# Formal Methods
# Module II: Model Checking
# Ch. 08: **Abstraction in Model Checking**

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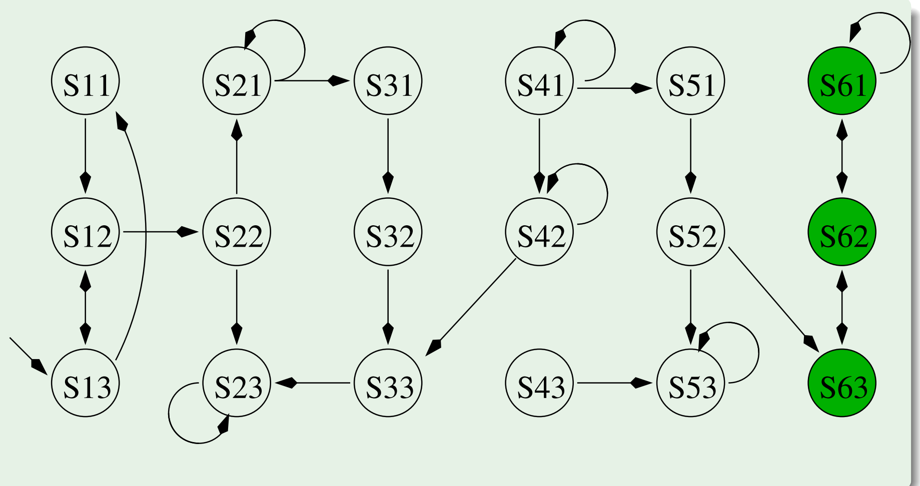
# Outline

1. Abstraction
2. Abstraction-Based Symbolic Model Cheching
    - Abstraction
    - Checking the counter-examples
    - Refinement
3. Exercises

# Outline

# Model Checking Safety Properties: $M \models \mathbf{G}\neg BAD$

Add reachable states until reaching a fixed-point or a “bad” state




Problem: too many states to handle! (even for symbolic MC)

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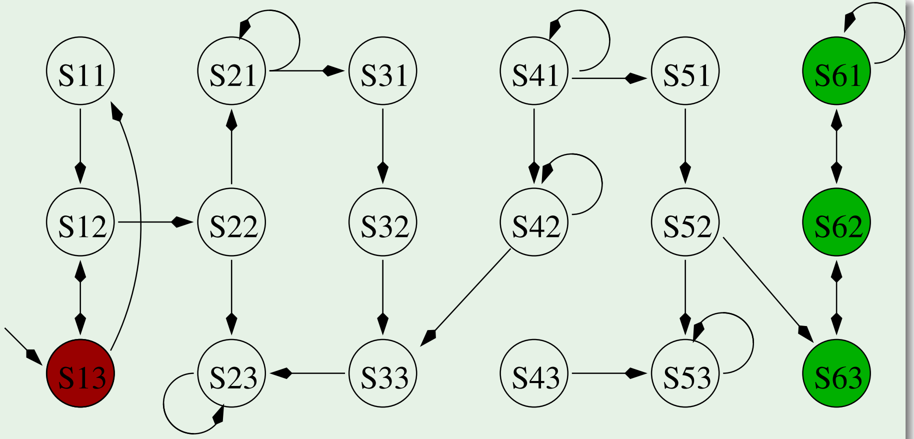


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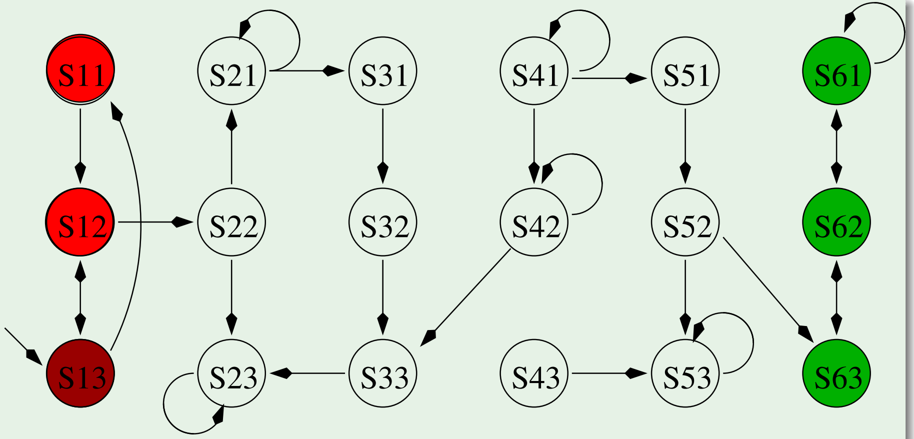


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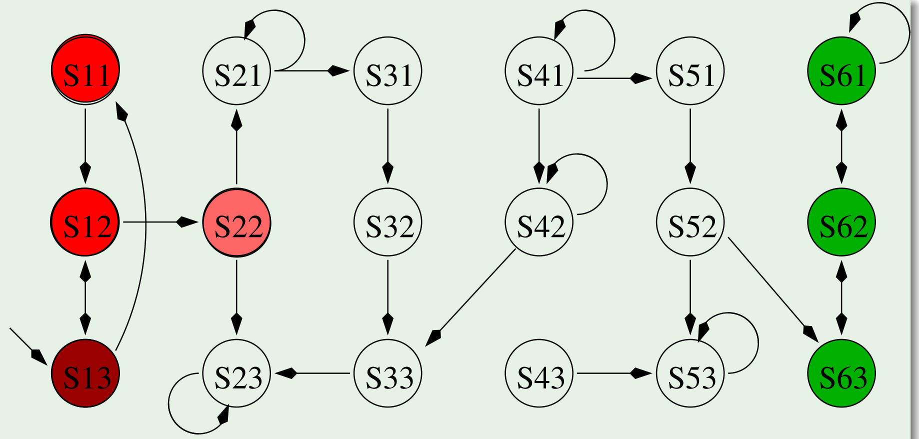


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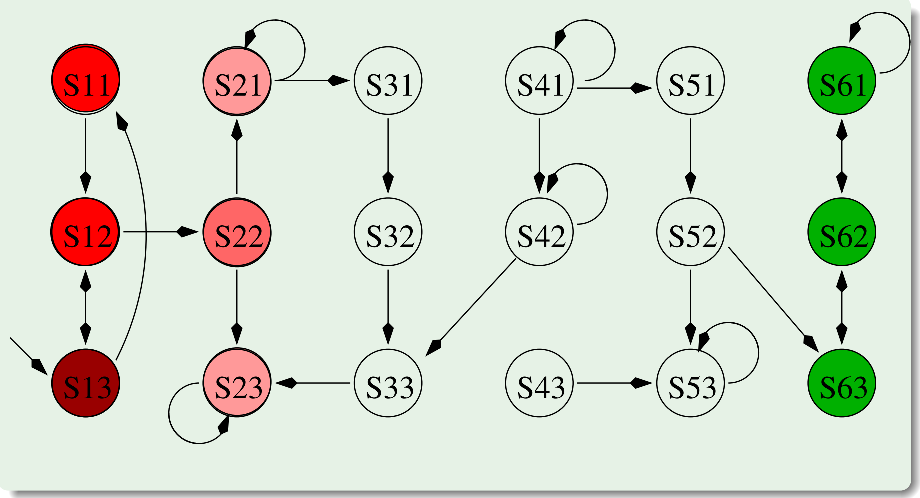


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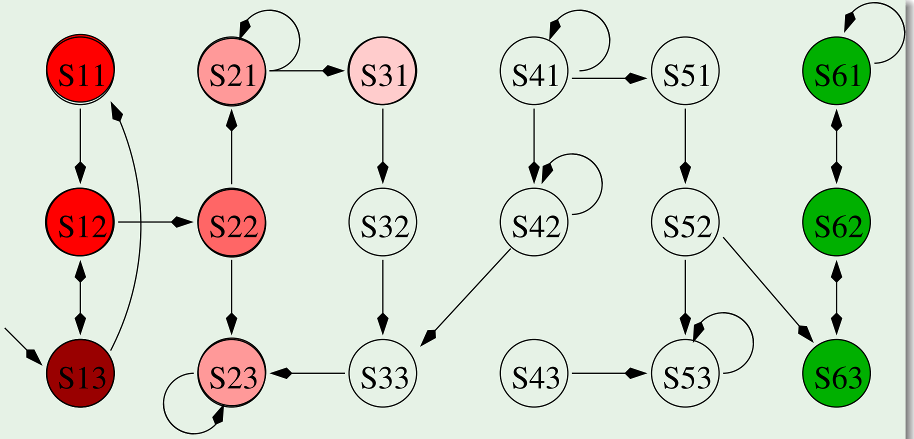


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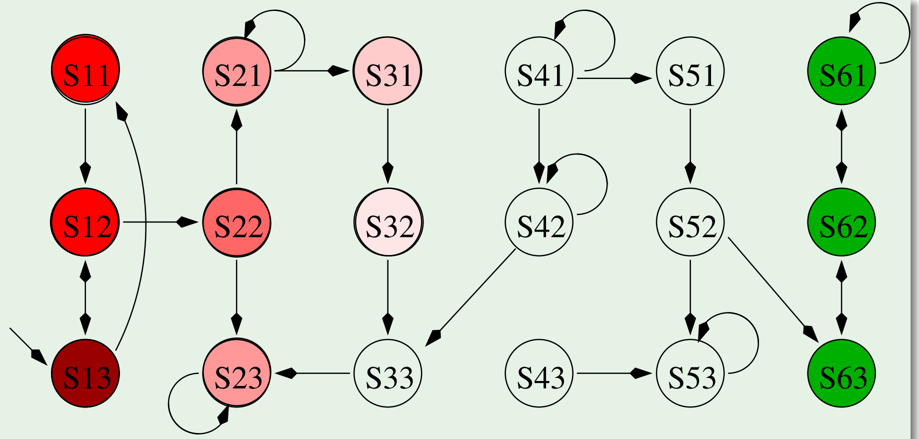


