Formal Methods:
Module I: Automated Reasoning
Ch. 04: **Automata-Theoretic LTL Reasoning**

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Outline

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Infinite Word Languages

Modeling infinite computations of reactive systems

Given an Alphabet Σ (e.g. $\Sigma \stackrel{\text{def}}{=} \{a, b\}$)

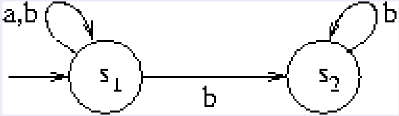
- An ω-word α over Σ is an infinite sequence
  $a_0,\ a_1,\ a_2 \ldots.$
  Formally, $\alpha : \mathbb{N} \rightarrow \Sigma$.
- The set of all infinite words is denoted by Σ^ω.
- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.
- Example: All words over $\{a, b\}$ with infinitely many a's.

Notation:

omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.
omega-languages $L, L_1 \subseteq \Sigma^\omega$
For $u \in \Sigma^+$, let $u^\omega = u.u.u \ldots$

Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\ldots$.
  There are several (infinite) possible runs.
  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
  Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \ldots$
- Acceptance Conditions: Büchi (Muller, Rabin, Street):
  Acceptance is based on states occurring infinitely often
- Notation Let $\rho \in Q^\omega$. Then,
  $$Inf(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}.\ \rho(i) = s\}.$$
  (The set of states occurring infinitely many times in ρ.)

Büchi Automata

Nondeterministic Büchi Automaton

- A Nondeterministic Büchi Automaton (NBA) is $(Q, \Sigma, \delta, I, F)$ s.t.
  - Q Finite set of states.
  - Σ is a finite alphabet
  - $I \subseteq Q$ set of initial states.
  - $F \subseteq Q$ set of accepting states.
  - $\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).
- A Deterministic Büchi Automaton (DBA) is an NBA s.t. the transition relation is functional: $\delta : Q \times \Sigma \longmapsto Q$

Runs and Language of NBAs

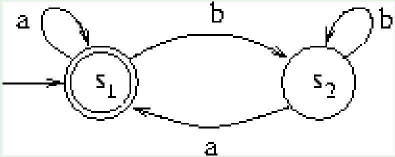
- A run ρ of A on ω-word $\alpha = a_0, a_1, a_2, \ldots$ is an infinite sequence $\rho = q_o, q_1, q_2, \ldots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
- The run ρ is accepting if

  $$Inf(\rho) \cap F \neq \emptyset.$$

- The language accepted by A
  $\mathcal{L}(A) = \{\alpha \in \Sigma^\omega \mid A \text{ has an accepting run on } \alpha\}$

Büchi Automaton: Example

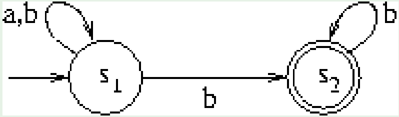
Let $\Sigma = \{a, b\}$.
Let a Deterministic Büchi Automaton (DBA) A_1 be



- With $F = \{s_1\}$ the automaton recognizes words with infinitely many a's.
- With $F = \{s_2\}$ the automaton recognizes words with infinitely many b's.

Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

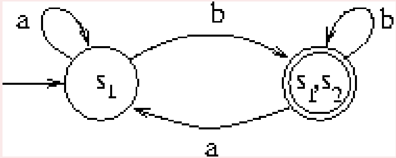
Theorem

*DBA*s are strictly less powerful than *NBA*s.

Remark:

The subset construction of standard Final-State automata does not work!

Let DA_2 be



- DA_2 is not equivalent to A_2 (e.g., it recognizes $(b.a)^\omega$)
- There is no DBA equivalent to A_2

Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$. $|A| \leq |A_1| \cdot |A_2| \cdot 2$.

Union of two NBAs

Definition: union of NBAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.
Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) \;=\; \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2
(same construction as with ordinary automata)

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
$$I = I_1 \times I_2 \times \{1\}.$$
$$F = F_1 \times Q_2 \times \{1\}.$$

$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \notin F_1$.
$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 2\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \in F_1$.
$\langle p, q, 2\rangle \xrightarrow{a} \langle p', q', 2\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \notin F_2$.
$\langle p, q, 2\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \in F_2$.

Theorem

- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

Synchronous Product of NBAs: Intuition

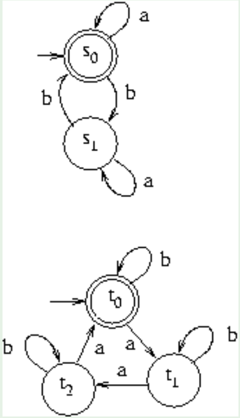
- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track

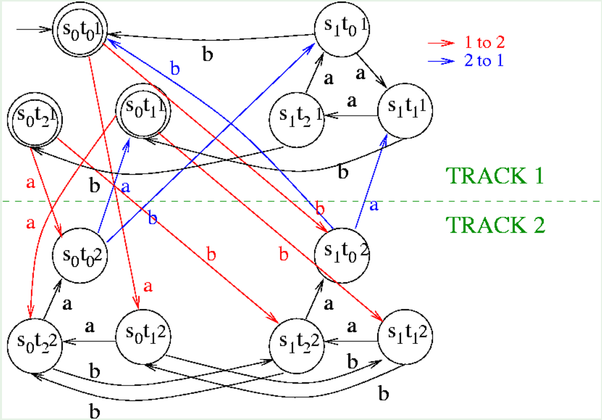
$\Longrightarrow$ to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2

- Important subcase: If $F_2 = Q_2$, then

$$Q = Q_1 \times Q_2.$$
$$I = I_1 \times I_2.$$
$$F = F_1 \times Q_2.$$

Synchronous Product of NBAs: Example





Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
$|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

(i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
(ii) determinize and Complement the Rabin automaton
(iii) convert the Rabin automaton into a Büchi automaton.

Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
The automaton remains in phase i till it visits a state in F_i. Then, it moves to $(i \; mod \; K) + 1$ mode.

De-generalization of a generalized NBA

Definition: De-generalization of a generalized NBA

Let $A \stackrel{\text{def}}{=} (Q, \Sigma, \delta, I, FT)$ a generalized BA s.f. $FT \stackrel{\text{def}}{=} \{F_1, ..., F_K\}$.
Then a language-equivalent BA $A' \stackrel{\text{def}}{=} (Q', \Sigma, \delta', I', F')$ is built as follows

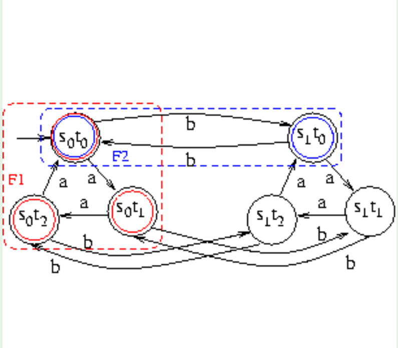
$Q' = Q_1 \times \{1, ..., K\}$.
$I' = I \times \{1\}$.
$F' = F_1 \times \{1\}$.
δ' is s.t., for every $i \in [1, ..., K]$:

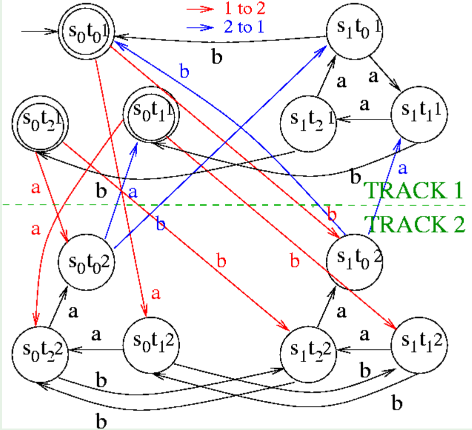
$$\langle p, i\rangle \xrightarrow{a} \langle q, i\rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \notin F_i.$$
$$\langle p, i\rangle \xrightarrow{a} \langle q, (i \ mod \ K) + 1\rangle \quad \text{iff} \quad p \xrightarrow{a} q \in \delta \quad \text{and} \quad p \in F_i.$$

Theorem

- $\mathcal{L}(A') = \mathcal{L}(A)$.
- $|A'| \leq K \cdot |A|$.

Degeneralizing a Büchi automaton: Example





Omega-regular Expressions

Definition

A language is called ω-regular if it has the form $\cup_{i=1}^{n} U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

Theorem

A language L is ω-regular iff it is NBA-recognizable.

