

Formal Methods

Module I: Automated Reasoning

Ch. 01: **Propositional Satisfiability (SAT)**

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Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Modern CDCL SAT Solvers
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 4 Ordered Binary Decision Diagrams – OBDDs
- 5 SAT Functionalities: proofs, unsat cores, interpolants, optimization

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Propositional Logic (aka Boolean Logic)



Basic Definitions

- **Propositional formula** (aka **Boolean formula**)
 - \top, \perp are formulas
 - a **propositional atom** A_1, A_2, A_3, \dots is a formula;
 - if φ_1 and φ_2 are formulas, then
 $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$
are formulas.
- Ex: $\varphi \stackrel{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4)))$
- **Atoms**(φ): the set $\{A_1, \dots, A_N\}$ of atoms occurring in φ .
 - Ex: $\text{Atoms}(\varphi) = \{A_1, A_2, A_3, A_4\}$
- **Literal**: a propositional atom A_i (**positive literal**) or its negation $\neg A_i$ (**negative literal**)
 - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- **Clause**: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee \dots)$)
- **Cube**: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge \dots)$)

Semantics of Boolean operators

Truth Table

α	β	$\neg\alpha$	$\alpha\wedge\beta$	$\alpha\vee\beta$	$\alpha\rightarrow\beta$	$\alpha\leftarrow\beta$	$\alpha\leftrightarrow\beta$	$\alpha\oplus\beta$
\perp	\perp	\top	\perp	\perp	\top	\top	\top	\perp
\perp	\top	\top	\perp	\top	\top	\perp	\perp	\top
\top	\perp	\perp	\perp	\top	\perp	\top	\perp	\top
\top	\top	\perp	\top	\top	\top	\top	\top	\perp

Semantics of Boolean operators (cont.)

Note

- \wedge , \vee , \leftrightarrow and \oplus are commutative:

$$(\alpha \wedge \beta) \iff (\beta \wedge \alpha)$$

$$(\alpha \vee \beta) \iff (\beta \vee \alpha)$$

$$(\alpha \leftrightarrow \beta) \iff (\beta \leftrightarrow \alpha)$$

$$(\alpha \oplus \beta) \iff (\beta \oplus \alpha)$$

- \wedge , \vee , \leftrightarrow and \oplus are associative:

$$((\alpha \wedge \beta) \wedge \gamma) \iff (\alpha \wedge (\beta \wedge \gamma)) \iff (\alpha \wedge \beta \wedge \gamma)$$

$$((\alpha \vee \beta) \vee \gamma) \iff (\alpha \vee (\beta \vee \gamma)) \iff (\alpha \vee \beta \vee \gamma)$$

$$((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) \iff (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) \iff (\alpha \leftrightarrow \beta \leftrightarrow \gamma)$$

$$((\alpha \oplus \beta) \oplus \gamma) \iff (\alpha \oplus (\beta \oplus \gamma)) \iff (\alpha \oplus \beta \oplus \gamma)$$

- \rightarrow , \leftarrow are neither commutative nor associative:

$$(\alpha \rightarrow \beta) \not\iff (\beta \rightarrow \alpha)$$

$$((\alpha \rightarrow \beta) \rightarrow \gamma) \not\iff (\alpha \rightarrow (\beta \rightarrow \gamma))$$

Syntactic Properties of Boolean Operators

$$\begin{aligned}\neg\neg\alpha &\iff \alpha \\ (\alpha \vee \beta) &\iff \neg(\neg\alpha \wedge \neg\beta) \\ \neg(\alpha \vee \beta) &\iff (\neg\alpha \wedge \neg\beta) \\ (\alpha \wedge \beta) &\iff \neg(\neg\alpha \vee \neg\beta) \\ \neg(\alpha \wedge \beta) &\iff (\neg\alpha \vee \neg\beta) \\ (\alpha \rightarrow \beta) &\iff (\neg\alpha \vee \beta) \\ \neg(\alpha \rightarrow \beta) &\iff (\alpha \wedge \neg\beta) \\ (\alpha \leftarrow \beta) &\iff (\alpha \vee \neg\beta) \\ \neg(\alpha \leftarrow \beta) &\iff (\neg\alpha \wedge \beta) \\ (\alpha \leftrightarrow \beta) &\iff ((\alpha \rightarrow \beta) \wedge (\alpha \leftarrow \beta)) \\ &\iff ((\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)) \\ \neg(\alpha \leftrightarrow \beta) &\iff (\neg\alpha \leftrightarrow \beta) \\ &\iff (\alpha \leftrightarrow \neg\beta) \\ &\iff ((\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta)) \\ (\alpha \oplus \beta) &\iff \neg(\alpha \leftrightarrow \beta)\end{aligned}$$

Boolean logic can be expressed in terms of $\{\neg, \wedge\}$ (or $\{\neg, \vee\}$) only!

Exercises

1 For every pair of formulas $\alpha \iff \beta$ below, show that α and β can be rewritten into each other by applying the syntactic properties of the previous slide

- $(A_1 \wedge A_2) \vee A_3 \iff (A_1 \vee A_3) \wedge (A_2 \vee A_3)$
- $(A_1 \vee A_2) \wedge A_3 \iff (A_1 \wedge A_3) \vee (A_2 \wedge A_3)$
- $A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow A_4)) \iff (A_1 \wedge A_2 \wedge A_3) \rightarrow A_4$
- $A_1 \rightarrow (A_2 \wedge A_3) \iff (A_1 \rightarrow A_2) \wedge (A_1 \rightarrow A_3)$
- $(A_1 \vee A_2) \rightarrow A_3 \iff (A_1 \rightarrow A_3) \wedge (A_2 \rightarrow A_3)$
- $A_1 \oplus A_2 \iff (A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$
- $\neg A_1 \leftrightarrow \neg A_2 \iff A_1 \leftrightarrow A_2$
- $A_1 \leftrightarrow A_2 \leftrightarrow A_3 \iff A_1 \oplus A_2 \oplus A_3$

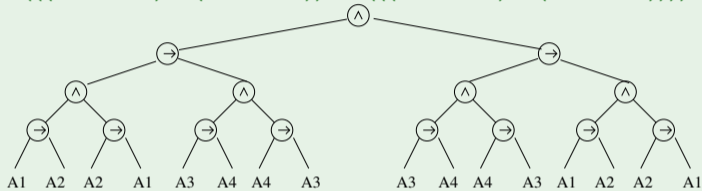
Tree & DAG Representations of Formulas

- Formulas can be represented either as **trees** or as **DAGS** (Directed Acyclic Graphs)
- **DAG representation can be up to exponentially smaller**
 - in particular, when \leftrightarrow 's are involved

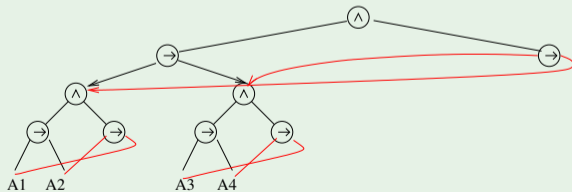
$$\begin{aligned} & (A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\ & \quad \Downarrow \\ & (((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge \\ & ((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2))) \\ & \quad \Downarrow \\ & (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge \\ & (((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)))) \end{aligned}$$

Tree & DAG Representations of Formulas: Example

$((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \wedge$
 $((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)))$



Tree Representation



DAG Representation

Semantics: Basic Definitions

- **Total truth assignment** μ for φ :
 $\mu : \mathit{Atoms}(\varphi) \mapsto \{\top, \perp\}$.
 - represents a **possible world** or a **possible state of the world**
- **Partial Truth assignment** μ for φ :
 $\mu : \mathcal{A} \mapsto \{\top, \perp\}, \mathcal{A} \subset \mathit{Atoms}(\varphi)$.
 - represents 2^k total assignments, k is # unassigned variables
- **Notation: set and formula representations of an assignment**
 - μ can be represented **as a set of literals**:
EX: $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies \{A_1, \neg A_2\}$
 - μ can be represented **as a formula (cube)**:
EX: $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies (A_1 \wedge \neg A_2)$

Semantics: Basic Definitions [cont.]

- A **total** truth assignment μ **satisfies** φ (μ is a model of φ , $\mu \models \varphi$):

$$\mu \models A_i \iff \mu(A_i) = \top$$

$$\mu \models \neg\varphi \iff \text{not } \mu \models \varphi$$

$$\mu \models \alpha \wedge \beta \iff \mu \models \alpha \text{ and } \mu \models \beta$$

$$\mu \models \alpha \vee \beta \iff \mu \models \alpha \text{ or } \mu \models \beta$$

$$\mu \models \alpha \rightarrow \beta \iff \text{if } \mu \models \alpha, \text{ then } \mu \models \beta$$

$$\mu \models \alpha \leftrightarrow \beta \iff \mu \models \alpha \text{ iff } \mu \models \beta$$

$$\mu \models \alpha \oplus \beta \iff \mu \models \alpha \text{ iff not } \mu \models \beta$$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$ (the set of models of φ)

- A **partial** truth assignment μ **satisfies** φ iff all total assignments extending μ satisfy φ

- Ex: $\{A_1\} \models (A_1 \vee A_2)$ because both $\{A_1, A_2\} \models (A_1 \vee A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$

- φ is **satisfiable** iff $\mu \models \varphi$ for some μ (i.e. $M(\varphi) \neq \emptyset$)

- α **entails** β ($\alpha \models \beta$): $\alpha \models \beta$ iff $\mu \models \alpha \implies \mu \models \beta$ for all μ s
(i.e., $M(\alpha) \subseteq M(\beta)$)

- φ is **valid** ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for all μ s
(i.e., $\mu \in M(\varphi)$ for all μ s)

Properties & Results

Property

φ is valid iff $\neg\varphi$ is not satisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

Equivalence and Equi-Satisfiability

- α and β are **equivalent** iff, for every μ , $\mu \models \alpha$ iff $\mu \models \beta$
(i.e., if $M(\alpha) = M(\beta)$)
- α and β are **equi-satisfiable** iff exists μ_1 s.t. $\mu_1 \models \alpha$ iff exists μ_2 s.t. $\mu_2 \models \beta$
(i.e., if $M(\alpha) \neq \emptyset$ iff $M(\beta) \neq \emptyset$)
- α, β equivalent
 $\Downarrow \Updownarrow$
 α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
 $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α : $\beta \stackrel{\text{def}}{=} T(\alpha)$:
 - T is **validity-preserving** [resp. **satisfiability-preserving**] iff
 $T(\alpha)$ and α are equivalent [resp. equi-satisfiable]

Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

- Intuition:

- $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \perp\}$ s.t. $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \perp\}$, $\mu \cup \{v := \text{truthvalue}\} \models \varphi$

- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

NP-Completeness of SAT

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is **NP-complete**

⇒ The most important logical problems (**validity**, **inference**, **entailment**, **equivalence**, ...) can be straightforwardly reduced to **(un)satisfiability**, and are thus **(co)NP-complete**.



No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- **Positive/negative occurrences**

- φ occurs positively in φ ;
- if $\neg\varphi_1$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ
- if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ ,
then φ_1 and φ_2 occur positively [negatively] in φ ;
- if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ ;
- if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ ,
then φ_1 and φ_2 occur positively and negatively in φ ;

Negative Normal Form (NNF)

- φ is in **Negative normal form** iff it is given only by the recursive applications of \wedge, \vee to literals.

- **every φ can be reduced into NNF:**

(i) substituting all \rightarrow 's and \leftrightarrow 's:

$$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

(ii) pushing down negations recursively:

$$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$$

$$\neg\neg\alpha \implies \alpha$$

- The reduction is **linear** if a DAG representation is used.
- Preserves the **equivalence** of formulas.

NNF: Example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

↓

$$\begin{aligned} & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \wedge \\ & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \end{aligned}$$

↓

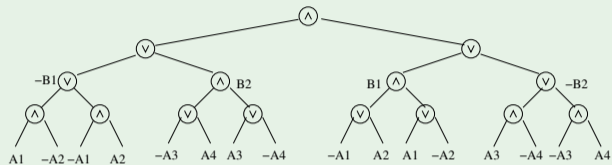
$$\begin{aligned} & ((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2))) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \end{aligned}$$

↓

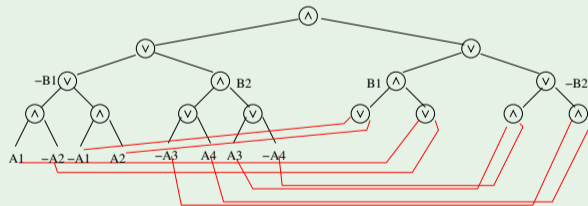
$$\begin{aligned} & (((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))) \end{aligned}$$

NNF: Example [cont.]

Note



Tree Representation



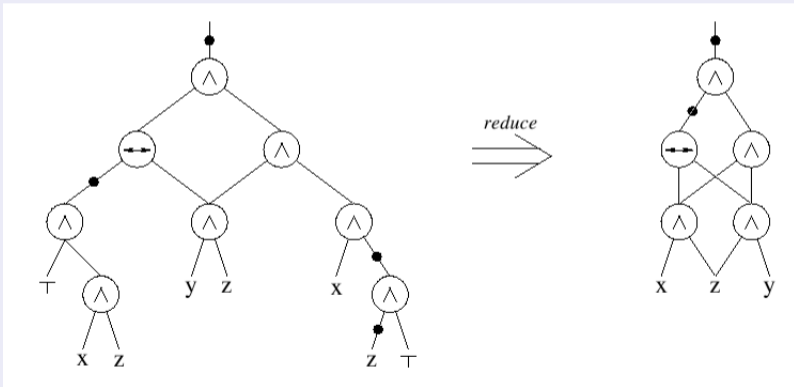
DAG Representation

For each non-literal subformula φ , φ and $\neg\varphi$ have different representations \implies they are not shared.

Optimized polynomial representations

And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations:
{ \wedge , \leftrightarrow , \neg }-only, negations on arcs, sorting of subformulae, lifting of \neg 's over \leftrightarrow 's,...



