Formal Methods:
Module I: Automated Reasoning
Ch. 05: **Automata-Theoretic LTL Reasoning**

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Outline

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1. Büchi Automata
2. The Automata-Theoretic Approach to LTL Reasoning
   - General Ideas
   - Language-Emptiness Checking of Büchi Automata
   - From Kripke Models to Büchi Automata
   - From LTL Formulas to Büchi Automata
   - Complexity
3. Exercises

Infinite Word Languages

Modeling infinite computations of reactive systems

Given an Alphabet Σ (e.g. $\Sigma \stackrel{\text{def}}{=} \{a, b\}$)

- An ω-word α over Σ is an infinite sequence

  $a_0, \ a_1, \ a_2 \ldots$.

  Formally, $\alpha : \mathbb{N} \rightarrow \Sigma$.
- The set of all infinite words is denoted by Σ^ω.
- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.
- Example: All words over $\{a, b\}$ with infinitely many a's.

Notation:

omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.

omega-languages $L, L_1 \subseteq \Sigma^\omega$

For $u \in \Sigma^+$, let $u^\omega = u.u.u\ldots$

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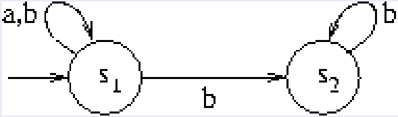
Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\ldots$.
  There are several (infinite) possible runs.
  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
  Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \ldots$
- Acceptance Conditions: Büchi (Muller, Rabin, Street):
  Acceptance is based on states occurring infinitely often
- Notation Let $\rho \in Q^\omega$. Then,
  $$Inf(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}.\ \rho(i) = s\}.$$
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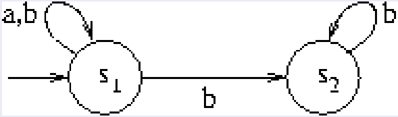

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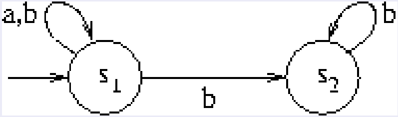

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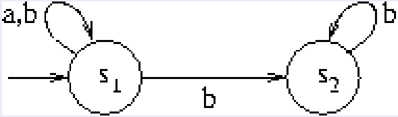

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Nondeterministic Büchi Automaton

- A Nondeterministic Büchi Automaton (NBA) is $(Q, \Sigma, \delta, I, F)$ s.t.
  - Q Finite set of states.
  - Σ is a finite alphabet
  - $I \subseteq Q$ set of initial states.
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  - $\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).
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Runs and Language of NBAs

- A run ρ of A on ω-word $\alpha = a_0, a_1, a_2, \ldots$ is an infinite sequence $\rho = q_0, q_1, q_2, \ldots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
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Büchi Automaton: Example

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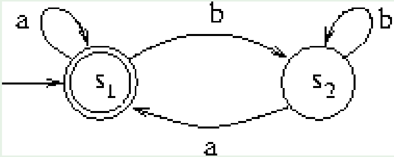

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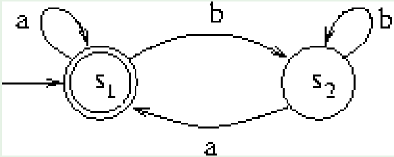

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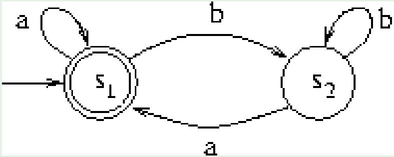

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Büchi Automaton: Example (2)

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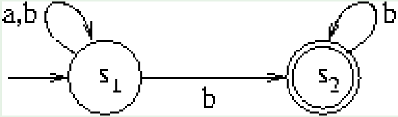

With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

Theorem

*DBA*s are strictly less powerful than *NBA*s.

The subset construction does not work!

Let DA_2 be

- DA_2 is not equivalent to A_2 (e.g., it recognizes $(b.a)^\omega$)
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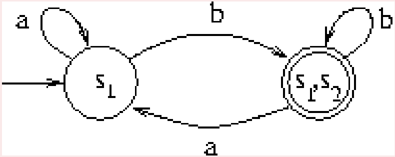

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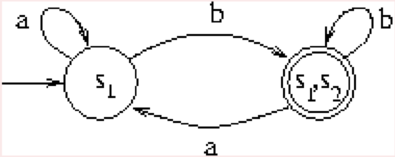

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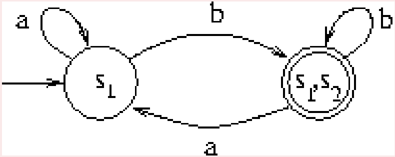

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Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
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Union of two NBAs

Definition: union of NBAs

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Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
$$I = I_1 \times I_2 \times \{1\}.$$
$$F = F_1 \times Q_2 \times \{1\}.$$

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Theorem

- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
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Synchronous Product of NBAs: Intuition

- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track
  $\Longrightarrow$ in order to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2
- Important subcase: If $F_2 = Q_2$, then

$$Q = Q_1 \times Q_2.$$
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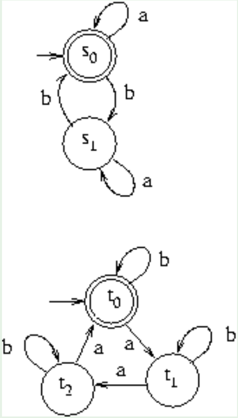