Outline

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Automata-Theoretic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$
\begin{aligned}
& \models \psi \quad \text{(LTL)} \\
\Longleftrightarrow\ & \neg\psi \text{ unsat} \\
\Longleftrightarrow\ & \mathcal{L}(A_{\neg\psi}) = \emptyset
\end{aligned}
$$

- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

$$
\begin{aligned}
& \varphi \models \psi \quad \text{(LTL)} \\
& \models \varphi \rightarrow \psi \quad \text{(LTL)} \\
\Longleftrightarrow\ & \varphi \wedge \neg\psi \text{ unsat} \\
\Longleftrightarrow\ & \mathcal{L}(A_{\varphi\wedge\neg\psi}) = \emptyset
\end{aligned}
$$

- $A_{\varphi\wedge\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\varphi \wedge \neg\psi$ (satisfy φ and do not satisfy ψ)

Automata-Theoretic LTL Satisfiability and Entailment

Two steps for checking $\models \psi$ [resp. $\varphi \models \psi$]

(i) Compute $A_{\neg\psi}$ [resp. $A_{\varphi\wedge\neg\psi}$]

(ii) Check the emptiness of $\mathcal{L}(A_{\neg\psi})$ [resp. $\mathcal{L}(A_{\varphi\wedge\neg\psi})$]

Automata-Theoretic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$
\begin{array}{ll}
 & M \models \psi \quad \text{(LTL)} \\
\Longleftrightarrow & \mathcal{L}(M) \subseteq \underline{\mathcal{L}(\psi)} \\
\Longleftrightarrow & \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset \\
\Longleftrightarrow & \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset \\
\Longleftrightarrow & \mathcal{L}(A_M) \cap \mathcal{L}(A_{\neg\psi}) = \emptyset \\
\Longleftrightarrow & \mathcal{L}(A_M \times A_{\neg\psi}) = \emptyset
\end{array}
$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\Longrightarrow$ $A_M \times A_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ.

Automata-Theoretic LTL Model Checking

Four steps

Let $\varphi \stackrel{\text{def}}{=} \neg\psi$:

(i) Compute A_M

(ii) Compute A_φ

(iii) Compute the product $A_M \times A_\varphi$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

Outline

NBA emptiness checking

- Find an accepting cycle reachable from an initial state.
- A naive algorithm:
  - (i) a DFS finds the final states f reachable from an initial state;
  - (ii) for each f, a second DFS finds if it can reach f (i.e., if there exists a loop)

  Complexity: $O(n^2)$
- SCC-based algorithm:
  - (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
  - (ii) drop all SCCs which do not have at least one arc, and which do not contain at least one accepting state f
  - (iii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.

  Complexity: $O(n)$
- Drawbacks: it stores too much information and does not find directly a counterexample.

Double Nested DFS algorithm

Double Nested DFS

- Two nested DFSs
  - `DFS1` finds the final states f reachable from an initial state
  - for each f, `DFS2` finds if it can reach f (i.e., if there exists a loop)
- Two Hash tables:
  - `T1`: reachable states
  - `T2`: states reachable from a reachable final state
- Two stacks:
  - `S1`: current branch of states reachable
  - `S2`: current branch of states reachable from final state f
- It stops as soon as it finds a counterexample.
- The counterexample is given by
  - the stack of `DFS2` (an accepting, preceded by cycle)
  - the stack of `DFS1` (a path from an initial state to the cycle)

- DFS1 invokes DFS2 on each f_i only after popping it (postorder)
- T2 passed by reference, is not reset at each call of DFS2 !

Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
   stack S1=I; stack S2=∅;
   Hashtable T1=I; Hashtable T2=∅;
   while S1!=∅ {
      v=top(S1);
      if ∃w s.t. w∈δ(v) && T1(w)==0 {
         hash(w,T1);
         push(w,S1);
      } else {
         pop(S1);
         if (v∈F && !DFS2(v,S2,T2,A))
            return False;
   }    }
   return True;
}
```

Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
   hash(f,T);
   S = {f}
   while S!=∅ {
      v=top(S);
      if f∈δ(v) return False;
      if ∃w s.t. w∈δ(v) && T(w)==0 {
         hash(w);
         push(w);
      } else pop(S);
   }
   return True;
}
```

Remark: `T` passed by reference, is not reset at each call of `DFS2` !

Double nested DFS: Intuition

DFS1 invokes DFS2 on each $f_1, ..., f_n$ only after popping it (postorder):

- suppose *DFS2* is invoked on f_j before than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_i, ...)$ it is encountered a state S which has already been explored by $DFS2(f_j, ...)$ for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

$\Longrightarrow$ It is safe to backtrack!

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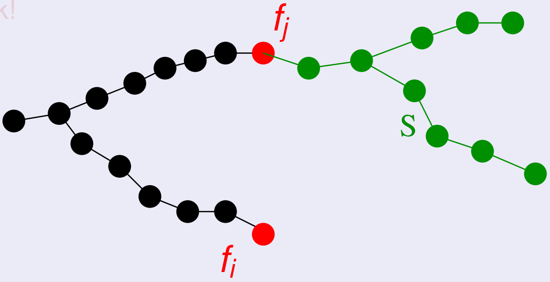

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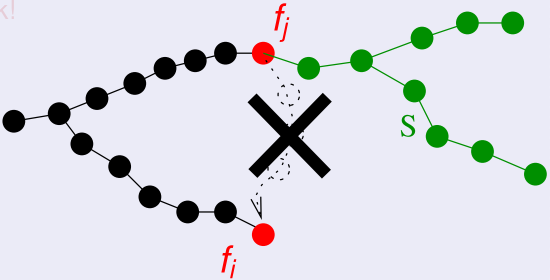

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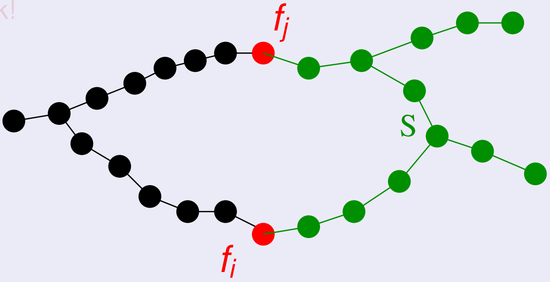

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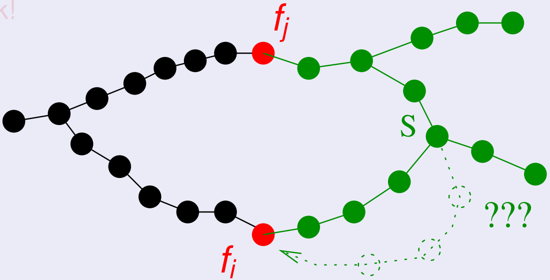

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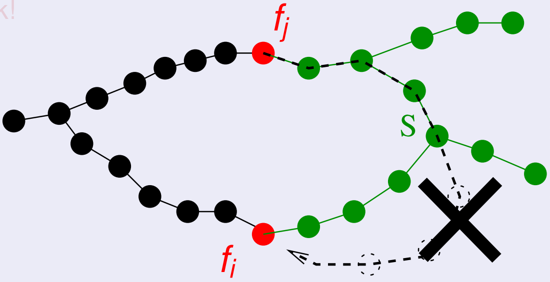

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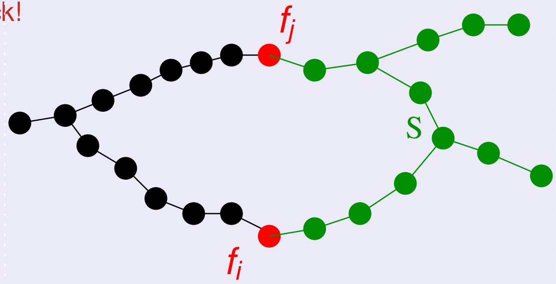
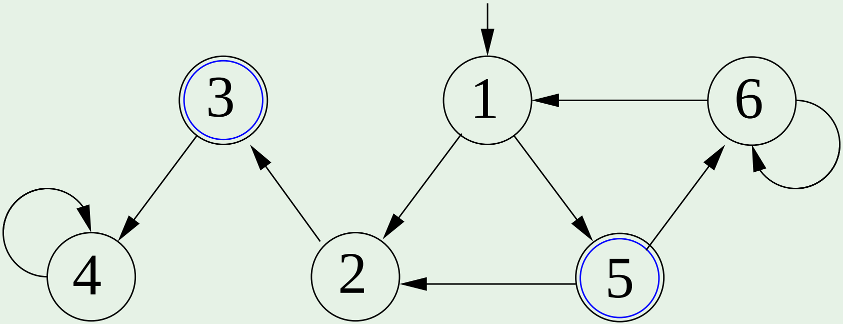