Conjunctive Normal Form (CNF)

- φ is in **Conjunctive normal form** iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^L \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called **clauses**
- Easier to handle: list of lists of literals.
 \implies no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,

(i) expanding implications and equivalences:

$$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

(ii) pushing down negations recursively:

$$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$$

$$\neg\neg\alpha \implies \alpha$$

(iii) applying recursively the DeMorgan's Rule: $(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$

- Resulting formula worst-case **exponential**:

- ex: $\|CNF(\bigvee_{i=1}^N (l_{i1} \wedge l_{i2}))\| = \| (l_{11} \vee l_{21} \vee \dots \vee l_{N1}) \wedge (l_{12} \vee l_{22} \vee \dots \vee l_{N2}) \wedge \dots \wedge (l_{1N} \vee l_{2N} \vee \dots \vee l_{NN}) \| = 2^N$

- $Atoms(CNF(\varphi)) = Atoms(\varphi)$

- $CNF(\varphi)$ is **equivalent** to φ .

- Rarely used in practice.

Labeling CNF conversion $CNF_{label}(\varphi)$

Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Tseitin's CNF-ization)

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$

$$\varphi \implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j))$$

$$\varphi \implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$$

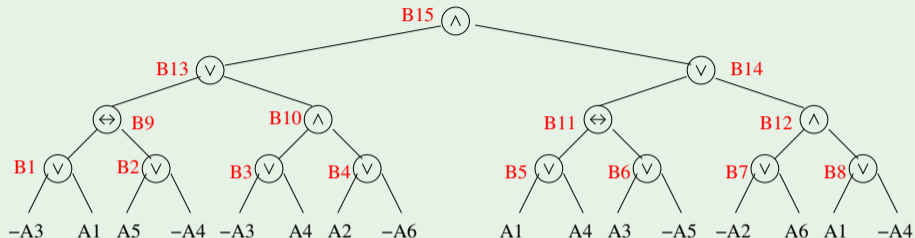
l_i, l_j being literals and B being a “new” variable.

- Worst-case **linear**!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$ is **equi-satisfiable** (but not equivalent) to φ .
 - moreover: $\exists B_1, \dots, B_k. CNF_{label}(\varphi)$ equivalent to φ , s.t. B_1, \dots, B_k all fresh variables introduced
- Much more used than classic conversion in practice

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	\iff	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	\iff	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	\iff	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF Conversion CNF_{label} – Example



$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \wedge$$

... \wedge

$$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \wedge$$

$$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \wedge$$

... \wedge

$$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \wedge$$

$$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \wedge$$

$$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \wedge$$

$$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \wedge$$

B_{15}

$$(\neg B_1 \vee \neg A_3 \vee A_1) \wedge (B_1 \vee A_3) \wedge (B_1 \vee \neg A_1) \wedge$$

... \wedge

$$(\neg B_8 \vee A_1 \vee \neg A_4) \wedge (B_8 \vee \neg A_1) \wedge (B_8 \vee A_4) \wedge$$

$$(\neg B_9 \vee \neg B_1 \vee B_2) \wedge (\neg B_9 \vee B_1 \vee \neg B_2) \wedge$$

$$(B_9 \vee B_1 \vee B_2) \wedge (B_9 \vee \neg B_1 \vee \neg B_2) \wedge$$

= ... \wedge

$$(B_{12} \vee \neg B_7 \vee \neg B_8) \wedge (\neg B_{12} \vee B_7) \wedge (\neg B_{12} \vee B_8) \wedge$$

$$(\neg B_{13} \vee B_9 \vee B_{10}) \wedge (B_{13} \vee \neg B_9) \wedge (B_{13} \vee \neg B_{10}) \wedge$$

$$(\neg B_{14} \vee B_{11} \vee B_{12}) \wedge (B_{14} \vee \neg B_{11}) \wedge (B_{14} \vee \neg B_{12}) \wedge$$

$$(B_{15} \vee \neg B_{13} \vee \neg B_{14}) \wedge (\neg B_{15} \vee B_{13}) \wedge (\neg B_{15} \vee B_{14}) \wedge$$

B_{15}

Labeling CNF conversion CNF_{label} (improved)

- As in the previous case, applying instead the rules:

$$\begin{aligned}\varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \rightarrow (l_i \vee l_j)) && \text{if } (l_i \vee l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF((l_i \vee l_j) \rightarrow B) && \text{if } (l_i \vee l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \rightarrow (l_i \wedge l_j)) && \text{if } (l_i \wedge l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF((l_i \wedge l_j) \rightarrow B) && \text{if } (l_i \wedge l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \rightarrow (l_i \leftrightarrow l_j)) && \text{if } (l_i \leftrightarrow l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF((l_i \leftrightarrow l_j) \rightarrow B) && \text{if } (l_i \leftrightarrow l_j) \text{ neg.}\end{aligned}$$

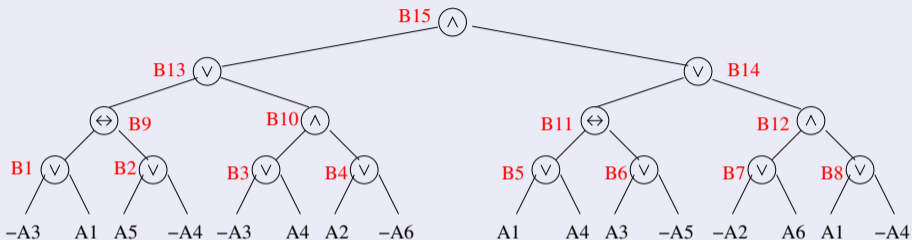
- Smaller in size:

$$\begin{aligned}CNF(B \rightarrow (l_i \vee l_j)) &= (\neg B \vee l_i \vee l_j) \\ CNF(((l_i \vee l_j) \rightarrow B)) &= (\neg l_i \vee B) \wedge (\neg l_j \vee B)\end{aligned}$$

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	\iff	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	\iff	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	\iff	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	\iff	$(B \vee \neg l_i \neg l_j)$
$CNF(B \rightarrow (l_i \leftrightarrow l_j))$	\iff	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j)$
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	\iff	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF conversion CNF_{label} – example



Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$
 $\dots \quad \wedge$
 $CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \quad \wedge$
 $CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$
 $\dots \quad \wedge$
 $CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \quad \wedge$
 $CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \quad \wedge$
 $CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \quad \wedge$
 $CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \quad \wedge$
 B_{15}

Improved

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$
 $\dots \quad \wedge$
 $CNF(B_8 \rightarrow (A_1 \vee \neg A_4)) \quad \wedge$
 $CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$
 $\dots \quad \wedge$
 $CNF(B_{12} \rightarrow (B_7 \wedge B_8)) \quad \wedge$
 $CNF(B_{13} \rightarrow (B_9 \vee B_{10})) \quad \wedge$
 $CNF(B_{14} \rightarrow (B_{11} \vee B_{12})) \quad \wedge$
 $CNF(B_{15} \rightarrow (B_{13} \wedge B_{14})) \quad \wedge$
 B_{15}

Labeling CNF conversion CNF_{label} – further improvements

- Do not apply CNF_{label} when not necessary:
(e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \implies CNF_{label}(\varphi_1) \wedge \varphi_2$, if φ_2 already in CNF)
- Apply DeMorgan's rules where it is more effective:
(e.g., $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \implies CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$)
- Exploit the associativity of \wedge 's and \vee 's:
$$\dots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \dots \implies \dots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3)) \dots$$
- Before applying CNF_{label} , rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Exercises

- 1 Consider the following Boolean formula φ :

$$\neg(((\neg A_1 \rightarrow A_2) \wedge (\neg A_3 \rightarrow A_4)) \vee ((A_5 \rightarrow A_6) \wedge (A_7 \rightarrow \neg A_8)))$$

Compute the Negative Normal Form of φ

- 2 Consider the following Boolean formula φ :

$$((\neg A_1 \wedge A_2) \vee (A_7 \wedge A_4) \vee (\neg A_3 \wedge \neg A_2) \vee (A_5 \wedge \neg A_4))$$

- 1 Produce the CNF formula $CNF(\varphi)$.
- 2 Produce the CNF formula $CNF_{label}(\varphi)$.
- 3 Produce the CNF formula $CNF_{label}(\varphi)$ (improved version)

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Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
 - AI, formal verification, circuit synthesis, operational research,....
- Important in AI: $KB \models \alpha$: entail fact α from knowledge base KB (aka **Model Checking**: $M(KB) \subseteq M(\alpha)$)
 - typically $KB \gg \alpha$
- All propositional reasoning tasks reduced to **satisfiability (SAT)**
 - $KB \models \alpha \implies \text{SAT}(KB \wedge \neg\alpha) = \text{false}$
 - input formula CNF-ized and fed to a **SAT solver**
- **Current SAT solvers dramatically efficient:**
 - handle industrial problems with $10^6 - 10^7$ variables & clauses!
 - used as backend engines in a variety of systems

Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
\perp	\perp	\perp	\perp	\top	\top
\perp	\top	\perp	\top	\top	\perp
\top	\perp	\perp	\top	\perp	\perp
\top	\top	\top	\top	\top	\top

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|\text{Atoms}(\varphi)|}$ lines.
- Never used in practice.

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The Resolution Rule

- **Resolution**: deduction of a new clause from a pair of clauses with exactly one incompatible variable (**resolvent**):

$$\frac{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l}_{\text{resolvent}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'}) \quad (\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{\neg l}_{\text{resolvent}} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}$$

- Ex:
$$\frac{(A \vee B \vee C \vee D \vee E) \quad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$$

- Note: many standard inference rules subcases of resolution:
(recall that $\alpha \rightarrow \beta \iff \neg\alpha \vee \beta$)

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \text{ (trans.)} \quad \frac{A \quad A \rightarrow B}{B} \text{ (m. ponens)} \quad \frac{\neg B \quad A \rightarrow B}{\neg A} \text{ (m. tollens)}$$

Improvements: Subsumption & Unit Propagation

Alternative “set” notation (Γ clause set):

$$\frac{\Gamma, \phi_1, \dots, \phi_n}{\Gamma, \phi'_1, \dots, \phi'_n} \quad \left(\text{e.g., } \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2} \right)$$

- Clause Subsumption (C clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

$$\frac{\Gamma \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma \wedge (l) \wedge (\bigvee_i l_i)}$$

- Unit Subsumption:

$$\frac{\Gamma \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma \wedge (l)}$$

- Unit Propagation = Unit Resolution + Unit Subsumption

“Deterministic” rule: applied **before** other “non-deterministic” rules!

Basic Propositional Inference: Resolution [33, 10]

- Assume input formula in CNF
 - if not, apply Tseitin CNF-ization first

⇒ φ is represented as a set of clauses

- **Search** for a refutation of φ (is φ unsatisfiable?)
 - recall: $\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ unsatisfiable
- Basic idea: **apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either**
 - a false clause is generated, or
 - the resolution rule is no more applicable
- **Correct:** if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- **Complete:** if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- **Time-inefficient**
- **Very Memory-inefficient (exponential in memory)**
- Many different strategies

Resolution: basic strategy [10]

```
function  $DP(\Gamma)$ 
  if  $\perp \in \Gamma$                                 /* unsat */
    then return False;
  if (Resolve() is no more applicable to  $\Gamma$ ) /* sat   */
    then return True;
  if {a unit clause ( $l$ ) occurs in  $\Gamma$ }      /* unit   */
    then  $\Gamma := Unit\_Propagate(l, \Gamma)$ ;
    return  $DP(\Gamma)$ 
   $A := select\_variable(\Gamma)$ ;                /* resolve */
   $\Gamma = \Gamma \cup \bigcup_{A \in C', \neg A \in C''} \{Resolve(C', C'')\} \setminus \bigcup_{A \in C', \neg A \in C''} \{C', C''\}$ ;
  return  $DP(\Gamma)$ 
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$\begin{array}{cccc} (A_1 \vee A_2) & (A_1 \vee \neg A_2) & (\neg A_1 \vee A_2) & (\neg A_1 \vee \neg A_2) \\ \Downarrow & & & \\ (A_2) & (A_2 \vee \neg A_2) & (\neg A_2 \vee A_2) & (\neg A_2) \\ \Downarrow & & & \\ \perp & & & \end{array}$$

\Rightarrow UNSAT

Resolution: Examples (cont.)

$$\begin{array}{c} (A \vee B \vee C) \quad (B \vee \neg C \vee \neg F) \quad (\neg B \vee E) \\ \Downarrow \\ (A \vee C \vee E) \quad (\neg C \vee \neg F \vee E) \\ \Downarrow \\ (A \vee E \vee \neg F) \end{array}$$

\Rightarrow SAT

Resolution: Examples

$$(A \vee B) \quad (A \vee \neg B) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(A) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(C) \quad (\neg C)$$
$$\Downarrow$$
$$\perp$$

\Rightarrow UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
 - ⇒ May require **exponential space**
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

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Semantic tableaux [39]

- **Search** for an assignment satisfying φ
- applies recursively **elimination rules** to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i , the branch is **closed**, otherwise it is **open**.
- if no rule can be applied to an open branch μ , then $\mu \models \varphi$;
- if all branches are **closed**, the formula is **not satisfiable**;

Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \rightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2}$$

(\wedge -elimination)

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi}$$

($\neg\neg$ -elimination)

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \quad \Gamma, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \neg\varphi_2}$$

$$\frac{\Gamma, (\varphi_1 \rightarrow \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \varphi_2}$$

(\vee -elimination)

$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \quad \Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \quad \Gamma, \neg\varphi_1, \varphi_2}$$

(\leftrightarrow -elimination).

Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

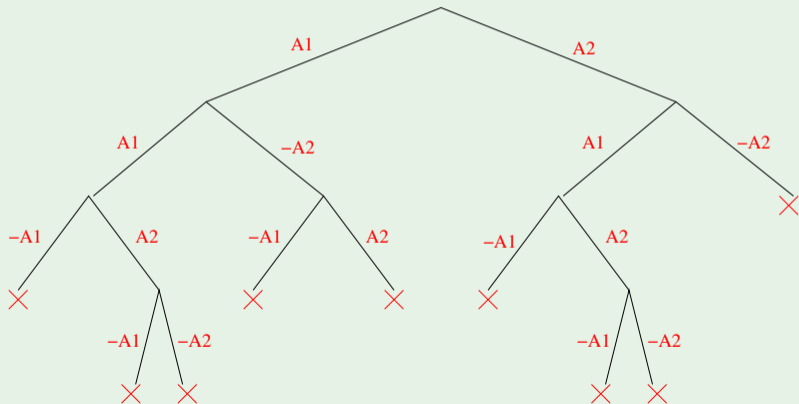
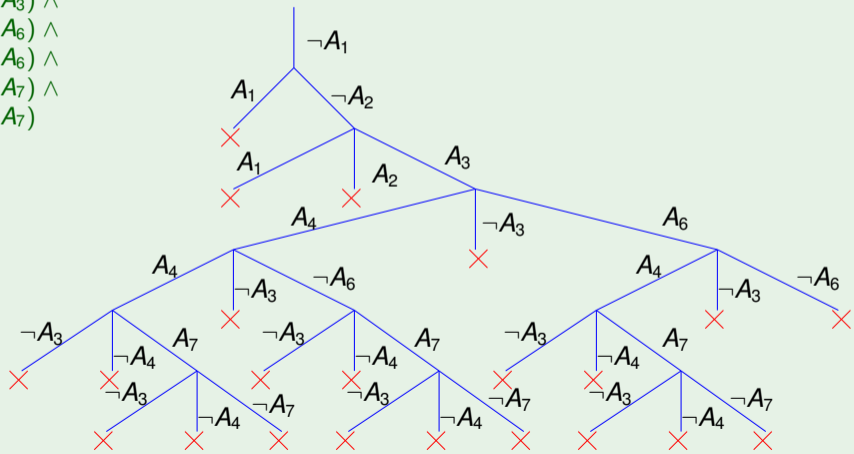


Tableau algorithm