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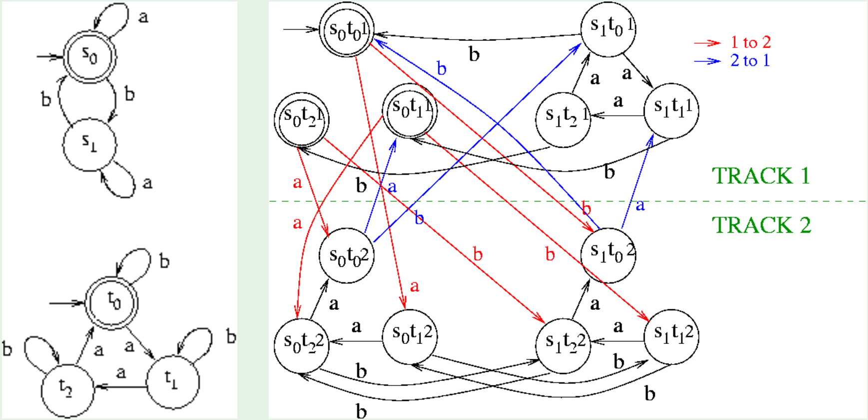

Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that
$\mathcal{L}(A_2) \;=\; \overline{\mathcal{L}(A_1)}$.
$|A_2| \;=\; O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

(i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
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Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
The automaton remains in phase i till it visits a state in F_i. Then, it moves to $(i \; mod \; K) + 1$ mode.

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De-generalization of a generalized NBA

Definition: De-generalization of a generalized NBA

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δ' is s.t., for every $i \in [1, ..., K]$:

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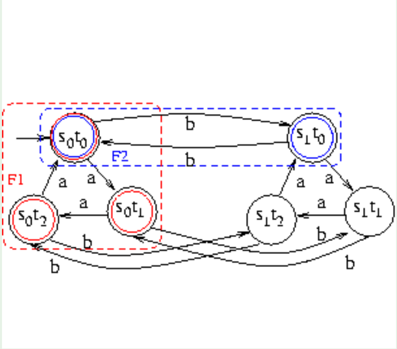

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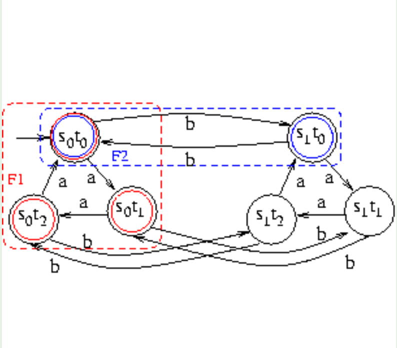

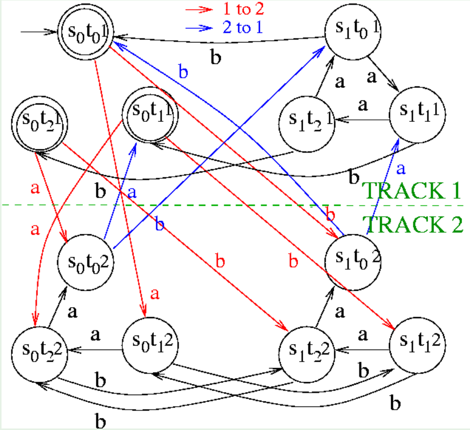

Omega-regular Expressions

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A language is called ω-regular if it has the form $\cup_{i=1}^{n} \; U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

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A language L is ω-regular iff it is NBA-recognizable.

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Outline

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Automata-Theoretic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$\models \psi \quad \text{(LTL)}$$
$$\Longleftrightarrow \neg\psi \text{ unsat}$$
$$\Longleftrightarrow \mathcal{L}(A_{\neg\psi}) = \emptyset$$

- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

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Automata-Theoretic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

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- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
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Automata-Theoretic LTL Model Checking

Four steps

Let $\varphi \stackrel{\text{def}}{=} \neg\psi$:

(i) Compute A_M
(ii) Compute A_φ
(iii) Compute the product $A_M \times A_\varphi$
(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

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NBA emptiness checking

- Find an accepting cycle reachable from an initial state.
- A naive algorithm:
  (i) a DFS finds the final states *f* reachable from an initial state;
  (ii) for each *f*, a second DFS finds if it can reach f (i.e., if there exists a loop)

  Complexity: $O(n^2)$
- SCC-based algorithm:
  (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
  (ii) drop all SCCs which do not have at least one arc, and which do not contain at least one accepting state f
  (iii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.

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  Complexity: $O(n)$
- Drawbacks: it stores too much information and does not find directly a counterexample.

NBA emptiness checking

- Find an accepting cycle reachable from an initial state.
- A naive algorithm:
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Double Nested DFS algorithm

Double Nested DFS

- Two nested DFSs
  - `DFS1` finds the final states f reachable from an initial state
  - for each f, `DFS2` finds if it can reach f (i.e., if there exists a loop)
- Two Hash tables:
  - `T1`: reachable states
  - `T2`: states reachable from a reachable final state
- Two stacks:
  - `S1`: current branch of states reachable
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- It stops as soon as it finds a counterexample.
- The counterexample is given by
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  - the stack of `DFS1` (a path from an initial state to the cycle)

- DFS1 invokes DFS2 on each f_i only after popping it (postorder)
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Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
   stack S1=I; stack S2=∅;
   Hashtable T1=I; Hashtable T2=∅;
   while S1!=∅ {
      v=top(S1);
      if ∃w s.t. w∈δ(v) && T1(w)==0 {
         hash(w,T1);
         push(w,S1);
      } else {
         pop(S1);
         if (v∈F && !DFS2(v,S2,T2,A))
            return False;
   }     }
   return True;
}
```

Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
    hash(f,T);
    S = {f}
    while S!=∅ {
        v=top(S);
        if f∈δ(v) return False;
        if ∃w s.t. w∈δ(v) && T(w)==0 {
            hash(w);
            push(w);
        } else pop(S);
    }
    return True;
}
```

Remark: `T` passed by reference, is not reset at each call of `DFS2` !