Double Nested DFS: example



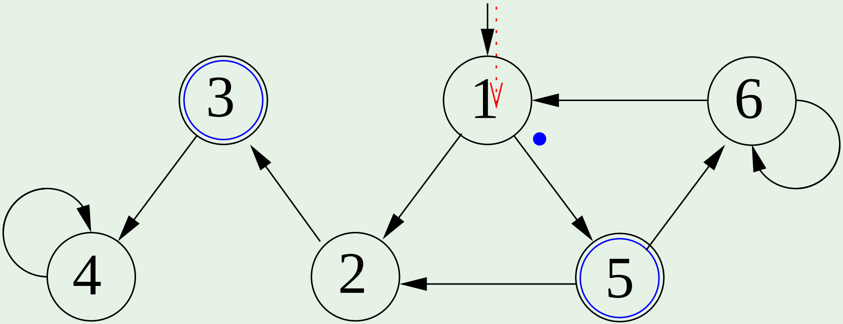
T1

S1

T2

S2

Double Nested DFS: example



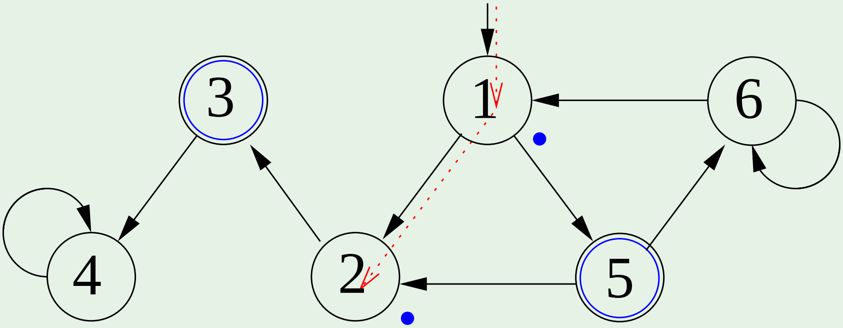
T1 1

S1 1

T2

S2

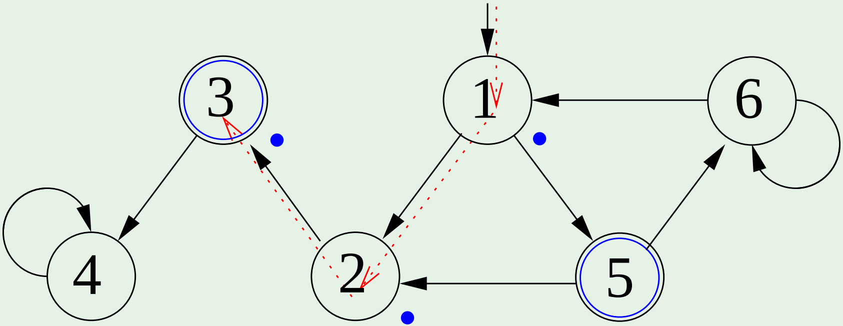
Double Nested DFS: example



T1 1 2 T2

S1 1 2 S2

Double Nested DFS: example



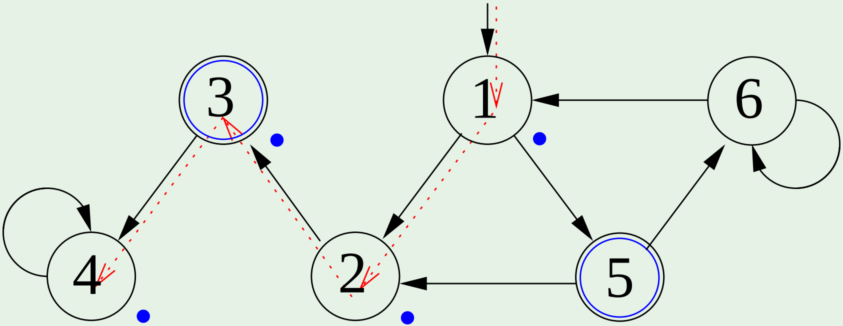
T1 1 2 3

S1 1 2 3

T2

S2

Double Nested DFS: example



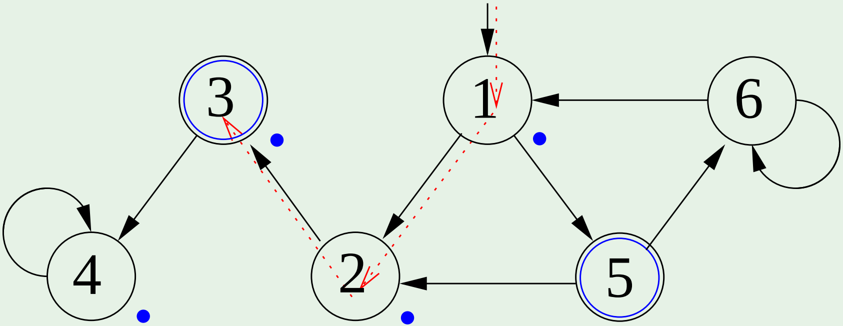
T1 1 2 3 4

S1 1 2 3 4

T2

S2

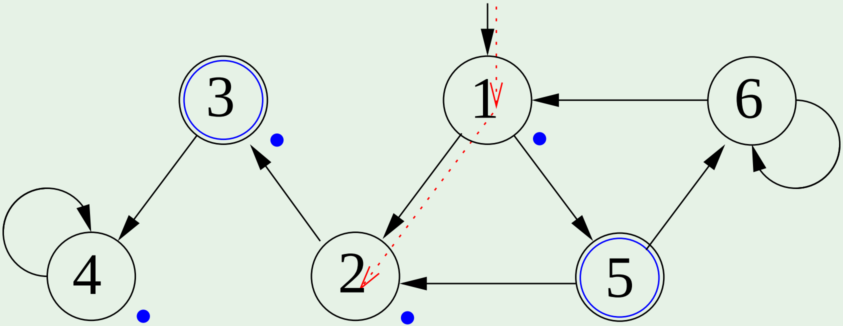
Double Nested DFS: example



T1 1 2 3 4 T2

S1 1 2 3 S2

Double Nested DFS: example



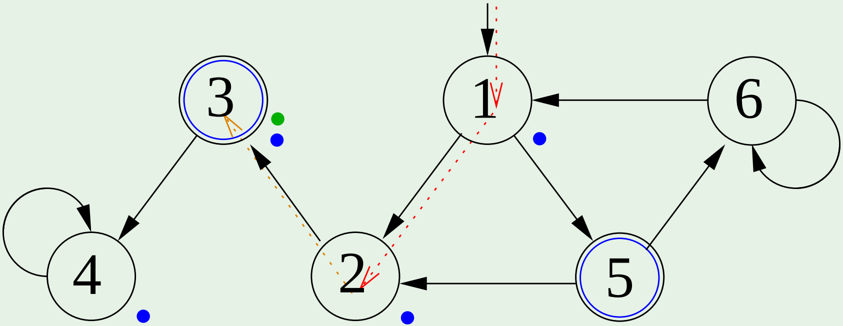
T1 1 2 3 4

S1 1 2

T2

S2

Double Nested DFS: example