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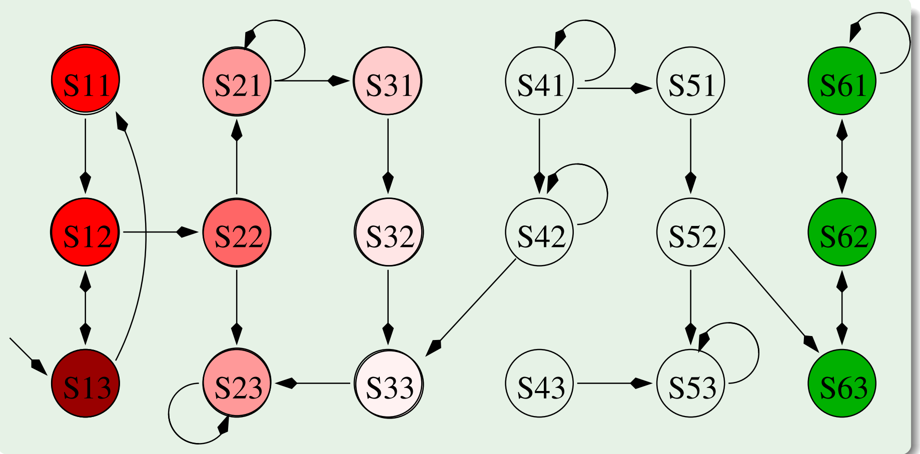


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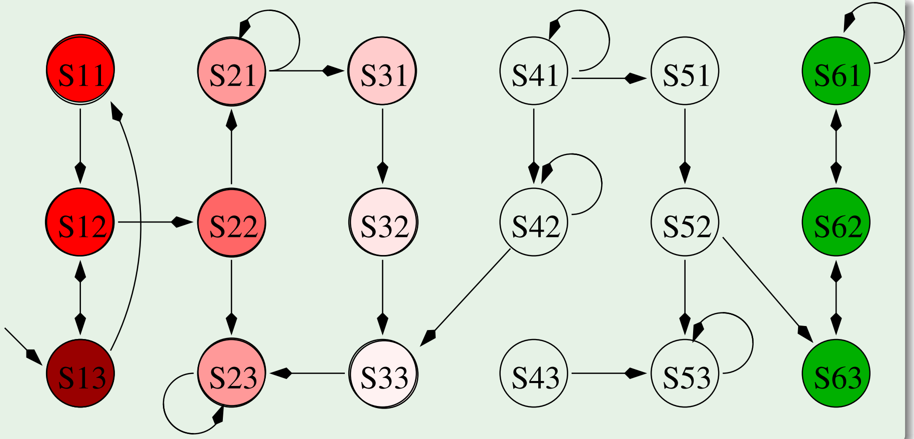


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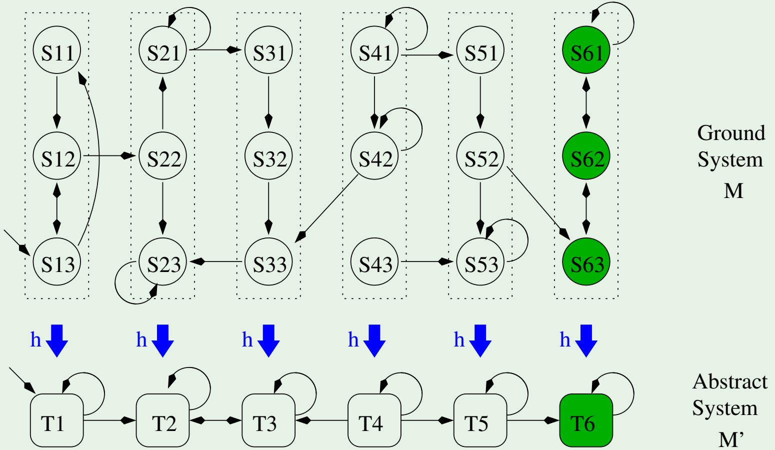
# Idea: Abstraction

Apply a (non-injective) Abstraction Function *h* to M
$\Longrightarrow$ Build an abstract (and much smaller) system M'



# Abstraction & Refinement

## Abstraction & Refinement

- Let $S$ be the ground (concrete) state space
- Let $S'$ be the abstract state space
- Abstraction: a (typically non-injective) map $h : S \longmapsto S'$
  - $h$ typically a many-to-one function
- Refinement: a map $r : S' \longmapsto 2^S$ s.t. $r(s') \stackrel{\text{def}}{=} \{s \in S \mid s' = h(s)\}$

# Simulation and Bisimulation

## Simulation

Let $M_1 \stackrel{\text{def}}{=} \langle S_1, I_1, R_1, AP_1, L_1 \rangle$ and $M_2 \stackrel{\text{def}}{=} \langle S_2, I_2, R_2, AP_2, L_2 \rangle$. Then $p \subseteq S_1 \times S_2$ is a simulation between $M_1$ and $M_2$ ($M_1$ simulates $M_2$) iff

- for every $s_2 \in I_2$ exists $s_1 \in I_1$ s.t. $\langle s_1, s_2 \rangle \in p$
- for every $\langle s_1, s_2 \rangle \in p$:
  - for every $\langle s_2, t_2 \rangle \in R_2$, exists $\langle s_1, t_1 \rangle \in R_1$ s.t. $\langle t_1, t_2 \rangle \in p$

(Intuitively, for every transition in $M_2$ there is a corresponding transition in $M_1$.)

Example of p (spy game): "follower $M_1$ keeps escaper $M_2$ at eyesight"

## Bisimulation

P is a bisimulation between $M$ and $M'$ iff it is both a simulation between $M$ and $M'$ and between $M'$ and $M$.
We say that $M$ and $M'$ bisimulate each other.

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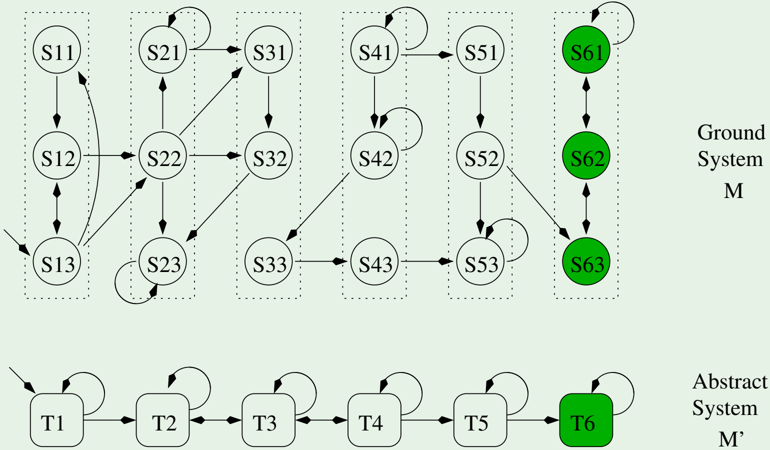
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Ground System M
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T6
Abstract System M'

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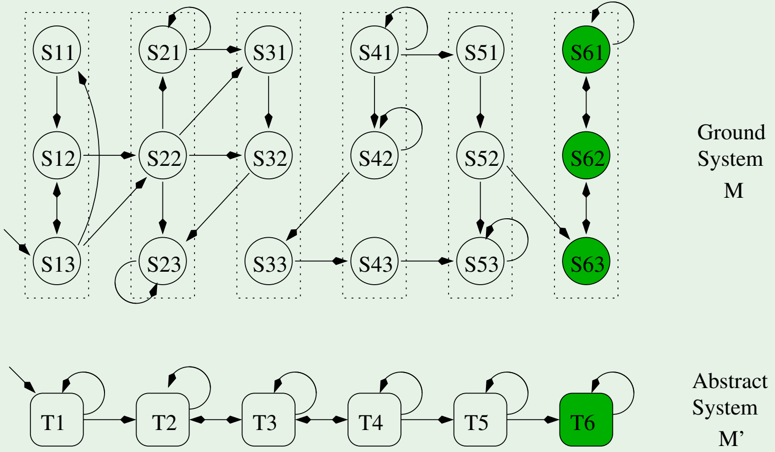


- Does M simulate M'?

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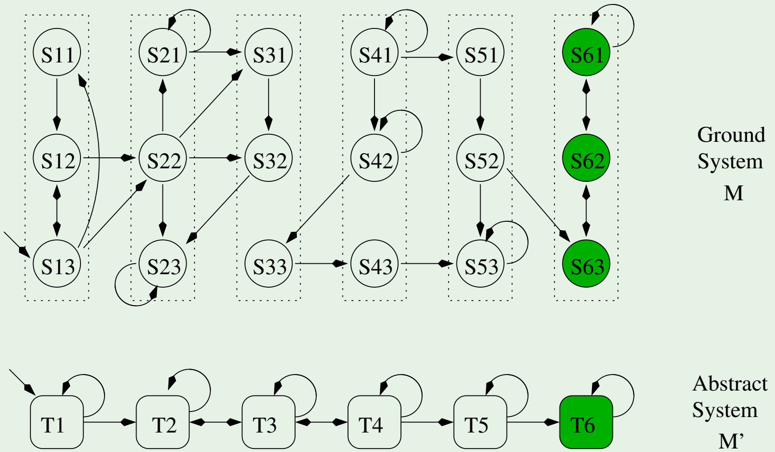


- Does M simulate M'? No: e.g., no arc from *S*23 to any *S*3*i*.

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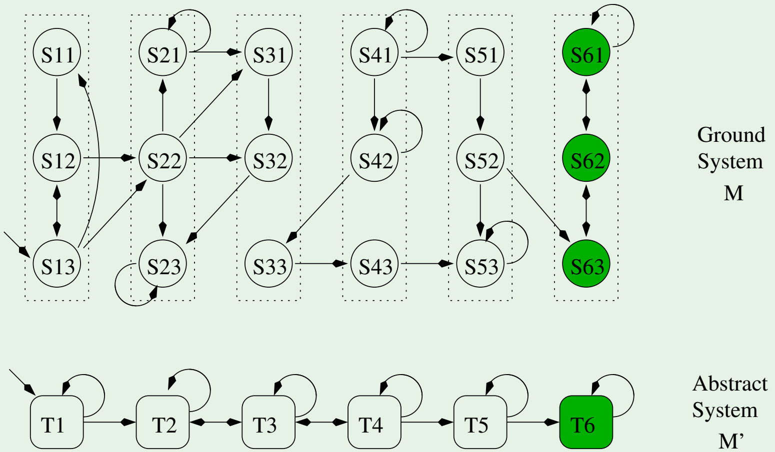


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- Does M' simulate M?