```
function Tableau( $\Gamma$ )  
  if  $A_i \in \Gamma$  and  $\neg A_i \in \Gamma$                                 /* branch closed */  
    then return False;  
  if  $(\varphi_1 \wedge \varphi_2) \in \Gamma$                                     /*  $\wedge$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1, \varphi_2\} \setminus \{(\varphi_1 \wedge \varphi_2)\}$ );  
  if  $(\neg\neg\varphi_1) \in \Gamma$                                         /*  $\neg\neg$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\neg\neg\varphi_1)\}$ );  
  if  $(\varphi_1 \vee \varphi_2) \in \Gamma$                                     /*  $\vee$ -elimination */  
    then return    Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ ) or  
                   Tableau( $\Gamma \cup \{\varphi_2\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ );  
  ...  
  return True;                                                /* branch expanded */
```

Semantic Tableaux: Example

$(\neg A_1) \wedge$
 $(A_1 \vee \neg A_2) \wedge$
 $(A_1 \vee A_2 \vee A_3) \wedge$
 $(A_4 \vee \neg A_3 \vee A_6) \wedge$
 $(A_4 \vee \neg A_3 \vee \neg A_6) \wedge$
 $(\neg A_3 \vee \neg A_4 \vee A_7) \wedge$
 $(\neg A_3 \vee \neg A_4 \vee \neg A_7)$



\Rightarrow unsat

Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- **Branches on disjunctions**
- **Intuitive, modular, easy to extend**
⇒ loved by logicians.
- **Rather inefficient**
⇒ avoided by computer scientists.
- Requires **polynomial space**

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- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ ;
- At each step assigns a truth value to (all instances of) **one atom**.
- Performs **deterministic choices** first.

$$\frac{\varphi_1 \wedge (I)}{\varphi_1[I|\top]} \text{ (Unit)}$$

$$\frac{\varphi}{\varphi[I|\top]} \text{ (I Pure)}$$

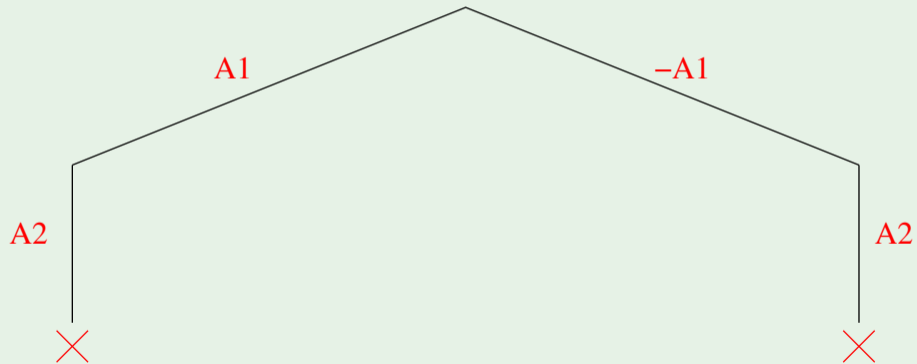
$$\frac{\varphi}{\varphi[I|\top] \quad \varphi[I|\perp]} \text{ (split)}$$

(I is a **pure literal** in φ iff it occurs **only positively**).

- Split applied **if and only if the others cannot be applied**.
- Richer formalisms described in [40, 29, 30]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



DPLL Algorithm

```
function  $DPLL(\varphi, \mu)$   
  if  $\varphi = \top$                                 /* base */  
    then return True;  
  if  $\varphi = \perp$                                 /* backtrack */  
    then return False;  
  if {a unit clause ( $l$ ) occurs in  $\varphi$ }          /* unit */  
    then return  $DPLL(assign(l, \varphi), \mu \wedge l)$ ;  
  if {a literal  $l$  occurs pure in  $\varphi$ }          /* pure */  
    then return  $DPLL(assign(l, \varphi), \mu \wedge l)$ ;  
   $l := choose\_literal(\varphi)$ ;                    /* split */  
  return  $DPLL(assign(l, \varphi), \mu \wedge l)$  or  
          $DPLL(assign(\neg l, \varphi), \mu \wedge \neg l)$ ;
```

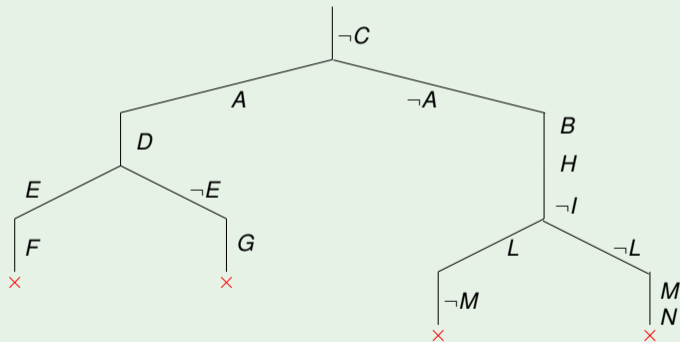
- The pure-literal rule is nowadays obsolete.
- $choose_literal(\varphi)$ picks only variables still occurring in the formula

DPLL – example

DPLL (without pure-literal rule)

Here “choose-literal” selects variable in alphabetic, selecting true first.

$(\neg C \vee A \vee B) \wedge$
 $(B \vee A \vee C) \wedge$
 $(\neg A \vee D \vee E) \wedge$
 $(\neg E \vee \neg A \vee F) \wedge$
 $(\neg E \vee \neg F \vee \neg A) \wedge$
 $(G \vee \neg A \vee E) \wedge$
 $(E \vee \neg G \vee \neg A) \wedge$
 $(A \vee H \vee C) \wedge$
 $(\neg H \vee \neg I \vee A) \wedge$
 $(I \vee L \vee M) \wedge$
 $(\neg L \vee C \vee \neg M) \wedge$
 $(A \vee \neg L \vee M) \wedge$
 $(L \vee N \vee \neg H) \wedge$
 $(I \vee L \vee \neg N)$



⇒ UNSAT

DPLL – summary

- Handles **CNF formulas** (non-CNF variant known [1, 15]).
- **Branches on truth values**
⇒ all instances of an atom assigned simultaneously
- **Postpones branching as much as possible.**
- Mostly ignored by logicians.
- (The grandfather of) **the most efficient SAT algorithms**
⇒ loved by computer scientists.
- Requires **polynomial space**
- **Choose_literal()** critical!
- Many very efficient implementations [42, 38, 2, 28].

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DPLL: “Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment /
- / is toggled, is labeled as “closed”, and the search proceeds.

DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible

⇒ **lots of useless search!**

DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

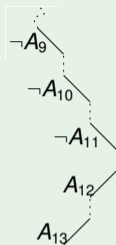
$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

...

DPLL Chronological Backtracking: Example

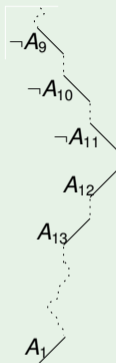
$C_1 : \neg A_1 \vee A_2$
 $C_2 : \neg A_1 \vee A_3 \vee A_9$
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
 $C_6 : \neg A_5 \vee \neg A_6$
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
 $C_8 : A_1 \vee A_8$
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
(initial assignment)

DPLL Chronological Backtracking: Example

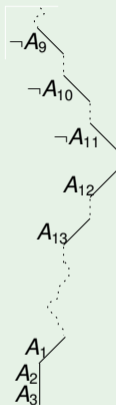
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$
... (branch on A_1)

DPLL Chronological Backtracking: Example

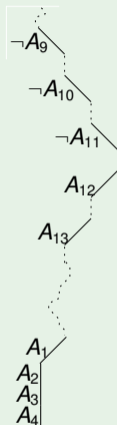
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$
(unit A_2, A_3)

DPLL Chronological Backtracking: Example

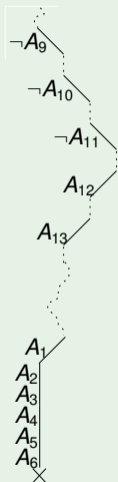
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4 \quad \checkmark$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$
(unit A_4)

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
 $C_6 : \neg A_5 \vee \neg A_6$ ✗
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
 $C_8 : A_1 \vee A_8$ ✓
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

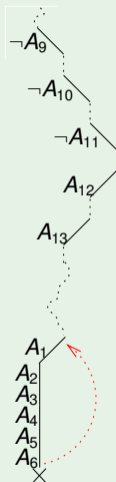


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) \implies conflict

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
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- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
 \implies backtrack up to A_1



DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2 \quad \checkmark$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

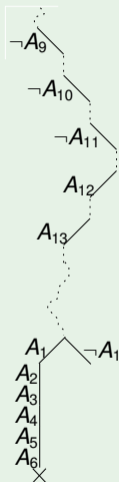
$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

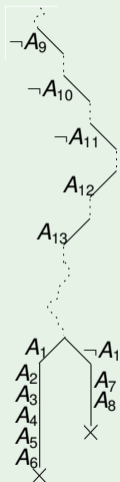
...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1\}$
(unit $\neg A_1$)



DPLL Chronological Backtracking: Example

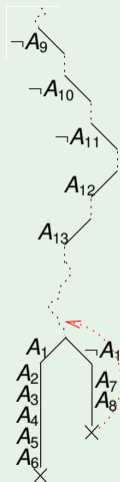
- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
 $C_6 : \neg A_5 \vee \neg A_6$
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
 $C_8 : A_1 \vee A_8$ ✓
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✗
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8\}$
(unit A_7, A_8) \implies conflict

DPLL Chronological Backtracking: Example

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- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

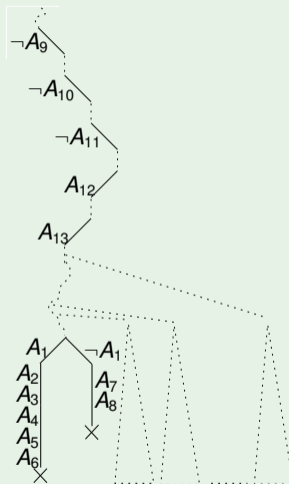
\implies backtrack to the most recent open branching point

DPLL Chronological Backtracking: Example

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- ...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

\Rightarrow lots of useless search before backtracking up to A_{13} !



Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers**
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 4 Ordered Binary Decision Diagrams – OBDDs
- 5 SAT Functionalities: proofs, unsat cores, interpolants, optimization

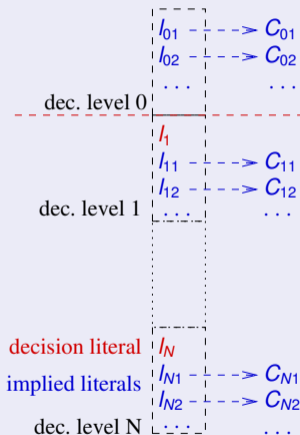
Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
 - inspired to conflict-driven backjumping and learning in CSPs
 - learns implied clauses as nogoods
- Random restarts
 - abandon the current search tree and restart on top level
 - previously-learned clauses maintained
- Smart literal selection heuristics (ex: VSIDS)
 - “static”: scores updated only at the end of a branch
 - “local”: privileges variable in recently learned clauses
- Smart preprocessing/inprocessing technique to simplify formulas
- Smart indexing techniques (e.g. 2-watched literals)
 - efficiently do/undo assignments and reveal unit clauses
- Allow Incremental Calls (stack-based interface)
 - allow for reusing previous search on “similar” problems

Can handle industrial problems with 10^6 – 10^7 variables and clauses!

Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into **decision levels**:
 - one **decision literal**
 - its **implied literals**
 - each implied literal tagged with the clause causing its unit-propagation (**antecedent clause**)
- equivalent to an **implication graph**

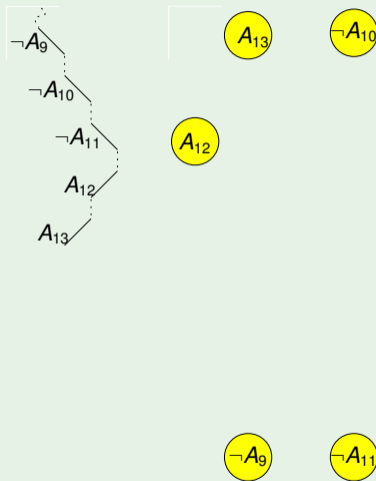


Implication graph

- An **implication graph** is a DAG s.t.:
 - each node represents a variable assignment (literal)
 - each edge $l_i \xrightarrow{c} l$ is labeled with a clause
 - the node of a decision literal has no incoming edges
 - all edges incoming into a node l are labeled with the same clause c , s.t. $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$ iff $c = \neg l_1 \vee \dots \vee \neg l_n \vee l$
(c is said to be the **antecedent clause** of l)
 - when both l and $\neg l$ occur in the graph, we have a **conflict**.
- Intuition:
 - representation of the dependencies between literals in μ
 - the graph contains $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$ iff l has been obtained from l_1, \dots, l_n by unit propagation on c
 - a partition of the graph with all decision literals on one side and the conflict on the other represents a **conflict set**

Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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- ...

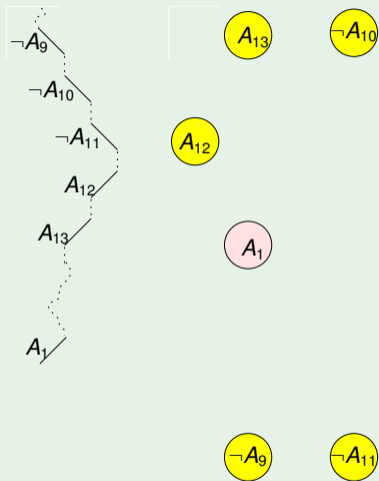


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

(Initial assignment. Note: c_1, \dots, c_9 inconsistent.)

Example

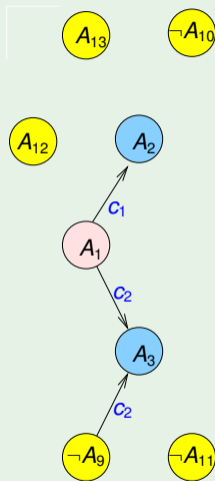
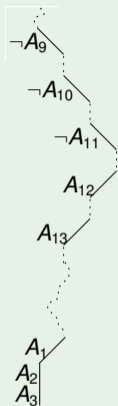
- $C_1 : \neg A_1 \vee A_2$
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- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$
... (decide A_1)

Example

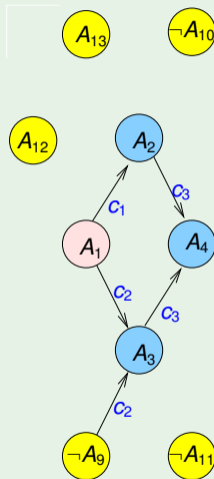
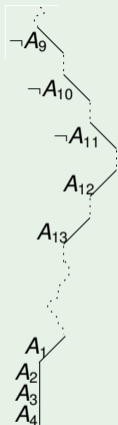
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- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$
(unit A_2, A_3)

Example

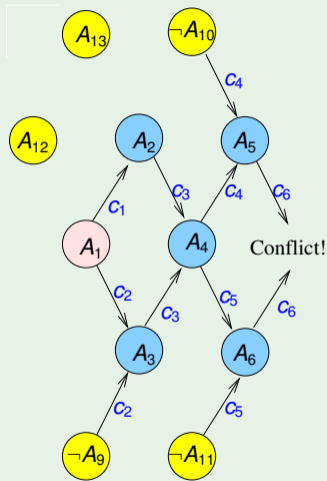
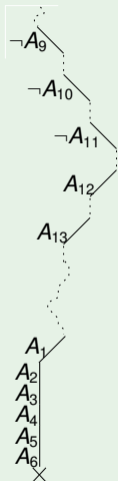
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- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
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- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$
(unit A_4)

Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
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- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
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- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, \neg A_{12}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4, A_5, A_6\}$
 (unit A_5, A_6) \implies conflict

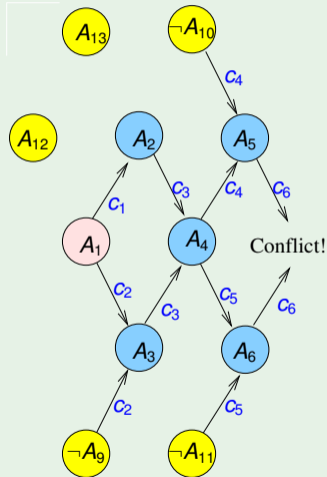
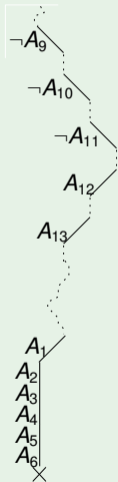
Unique implication point - UIP [44]

- A node l in an implication graph is an **unique implication point** (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through l .
 - the most recent decision node is an UIP (**last UIP**)
 - all other UIP's have been assigned after the most recent decision

Unique implication point - UIP - example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...

- A_1 is the last UIP
- A_4 is the 1st UIP



Schema of a CDCL DPLL solver [38, 45]