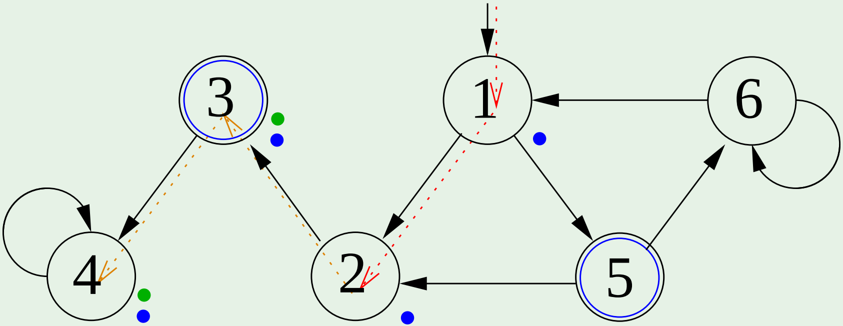
T1 1 2 3 4

S1 1 2

T2 3

S2 3

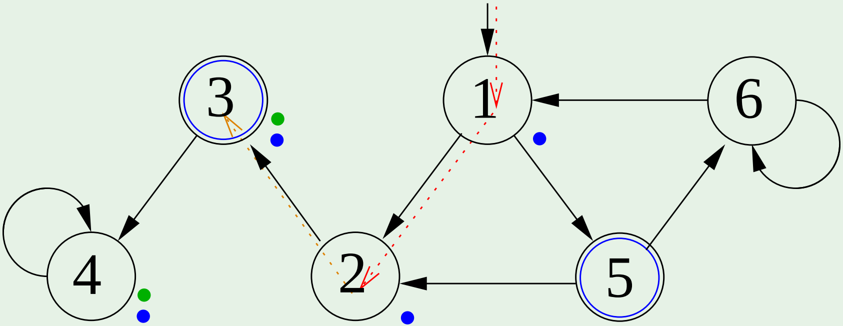
Double Nested DFS: example



T1 1 2 3 4　　T2 3 4

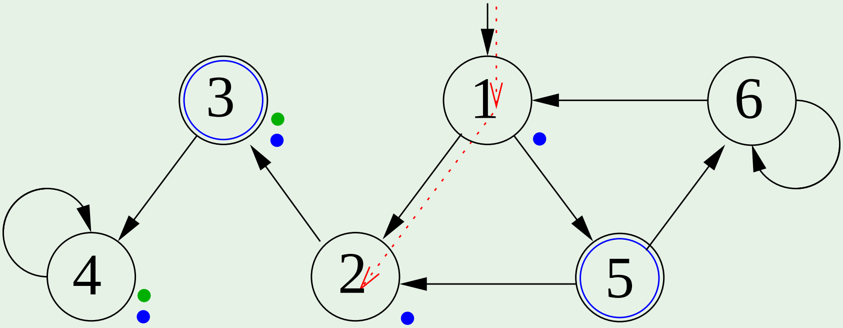
S1 1 2　　S2 3 4

Double Nested DFS: example



T1 1 2 3 4
T2 3 4
S1 1 2
S2 3

Double Nested DFS: example



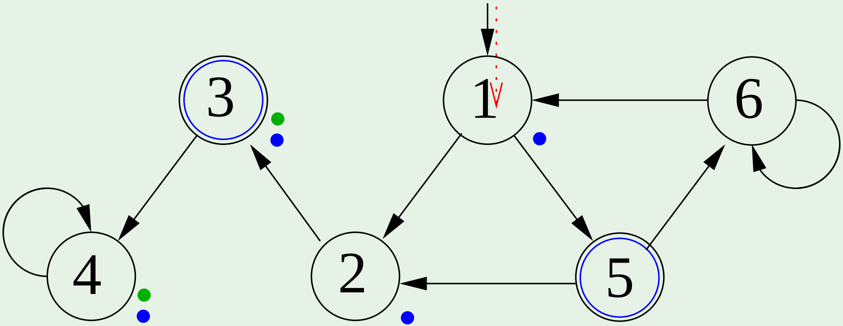
T1 1 2 3 4

S1 1 2

T2 3 4

S2

Double Nested DFS: example



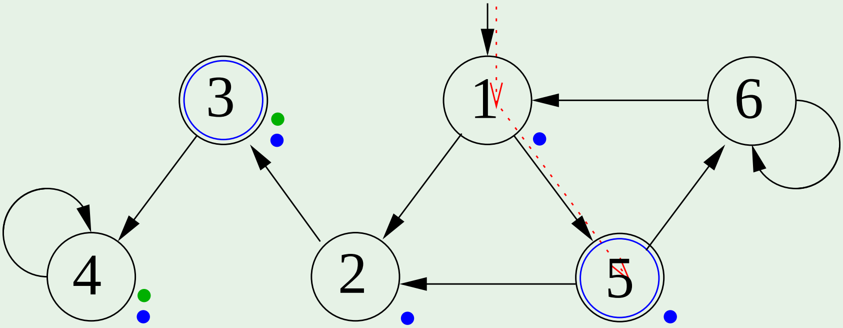
T1 1 2 3 4

S1 1

T2 3 4

S2

Double Nested DFS: example



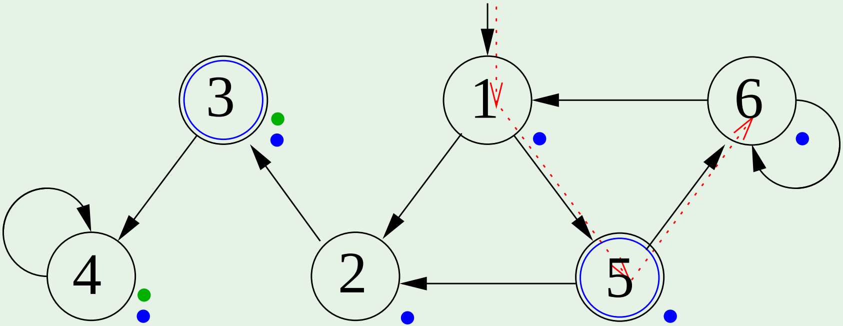
T1 1 2 3 4 5

S1 1 5

T2 3 4

S2

Double Nested DFS: example



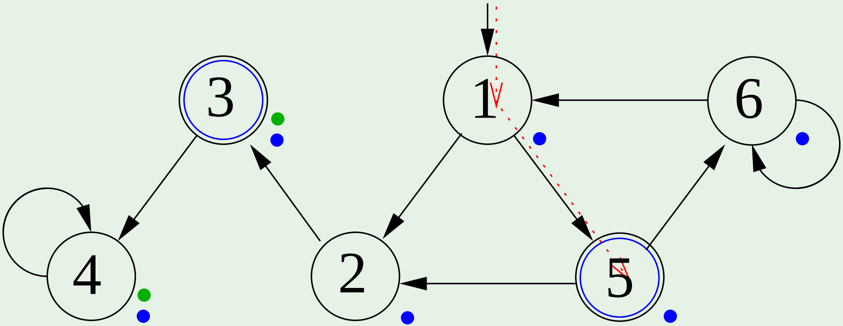
T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