Double nested DFS: intuition

DFS1 invokes DFS2 on each $f_1, ..., f_n$ only after popping it (postorder):

- suppose *DFS2* is invoked on f_j before than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_i, ...)$ it is encountered a state S which has already been explored by $DFS2(f_j, ...)$ for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

$\Longrightarrow$ it is safe to backtrack.

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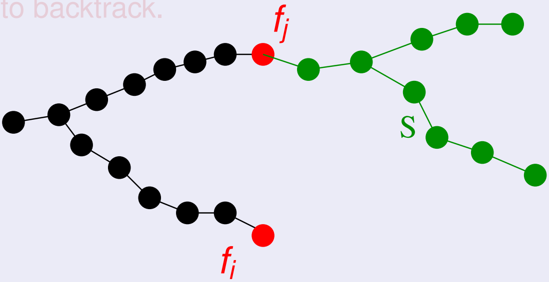

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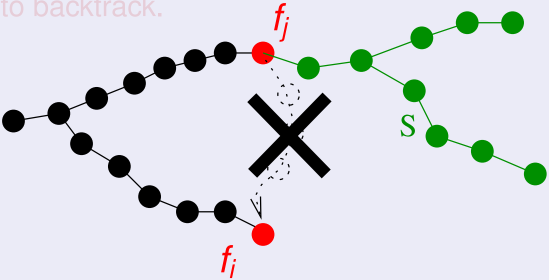

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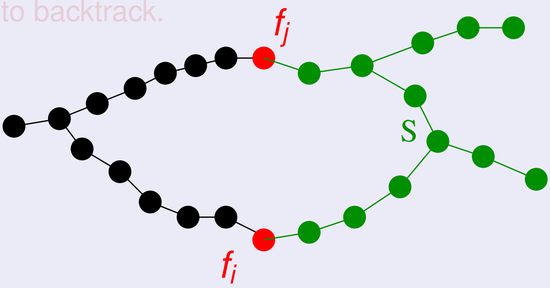

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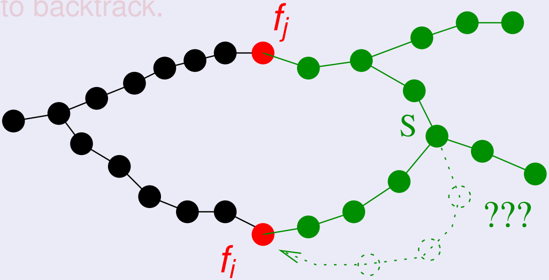

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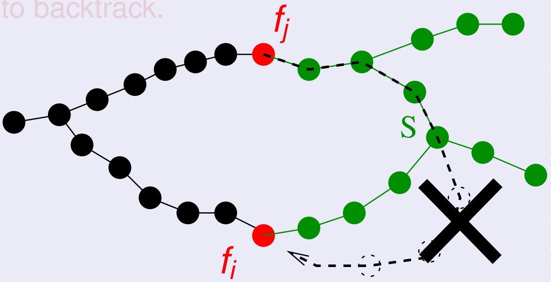

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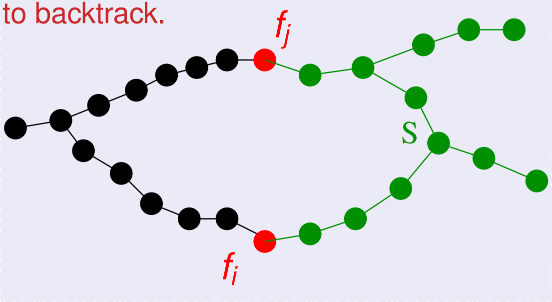

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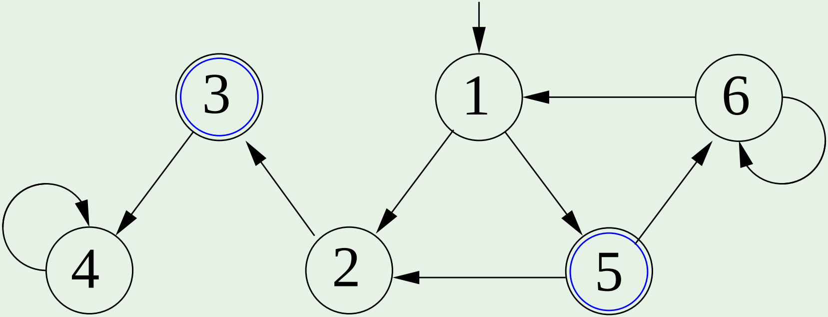