```
Function CDCL-SAT (formula:  $\varphi$ , assignment &  $\mu$ ) {
  status := preprocess( $\varphi, \mu$ );
  while (1) {
    while (1) {
      status := deduce( $\varphi, \mu$ );
      if (status == Sat)
        return Sat;
      if (status == Conflict) {
         $\langle \text{blevel}, \eta \rangle := \text{analyze\_conflict}(\varphi, \mu)$ ;
        //  $\eta$  is a conflict set
        if (blevel == 0)
          return Unsat;
        else backtrack(blevel,  $\varphi, \mu$ );
      }
      else break;
    }
    decide_next_branch( $\varphi, \mu$ );
  }
}
```

Schema of a CDCL DPLL solver [38, 45] (cont.)

- `preprocess` (φ, μ) simplifies φ into an easier equisatisfiable formula, updating μ .
- `decide_next_branch` (φ, μ) chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce` (φ, μ) performs all deterministic assignments (unit-propagations plus others), and updates φ, μ accordingly.
- `analyze_conflict` (φ, μ) Computes the subset η of μ causing the conflict (conflict set), and returns the “wrong-decision” level suggested by η (“0” means that η is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack` (`blevel`, φ, μ) undoes the branches up to `blevel`, and updates φ, μ accordingly

Backjumping and learning: general ideas [2, 38]

- When a branch μ fails:
 - (i) **conflict analysis**: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (**conflict set η**)
 - (ii) **learning**: add the **conflict clause** $C \stackrel{\text{def}}{=} \neg\eta$ to the clause set
 - (iii) **backjumping**: use η to decide the point where to backtrack
- Jump back up much more than one decision level in the stack
 \implies **may avoid lots of redundant search!!**.
- We illustrate two main backjumping & learning strategies:
 - the original strategy presented in [38]
 - the state-of-the-art 1stUIP strategy of [44]

Conflict analysis

1. $C :=$ falsified clause (**conflicting clause**)
2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in Cuntil C verifies some given termination criteria

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criterion: **decision**

...until C contains only decision literals

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\frac{\frac{\neg A_1 \vee A_2}{\neg A_1 \vee A_3 \vee A_9}}{\neg A_2 \vee \neg A_3 \vee A_4}}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}}{\neg A_4 \vee A_5 \vee A_{10}}}{\neg A_4 \vee A_6 \vee A_{11}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}}{\neg A_4 \vee A_{10} \vee A_{11}}}{\neg A_5 \vee \neg A_6} \text{ (A}_6\text{)} \\
 \text{Conflicting cl.} \\
 \text{(A}_5\text{)} \\
 \text{(A}_4\text{)} \\
 \text{(A}_3\text{)} \\
 \text{(A}_2\text{)}
 \end{array}$$

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 until C verifies some given termination criteria

criterion: **last UIP**

... until C contains only one literal assigned at current decision level, and it is the decision literal (**last UIP**)

$$\begin{array}{r}
 \neg A_1 \vee A_2 \\
 \hline
 \neg A_1 \vee A_3 \vee A_9 \quad \neg A_2 \vee \neg A_3 \vee A_4 \\
 \hline
 \neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11} \quad \neg A_4 \vee \neg A_3 \vee A_{10} \vee A_{11} \\
 \hline
 \neg A_1 \vee A_9 \vee A_{10} \vee A_{11} \quad \neg A_4 \vee A_{10} \vee A_{11} \\
 \hline
 \neg A_2 \vee \neg A_3 \vee A_4 \quad \neg A_4 \vee A_5 \vee A_{10} \\
 \hline
 \neg A_4 \vee \neg A_3 \vee A_4 \quad \neg A_4 \vee \neg A_5 \vee A_{11} \\
 \hline
 \neg A_4 \vee A_5 \vee A_{10} \quad \neg A_4 \vee A_{10} \vee A_{11} \\
 \hline
 \neg A_4 \vee A_6 \vee A_{11} \quad \overbrace{\neg A_5 \vee \neg A_6}^{\text{Conflicting cl.}} \\
 \hline
 \neg A_4 \vee A_6 \vee A_{11} \vee \neg A_5 \vee \neg A_6 \quad (A_6)
 \end{array}$$

Conflict analysis

1. $C :=$ falsified clause (**conflicting clause**)
2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C
 until C verifies some given termination criteria

critterion: **1st UIP**

... until C contains only one literal assigned at current decision level (**1st UIP**)

$$\begin{array}{c}
 \text{Conflicting cl.} \\
 \overline{\neg A_4 \vee A_6 \vee A_{11}} \quad \overline{\neg A_5 \vee \neg A_6} \quad (A_6) \\
 \hline
 \neg A_4 \vee A_5 \vee A_{10} \quad \neg A_4 \vee \neg A_5 \vee A_{11} \quad (A_5) \\
 \hline
 \underbrace{\neg A_4}_{\text{1st UIP}} \vee A_{10} \vee A_{11}
 \end{array}$$

Conflict analysis

1. C := falsified clause (**conflicting clause**)
2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in Cuntil C verifies some given termination criteria

Note:

$\varphi \models C$, so that C can be safely added to φ .

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

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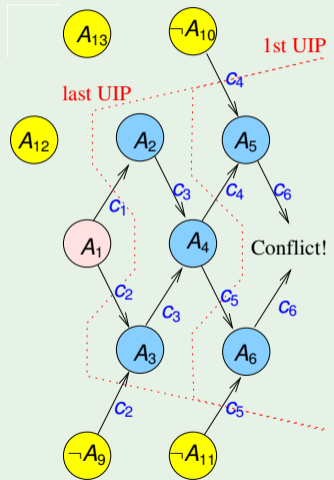
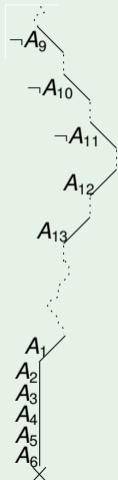
Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis and implication graph - example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓✓✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓✓✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓✓✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓✓✓
- $C_8 : A_1 \vee A_8$ ✓✓✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓✓✓
- ...

Note: in

this case decision and last-UIP criteria produce the same partition

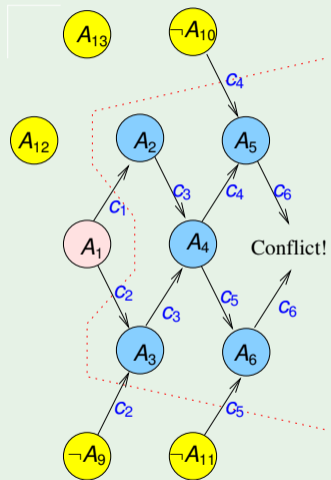
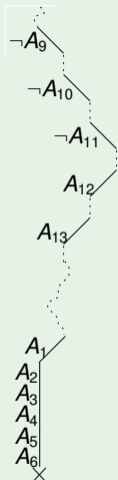


The original backjumping and learning strategy of [38]

- Idea: when a branch μ fails,
 - (i) **conflict analysis**: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the “Decision” criterion;
 - (ii) **learning**: add the conflict clause C to the clause set
 - (iii) **backjumping**: backtrack to the most recent branching point s.t. the stack does not fully contain η , and then unit-propagate the unassigned literal on C

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



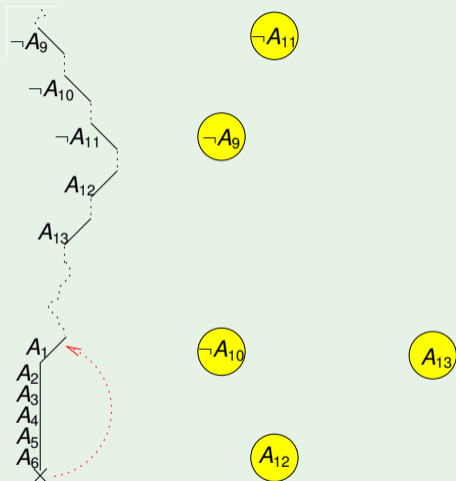
⇒ **Conflict set:** $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ ("decision" schema)

⇒ learn the conflict clause $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
- ...

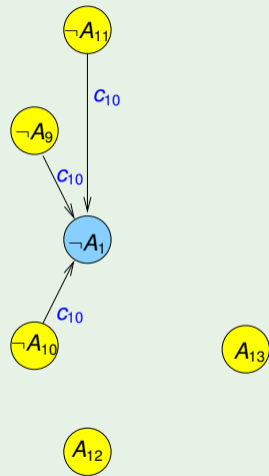
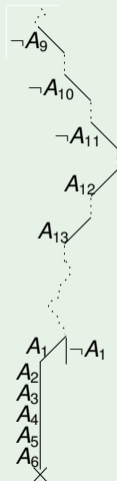
$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
 \implies backtrack up to A_1



The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- ...

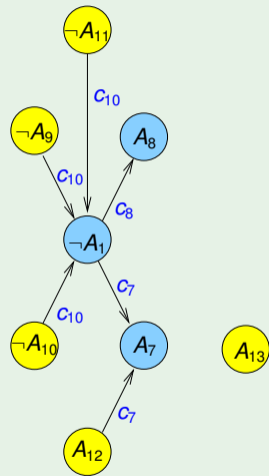
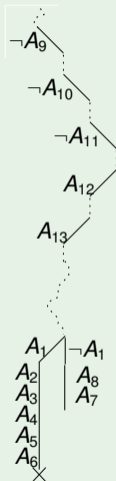
{ ..., $\neg A_9$, $\neg A_{10}$, $\neg A_{11}$, A_{12} , A_{13} , ..., $\neg A_1$ }
 (unit $\neg A_1$)



The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- ...

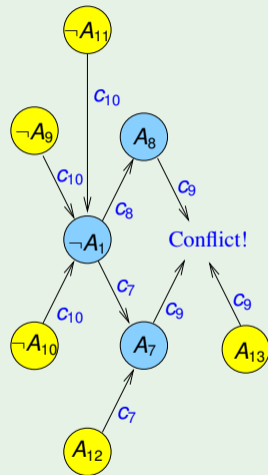
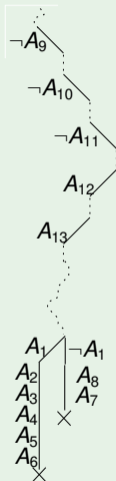
{ ..., $\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8$ }
 (unit A_7, A_8)



The Original Backjumping Strategy: Example

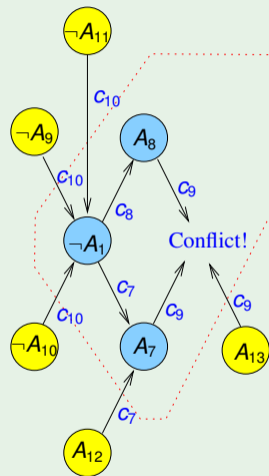
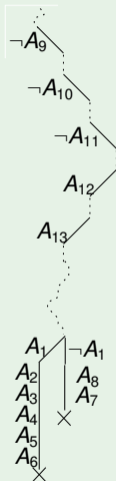
- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
 $C_6 : \neg A_5 \vee \neg A_6$
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
 $C_8 : A_1 \vee A_8$ ✓
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✗
 $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
...

{..., $\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8$ }
Conflict!



The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✗
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- ...

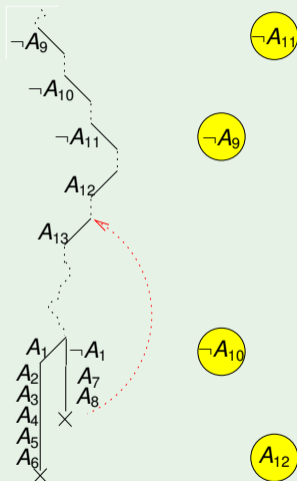


⇒ conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.

⇒ learn $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

The Original Backjumping Strategy: Example

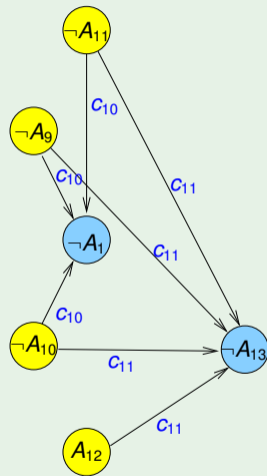
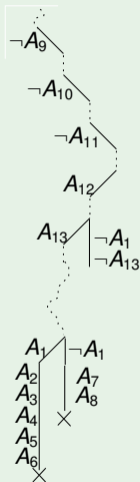
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$
- ...



⇒ backtrack to A_{13} ⇒ Lots of search saved!

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ ✓
- ...



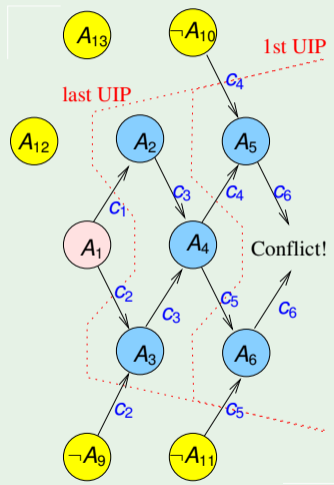
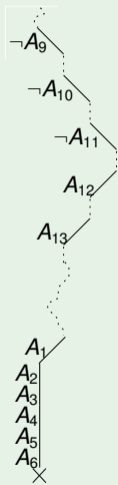
⇒ backtrack to A_{13} , then set A_{13} and A_1 to \perp, \dots

State-of-the-art backjumping and learning [44]

- Idea: when a branch μ fails,
 - (i) **conflict analysis**: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the **1stUIP strategy**
 - (ii) **learning**: add the conflict clause C to the clause set
 - (iii) **backjumping**: **backtrack to the highest branching point s.t. the stack contains all-but-one literals in η , and then unit-propagate the unassigned literal on C**

1st UIP strategy – example (7)

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



⇒ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping [44]

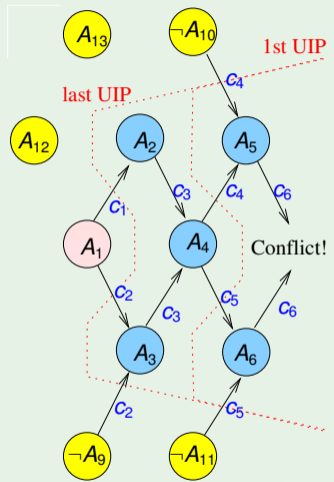
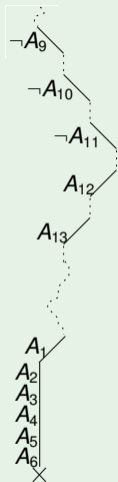
- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

\implies backtrack to A_{11} , then assign $\neg A_4$

1st UIP strategy – example (7)