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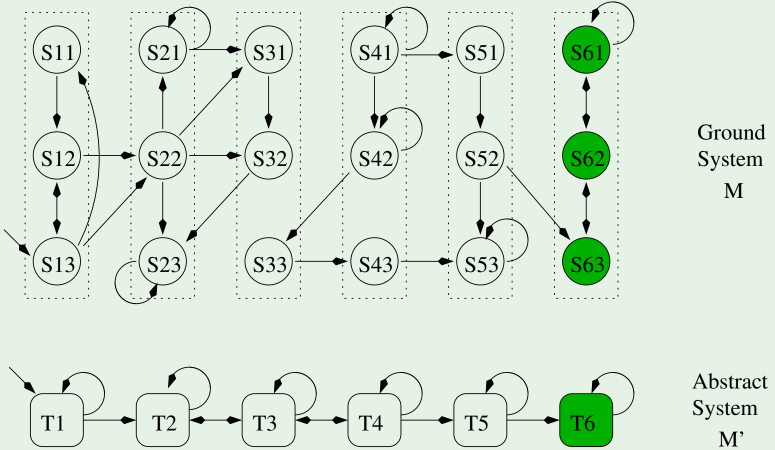


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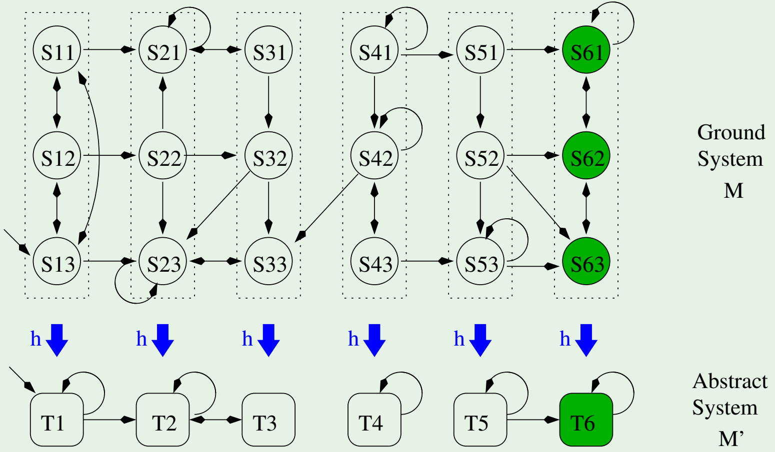
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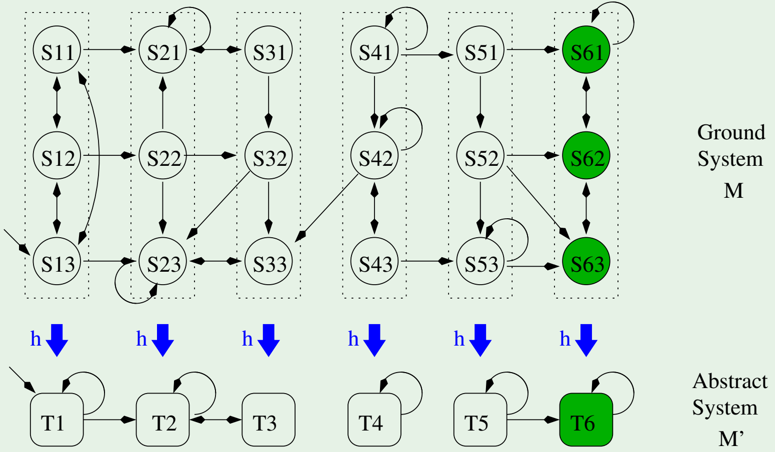


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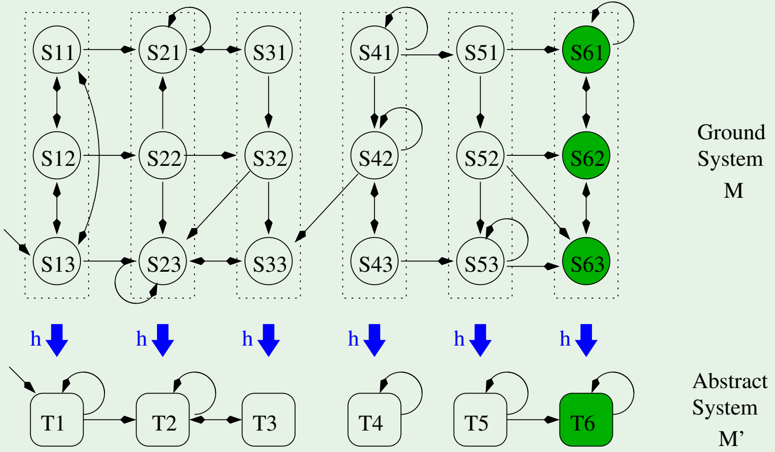


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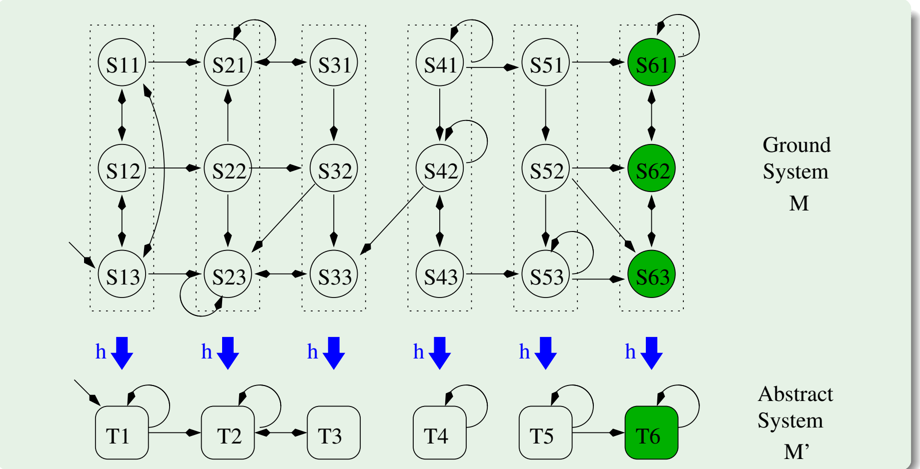


- Does M simulate M'? Yes
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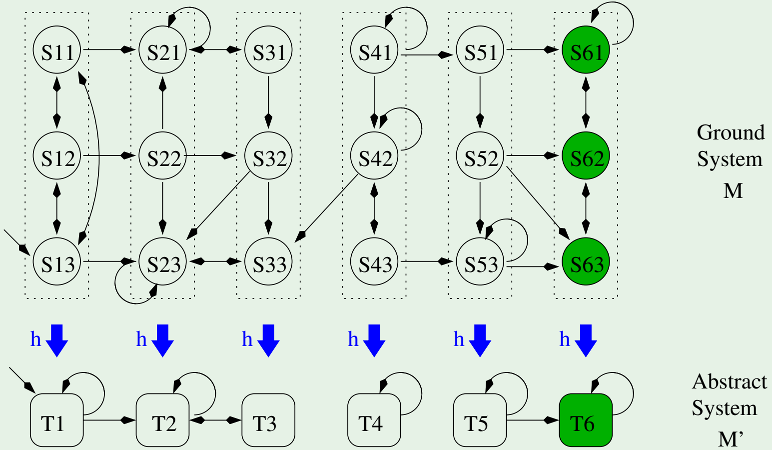


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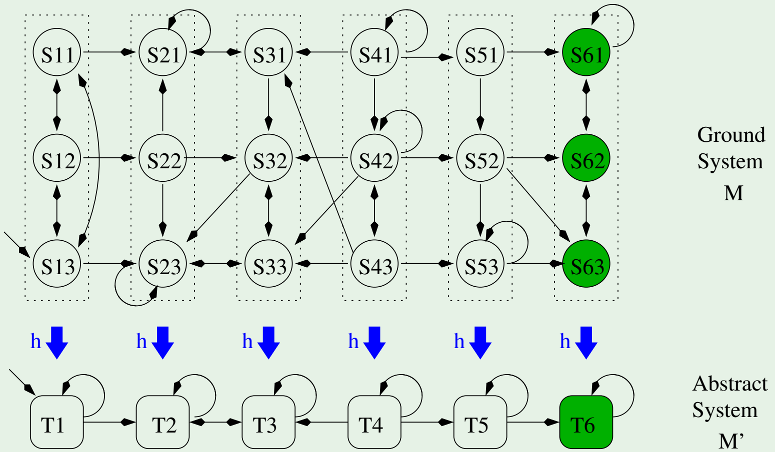
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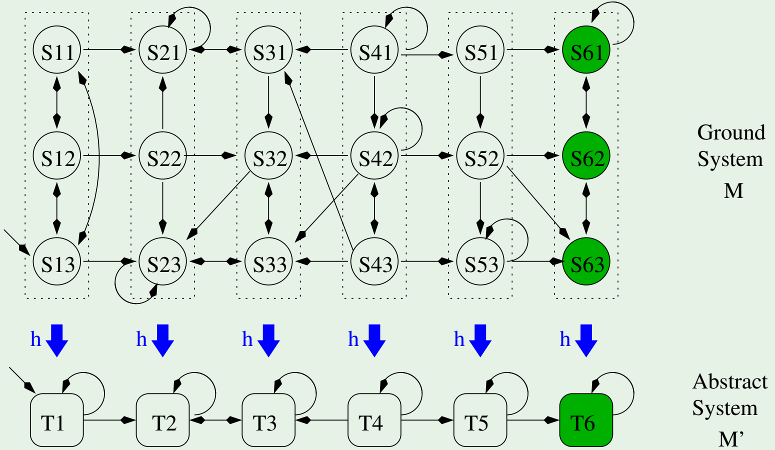