Double Nested DFS: example



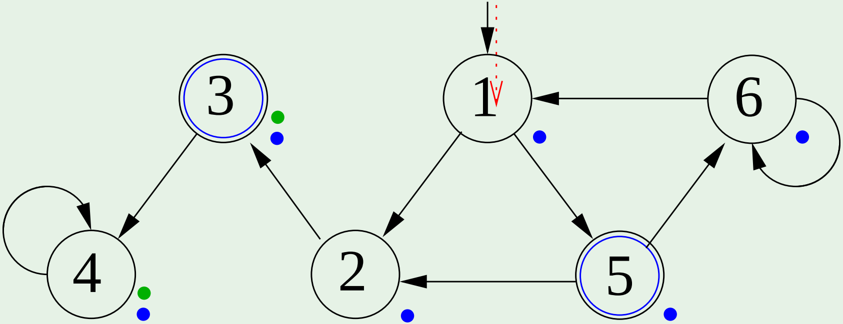
T1 1 2 3 4 5 6

T2 3 4

S1 1 5

S2

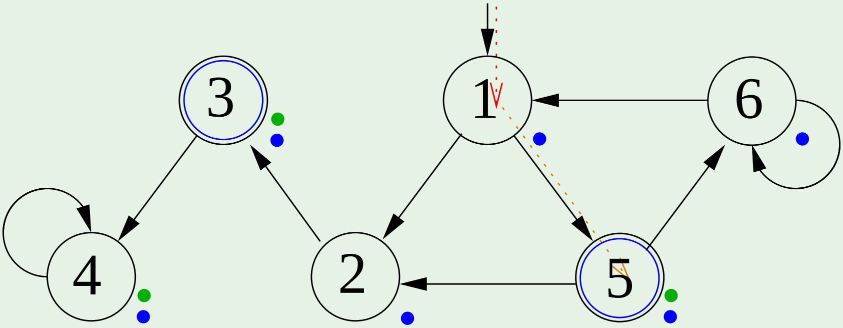
Double Nested DFS: example



T1 1 2 3 4 5 6 T2 3 4

S1 1 S2

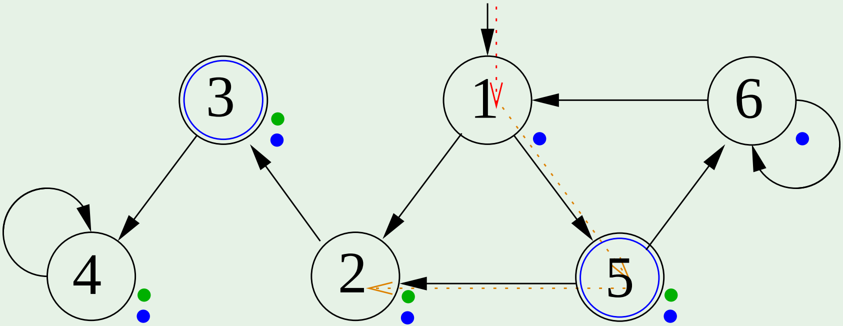
Double Nested DFS: example



T1 1 2 3 4 5 6 T2 3 4 5

S1 1 S2 5

Double Nested DFS: example



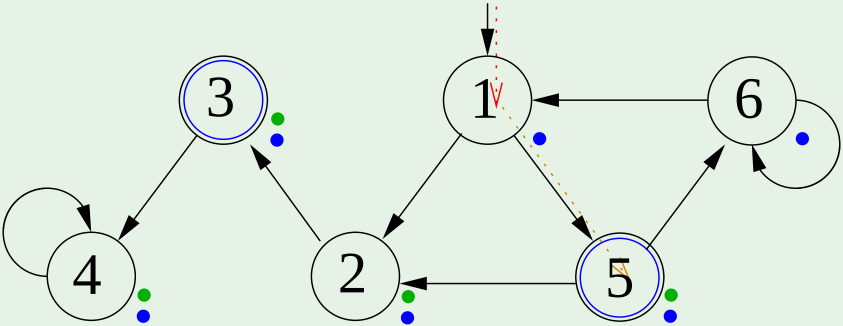
T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2

S2 5 2

Double Nested DFS: example



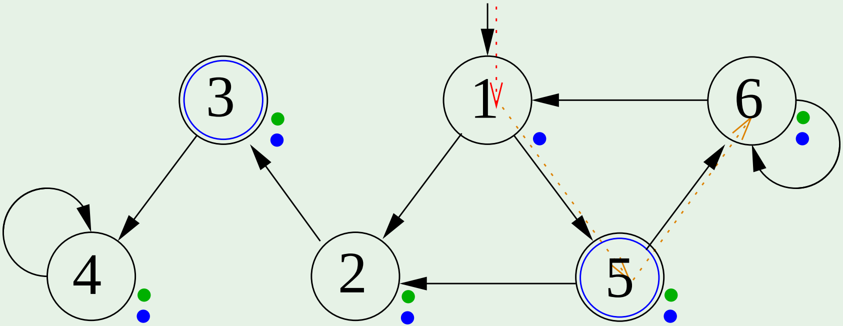
T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2

S2 5

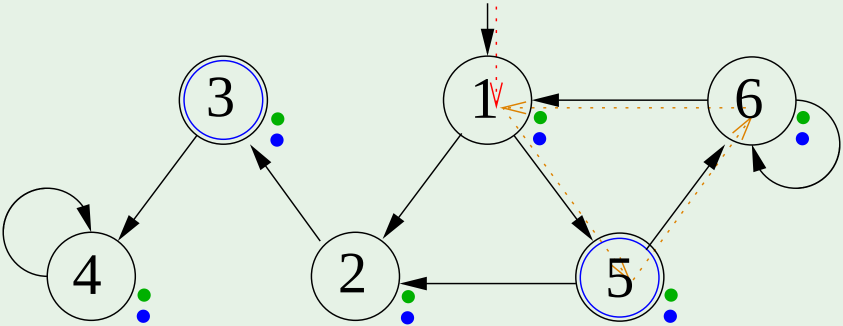
Double Nested DFS: example



T1 1 2 3 4 5 6   T2 3 4 5 2 6

S1 1   S2 5 6

Double Nested DFS: example



T1 1 2 3 4 5 6 T2 3 4 5 2 6 1

S1 1 S2 5 6 1

Outline

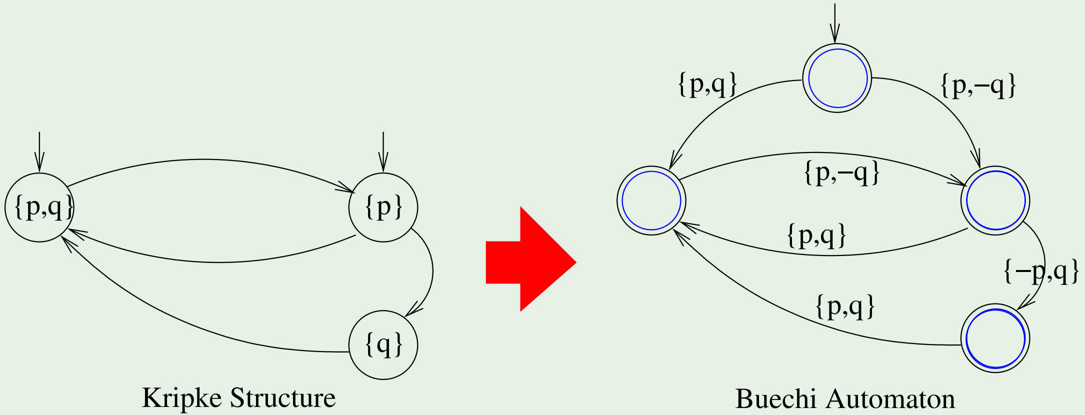
Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke model $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$
  - Initial State: $I := \{init\}$
  - Accepting States: $F := Q = S \cup \{init\}$
  - Transitions:

$$\delta : \quad \begin{array}{l} q \xrightarrow{a} q' \text{ iff } (q, q') \in R \textit{ and } L(q') = a \\ init \xrightarrow{a} q \text{ iff } q \in S_0 \textit{ and } L(q) = a \end{array}$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

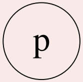
Computing a NBA A_M from a Kripke Structure M: Example



$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



- in a Kripke Structure, it means that p is true and all other propositions are false;
- in a Büchi Automaton, it means that p is true and all other propositions are irrelevant ("don't care"), i.e. they can be either true or false.

Outline

Translation problem

Problem

Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL validity/satisfiability/entailment e model checking
- We translate an LTL formula into a Generalized Büchi Automata (GBA), then into an NBA

LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, $\mathbf{X}$, $\mathbf{U}$, $\mathbf{R}$ on propositional literals.
- Done by pushing negations down to literal level:

$$
\begin{array}{lcl}
\neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & (\neg\varphi_1 \wedge \neg\varphi_2) \\
\neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & (\neg\varphi_1 \vee \neg\varphi_2) \\
\neg\mathbf{X}\varphi_1 & \Longrightarrow & \mathbf{X}\neg\varphi_1 \\
\neg(\varphi_1 \mathbf{U} \varphi_2) & \Longrightarrow & (\neg\varphi_1 \mathbf{R} \neg\varphi_2) \\
\neg(\varphi_1 \mathbf{R} \varphi_2) & \Longrightarrow & (\neg\varphi_1 \mathbf{U} \neg\varphi_2)
\end{array}
$$

$\Longrightarrow$ The resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).

- encoding linear if a DAG representation is used

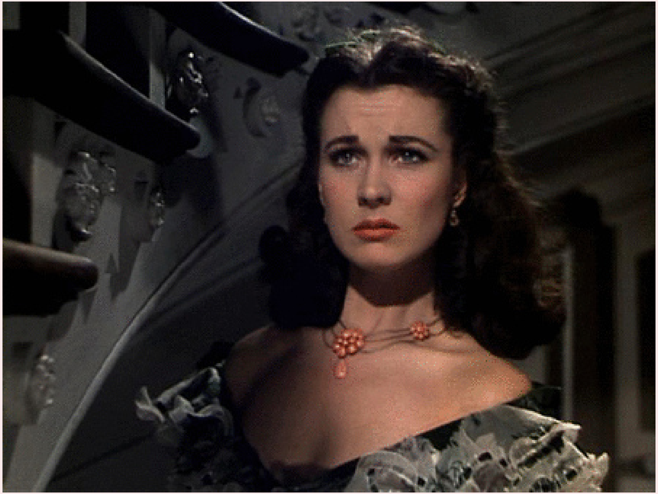
- In the construction of A_φ we now assume that φ is in NNF.
  $\Longrightarrow$ every non-atomic subformula occurs positively in φ
- For convenience, we still use **F**'s and **G**'s as shortcuts: $\mathbf{F}\varphi$ for $\top\mathbf{U}\varphi$ and $\mathbf{G}\varphi$ for $\bot\mathbf{R}\varphi$

On-the-fly Construction of A_φ (Intuition)

Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ:
$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$ [and $\mathbf{F}\psi \Longrightarrow \psi \vee \mathbf{XF}\psi$]
$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$ [and $\mathbf{G}\psi \Longrightarrow \psi \wedge \mathbf{XG}\psi$]
until we get a Boolean combination of elementary subformulas of φ
(An elementary formula is a proposition or a **X**-formula.)

Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, and then push the conjunctions inside the next:

$$\varphi \implies \bigvee_i (\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X}\psi_{ik}) \implies \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}).$$

- Each disjunct $(\overbrace{\bigwedge_j l_{ij}}^{labels} \wedge \overbrace{\mathbf{X} \bigwedge_k \psi_{ik}}^{next\ part})$ represents a state:
  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
  - $\mathbf{X} \bigwedge_k \psi_{ik}$ represents the next part of the state (obbligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 3: For every state S_i represented by $(\bigwedge_j l_{ij} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\varphi_i})$

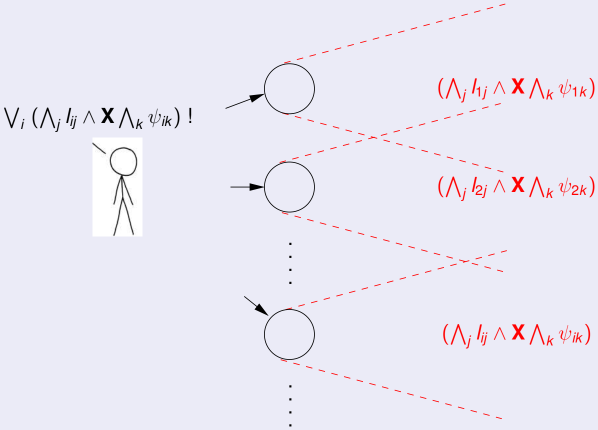
- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
- mark that the state S_i satisfies φ
- apply recursively steps 1-2-3 to $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$,
  - rewrite φ_i into $\bigvee_{i'}(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$
  - from each disjunct $(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$ generate a new state $S_{ii'}$ (if not already present) and label it as satisfying $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$
- draw an edge from S_i to all states $S_{ii'}$ which satisfy $\bigwedge_k \psi_{ik}$
- (if no next part occurs, $\mathbf{X}\top$ is implicitly assumed, so that an edge to a “true” node is drawn)

On-the-fly Construction of A_φ (Intuition) [cont.]

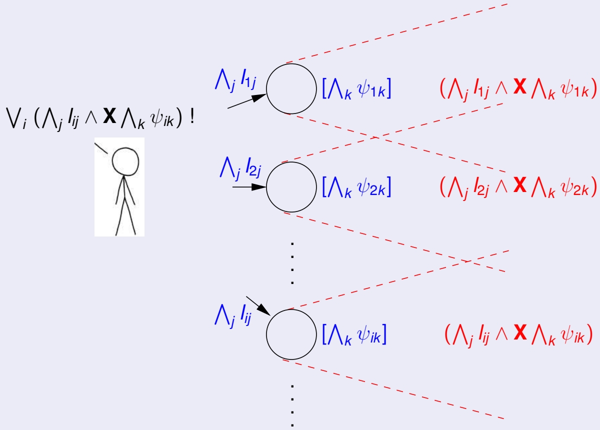
On-the-fly Construction of A_φ (Intuition) [cont.]

$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

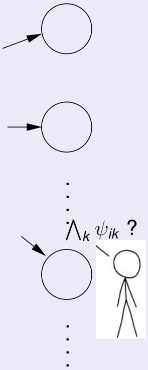
On-the-fly Construction of A_φ (Intuition) [cont.]



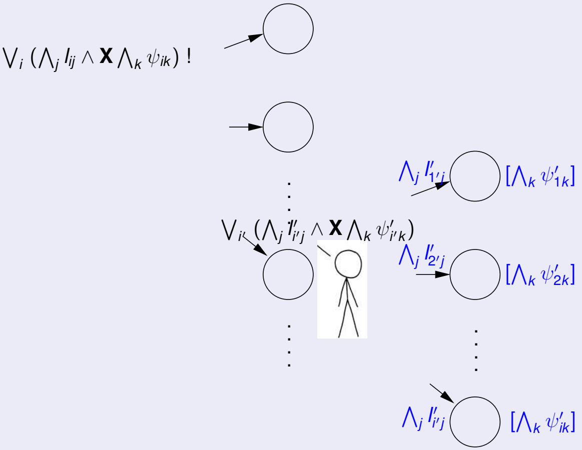
On-the-fly Construction of A_φ (Intuition) [cont.]



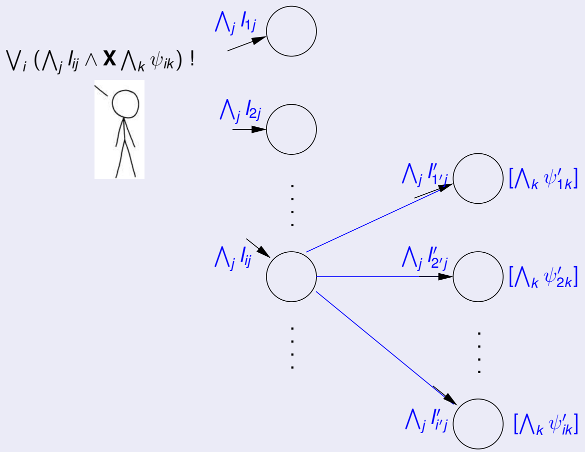
On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

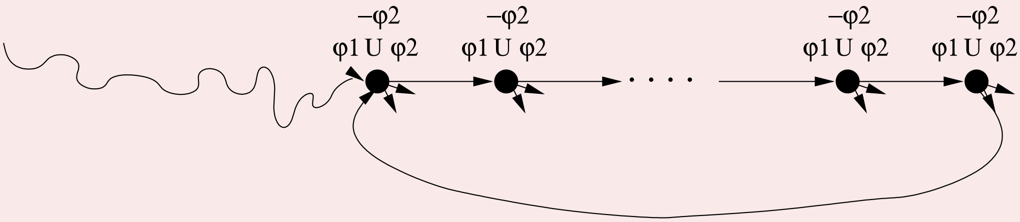
Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j, mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no **U**-subformulas, then mark all states with F_1
—i.e., $FT \stackrel{\text{def}}{=} \{Q\}$).

Remark

The fact that we initially converted the formula into NNF guarantees that only positive **U**/**F**-subformulas and negative **R**-/**G**-subformulas are considered here

Dealing with **U**-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \Longleftrightarrow (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X} \varphi_1 \mathbf{U} \varphi_2))$ are a property, not a definition of **U**:
  $\Longrightarrow$ they implicitly admit a "weaker" semantics of $\varphi_1 \mathbf{U} \varphi_2$, in which $\varphi_1 \mathbf{U} \varphi_2$ always holds and φ_2 never holds
- It cannot happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.



$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ holds

- In LTL: $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ ("avoid bad loop")

On-the-fly Construction of A_φ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of φ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $\mathit{Cover}(\Psi) \stackrel{\text{def}}{=} \mathit{Expand}(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - $\mathit{Expand}(\Psi, s)$ takes as input:
    - a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ to be expanded
    - a state $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$ under construction

    and returns a set of states $\{\langle \lambda_i, \chi_i, \sigma_i \rangle\}_i$ representing te expansion of Ψ
  - Combines steps 1. and 2. of previous slides

On-the-fly Construction of A_φ - Expand

Given $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \stackrel{\text{def}}{=} \langle\lambda, \chi, \sigma\rangle$, we define $Expand(\Psi, s)$ recursively as follows:

- if $\Psi = \emptyset$, $Expand(\Psi, s) = \{s\}$
- if $\bot \in \Psi$, $Expand(\Psi, s) = \emptyset$
- if $\top \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $Expand(\Psi, s) = Expand(\Psi \backslash \{\top\}, \langle\lambda, \chi, \sigma \cup \{\top\}\rangle)$
- if $l \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$, l propositional literal
  $Expand(\Psi, s) = Expand(\Psi \backslash \{l\}, \langle\lambda \cup \{l\}, \chi, \sigma \cup \{l\}\rangle)$
  (add l to the labels of s and to set of satisfied formulas)
- if $\mathbf{X}\psi \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $Expand(\Psi, s) = Expand(\Psi \backslash \{X\psi\}, \langle\lambda, \chi \cup \{\psi\}, \sigma \cup \{\mathbf{X}\psi\}\rangle)$
  (add ψ to the next part of s and $\mathbf{X}\psi$ to set of satisfied formulas)
- if $\psi_1 \wedge \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $Expand(\Psi, s) = Expand(\Psi \cup \{\psi_1, \psi_2\} \backslash \{\psi_1 \wedge \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \wedge \psi_2\}\rangle)$
  (process both ψ_1 and ψ_2 and add $\psi_1 \wedge \psi_2$ to σ)
- ...