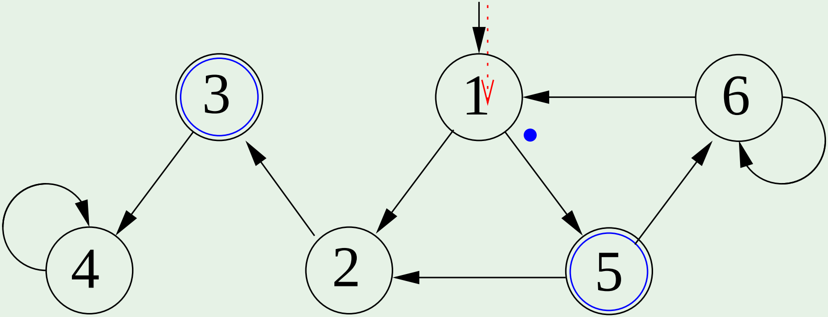
T1

S1

T2

S2

Double Nested DFS: example



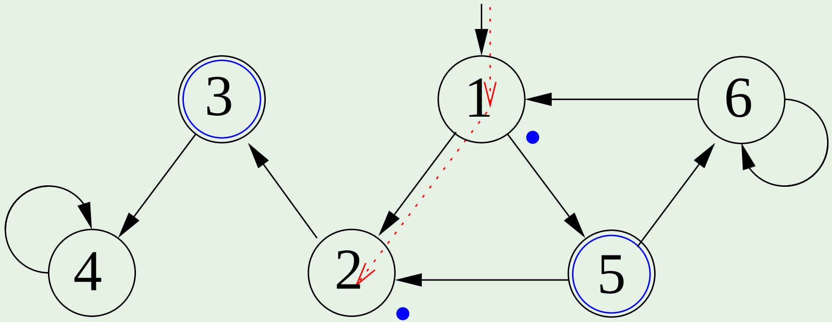
T1 1

S1 1

T2

S2

Double Nested DFS: example



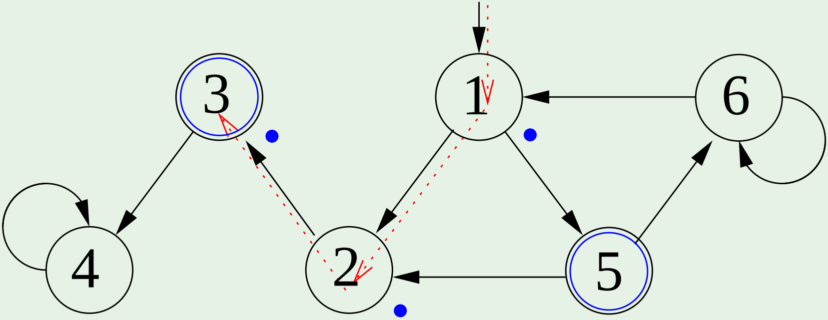
T1 1 2

S1 1 2

T2

S2

Double Nested DFS: example



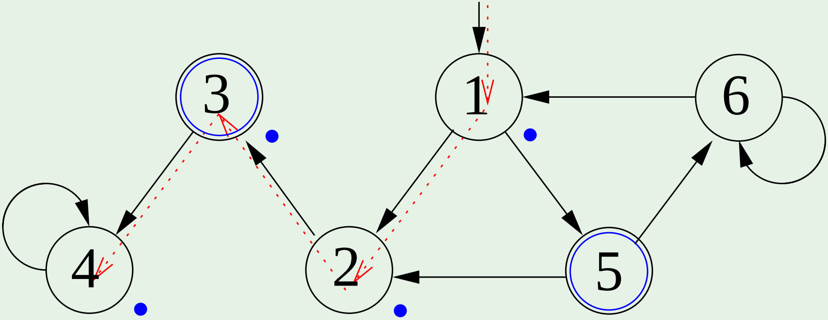
T1 1 2 3

S1 1 2 3

T2

S2

Double Nested DFS: example



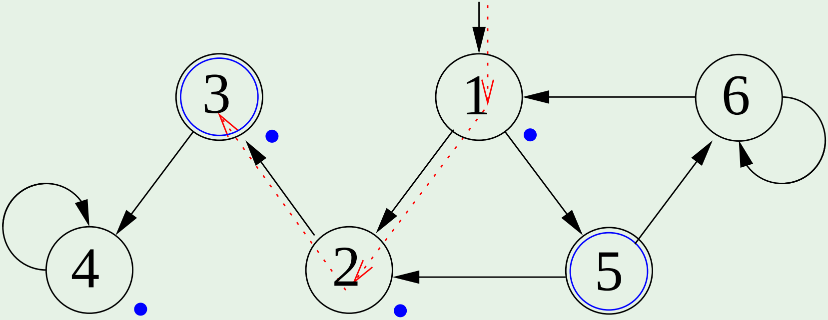
T1 1 2 3 4

S1 1 2 3 4

T2

S2

Double Nested DFS: example



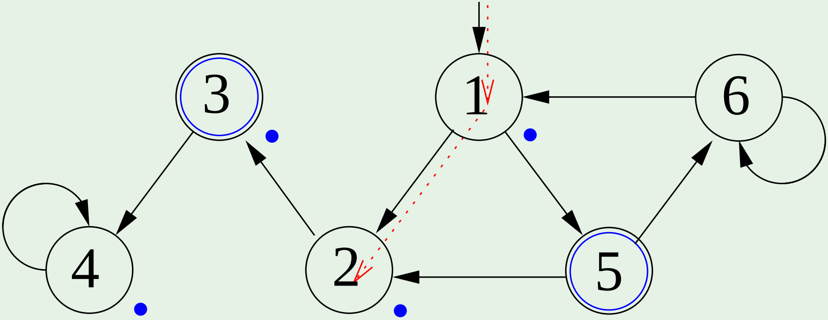
T1 1 2 3 4

S1 1 2 3

T2

S2

Double Nested DFS: example



T1 1 2 3 4

S1 1 2

T2

S2

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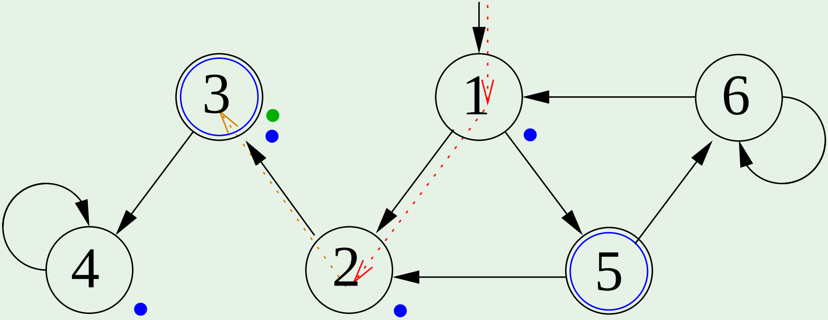