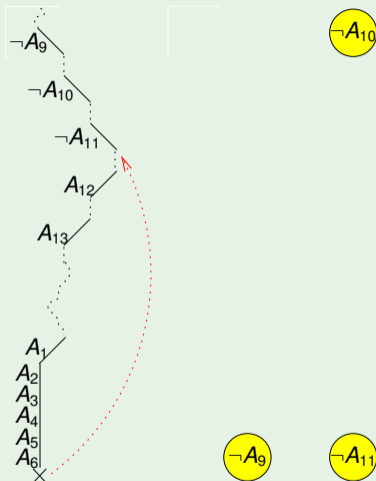
- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



⇒ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy – example (8)

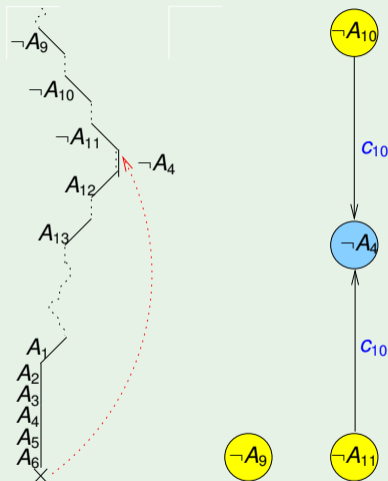
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4$
- ...



\Rightarrow backtrack up to $A_{11} \Rightarrow \{\dots, \neg A_9, \neg A_{10}, \neg A_{11}\}$

1st UIP strategy – example (9)

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10} \quad \checkmark$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11} \quad \checkmark$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4 \quad \checkmark$
- ...



\Rightarrow unit propagate $\neg A_4 \Rightarrow \{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_4\} \dots$

1st UIP strategy and backjumping – intuition

- An UIP is a **single** reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
 - does not enlarge the conflict
 - requires less resolution steps to compute C
 - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
 - jump higher
 - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [2, 38]

Idea: When a conflict set η is revealed, then $C \stackrel{\text{def}}{=} \neg\eta$ added to φ

\implies the solver will no more generate an assignment containing η : when $|\eta| - 1$ literals in η are assigned, the other is set \perp by unit-propagation on C

\implies **Drastic pruning of the search!**

Learning – example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

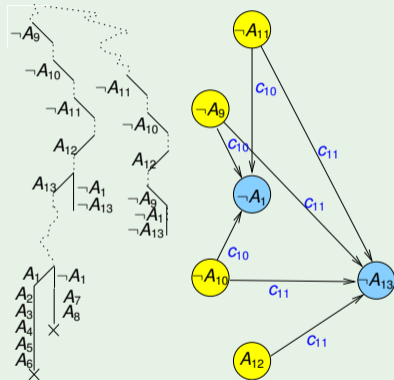
$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13} \quad \checkmark$$

$$C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1 \quad \checkmark$$

$$C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13} \quad \checkmark$$

...

⇒ Unit: $\{\neg A_1, \neg A_{13}\}$



Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their **activity**.

A clause is currently **active** if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [30])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.

⇒ CDCL solvers require polynomial space

“Lazy” Strategy

- when a clause is involved in conflict analysis, increase its activity
- when needed, drop the least-active clauses

State-of-the-art backjumping and learning: intuitions

- **Backjumping:** allows for climbing up to many decision levels in the stack
 - intuition: “go back to the oldest decision where you’d have done something different if only you had known C ”
⇒ may avoid lots of redundant search
- **Learning:** in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
 - intuition: “when you’re about to repeat the mistake, do the opposite of the last step”
⇒ avoid finding the same conflict again

Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
 - Backjumping: if causes the highest backjump (“local” role)
 - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented (see, e.g., [2, 38, 44])

Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements**
 - SAT Under Assumptions & Incremental SAT
- 4 Ordered Binary Decision Diagrams – OBDDs
- 5 SAT Functionalities: proofs, unsat cores, interpolants, optimization

Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
 - detect and remove subsumed clauses
 - detect & collapse equivalent literals
 - apply basic resolution steps
 - ...

Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\begin{aligned} \varphi_1 \wedge (b_2 \vee h_1) \wedge \varphi_2 \wedge (b_2 \vee b_3 \vee h_1) \wedge \varphi_3 \\ \Downarrow \\ \varphi_1 \wedge (h_1 \vee b_2) \wedge \varphi_2 \wedge \varphi_3 \end{aligned}$$

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [7]

Repeat:

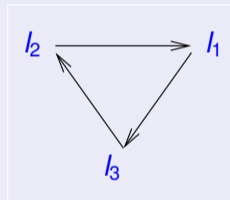
- (i) build the implication graph induced by binary clauses
- (ii) detect **strongly connected cycles** \implies **equivalence classes of literals**
- (iii) perform substitutions
- (iv) perform unit and pure literal.

Until (no more simplification is possible).

• Ex:

$$\begin{aligned} & \varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4 \\ & \quad \downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3} \\ & (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;] \end{aligned}$$

• Very effective in many application domains.



Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$

\Downarrow *resolve*

$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$

\Downarrow *unit-propagate*

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- **Branch** is the source of non-determinism for DPLL
 - ⇒ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

Some Heuristics

- **MOMS** heuristics (DPLL): pick the literal occurring **m**ost **o**ften in the **m**inimal **s**ize clauses
⇒ fast and simple, many variants
- **Jeroslow-Wang** (DPLL): choose the literal with maximum

$$\text{score}(l) := \sum_{l \in c \ \& \ c \in \varphi} 2^{-|c|}$$

⇒ estimates l 's contribution to the satisfiability of φ

- **Satz** [21] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
⇒ maximizes the effects of unit propagation
- **VSIDS** [28] (CDCL+): **v**ariable **s**tate **i**ndependent **d**ecaying **s**um
 - “static”: scores updated only at the end of a branch
 - “local”: privileges variable in recently learned clauses

Restarts [16]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- avoid getting stuck in certain areas of the search space

⇒ may significantly reduce the overall search space

Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT**
- 4 Ordered Binary Decision Diagrams – OBDDs
- 5 SAT Functionalities: proofs, unsat cores, interpolants, optimization

SAT under assumptions: $SAT(\varphi, \{l_1, \dots, l_n\})$ [12]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals

$\mathcal{A} \stackrel{\text{def}}{=} \{l_1, \dots, l_n\}$: $SAT(\varphi, \{l_1, \dots, l_n\})$

- $SAT(\varphi, \{l_1, \dots, l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^n l_i)$
- often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1)$, $SAT(\varphi, \mathcal{A}_2)$, ...
- Idea:
 - l_1, \dots, l_n “decided” at decision level 0 before starting the search
 - if backtrack to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT

Property

If the “decision” strategy for conflict analysis is used,
then η is the subset of assumptions causing the inconsistency

Selection of sub-formulas

Idea: select clauses [12, 23]

Let φ be $\bigwedge_{i=1}^n C_i$.

- let $S_1 \dots S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$

\Rightarrow $\text{SAT}(\bigwedge_{i=1}^n (\neg S_i \vee C_i), \mathcal{A})$: same as $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (C_i))$

- if S_i is not assumed, then $\neg S_i \vee C_i$ does not contribute to search

\Rightarrow “Select” (activate) only a subset of the clauses in φ at each call.

Generalised Idea: select blocks of clauses

Let φ be $\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} C_{ij})$.

- let $S_1 \dots S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$
- $\text{SAT}(\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} (\neg S_i \vee C_{ij})), \mathcal{A})$: same as $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (\bigwedge_{j=1}^{n_i} C_{ij}))$

\Rightarrow Allows for “selecting” block of clauses at each call.

Example

- Initial formula φ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

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$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

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$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

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$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Incremental SAT solving [12, 11]

- Many CDCL solvers provide a **stack-based incremental interface**
 - it is possible to push/pop ϕ_i into a stack of subformulas $\{\phi_1, \dots, \phi_k\}$
 - check incrementally the satisfiability of $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^k \phi_i$.
 - Maintains the **status** of the search from one call to the other
 - in particular it records the **learned clauses** (plus other information)
 - \Rightarrow **reuses search from one call to another**
 - Very useful in many applications (in particular in FV)
-
- Idea: **incremental** calls $SAT(\varphi', \mathcal{A}_1), SAT(\varphi', \mathcal{A}_2), \dots$
 - $\varphi' \stackrel{\text{def}}{=} \bigwedge_i (\neg S_i \vee \phi_i)$, $\mathcal{A}_j \subseteq \{S_1, \dots, S_k\}$, $(\neg S_i \vee \bigwedge_j C_{ij}) \stackrel{\text{def}}{=} \bigwedge_j (\neg S_i \vee C_{ij})$
 - push/pop selection variables S_i
 - in practice, also subformulas ϕ_i can be pushed/popped
 - Key efficiency issue: **learned clauses safely reused from call to call (even if assumptions have been popped)**
