following facts, say if it is true of false in LTL.

(*a*) $\pi, s_0 \models \mathbf{GF}q$
[Solution: true]

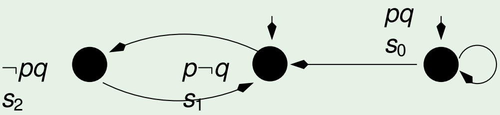
(*b*) $\pi, s_0 \models \mathbf{FG}(q \leftrightarrow \neg p)$
[Solution: true]

(*c*) $\pi, s_2 \models \mathbf{G}p$
[Solution: false]

(*d*) $\pi, s_2 \models p\mathbf{U}q$
[Solution: true]

Ex: LTL Model Checking

Consider the following Kripke Model M:



For each of the following facts, say if it is true or false in LTL.

(a) $M \models (p\mathbf{U}q)$
[Solution: true]

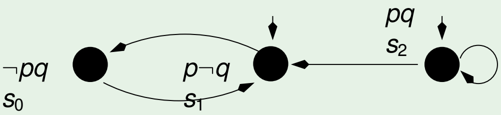
(b) $M \models \mathbf{G}(\neg p \rightarrow F\neg q)$
[Solution: true]

(c) $M \models \mathbf{G}p \rightarrow \mathbf{G}q$
[Solution: true]

(d) $M \models \mathbf{FG}p$
[Solution: false]

Ex: CTL Model Checking

Consider the following Kripke Model M:



For each of the following facts, say if it is true or false in CTL.

(a) $M \models \mathbf{AF}\neg p$
[Solution: false]

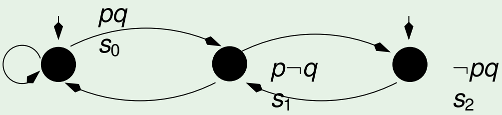
(b) $M \models \mathbf{EG}p$
[Solution: false]

(c) $M \models \mathbf{A}(p\mathbf{U}q)$
[Solution: true]

(d) $M \models \mathbf{E}(p\mathbf{U}\neg q)$
[Solution: true]

Ex: CTL Model Checking

Consider the following Kripke Model M:



For each of the following facts, say if it is true or false in CTL.

(*a*) $M \models \mathbf{AF}\neg q$
[Solution: false]

(*b*) $M \models \mathbf{EG}q$
[Solution: false]

(*c*) $M \models ((\mathbf{AGAF}p \lor \mathbf{AGAF}q) \land (\mathbf{AGAF}\neg p \lor \mathbf{AGAF}\neg q)) \rightarrow q$
[Solution: true]

(*d*) $M \models \mathbf{AFEG}(p \land q)$
[Solution: false]