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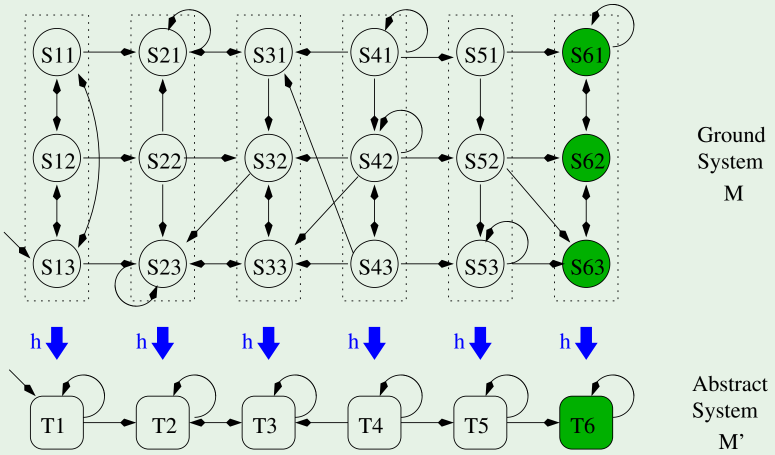


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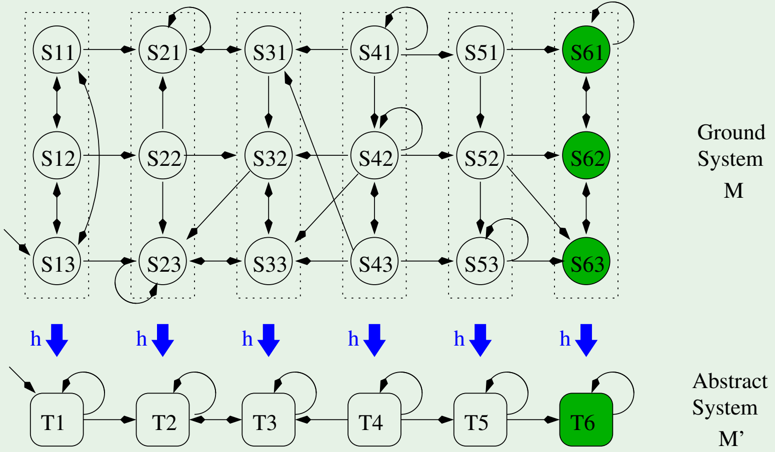


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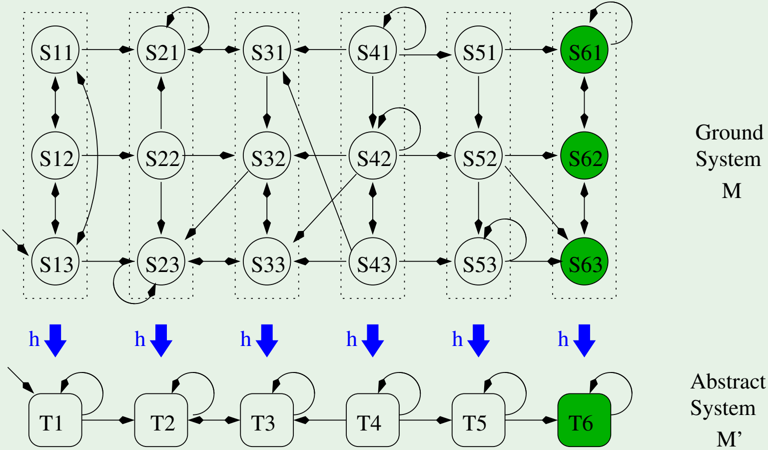


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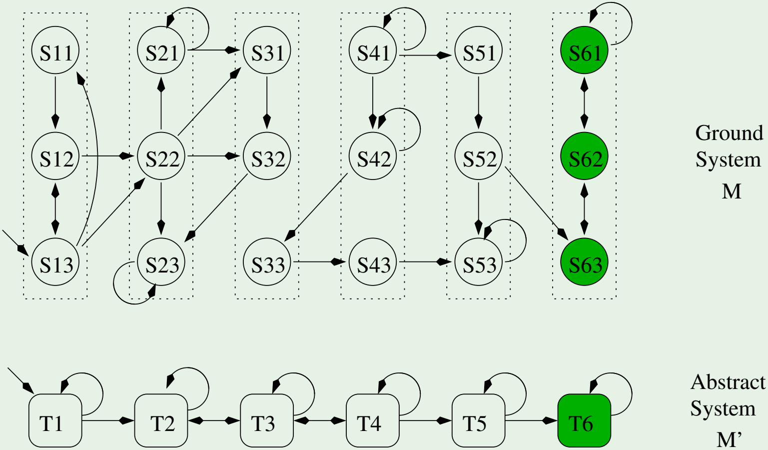
# Existential Abstraction (Over-Approximation)

An Abstraction from M to M’ is an Existential Abstraction (aka Over-Approximation) iff $M'$ simulates $M$



# Model Checking with Existential Abstractions

## Preservation Theorem

- Let $\varphi$ be a universally-quantified property (e.g., in LTL or ACTL)
- Let $M'$ simulate $M$

Then we have that

$$M' \models \varphi \Longrightarrow M \models \varphi$$

- Intuition: if M has a countermodel, then M' simulates it
- The converse does not hold

$$M \models \varphi \not\Longrightarrow M' \models \varphi$$

$\Longrightarrow$ The abstract counter-example may be spurious (e.g., in previous figure, $T1 \rightarrow T2 \rightarrow T3 \rightarrow T4 \rightarrow T5 \rightarrow T6$)

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# Bisimulation Abstraction

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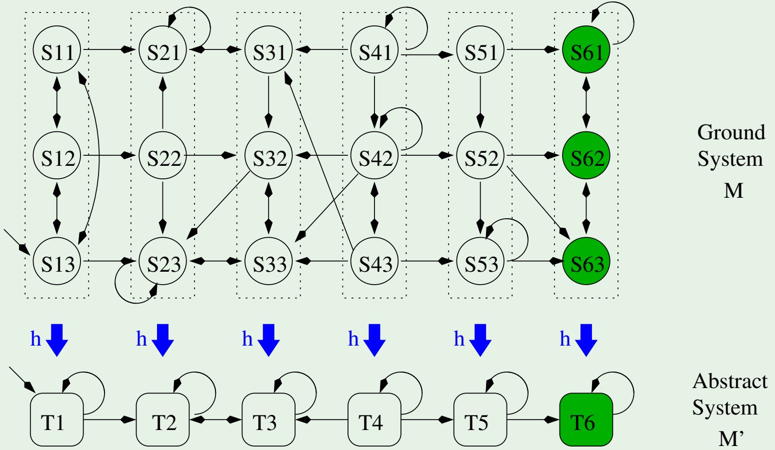
$M$ simulates $M'$ and $M'$ simulates $M$



# Model Checking with Bisimulation Abstractions

## Preservation Theorem

- Let $\varphi$ be any ACTL/LTL property
- Let $M$ simulate $M'$ and $M'$ simulate $M$

Then we have that

$$M' \models \varphi \Longleftrightarrow M \models \varphi$$

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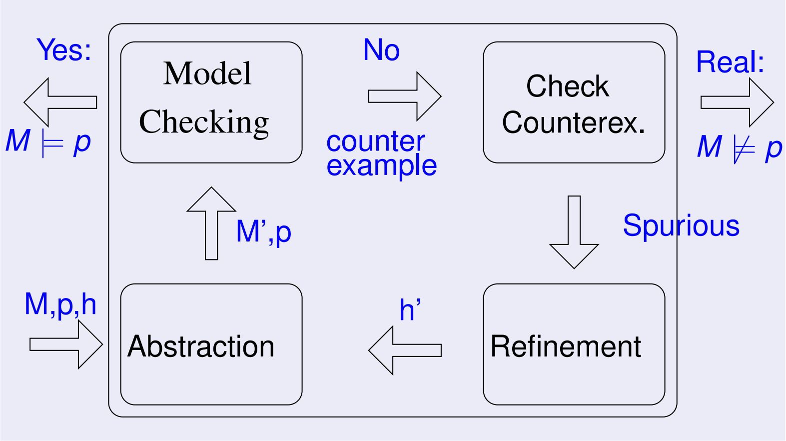
# Counter-Example Guided Abstraction Refinement - CEGAR

**General Schema:**



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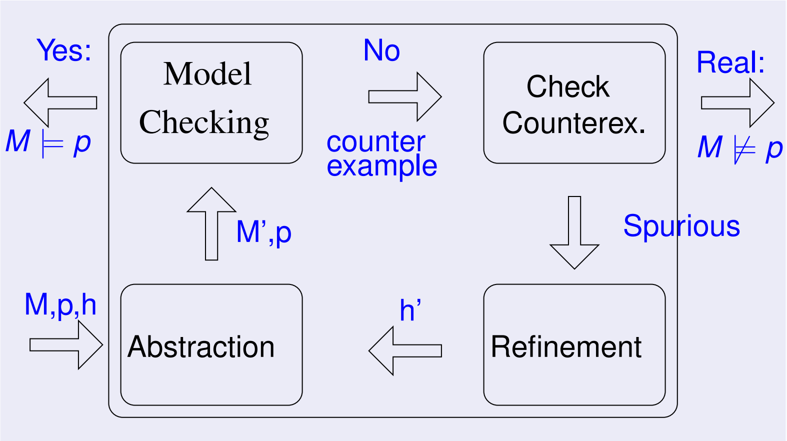

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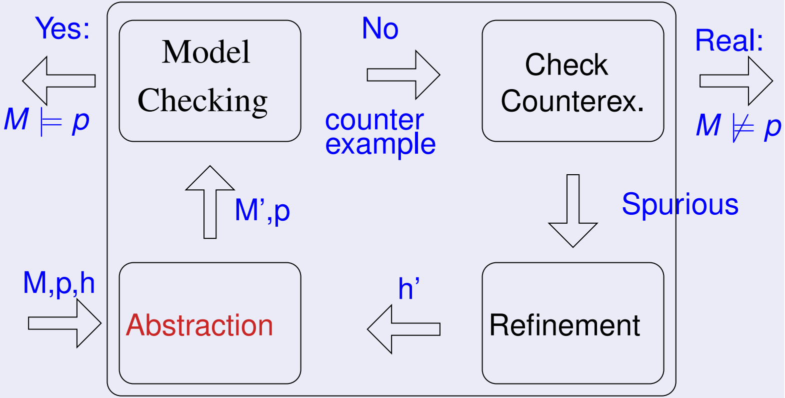

# A Popular Abstraction for Symbolic MC of $\mathbf{G}\neg BAD$ I

- A.k.a. "Localization Reduction"
- Partition Boolean variables into visible (V) and invisible (I) ones
  - The abstract model built on visible variables only.
  - Invisible variables are made inputs (no updates in the transition relation)
  - All variables occurring in "$\neg BAD$" must be visible
- The abstraction function maps each state to its projection over V.