 - a learned clause $C \stackrel{\text{def}}{=} \bigvee_j \neg S_j \vee C'$ is s.t. $\bigwedge_j (\neg S_j \vee \phi_j) \models C$
 - \Rightarrow C contains the vars selecting the subformulas it is derived from
 - \Rightarrow in $SAT(\varphi', \mathcal{A})$, if some $S_j \notin \mathcal{A}$, then C is not active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

[push(S_1)]: $SAT(\varphi', \{\dots, S_1\})$: ϕ_1 active \implies learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

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$$(\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1$$

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- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \end{array}$$

[push(S_2): SAT(φ' , $\{\dots, S_1, S_2\}$): ϕ_1, ϕ_2 active \implies learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \\ (\neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_2, \text{ inactive} \end{array}$$

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$$\begin{array}{l} \dots \\ (\neg A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

$$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3$$

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$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2, \textit{inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2, \textit{inactive} \\ (\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3 \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_2, \textit{inactive} \end{array}$$

[pop(S_2);push(S_3): SAT(φ' , { \dots , S_1 , S_3 })]: ϕ_1, ϕ_3 active $\implies \dots$

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$$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3$$

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$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2, \textit{inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2, \textit{inactive} \\ (\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3 \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_2, \textit{inactive} \end{array}$$

[pop(S_2);push(S_3): SAT(φ' , { \dots , S_1 , S_3 })]: ϕ_1, ϕ_3 active $\implies \dots$

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Ordered Binary Decision Diagrams (OBDDs) [8]

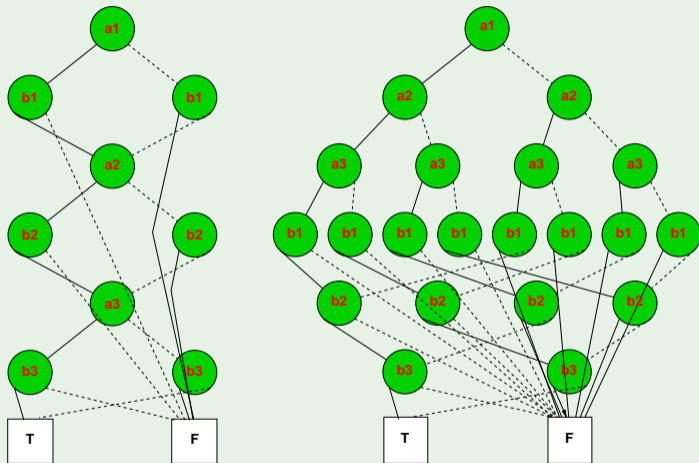
Canonical representation of Boolean formulas

- “If-then-else” binary direct acyclic graphs (DAGs) with one root and two leaves: **1**, **0** (or **T**, **⊥**; or **T**, **F**)
- **Variable ordering** A_1, A_2, \dots, A_n imposed a priori.
- Paths leading to **1** represent **models**
Paths leading to **0** represent **counter-models**

Note

Some authors call them **Reduced** Ordered Binary Decision Diagrams (**ROBDDs**)

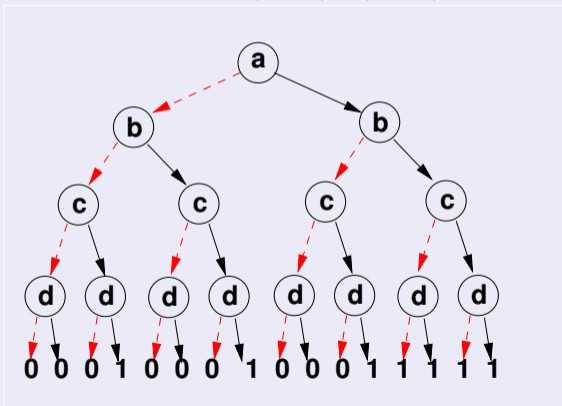
OBDD - Examples



OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- **Ordered Decision Tree:**
from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

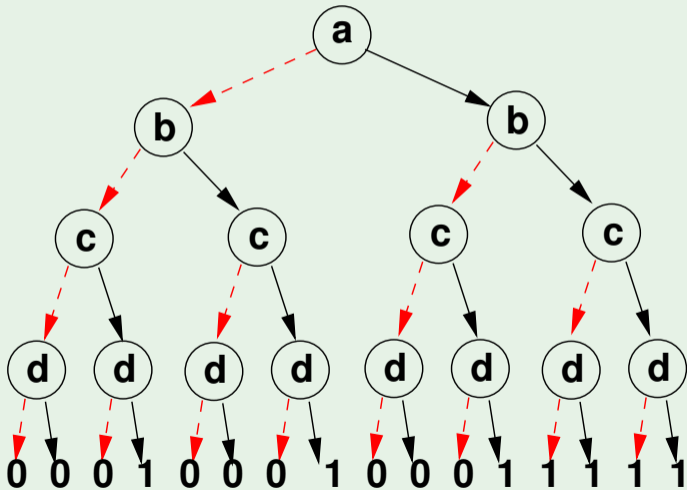


From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following **reductions**:
 - **share subnodes**: point to the same occurrence of a subtree (via **hash consing**)
 - **remove redundancies**: nodes with same left and right children can be eliminated:
"if A then B else B " \implies " B "

Reduction: example

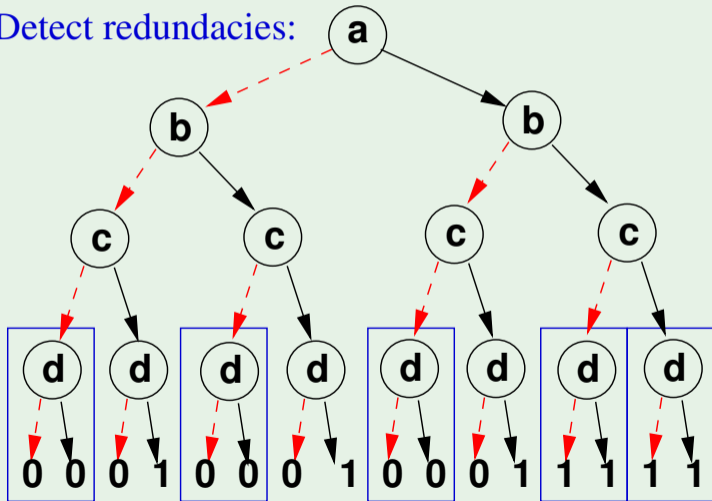
$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

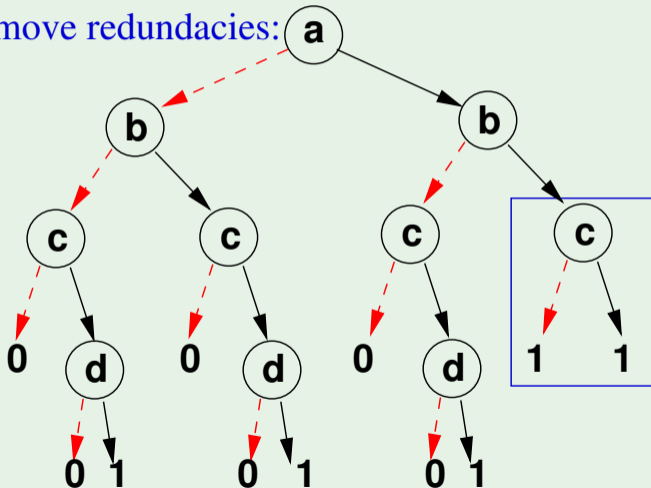
Detect redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

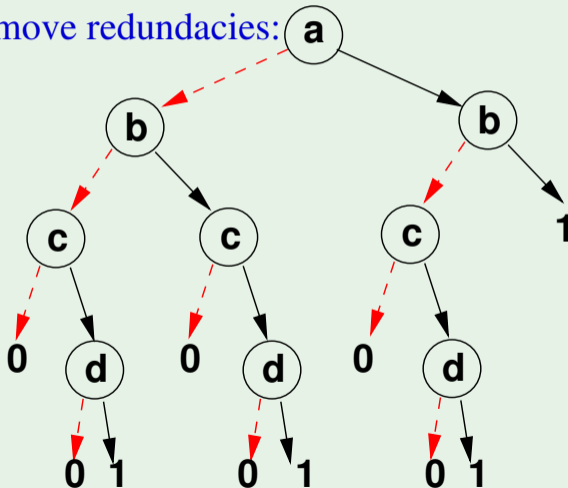
Remove redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

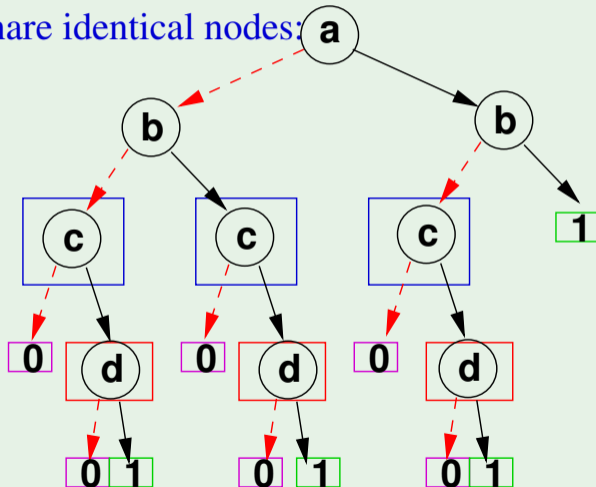
Remove redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

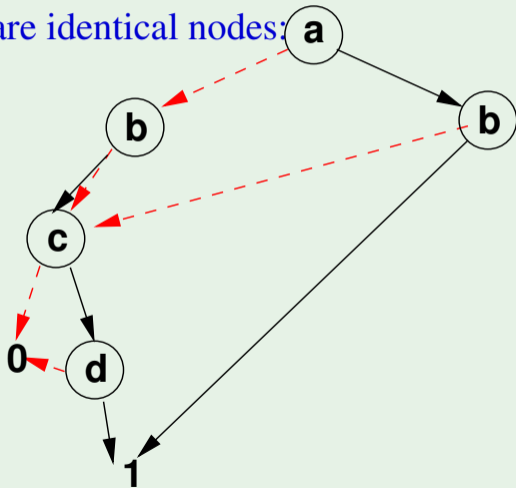
Share identical nodes:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

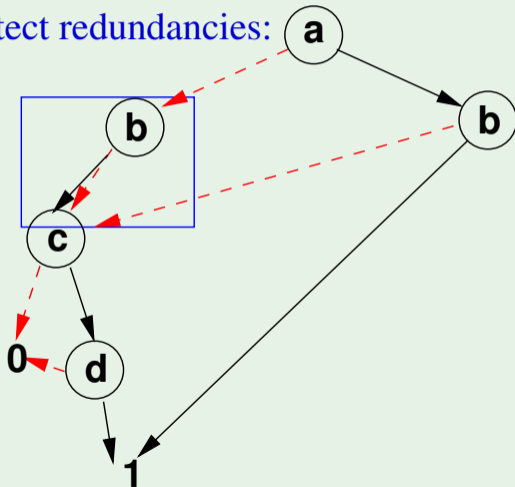
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Reduction: example

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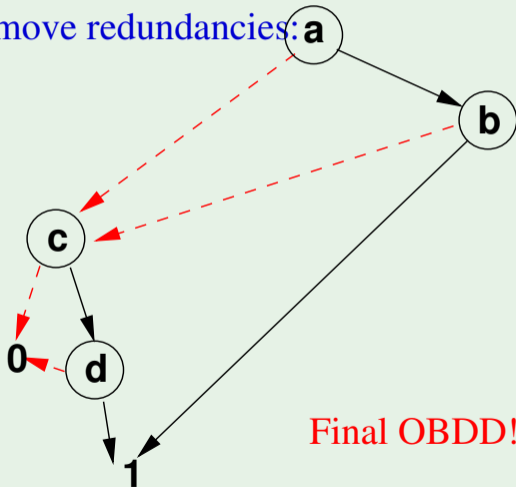
Detect redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

Remove redundancies:



If-Then-Else Operators: “*ite(...)*”

If-Then-Else Operators: “*ite(...)*”

● *ite*($\phi, \varphi^\top, \varphi^\perp$): “If ϕ Then φ^\top Else φ^\perp ”

● $\text{ite}(\phi, \varphi^\top, \varphi^\perp) \stackrel{\text{def}}{=} ((\neg\phi \vee \varphi^\top) \wedge (\phi \vee \varphi^\perp)) \iff ((\phi \wedge \varphi^\top) \vee (\neg\phi \wedge \varphi^\perp))$

● properties:

$$\neg \text{ite}(\phi, \varphi^\top, \varphi^\perp) = \text{ite}(\phi, \neg\varphi^\top, \neg\varphi^\perp)$$

$$\text{ite}(\phi, \varphi_1^\top, \varphi_1^\perp) \text{ op } \text{ite}(\phi, \varphi_2^\top, \varphi_2^\perp) = \text{ite}(\phi, (\varphi_1^\top \text{ op } \varphi_2^\top), (\varphi_1^\perp \text{ op } \varphi_2^\perp))$$

$$\text{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \text{ op } \text{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp) = \text{ite}(\phi_1, (\varphi_1^\top \text{ op } \text{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp)), (\varphi_1^\perp \text{ op } \text{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp)))$$

$$= \text{ite}(\phi_2, (\text{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \text{ op } \varphi_2^\top), (\text{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \text{ op } \varphi_2^\perp))$$

$\text{op} \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Recursive structure of an OBDD

Assume the variable ordering A_1, A_2, \dots, A_n :

$$\begin{aligned} \text{OBDD}(\top, \{A_1, A_2, \dots, A_n\}) &= 1 \\ \text{OBDD}(\perp, \{A_1, A_2, \dots, A_n\}) &= 0 \\ \text{OBDD}(\varphi, \{A_1, A_2, \dots, A_n\}) &= \begin{aligned} &\text{if } A_1 \\ &\text{then } \text{OBDD}(\varphi[A_1|\top], \{A_2, \dots, A_n\}) \\ &\text{else } \text{OBDD}(\varphi[A_1|\perp], \{A_2, \dots, A_n\}) \end{aligned} \end{aligned}$$

Incrementally building an OBDD

- $obdd_build(\top, \{\dots\}) := \top$,
- $obdd_build(\perp, \{\dots\}) := \perp$,
- $obdd_build(A_i, \{\dots\}) := ite(A_i, \top, \perp)$,
- $obdd_build((\neg\varphi), \{A_1, \dots, A_n\}) := apply(\neg, obdd_build(\varphi, \{A_1, \dots, A_n\}))$
- $obdd_build((\varphi_1 \text{ op } \varphi_2), \{A_1, \dots, A_n\}) :=$
 $reduce($
 $apply($ $op,$
 $obdd_build(\varphi_1, \{A_1, \dots, A_n\}),$ $op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
 $obdd_build(\varphi_2, \{A_1, \dots, A_n\})$
 $)$
 $)$

Incrementally building an OBDD (cont.)

- $apply(op, O_i, O_j) := (O_i op O_j)$ **if** $(O_i \in \{\top, \perp\}$ or $O_j \in \{\top, \perp\})$
- $apply(\neg, ite(A_i, \varphi_i^\top, \varphi_i^\perp)) :=$
 $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\perp))$
- $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), ite(A_j, \varphi_j^\top, \varphi_j^\perp)) :=$
if $(A_i = A_j)$ **then** $ite(A_i, apply(op, \varphi_i^\top, \varphi_j^\top),$
 $apply(op, \varphi_i^\perp, \varphi_j^\perp))$
if $(A_i < A_j)$ **then** $ite(A_i, apply(op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\perp)),$
 $apply(op, \varphi_i^\perp, ite(A_j, \varphi_j^\top, \varphi_j^\perp)))$
if $(A_i > A_j)$ **then** $ite(A_j, apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\top),$
 $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\perp))$

$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

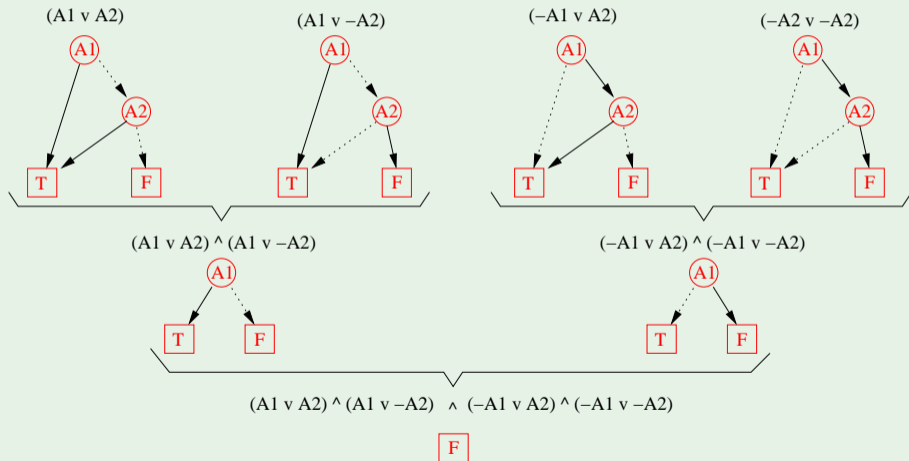
$$\begin{aligned} & \text{apply}(\vee, \overbrace{\text{ite}(A_1, \top, \perp)}^{A_1}, \overbrace{\text{ite}(A_2, \top, \perp)}^{A_2}) \\ &= \text{ite}(A_1, \text{apply}(\vee, \top, \text{ite}(A_2, \top, \perp)), \text{apply}(\vee, \perp, \text{ite}(A_2, \top, \perp))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp)) \end{aligned}$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2), (A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$\begin{aligned} & \text{apply}(\wedge, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp))}^{(A_1 \vee A_2)}, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \perp, \top))}^{(A_1 \vee \neg A_2)}), \\ &= \text{ite}(A_1, \text{apply}(\wedge, \top, \top), \text{apply}(\wedge, \text{ite}(A_2, \top, \perp), \text{ite}(A_2, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \text{apply}(\wedge, \top, \perp), \text{apply}(\wedge, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \perp, \perp)) \\ &= \text{ite}(A_1, \top, \perp) \end{aligned}$$

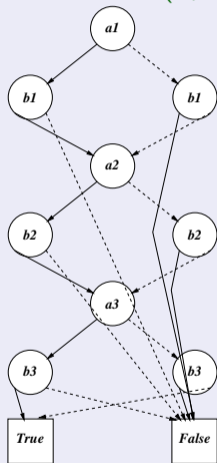
OBDD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

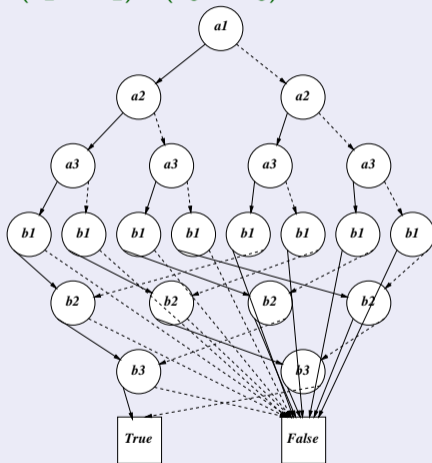


Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$



Linear size



Exponential size

OBDD's as canonical representation of Boolean formulas