T1 1 2 3 4

S1 1 2

T2 3

S2 3

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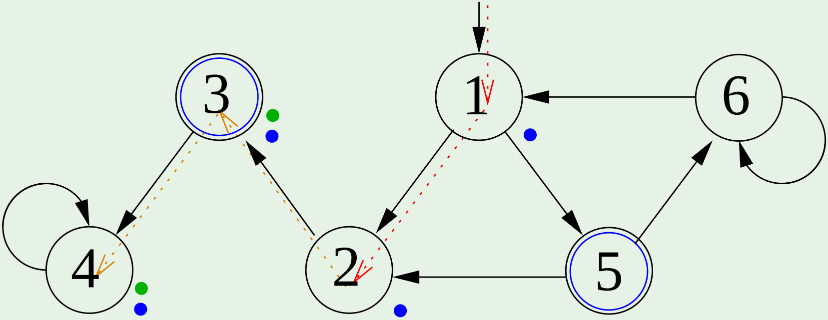

T1 1 2 3 4

S1 1 2

T2 3 4

S2 3 4

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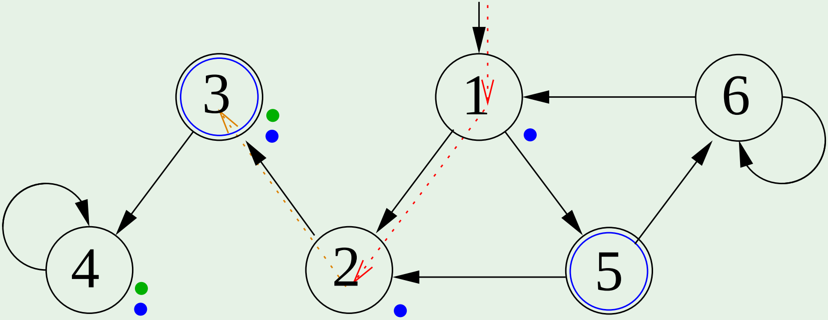

T1 1 2 3 4

S1 1 2

T2 3 4

S2 3

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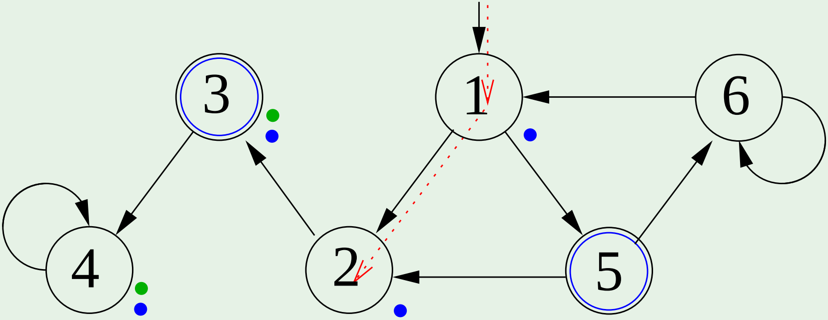

T1 1 2 3 4

S1 1 2

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S2

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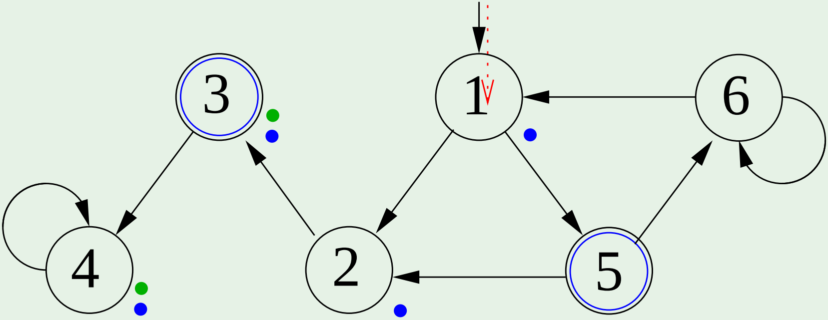

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S1 1

T2 3 4

S2

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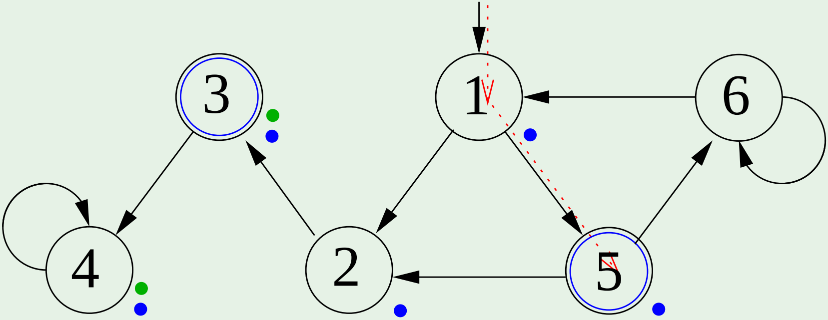