$\Longrightarrow$ Group ground states with same visible part to a single abstract state.

$$\left[\begin{array}{c|cccc} & \multicolumn{2}{c}{\textit{visible}} & \multicolumn{2}{c}{\textit{invisible}} \\ & x_1 & x_2 & x_3 & x_4 \\ \hline S_{11}: & 0 & 0 & 0 & 0 \\ S_{12}: & 0 & 0 & 0 & 1 \\ S_{13}: & 0 & 0 & 1 & 0 \\ S_{14}: & 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc} T_1: & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{c|cccc} & \multicolumn{2}{c}{visible} & \multicolumn{2}{c}{invisible} \\ & x_1 & x_2 & x_3 & x_4 \\ \hline S_{11}: & 0 & 0 & 0 & 0 \\ S_{12}: & 0 & 0 & 0 & 1 \\ S_{13}: & 0 & 0 & 1 & 0 \\ S_{14}: & 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \left[\, T_1 : \; 0 \quad 0 \,\right]$$

# A Popular Abstraction for Symbolic MC of $\mathbf{G}\neg BAD$ I

- A.k.a. "Localization Reduction"
- Partition Boolean variables into visible (V) and invisible (I) ones
  - The abstract model built on visible variables only.
  - Invisible variables are made inputs (no updates in the transition relation)
  - All variables occurring in "$\neg BAD$" must be visible
- The abstraction function maps each state to its projection over V.

$\Longrightarrow$ Group ground states with same visible part to a single abstract state.

$$
\left[\begin{array}{lcccc}
 & \multicolumn{2}{c}{\textit{visible}} & \multicolumn{2}{c}{\textit{invisible}} \\
 & x_1 & x_2 & x_3 & x_4 \\
\hline
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$$

# A Popular Abstraction for Symbolic MC of $\mathbf{G}\neg BAD$ II

M' can be computed efficiently if M is in functional form
(e.g. sequential circuits).

$$\begin{bmatrix} next(x_1) := f_1(x_1, x_2, x_3, x_4) \\ next(x_2) := f_2(x_1, x_2, x_3, x_4) \\ next(x_3) := f_3(x_1, x_2, x_3, x_4) \\ next(x_4) := f_4(x_1, x_2, x_3, x_4) \end{bmatrix} \Longrightarrow \begin{bmatrix} next(x_1) := f_1(x_1, x_2, x_3, x_4) \\ next(x_2) := f_2(x_1, x_2, x_3, x_4) \end{bmatrix}$$

Note: The next values of invisible variables, $next(x_3)$ and $next(x_4)$, can assume every value nondeterministically
$\Longrightarrow$ do not constrain the transition relation

Since $M'$ obviously simulates $M$, this is an Existential Abstraction

- $M' \models \varphi \Longrightarrow M \models \varphi$
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# Outline

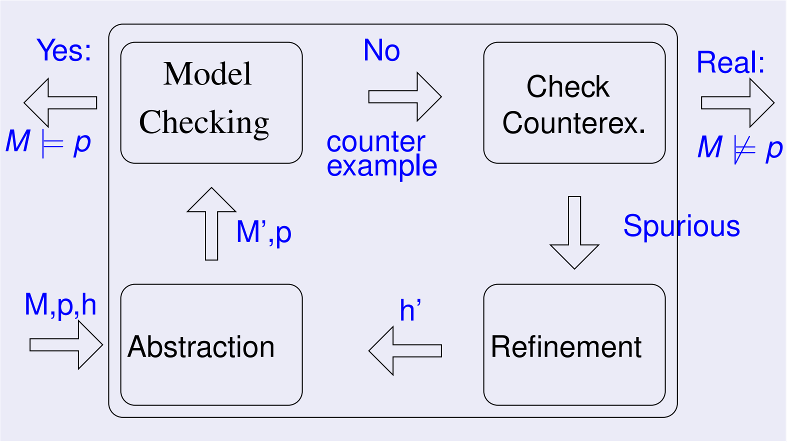
1 Abstraction

2 Abstraction-Based Symbolic Model Cheching
- Abstraction
- Checking the counter-examples
- Refinement

3 Exercises

# Counter-Example Guided Abstraction Refinement

## General Schema:



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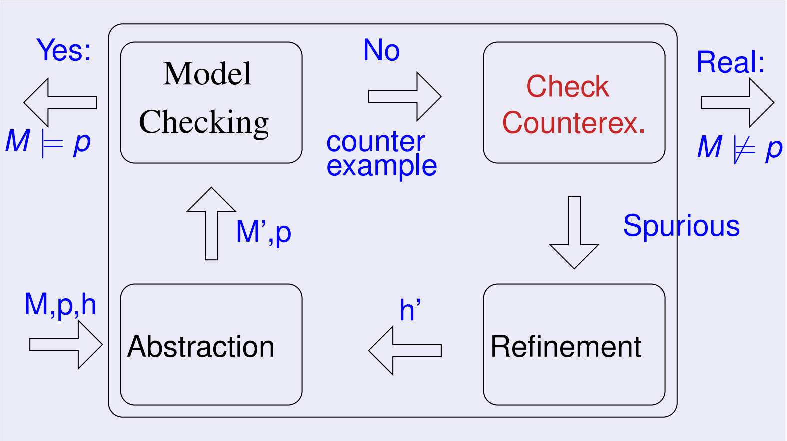

# Checking the Abstract Counter-Example I

## The problem

- Let $c_0, ..., c_m$ counter-example in the abstract space
  - Note: each $c_i$ is a truth assignment on the visible variables
- Problem: check if there exist a corresponding ground counterexample $s_0, ..., s_m$ s.t. $c_i = h(s_i)$, for every $i$

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# Checking the Abstract Counter-Example II

## Idea

- Simulate the counterexample on the concrete model
- Use Bounded Model Checking:

$$\Phi \stackrel{\text{def}}{=} I(s_0) \wedge \bigwedge_{i=0}^{m-1} R(s_i, s_{i+1}) \wedge \bigwedge_{i=0}^{m} visible(s_i) = c_i$$

If satisfiable, the counter example is real, otherwise it is spurious

Note: much more efficient than the direct BMC problem:

$$\Phi \stackrel{\text{def}}{=} I(s_0) \wedge \bigwedge_{i=0}^{m-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{m} \neg BAD_i$$

$\Longrightarrow$ cuts a $2^{(m+1)\cdot|V|}$ factor from the Boolean search space.

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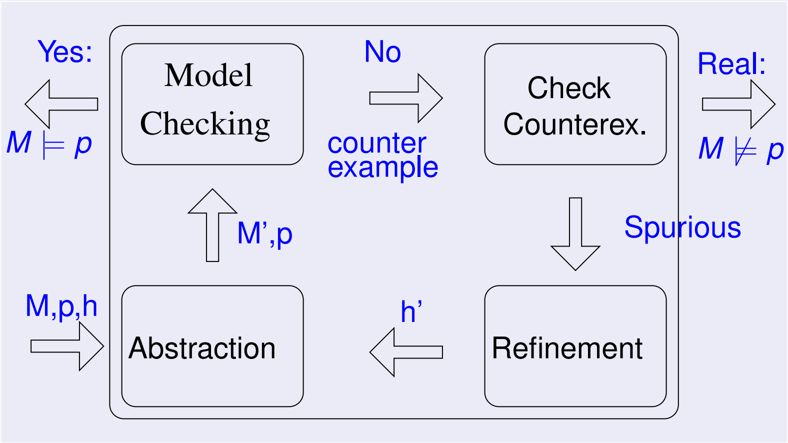

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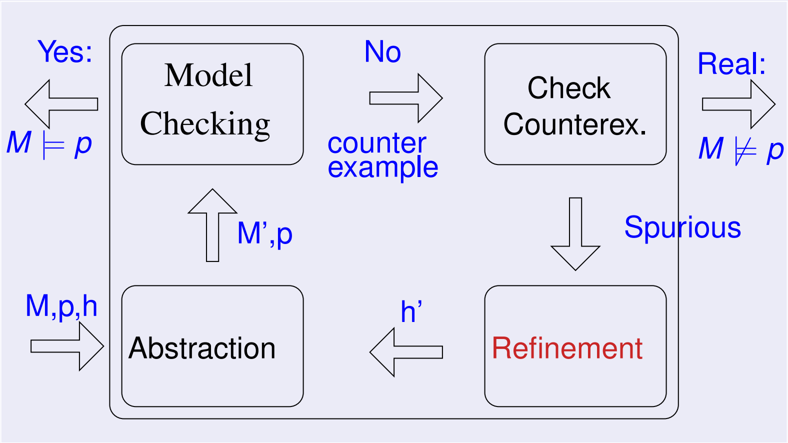

# The cause of spurious counter-examples I

## Problem

There is a state in the abstract counter-example (failure state) s.t. two different and un-connected kinds of ground states are mapped into it:

- Deadend states: reachable states which do not allow to proceed along a refinement of the abstract counter-example
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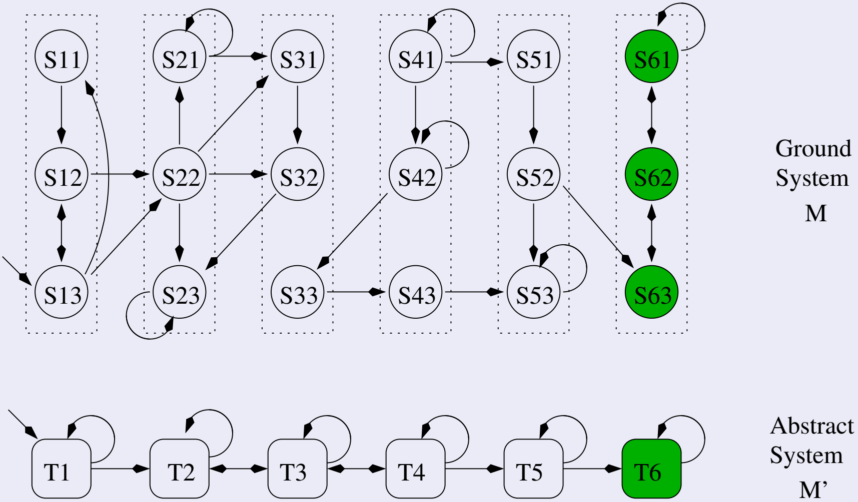
# The cause of spurious counter-examples II

For the spurious counter-example:

$T1 \rightarrow T2 \rightarrow T3 \rightarrow T4 \rightarrow T5 \rightarrow T6$



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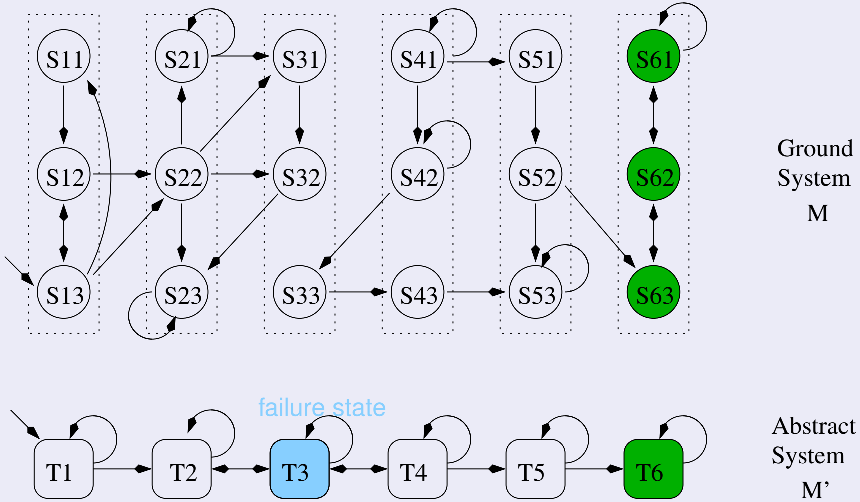

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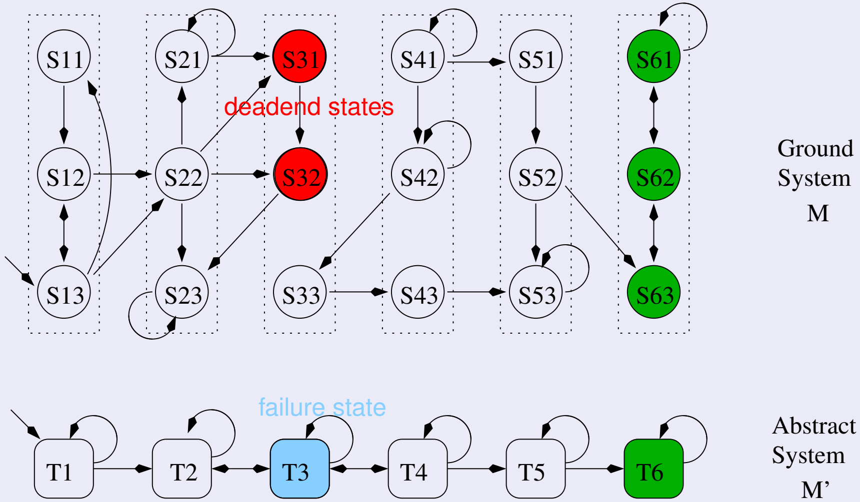

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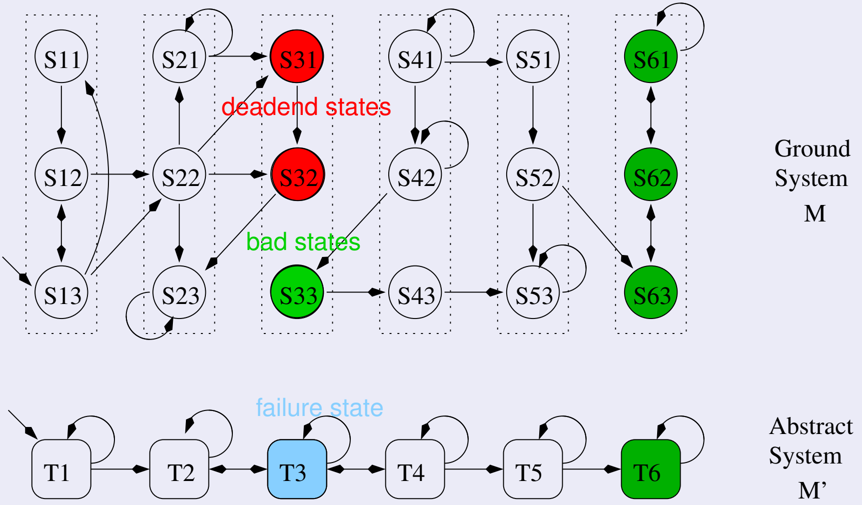

# The cause of spurious counter-examples III

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There is a state in the abstract counter-example (failure state) s.t. two different and un-connected kinds of ground states are mapped into it:

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### Solution: Refine the abstraction function.

1. identify the failure state and its deadend and bad states
2. refine the abstraction function s.t. deadend and bad states are mapped into different abstract state

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# Identify the failure state and its deadend & bad states

- The failure state is the state of maximum index $f$ in the abstract counter-example s.t. the following formula is satisfiable:

$$\Phi_D \stackrel{\text{def}}{=} I(s_0) \wedge \bigwedge_{i=0}^{f-1} R(s_i, s_{i+1}) \wedge \bigwedge_{i=0}^{f} visible(s_i) = c_i$$

- The (restriction on index $f$ of the) models of $\Phi_D$ identify the deadend states $\{d_1, ..., d_k\}$
- The bad states $\{b_1, ..., b_n\}$ are identified by the (restriction on index $f$ of the) models of the following formula:

$$\Phi_B \stackrel{\text{def}}{=} R(s_f, s_{f+1}) \wedge visible(s_f) = c_f \wedge visible(s_{f+1}) = c_{f+1}$$

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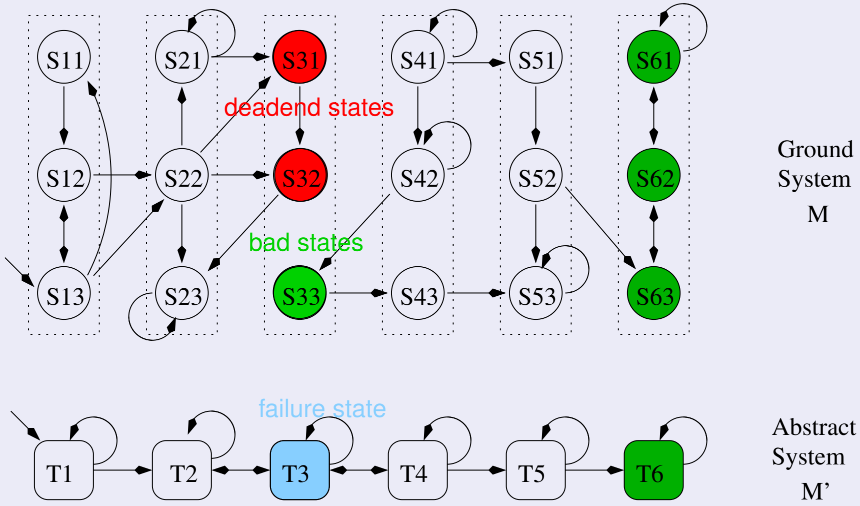

# Refinement: Separate deadend & bad states

## The state separation problem

- Input: sets $D \stackrel{\text{def}}{=} \{d_1, ..., d_k\}$ and $B \stackrel{\text{def}}{=} \{b_1, ..., b_n\}$ of states
- Output: (possibly smallest) set $U \in I$ of invisible variables s.t.

$$\forall d_i \in D,\ \forall b_j \in B,\ \exists u \in U\ s.t.\ d_i(u) \neq b_j(u)$$

$\Longrightarrow$ the truth values of $U$ allow for separating each pair $\langle d_i, b_j \rangle$

$\Longrightarrow$ The refinement $h'$ is obtained by adding U to V.

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# Example

visible, invisible

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
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- differentiating $d_1, b_1$: make $x_4$ visible
- differentiating $d_1, b_2$: make $x_5$ visible
- differentiating $d_2, b_1$: make $x_7$ visible
- differentiating $d_2, b_2$: already different

$\Longrightarrow$ $U = \{x_4, x_5, x_7\}$, $h'$ keeps only $x_6$ invisible

Goal: Keep $U$ as small as possible!

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# Two Separation Methods

- Separation based on Decision-Tree Learning
  - Not optimal.
  - Polynomial.
- ILP-based separation
  - Minimal separating set.
  - Computationally expensive.

# Separation with decision tree (Example)

Idea: expand the decision tree until no $\langle d_i, b_j \rangle$ pair belongs to set.