- An OBDD is a **canonical representation** of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff \text{OBDD}(\varphi_1) = \text{OBDD}(\varphi_2)$$

- equivalence check requires **constant time!**
 - ⇒ validity check requires constant time! ($\varphi \leftrightarrow \top$)
 - ⇒ (un)satisfiability check requires constant time! ($\varphi \leftrightarrow \perp$)
- the set of the paths from the root to 1 represent all the **models** of the formula
- the set of the paths from the root to 0 represent all the **counter-models** of the formula

Exponentiality of OBDD's

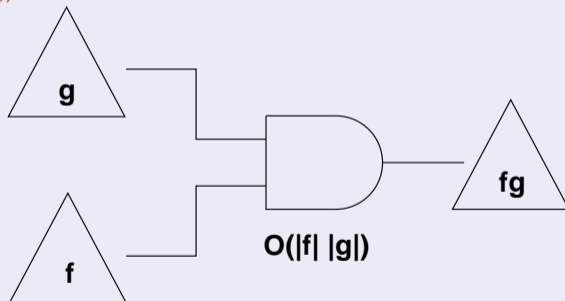
- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless $P = \text{co-NP}$)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

Note

The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

Useful Operations over OBDDs

- the **equivalence check** between two OBDDs is simple
 - are they the same OBDD? (\implies constant time)
- the size of a **Boolean composition** is up to the product of the size of the operands:
 $|f \text{ op } g| = O(|f| \cdot |g|)$



(but typically much smaller on average).

[Recall] Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

• Intuition:

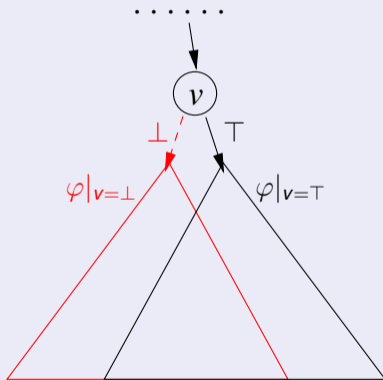
- $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \perp\}$ s.t. $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \perp\}$, $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
 - if f is a sub-OBDD labeled by variable v , then $\varphi|_{v=\top}$ and $\varphi|_{v=\perp}$ are the “then” and “else” branches of f

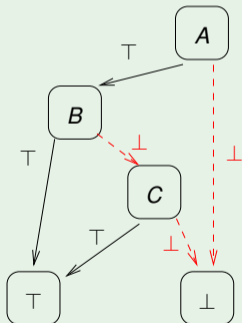


⇒ lots of sharing of subformulae!

Example

Let $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. \varphi$. Using the variable ordering “A, B, C”, draw the OBDD corresponding to the formulas φ and φ' .

$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$

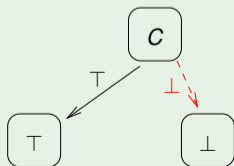


Example (cont.)

$$\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. (A \wedge (B \vee C))$$

$$\begin{aligned} \varphi' &\stackrel{\text{def}}{=} \exists A. \forall B. \varphi \\ &= \forall B. (A \wedge (B \vee C)) [A := \top] && \vee (\forall B. (A \wedge (B \vee C))) [A := \perp] \\ &= \forall B. (B \vee C) && \vee \forall B. \perp \\ &= ((B \vee C) [B := \top]) \quad \wedge \quad (B \vee C) [B := \perp] && \vee \quad \perp \\ &= (\top \quad \wedge \quad C) \\ &= C \end{aligned}$$

which corresponds to the following OBDD:



OBDD – summary

- **Factorize** common parts of the search tree (DAG)
- Require setting a **variable ordering** a priori (**critical!**)
- **Canonical representation** of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents **all** models and counter-models of the formula.
- Require **exponential space** in worst-case
- **Very efficient** for some practical problems (circuits, symbolic model checking).

Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Modern CDCL SAT Solvers
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 4 Ordered Binary Decision Diagrams – OBDDs
- 5 **SAT Functionalities: proofs, unsat cores, interpolants, optimization**

Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

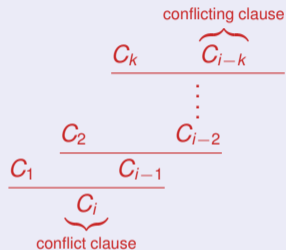
- Building **proofs of unsatisfiability**
- Extracting **unsatisfiable Cores**
- Computing **Craig Interpolants**
- Enumeration in SAT: **AIISAT** (hints)
- Optimization in SAT: **MaxSAT** (hints)

Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
 - to verify the result of the solver
 - to understand a “reason” for unsatisfiability
 - to build unsatisfiable cores and interpolants
- Can be built by **keeping track of the resolution steps performed when constructing the conflict clauses.**

Building Proofs of Unsatisfiability

- Recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal



- C_1, \dots, C_k , and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

$k \quad i-k \quad \rightarrow \quad i-k-1$

\dots

$2 \quad i-2 \quad \rightarrow \quad i-1$

$1 \quad i-1 \quad \rightarrow \quad i$

Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces “false” as conflict clause:

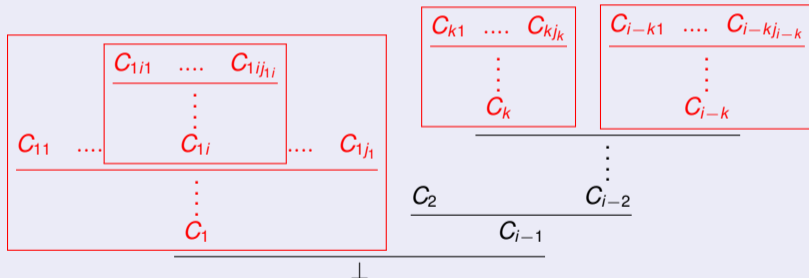
$$\begin{array}{c} \text{conflicting clause} \\ C_k \quad \overbrace{C_{i-k}} \\ \hline \vdots \\ C_2 \quad C_{i-2} \\ \hline C_1 \quad C_{i-1} \\ \hline \perp \end{array}$$

- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- C_1, \dots, C_k , and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

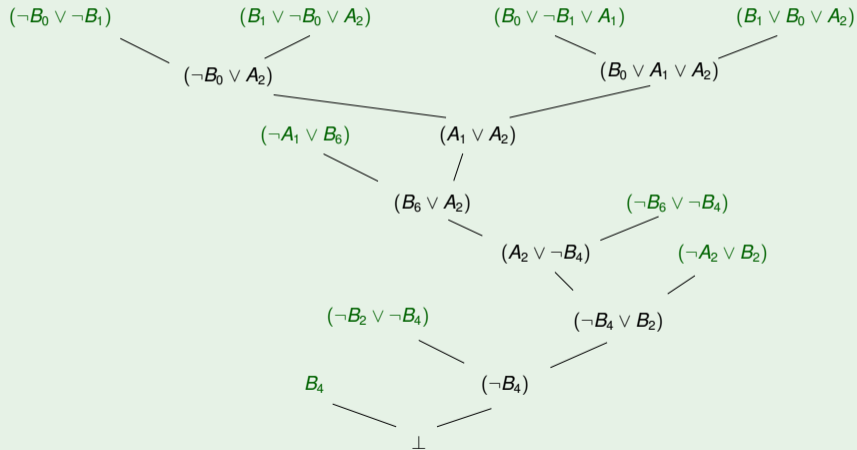
- for every **learned** leaf clause C_i , substitute C_i with the resolution proof generating it until all leaf clauses are original clauses



\Rightarrow We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$
 $(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$



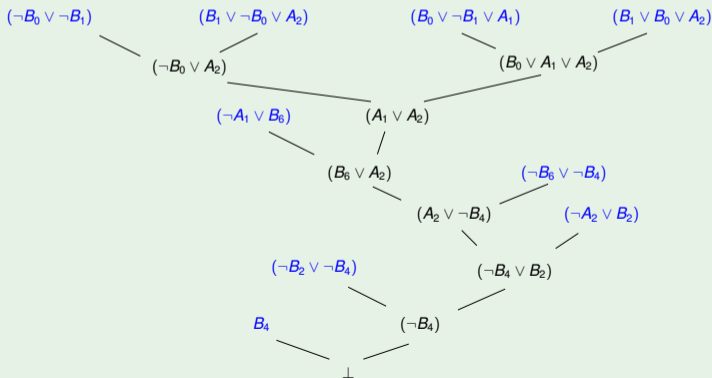
Extraction of unsatisfiable cores

- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset
 - ⇒ **unsatisfiable cores** (aka **(Minimal) Unsatisfiable Subsets, (M)US**)
- Lots of literature on the topic [46, 24, 26, 31, 43, 19, 13, 6]
- We recognize two main approaches:
 - **Proof-based** approach [46]: byproduct of finding a resolution proof
 - **Assumption-based** approach [24]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
 - repeat the process up to fixpoint
 - remove clauses one-by one, until satisfiability is obtained
 - combinations of the two processed above
 - ...

The proof-based approach to core extraction [46]

Unsat core: the set of leaf clauses of a resolution proof

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge \\ (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$



The assumption-based approach to core extraction [24]

Based on the following process:

- (i) each clause C_i is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh “selector” variable
- (ii) before starting the search each S_i is forced to true.
- (iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$
 $\implies \{C_j\}_j$ is the unsat core!

The assumption-based approach to core extraction

Example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge \\ B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$

(i) add selector variables:

$$(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge \\ (\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge \\ (\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7)$$

(ii) The conflict analysis returns: $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$,

(iii) corresponding to the unsat core:

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge \\ B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$$

Computing Craig Interpolants in SAT

Notation: Let " $X \preceq Y$ ", X, Y being Boolean formulas, denote the fact that all Boolean atoms in X occur also in Y .

Definition: Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models \perp$, a *Craig interpolant* is a formula I s.t.:

- $A \models I$,
- $I \wedge B \models \perp$,
- $I \preceq A$ and $I \preceq B$.

- Very important in many Formal Verification applications
- A few works presented [32, 25, 27]

Algorithm: Interpolant generation (for SAT)

- (i) Generate a resolution proof of unsatisfiability \mathcal{P} for $A \wedge B$.
- (ii) ...
- (iii) For every leaf clause C in \mathcal{P} ,
 - set $I_C \stackrel{\text{def}}{=} C \downarrow B$ if $C \in A$,
 - set $I_C \stackrel{\text{def}}{=} \top$ if $C \in B$.
- (iv) For every inner node C of \mathcal{P} obtained by resolution from $C_1 \stackrel{\text{def}}{=} p \vee \phi_1$ and $C_2 \stackrel{\text{def}}{=} \neg p \vee \phi_2$,
 - set $I_C \stackrel{\text{def}}{=} I_{C_1} \wedge I_{C_2}$ if p occurs in B ,
 - set $I_C \stackrel{\text{def}}{=} I_{C_1} \vee I_{C_2}$ if p does not occur in B .
- (v) Output I_{\perp} as an interpolant for (A, B) .

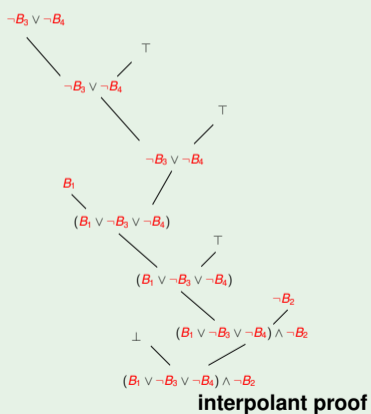
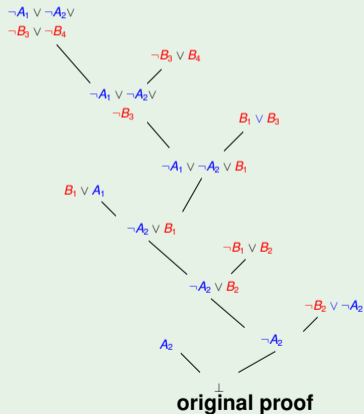
“ $\eta \downarrow B$ ” [resp. “ $\eta \setminus B$ ”] is the set of literals in η whose atoms do [resp. do not] occur in B .

- optimized versions for the purely-propositional case [25, 27]

Computing Craig Interpolants in SAT: example

$$A \stackrel{\text{def}}{=} (B_1 \vee A_1) \wedge A_2 \wedge (\neg B_2 \vee \neg A_2) \wedge (\neg A_1 \vee \neg A_2 \vee \neg B_3 \vee \neg B_4)$$

$$B \stackrel{\text{def}}{=} (\neg B_3 \vee B_4) \wedge (\neg B_1 \vee B_2) \wedge (B_1 \vee B_3)$$



$\Rightarrow (B_1 \vee \neg B_3 \vee \neg B_4) \wedge \neg B_2$ is an interpolant

All-SAT (hints)

- **All-SAT**: enumerate all truth assignments satisfying φ
- **All-SAT over an “important” subset of atoms $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$** : enumerate all assignments over \mathbf{P} which can be extended to satisfiable truth assignments propositionally satisfying φ