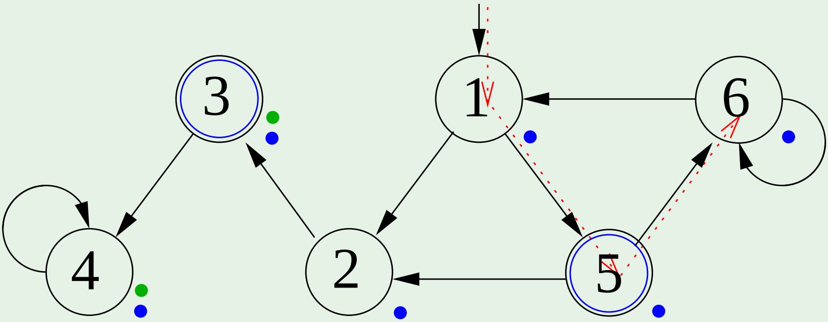
T1 1 2 3 4 5

T2 3 4

S1 1 5

S2

Double Nested DFS: example



T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

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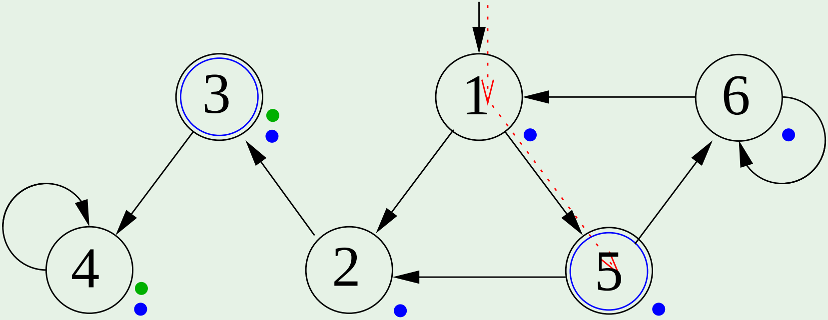

T1 1 2 3 4 5 6

T2 3 4

S1 1 5

S2

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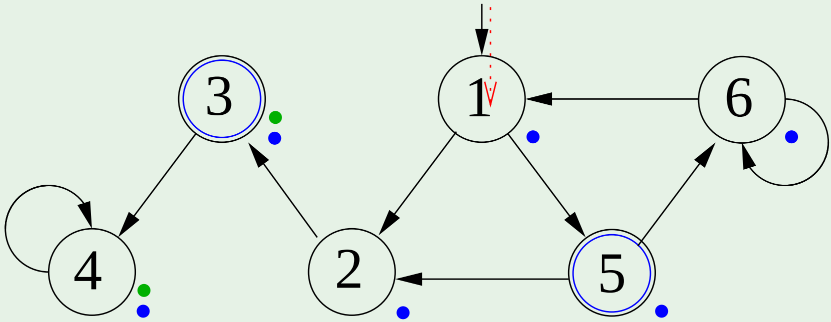

T1 1 2 3 4 5 6   T2 3 4

S1 1   S2

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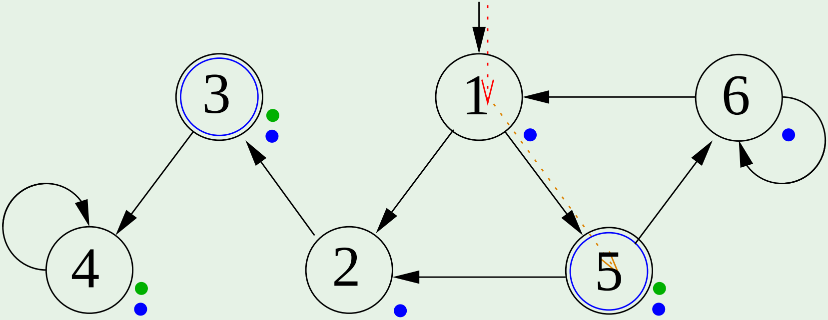

T1 1 2 3 4 5 6

T2 3 4 5

S1 1

S2 5

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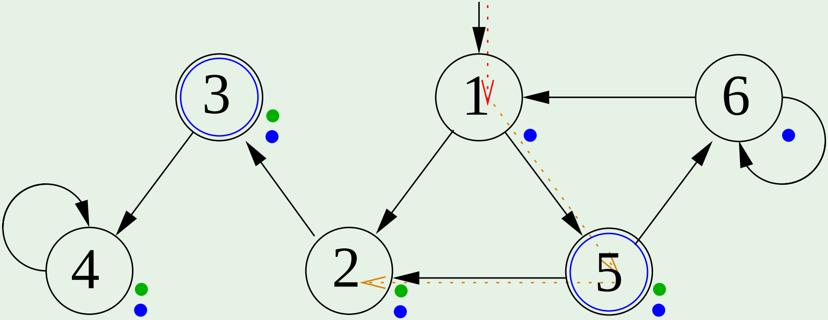

T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2

S2 5 2

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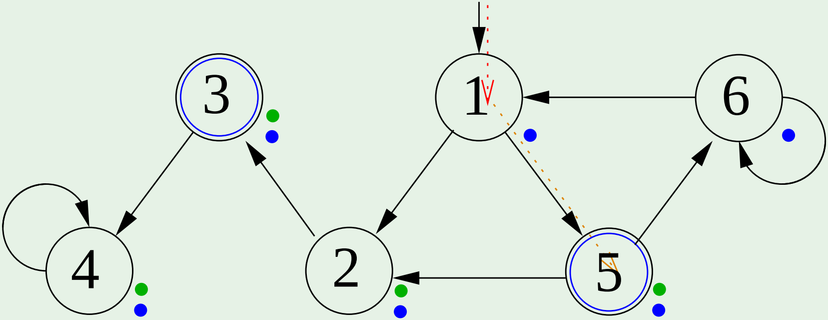