On-the-fly Construction of A_φ - Expand

Given $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \stackrel{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$, we define *Expand*$(\Psi, s)$ recursively as follows:

- ...
- if $\psi_1 \vee \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$\begin{aligned} Expand(\Psi, s) = & \ Expand(\Psi \cup \{\psi_1\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle) \\ & \cup Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle) \end{aligned}$$
  (split s into two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
- if $\psi_1 \mathbf{U} \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$\begin{aligned} Expand(\Psi, s) = & \ Expand(\Psi \cup \{\psi_1\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{U} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle) \\ & \cup Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\} \rangle) \end{aligned}$$
  (split s into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{U} \psi_2$ to σ)
- if $\psi_1 \mathbf{R} \psi_2 \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$\begin{aligned} Expand(\Psi, s) = & \ Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi \cup \{\psi_1 \mathbf{R} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle) \\ & \cup Expand(\Psi \cup \{\psi_1, \psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\} \rangle) \end{aligned}$$
  (split s into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{R} \psi_2$ to σ)

On-the-fly Construction of A_φ - Expand

Two relevant subcases: $\mathbf{F}\psi \stackrel{\text{def}}{=} \top\mathbf{U}\psi$ and $\mathbf{G}\psi \stackrel{\text{def}}{=} \bot\mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$Expand(\Psi, s) = Expand(\Psi \backslash \{\mathbf{F}\psi\}, \langle \lambda, \chi \cup \{\mathbf{F}\psi\}, \sigma \cup \{\mathbf{F}\psi\}\rangle) \cup Expand(\Psi \cup \{\psi\} \backslash \{\mathbf{F}\psi\}, \langle \lambda, \chi, \sigma \cup \{\mathbf{F}\psi\}\rangle)$$
- if $\mathbf{G}\psi \in \Psi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
  $$Expand(\Psi, s) = Expand(\Psi \cup \{\psi\} \backslash \{\mathbf{G}\psi\}, \langle \lambda, \chi \cup \{\mathbf{G}\psi\}, \sigma \cup \{\mathbf{G}\psi\}\rangle)$$
  (Note: $Expand(\Psi \cup \{\bot, \psi\} \backslash \{\mathbf{G}\psi\}, \ldots) = \emptyset$.)

Definition of A_φ

Given a set of LTL formulas Ψ, we define $Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.
For an LTL formula φ, we construct a Generalized NBA $A_\varphi = (Q, \Sigma, \delta, I, FT)$ as follows:

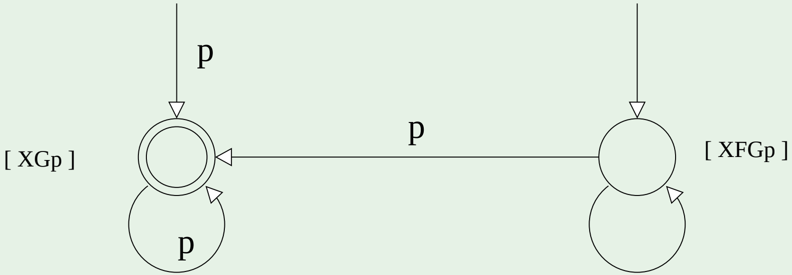
- $\Sigma = 3^{vars(\varphi)}$ ($v \in \{\top, \bot, *\}$, "$*$" is "don't care")
- Q is the smallest set such that
  - $Cover(\{\varphi\}) \subseteq Q$
  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
- $Q_0 = Cover(\{\varphi\})$.
- $s \xrightarrow{\lambda'} s' \in \delta$ iff, $s = \langle \lambda, \chi, \sigma \rangle$, $s' = \langle \lambda', \chi', \sigma' \rangle$ and $s' \in Cover(\chi)$
- $FT = \langle F_1, F_2, ..., F_k \rangle$ where, for all $(\psi_i \mathbf{U} \varphi_i)$ occurring positively in φ,
  $F_i = \{\langle \lambda, \chi, \sigma \rangle \in Q \mid (\psi_i \mathbf{U} \varphi_i) \notin \sigma \text{ or } \varphi_i \in \sigma\}$.
  (If there is no **U**-subformulas, then $FT \stackrel{\text{def}}{=} \{Q\}$).

Example: $\varphi = \mathbf{FG}p$

- $\mathit{Cover}(\{\mathbf{FG}p\})$
  $= \mathit{Expand}(\{\mathbf{FG}p\}, \langle\emptyset, \emptyset, \emptyset\rangle)$
  $= \mathit{Expand}(\emptyset, \langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle) \cup \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset, \emptyset, \{\mathbf{FG}p\}\rangle)$
  $= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\}\rangle)$
  $= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle)$
  $= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle\}$
- $\mathit{Cover}(\{\mathbf{G}p\}) = \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset, \emptyset, \emptyset\rangle)$
  $= \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\}\rangle)$
  $= \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle)$
  $= \{\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle\}$
- Optimization:
  merge $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle$ and $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle$

Example: $\varphi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
- $Q_0 = \{s_1, s_2\}$.
- $T: \quad s_1 \to \{s_1, s_2\},$
  $\quad\quad s_2 \to \{s_2\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2\}$.

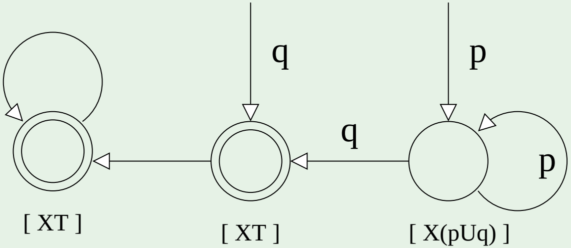


Example: $\varphi = p\mathbf{U}q$

- $$\begin{aligned}&\mathit{Cover}(\{p\mathbf{U}q\})\\ &= \mathit{Expand}(\{p\mathbf{U}q\}, \langle \emptyset, \emptyset, \emptyset\rangle)\\ &= \mathit{Expand}(\{p\}, \langle \emptyset, \{p\mathbf{U}q\}, \{p\mathbf{U}q\}\rangle) \cup \mathit{Expand}(\{q\}, \langle \emptyset, \emptyset, \{p\mathbf{U}q\}\rangle)\\ &= \mathit{Expand}(\emptyset, \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle) \cup \mathit{Expand}(\emptyset, \langle \{q\}, \emptyset, \{p\mathbf{U}q, q\}\rangle)\\ &= \{\langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle\} \cup \{\langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\}\rangle\}\end{aligned}$$
- $\mathit{Cover}(\{\top\}) = \{\langle \emptyset, \{\top\}, \{\top\}\rangle\}$

Example: $\varphi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle\{p\},\{p\mathbf{U}q\},\{p\mathbf{U}q, p\}\rangle$, $s_2 =_{def} \langle\{q\},\{\top\},\{p\mathbf{U}q, q\}\rangle$, $s_3 =_{def} \langle\emptyset,\{\top\},\{\top\}\rangle$.
- $Q = \{s_1, s_2, s_3\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T: \quad s_1 \to \{s_1, s_2\},$
  $s_2 \to \{s_3\}$
  $s_3 \to \{s_3\}$
- $FT = \langle F_1\rangle$ where $F_1 = \{s_2, s_3\}$.