- **Algorithms**
 - **BCLT [Lahiri et al, CAV'06]**:
each time a satisfiable assignment $\{l_1, \dots, l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$ belonged to the clause set
 - **MathSAT/NuSMV [Cavada et al, FMCAD'07]**:
As above, plus the Boolean search of the SAT solver is driven by an OBDD.

MaxSAT (hints)

- **MaxSAT**: given a pair of CNF formulas $\langle \varphi_h, \varphi_s \rangle$ s.t. $\varphi_h \wedge \varphi_s \models \perp$, $\varphi_s \stackrel{\text{def}}{=} \{C_1, \dots, C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s .
- **Weighted MaxSAT**: given also the positive integer **penalties** $\{w_1, \dots, w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to **optimization**
 \implies much harder than SAT
- Many different approaches (see e.g. [22])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \quad \varphi_s \stackrel{\text{def}}{=} \left(\begin{array}{l} (A_1 \vee \neg A_2) \wedge [4] \\ (\neg A_1 \vee A_2) \wedge [3] \\ (\neg A_1 \vee \neg A_2) \wedge [2] \end{array} \right)$$

$$\implies \mu = \{A_1, A_2\} \text{ (penalty} = 2)$$

References I



A. Armando and E. Giunchiglia.

Embedding Complex Decision Procedures inside an Interactive Theorem Prover.
Annals of Mathematics and Artificial Intelligence, 8(3–4):475–502, 1993.



R. J. Bayardo, Jr. and R. C. Schrag.

Using CSP Look-Back Techniques to Solve Real-World SAT instances.
In *Proc. AAAI'97*, pages 203–208. AAAI Press, 1997.



A. Belov and Z. Stachniak.

Improving variable selection process in stochastic local search for propositional satisfiability.
In *SAT'09*, LNCS. Springer, 2009.



A. Belov and Z. Stachniak.

Improved local search for circuit satisfiability.
In *SAT*, volume 6175 of *LNCS*, pages 293–299. Springer, 2010.



A. Biere, M. J. H. Heule, H. van Maaren, and T. Walsh, editors.

Handbook of Satisfiability.
IOS Press, February 2009.



Booleforce, <http://fmv.jku.at/booleforce/>.



R. Brafman.

A simplifier for propositional formulas with many binary clauses.
In *Proc. IJCAI'01*, 2001.



R. E. Bryant.

Graph-Based Algorithms for Boolean Function Manipulation.
IEEE Transactions on Computers, C-35(8):677–691, Aug. 1986.

References II



M. Davis, G. Longemann, and D. Loveland.

A machine program for theorem proving.
Journal of the ACM, 5(7), 1962.



M. Davis and H. Putnam.

A computing procedure for quantification theory.
Journal of the ACM, 7:201–215, 1960.



N. Eén and N. Sörensson.

Temporal induction by incremental sat solving.
Electr. Notes Theor. Comput. Sci., 89(4):543–560, 2003.



N. Eén and N. Sörensson.

An extensible SAT-solver.
In *Theory and Applications of Satisfiability Testing (SAT 2003)*, volume 2919 of *LNCS*, pages 502–518. Springer, 2004.



R. Gershman, M. Koifman, and O. Strichman.

Deriving Small Unsatisfiable Cores with Dominators.
In *Proc. CAV'06*, volume 4144 of *LNCS*. Springer, 2006.



E. Giunchiglia, M. Narizzano, A. Tacchella, and M. Vardi.

Towards an Efficient Library for SAT: a Manifesto.
In *Proc. SAT 2001*, *Electronics Notes in Discrete Mathematics*. Elsevier Science., 2001.



E. Giunchiglia and R. Sebastiani.

Applying the Davis-Putnam procedure to non-clausal formulas.
In *Proc. AI'IA'99*, volume 1792 of *LNAI*. Springer, 1999.

References III



C. Gomes, B. Selman, and H. Kautz.

Boosting Combinatorial Search Through Randomization.
In *Proceedings of the Fifteenth National Conference on Artificial Intelligence*, 1998.



C. P. Gomes, A. Sabharwal, and B. Selman.

Model Counting, chapter 20, pages 633–654.
In Biere et al. [5], February 2009.



H. H. Hoos and T. Stutzle.

Stochastic Local Search Foundation And Application.
Morgan Kaufmann, 2005.



J. Huang.

MUP: a minimal unsatisfiability prover.
In *Proc. ASP-DAC '05*. ACM Press, 2005.



H. A. Kautz, A. Sabharwal, and B. Selman.

Incomplete Algorithms, chapter 6, pages 185–203.
In Biere et al. [5], February 2009.



C. M. Li and Anbulagan.

Heuristics based on unit propagation for satisfiability problems.
In *Proceedings of the 15th International Joint Conference on Artificial Intelligence (IJCAI-97)*, pages 366–371, 1997.



C. M. Li and F. Manyà.

MaxSAT, Hard and Soft Constraints, chapter 19, pages 613–631.
In Biere et al. [5], February 2009.

References IV



I. Lynce and J. Marques-Silva.
On Computing Minimum Unsatisfiable Cores.
In *7th International Conference on Theory and Applications of Satisfiability Testing*, 2004.



I. Lynce and J. P. Marques-Silva.
On computing minimum unsatisfiable cores.
In *SAT*, 2004.



K. McMillan.
Interpolation and SAT-based model checking.
In *Proc. CAV*, 2003.



K. McMillan and N. Amla.
Automatic abstraction without counterexamples.
In *Proc. of TACAS*, 2003.



K. L. McMillan.
An interpolating theorem prover.
Theor. Comput. Sci., 345(1):101–121, 2005.



M. W. Moskewicz, C. F. Madigan, Y. Z., L. Zhang, and S. Malik.
Chaff: Engineering an efficient SAT solver.
In *Design Automation Conference*, 2001.



R. Nieuwenhuis, A. Oliveras, and C. Tinelli.
Abstract DPLL and abstract DPLL modulo theories.
In F. Baader and A. Voronkov, editors, *Proceedings of the 11th International Conference on Logic for Programming, Artificial Intelligence and Reasoning (LPAR'04)*, Montevideo, Uruguay, volume 3452 of *LNCS*, pages 36–50. Springer, 2005.

References V



R. Nieuwenhuis, A. Oliveras, and C. Tinelli.

Solving SAT and SAT Modulo Theories: from an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).
Journal of the ACM, 53(6):937–977, November 2006.



Y. Oh, M. N. Mneimneh, Z. S. Andraus, K. A. Sakallah, and I. L. Markov.

Amuse: A Minimally-Unsatisfiable Subformula Extractor.
In *Proc. DAC'04. ACM/IEEE*, 2004.



P. Pudlák.

Lower bounds for resolution and cutting planes proofs and monotone computations.
J. of Symb. Logic, 62(3), 1997.



A. Robinson.

A machine-oriented logic based on the resolution principle.
Journal of the ACM, 12:23–41, 1965.



R. Sebastiani.

Applying GSAT to Non-Clausal Formulas.
Journal of Artificial Intelligence Research, 1:309–314, 1994.



B. Selman and H. Kautz.

Domain-Independent Extension to GSAT: Solving Large Structured Satisfiability Problems.
In *Proc. of the 13th International Joint Conference on Artificial Intelligence*, pages 290–295, 1993.



B. Selman, H. Kautz, and B. Cohen.

Local Search Strategies for Satisfiability Testing.
In *Cliques, Coloring, and Satisfiability*, volume 26 of *DIMACS*, pages 521–532, 1996.

References VI



B. Selman, H. Levesque., and D. Mitchell.

A New Method for Solving Hard Satisfiability Problems.

In *Proc. of the 10th National Conference on Artificial Intelligence*, pages 440–446, 1992.



J. P. M. Silva and K. A. Sakallah.

GRASP - A new Search Algorithm for Satisfiability.

In *Proc. ICCAD'96*, 1996.



R. M. Smullyan.

First-Order Logic.

Springer-Verlag, NY, 1968.



C. Tinelli.

A DPLL-based Calculus for Ground Satisfiability Modulo Theories.

In *Proc. JELIA-02*, volume 2424 of *LNAI*, pages 308–319. Springer, 2002.



D. Tompkins and H. Hoos.

UBCSAT: An Implementation and Experimentation Environment for SLS Algorithms for SAT and MAX-SAT.

In *SAT*, volume 3542 of *LNCS*. Springer, 2004.



H. Zhang and M. Stickel.

Implementing the Davis-Putnam algorithm by tries.

Technical report, University of Iowa, August 1994.



J. Zhang, S. Li, and S. Shen.

Extracting Minimum Unsatisfiable Cores with a Greedy Genetic Algorithm.

In *Proc. ACAI*, volume 4304 of *LNCS*. Springer, 2006.

References VII



L. Zhang, C. F. Madigan, M. H. Moskewicz, and S. Malik.

Efficient conflict driven learning in a boolean satisfiability solver.

In *ICCAD '01: Proceedings of the 2001 IEEE/ACM international conference on Computer-aided design*, pages 279–285, Piscataway, NJ, USA, 2001. IEEE Press.



L. Zhang and S. Malik.

The quest for efficient boolean satisfiability solvers.

In *Proc. CAV'02*, number 2404 in LNCS, pages 17–36. Springer, 2002.



L. Zhang and S. Malik.

Extracting small unsatisfiable cores from unsatisfiable boolean formula.

In *Proc. of SAT*, 2003.

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The list of references above is by no means intended to be all-inclusive. The author of these slides apologizes both with the authors and with the readers for all the relevant works which are not cited here.

The papers (co)authored by the author of these slides are available at:

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- **Combination Methods in Automated Reasoning**
<http://combination.cs.uiowa.edu/>
- **The SAT Association**
<http://satassociation.org/>
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<http://www.satlive.org/index.jsp>
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