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$\{d_1, d_2, b_1, b_2\}$

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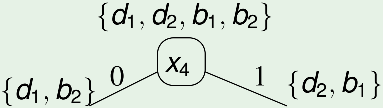
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

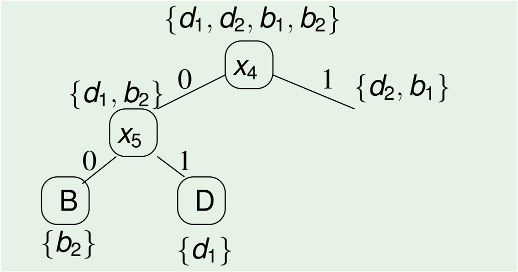



- differentiating $d_1, b_1$: $x_4$
- differentiating $d_1, b_2$: $x_5$
- differentiating $d_2, b_1$: $x_7$
  $\Longrightarrow U = \{x_4, x_5, x_7\}$

# Separation with decision tree (Example)

Idea: expand the decision tree until no $\langle d_i, b_j \rangle$ pair belongs to set.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

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- differentiating $d_2, b_1$: $x_7$

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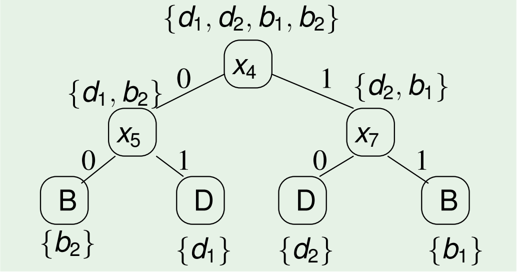
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |




- differentiating $d_1, b_1$: $x_4$
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- differentiating $d_2, b_1$: $x_7$

$\Longrightarrow U = \{x_4, x_5, x_7\}$

# Separation with 0-1 ILP

## Idea

- Encode the problem as a 0-1 ILP problem

$$\min \sum_{x_k \in I} v_k, \qquad \text{subject to}:$$

$$\sum_{\substack{x_k \in I \\ d(x_k) \neq b(x_k)}} v_k \geq 1 \qquad \forall d \in D,\ \forall b \in B,$$

- intuition: $v_k = \top$ iff $x_k$ must me made visible
- one constraint for every pair $\langle d_i, b_j \rangle$

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# Separation with 0-1 ILP: Example

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

$$min \, \{v_4 + v_5 + v_6 + v_7\} \qquad \textit{subject to}:$$

$$\begin{cases} v_4 + \quad\quad v_6 \quad\quad \geq 1 & \text{// separating } d_1, b_1 \\ \quad\quad v_5 \quad\quad\quad\quad \geq 1 & \text{// separating } d_1, b_2 \\ \quad\quad\quad\quad\quad\quad v_7 \geq 1 & \text{// separating } d_2, b_1 \\ v_4 + v_5 + v_6 + v_7 \geq 1 & \text{// separating } d_2, b_2 \end{cases}$$

$\Longrightarrow$ return $\{v_4, v_5, v_7\} \Longrightarrow U = \{x_4, x_5, x_7\}$
or return $\{v_5, v_6, v_7\} \Longrightarrow U = \{x_5, x_6, x_7\}$

# Separation with 0-1 ILP: Example

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

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| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $d_2$ | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $b_2$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

$$min\ \{v_4 + v_5 + v_6 + v_7\} \qquad subject\ to:$$

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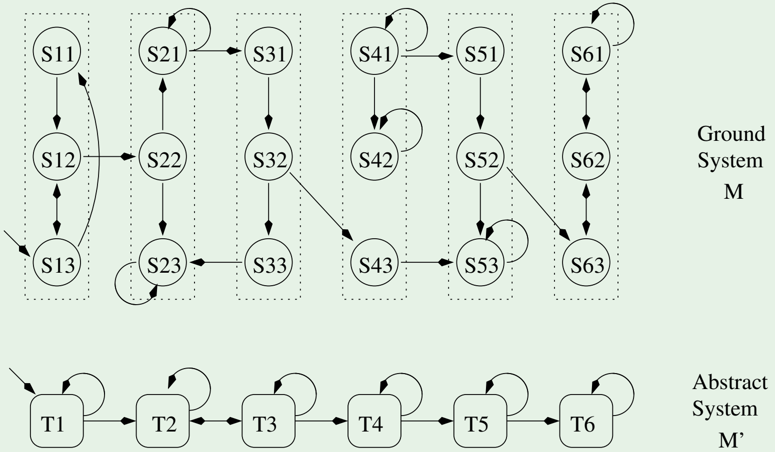
# Outline

# Ex: Simulation

Consider the following pair of ground and abstract machines $M$ and $M'$, and the abstraction $\alpha : M \longmapsto M'$ which, for every $j \in \{1, ..., 6\}$, maps $Sj1$, $Sj2$, $Sj3$ into $Tj$.



# Ex: Simulation [cont.]

For each of the following facts, say which is true and which is false.

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$(a)$ $M$ simulates $M'$.

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[ Solution: False. E.g.,: if $M$ is in $S23$, $M'$ is in $T2$ and $M'$ switches to $T3$, there is no transition in $M$ from $S23$ to any state $S3i$, $i \in \{1, 2, 3\}$. ]

# Ex: Simulation [cont.]

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($b$) $M'$ simulates $M$.

# Ex: Simulation [cont.]

For each of the following facts, say which is true and which is false.

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($b$) $M'$ simulates $M$.

[ Solution: true ]

# Ex: Simulation [cont.]

For each of the following facts, say which is true and which is false.

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(*b*) $M'$ simulates $M$.

[ Solution: true ]

(*c*) for every $j \in \{1, ..., 6\}$ and $i \in \{1, ..., 3\}$, if $Tj$ is reachable in $M'$, then $Sji$ is reachable in $M$

# Ex: Simulation [cont.]

For each of the following facts, say which is true and which is false.

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[ Solution: False. E.g., $T4$ is reachable but $S42$ is not. ]

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For each of the following facts, say which is true and which is false.

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[ Solution: true ]

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[ Solution: False. E.g., $T4$ is reachable but $S42$ is not. ]

($d$) for every $j \in \{1, ..., 6\}$ and $i \in \{1, ..., 3\}$, if $Sji$ is reachable in $M$, then $Tj$ is reachable in $M'$.

# Ex: Simulation [cont.]

For each of the following facts, say which is true and which is false.

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($d$) for every $j \in \{1, ..., 6\}$ and $i \in \{1, ..., 3\}$, if $Sji$ is reachable in $M$, then $Tj$ is reachable in $M'$.

[ Solution: true ]

# Ex: Abstraction-based MC

Consider the following pair of ground and abstract machines $M$ and $M'$, and the abstraction $\alpha : M \longmapsto M'$ which makes the variable `z` invisible.

$M$:

```
MODULE main
VAR
  x : boolean;
  y : boolean;
  z : boolean;
ASSIGN
  init(x) := FALSE;
  init(y) := FALSE;
  init(z) := TRUE;
TRANS
  (next(x) <-> y) &
  (next(y) <-> z) &
  (next(z) <-> x)
```

$M'$:

```
MODULE main
VAR
  x : boolean;
  y : boolean;
  z : boolean;
ASSIGN
  init(x) := FALSE;
  init(y) := FALSE;

TRANS
  (next(x) <-> y) &
  (next(y) <-> z)
```

# Ex: Abstraction-based MC [cont.]

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(*a*) Draw the FSM's for $M$ and $M'$ (n.b.: in $M'$ only $v_1$ and $v_2$ are state variables).

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[ Solution: (We label states with $xyz$ and $xy$. respectively. "$z = 0$" and "$z = 1$" are comments.)

]

# Ex: Abstraction-based MC [cont.]

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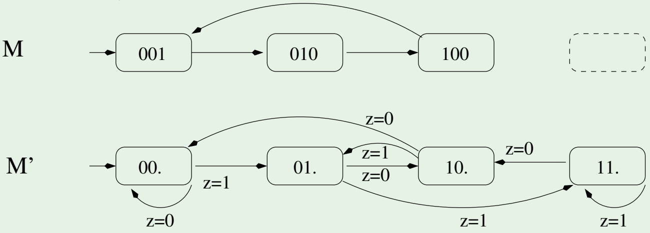


]

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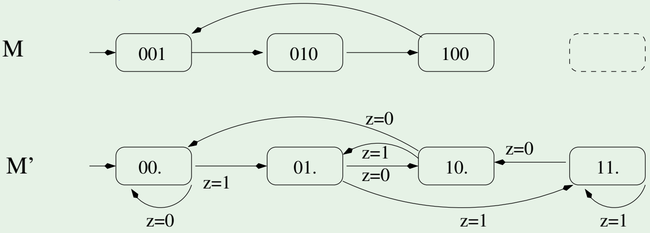


]

(*b*) Does $M$ simulate $M'$?

# Ex: Abstraction-based MC [cont.]

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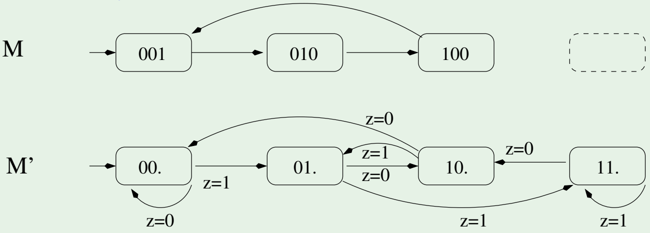

]

(*b*) Does $M$ simulate $M'$? [ Solution: No. E.g. the $M'$ execution looping on (00) cannot be simulated in $M$. ]

# Ex: Abstraction-based MC [cont.]

(*a*) Draw the FSM's for $M$ and $M'$ (n.b.: in $M'$ only $v_1$ and $v_2$ are state variables).
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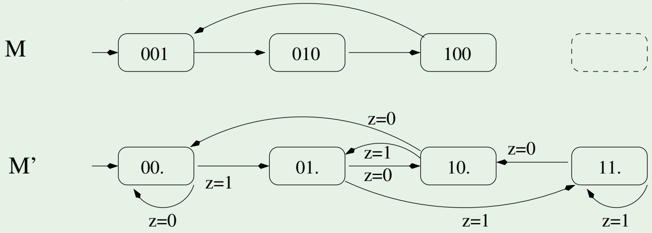

]

(*b*) Does $M$ simulate $M'$? [ Solution: No. E.g. the $M'$ execution looping on (00) cannot be simulated in $M$. ]

(*c*) Does $M'$ simulate $M$?