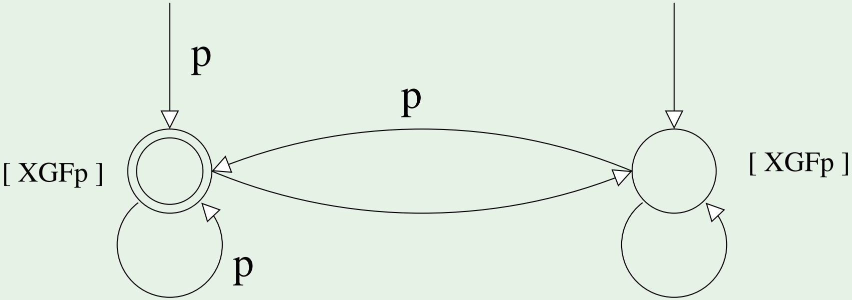
Example: $\varphi = \mathbf{GF}p$

$$
\begin{aligned}
&\mathit{Cover}(\{\mathbf{GF}p\})\\
&\quad = E(\{\mathbf{GF}p\}, \langle\emptyset, \emptyset, \emptyset\rangle)\\
&\quad = E(\{\mathbf{F}p\}, \langle\emptyset, \{\mathbf{GF}p\}, \{\mathbf{GF}p\}\rangle)\\
&\quad = E(\{\}, \langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle) \cup E(\{p\}, \langle\{\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle)\\
&\quad = E(\{\}, \langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle) \cup E(\{\}, \langle\{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\}\rangle)\\
&\quad = \{\langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle\} \cup \{\langle\{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\}\rangle\}
\end{aligned}
$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \iff \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: GFp

- Let $s_1 =_{def} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$, $s_2 =_{def} \langle \emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\} \rangle$,
- $Q = \{s_1, s_2\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T: \begin{array}{l} s_1 \to \{s_1, s_2\}, \\ s_2 \to \{s_1, s_2\} \end{array}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_1\}$.



NBAs of disjunctions of formulas

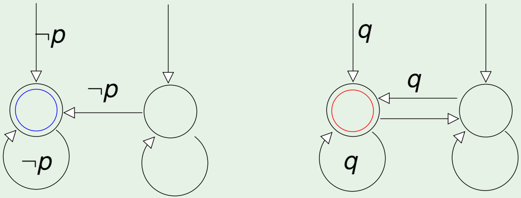
Remark

If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \stackrel{\text{def}}{=} (\mathbf{GF}p \to \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.
Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ:



Suggested Exercises:

- Find an NBA encoding:
  - p
  - $(p \wedge q) \vee (\neg p \wedge \neg q)$
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{G}r$

Outline

Automata-Theoretic LTL Model Checking: Complexity

Four steps:

(i) Compute A_M:
$|A_M| = O(|M|)$

(ii) Compute A_φ:
$|A_\varphi| = O(2^{|\varphi|})$

(iii) Compute the product $A_M \times A_\varphi$:
$|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$:
$O(|A_M \times A_\varphi|) = O(|M| \cdot 2^{|\varphi|})$

$\Longrightarrow$ The complexity of LTL M.C. grows linearly wrt. the size of the model M and exponentially wrt. the size of the property φ

Final Remarks

- Büchi automata are in general more expressive than LTL!

$\Longrightarrow$ some tools (e.g., Spin) allow specifications to be expressed directly as NBAs

$\Longrightarrow$ complementation of NBA relevanant in general

- For every LTL formula, there are many possible equivalent NBAs

$\Longrightarrow$ lots of research for finding “the best” conversion algorithm

- Performing the product and checking emptiness very relevant

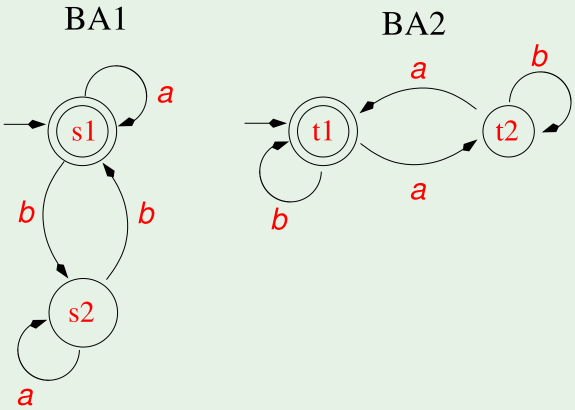
$\Longrightarrow$ lots of techniques developed (e.g., partial order reduction)

$\Longrightarrow$ lots on ongoing research

Outline

Ex: Product of Büchi automata

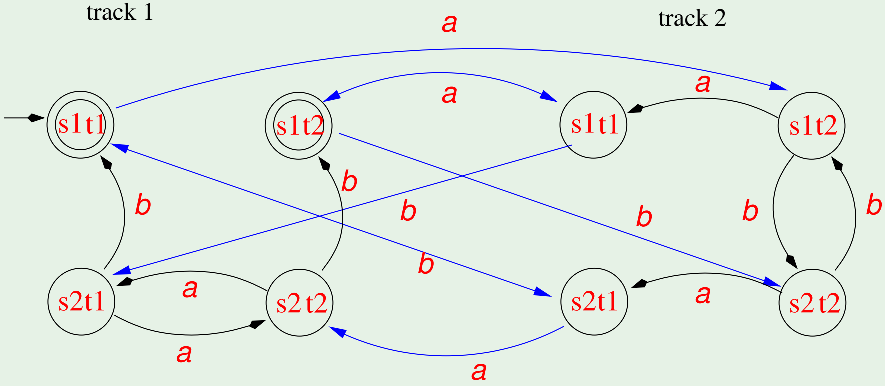
Given the following two Büchi automata (doubly-circled states represent accepting states, *a*, *b* are labels):



Write the product Büchi automaton $BA1 \times BA2$.

Ex: Product of Büchi automata

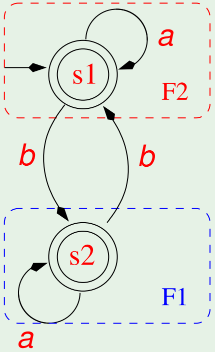
[Solution: The product is:



]

Ex: De-generalization of Büchi Automata

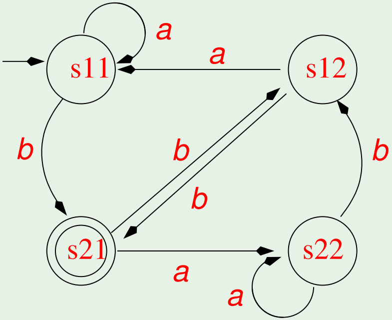
Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \stackrel{\text{def}}{=} \{s2\}, F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

Ex: De-generalization of Büchi Automata

[Solution: The result is:



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Ex: Construction of Büchi Automata

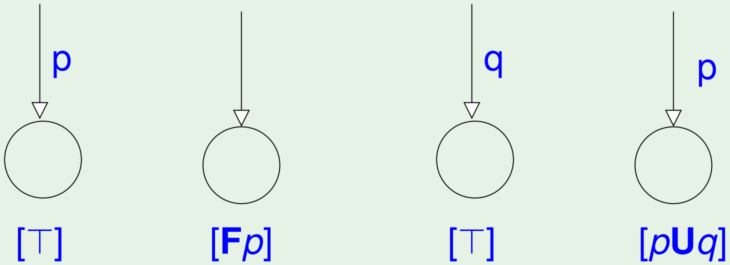
Consider the LTL formula $\varphi \stackrel{\text{def}}{=} (\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q)$.

(a) rewrite φ into Negative Normal Form

[Solution: $(\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q) \Longrightarrow (\neg\mathbf{G}\neg p) \vee (p\mathbf{U}q) \Longrightarrow (\mathbf{F}p) \vee (p\mathbf{U}q)$]

(b) find the initial states of a corresponding Buchi automaton (for each state, define the labels of the incoming arcs and the "next" section.)

[Solution: Applying tableaux rules we obtain: $p \vee \mathbf{XF}p \vee q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$, which is already in disjunctive normal form. This correspond to the following four initial states:



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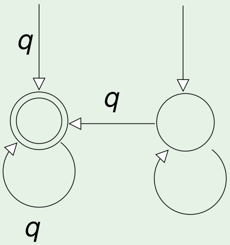
Ex: Büchi automaton

Given the following Büchi automaton BA (doubly-circled states represent accepting states):



Say which of the following sentences are true and which are false.

(*a*) BA accepts all and only the paths verifying **GF**q. [Solution: false]

(*b*) BA accepts all and only the paths verifying **FG**q. [Solution: true]

(*c*) BA accepts only paths verifying **F**q, but not all of them. [Solution: true]

(*d*) BA accepts all the paths verifying **F**q, but not only them. [Solution: false]