# Ex: Abstraction-based MC [cont.]

($a$) Draw the FSM's for $M$ and $M'$ (n.b.: in $M'$ only $v_1$ and $v_2$ are state variables).
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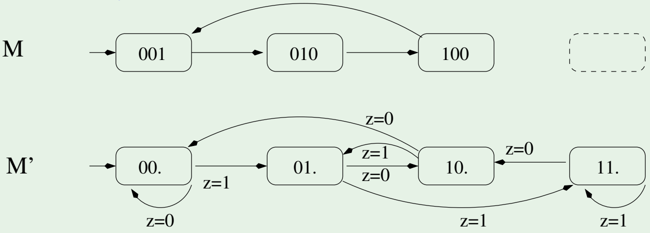

]

($b$) Does $M$ simulate $M'$? [ Solution: No. E.g. the $M'$ execution looping on (00) cannot be simulated in $M$. ]

($c$) Does $M'$ simulate $M$? [ Solution: Yes ]

# Ex: Abstraction-based MC [cont.]

(*a*) Draw the FSM's for $M$ and $M'$ (n.b.: in $M'$ only $v_1$ and $v_2$ are state variables).
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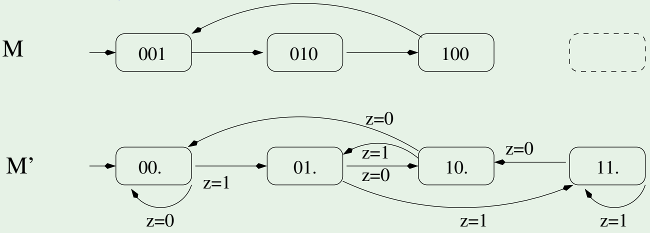


]

(*b*) Does $M$ simulate $M'$? [ Solution: No. E.g. the $M'$ execution looping on (00) cannot be simulated in $M$. ]

(*c*) Does $M'$ simulate $M$? [ Solution: Yes ]

(*d*) Is $\alpha$ a suitable abstraction for solving the MC problem $M \models \mathbf{G}\neg(v_1 \wedge v_2)$?
If yes, explain why. If no, produce a spurious counter-example.

# Ex: Abstraction-based MC [cont.]

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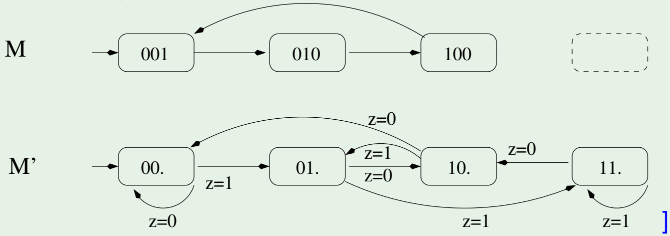

]

(*b*) Does $M$ simulate $M'$? [ Solution: No. E.g. the $M'$ execution looping on (00) cannot be simulated in $M$. ]

(*c*) Does $M'$ simulate $M$? [ Solution: Yes ]

(*d*) Is $\alpha$ a suitable abstraction for solving the MC problem $M \models \mathbf{G}\neg(v_1 \wedge v_2)$?
If yes, explain why. If no, produce a spurious counter-example.
[ Solution: No, since $M \models \mathbf{G}\neg(v_1 \wedge v_2)$ but $M' \not\models \mathbf{G}\neg(v_1 \wedge v_2)$. A spurious counter-example is $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$. ]

# Ex: Abstraction-based MC [cont.]

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(e) Use the SAT-based refinement technique to show that the abstract counter-example $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$ is spurious.

# Ex: Abstraction-based MC [cont.]

(e) Use the SAT-based refinement technique to show that the abstract counter-example $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$ is spurious.
[ Solution: We generate the following formula and feed it to a SAT solver:

$$
\begin{array}{lcl}
(\neg x_0 \land \neg y_0 \land z_0) & \land & // \; I(x_0, y_0, z_0) \land \\
((x_1 \leftrightarrow y_0) \land (y_1 \leftrightarrow z_0) \land (z_1 \leftrightarrow x_0)) & \land & // \; T(x_0, y_0, z_0, x_1, y_1, z_1) \land \\
((x_2 \leftrightarrow y_1) \land (y_2 \leftrightarrow z_1) \land (z_2 \leftrightarrow x_1)) & \land & // \; T(x_1, y_1, z_1, x_2, y_2, z_2) \land \\
(\neg x_0 \land \neg y_0) & \land & // \; (visible(s_0) = c_0) \land \\
(\neg x_1 \land \;\; y_1) & \land & // \; (visible(s_1) = c_1) \land \\
(\;\; x_2 \land \;\; y_2) & & // \; (visible(s_2) = c_2)
\end{array}
$$

]

# Ex: Abstraction-based MC [cont.]

(e) Use the SAT-based refinement technique to show that the abstract counter-example $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$ is spurious.
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$$
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(\neg x_0 \wedge \neg y_0 \wedge z_0) & \wedge & // \; I(x_0, y_0, z_0) \wedge \\
((x_1 \leftrightarrow y_0) \wedge (y_1 \leftrightarrow z_0) \wedge (z_1 \leftrightarrow x_0)) & \wedge & // \; T(x_0, y_0, z_0, x_1, y_1, z_1) \wedge \\
((x_2 \leftrightarrow y_1) \wedge (y_2 \leftrightarrow z_1) \wedge (z_2 \leftrightarrow x_1)) & \wedge & // \; T(x_1, y_1, z_1, x_2, y_2, z_2) \wedge \\
(\neg x_0 \wedge \neg y_0) & \wedge & // \; (visible(s_0) = c_0) \wedge \\
(\neg x_1 \wedge \quad y_1) & \wedge & // \; (visible(s_1) = c_1) \wedge \\
( \quad x_2 \wedge \quad y_2) & & // \; (visible(s_2) = c_2)
\end{array}
$$

$\Longrightarrow \{\neg x_0, \neg y_0, \; z_0, \neg x_1, \; y_1, \neg z_1, \; x_2, \neg y_2, \neg z_2\}$ are unit-propagated due to the first three rows

]

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((x_1 \leftrightarrow y_0) \wedge (y_1 \leftrightarrow z_0) \wedge (z_1 \leftrightarrow x_0)) & \wedge & // \ T(x_0, y_0, z_0, x_1, y_1, z_1) \wedge \\
((x_2 \leftrightarrow y_1) \wedge (y_2 \leftrightarrow z_1) \wedge (z_2 \leftrightarrow x_1)) & \wedge & // \ T(x_1, y_1, z_1, x_2, y_2, z_2) \wedge \\
(\neg x_0 \wedge \neg y_0) & \wedge & // \ (visible(s_0) = c_0) \wedge \\
(\neg x_1 \wedge \ \ y_1) & \wedge & // \ (visible(s_1) = c_1) \wedge \\
(\ \ x_2 \wedge \ \ y_2) & & // \ (visible(s_2) = c_2)
\end{array}
$$

$\Longrightarrow \{\neg x_0, \neg y_0, \ z_0, \neg x_1, \ y_1, \neg z_1, \ x_2, \neg y_2, \neg z_2\}$ are unit-propagated due to the first three rows

]

# Ex: Abstraction-based MC [cont.]

(e) Use the SAT-based refinement technique to show that the abstract counter-example $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$ is spurious.

[ Solution: We generate the following formula and feed it to a SAT solver:

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\begin{array}{lcl}
(\neg x_0 \land \neg y_0 \land z_0) & \land & // \; I(x_0, y_0, z_0) \land \\
((x_1 \leftrightarrow y_0) \land (y_1 \leftrightarrow z_0) \land (z_1 \leftrightarrow x_0)) & \land & // \; T(x_0, y_0, z_0, x_1, y_1, z_1) \land \\
((x_2 \leftrightarrow y_1) \land (y_2 \leftrightarrow z_1) \land (z_2 \leftrightarrow x_1)) & \land & // \; T(x_1, y_1, z_1, x_2, y_2, z_2) \land \\
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\end{array}
$$

$\Longrightarrow$ $\{\neg x_0, \neg y_0, \; z_0, \neg x_1, \; y_1, \neg z_1, \; x_2, \neg y_2, \neg z_2\}$ are unit-propagated due to the first three rows

$\Longrightarrow$ UNSAT

]

# Ex: Abstraction-based MC [cont.]

(e) Use the SAT-based refinement technique to show that the abstract counter-example $C \stackrel{\text{def}}{=} (00) \Longrightarrow (01) \Longrightarrow (11)$ is spurious.

[ Solution: We generate the following formula and feed it to a SAT solver:

$$
\begin{array}{lcl}
(\neg x_0 \land \neg y_0 \land z_0) & \land & //\ I(x_0, y_0, z_0) \land \\
((x_1 \leftrightarrow y_0) \land (y_1 \leftrightarrow z_0) \land (z_1 \leftrightarrow x_0)) & \land & //\ T(x_0, y_0, z_0, x_1, y_1, z_1) \land \\
((x_2 \leftrightarrow y_1) \land (y_2 \leftrightarrow z_1) \land (z_2 \leftrightarrow x_1)) & \land & //\ T(x_1, y_1, z_1, x_2, y_2, z_2) \land \\
(\neg x_0 \land \neg y_0) & \land & //\ (visible(s_0) = c_0) \land \\
(\neg x_1 \land \quad y_1) & \land & //\ (visible(s_1) = c_1) \land \\
(\quad x_2 \land \quad y_2) & & //\ (visible(s_2) = c_2)
\end{array}
$$

$\Longrightarrow$ $\{\neg x_0, \neg y_0, \ z_0, \neg x_1, \ y_1, \neg z_1, \ x_2, \neg y_2, \neg z_2\}$ are unit-propagated due to the first three rows

$\Longrightarrow$ UNSAT

$\Longrightarrow$ spurious counter-example.

]

# Ex: Separation problem

In a counter-example-guided-abstraction-refinement model checking process using localization reduction, variables $x_3, x_4, x_5, x_6, x_7, x_8$ are made invisible.
Suppose the process has identified a spurious counterexample with an abstract failure state [00], two ground deadend states $d_1, d_2$ and two ground bad states $b_1, b_2$ as described in the following table:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $d_2$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| $b_2$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

Identify a minimum-size subset of invisible variables which must be made visible in the next abstraction to avoid the above failure. Briefly explain why.

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|---|---|---|---|---|---|---|---|---|
| $d_1$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $d_2$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| $b_1$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| $b_2$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

Identify a minimum-size subset of invisible variables which must be made visible in the next abstraction to avoid the above failure. Briefly explain why.
[ Solution: The minimum-size subset is $\{x_7\}$. In fact, if $x_7$ is made visible, then both $d_1, d_2$ are made different from both $b_1, b_2$. ]