T1 1 2 3 4 5 6 T2 3 4 5 2

S1 1 S2 5

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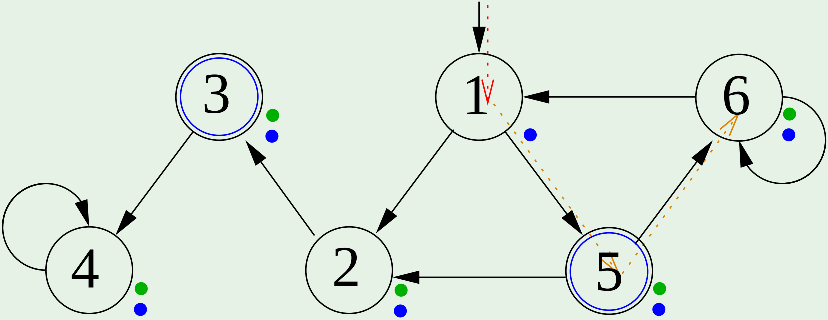

T1 1 2 3 4 5 6

T2 3 4 5 2 6

S1 1

S2 5 6

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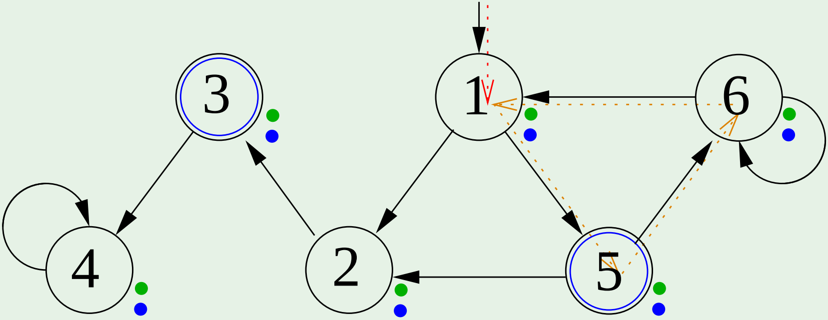

T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2 6 1

S2 5 6 1

Outline

Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke model $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$
  - Initial State: $I := \{init\}$
  - Accepting States: $F := Q = S \cup \{init\}$
  - Transitions:

$$\delta : \quad q \xrightarrow{a} q' \text{ iff } (q, q') \in R \; and \; L(q') = a$$
$$init \xrightarrow{a} q \text{ iff } q \in S_0 \; and \; L(q) = a$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing an NBA A_M from a Kripke Structure M

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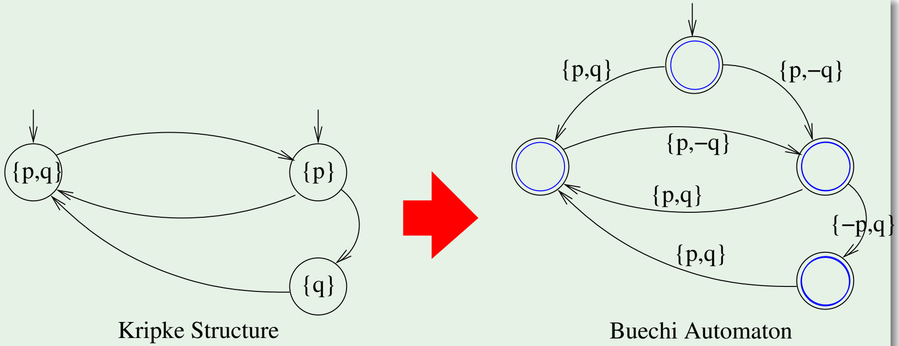
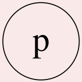
Computing a NBA A_M from a Kripke Structure M: Example



$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



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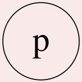

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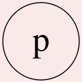

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Outline

Translation problem

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Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL model checking (in other words, every model checking algorithm that verifies the correctness of an LTL formula translates it in some sort of finite-state machine).
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LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, **X**, **U**, **R** on propositional literals.
- Done by pushing negations down to literal level:

$$\begin{array}{lcl} \neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & (\neg\varphi_1 \wedge \neg\varphi_2) \\ \neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & (\neg\varphi_1 \vee \neg\varphi_2) \\ \neg\mathbf{X}\varphi_1 & \Longrightarrow & \mathbf{X}\neg\varphi_1 \\ \neg(\varphi_1 \mathbf{U} \varphi_2) & \Longrightarrow & (\neg\varphi_1 \mathbf{R} \neg\varphi_2) \\ \neg(\phi_1 \mathbf{R} \phi_2) & \Longrightarrow & (\neg\phi_1 \mathbf{U} \neg\phi_2) \end{array}$$

  $\Longrightarrow$ the resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).
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On-the-fly Construction of A_φ (Intuition)

Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ

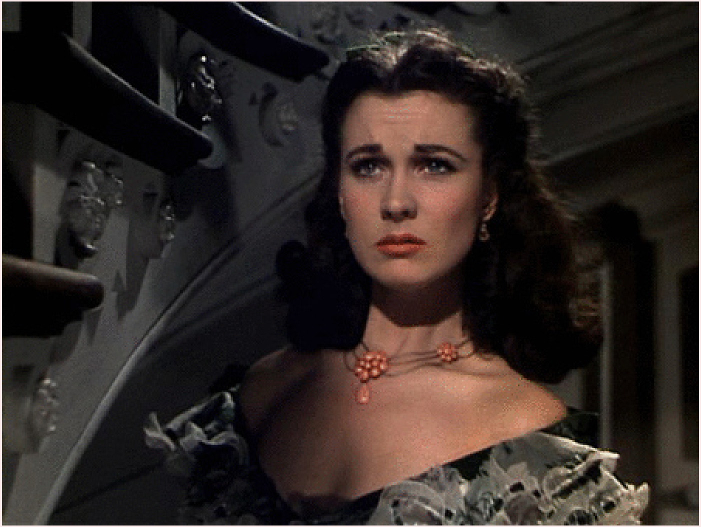
$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$ [and $\mathbf{F}\psi \Longrightarrow \psi \vee \mathbf{XF}\psi$]

$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$ [and $\mathbf{G}\psi \Longrightarrow \psi \wedge \mathbf{XG}\psi$]

until we get a Boolean combination of elementary subformulas of φ

(An elementary formula is a proposition or a **X**-formula.)

Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, and then push the conjunctions inside the next:

$$\varphi \Longrightarrow \bigvee_i \left(\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X}\psi_{ik}\right) \Longrightarrow \bigvee_i \left(\bigwedge_j l_{ij} \wedge \mathbf{X}\bigwedge_k \psi_{ik}\right).$$

- Each disjunct $(\underbrace{\bigwedge_j l_{ij}}_{labels} \wedge \underbrace{\mathbf{X}\bigwedge_k \psi_{ik}}_{next\ part})$ represents a state:
  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
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On-the-fly Construction of A_φ (Intuition) [cont.]

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- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
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φ ??

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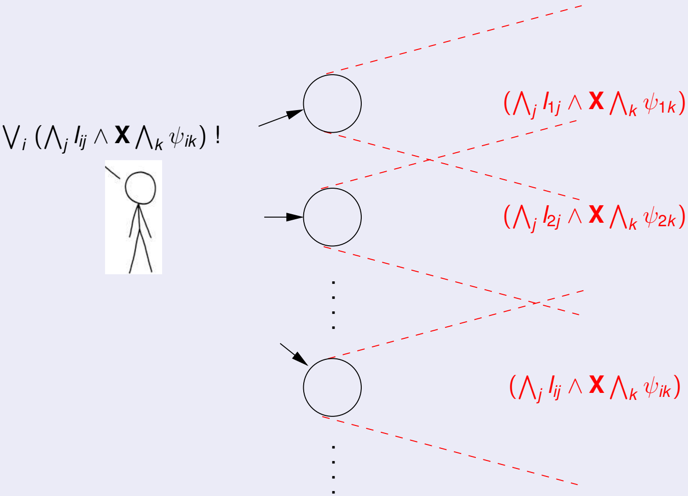
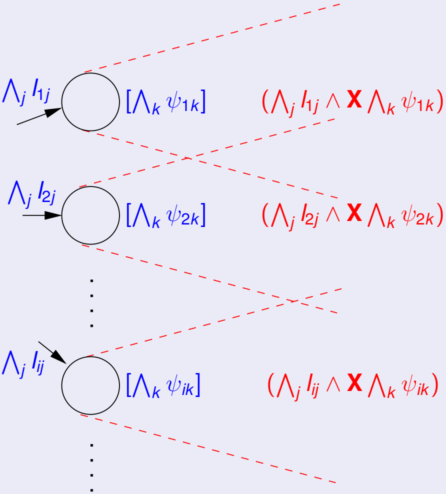
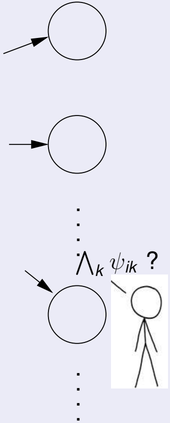


On-the-fly Construction of A_φ (Intuition) [cont.]

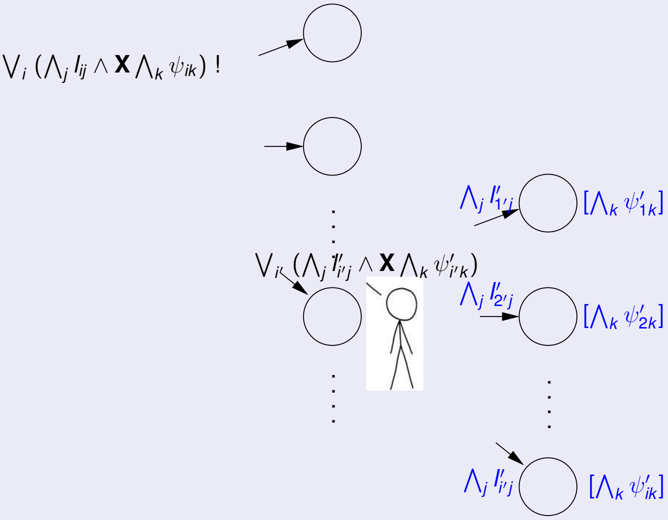
$\bigvee_i \left(\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}\right)$!



On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



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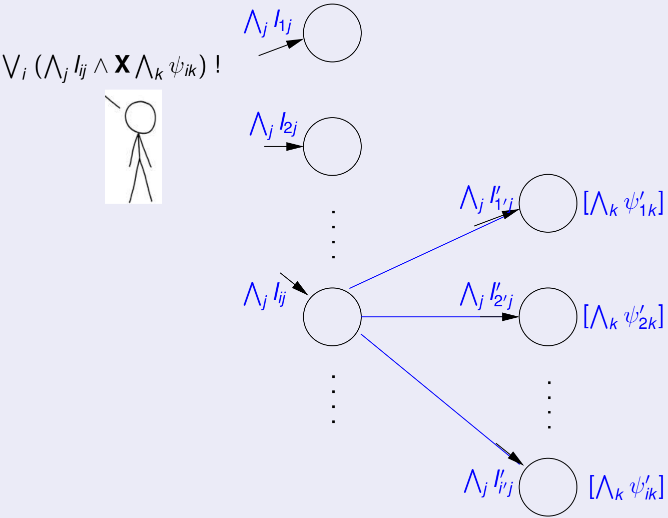

On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

Step 4: For every $\psi_i\mathbf{U}\varphi_i$, for every state q_j, mark q_j with F_i iff
$(\psi_i\mathbf{U}\varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no **U**-subformulas, then mark all states with F_1
—i.e., $FT \stackrel{\text{def}}{=} \{Q\}$).

Remark

The fact that we initially converted the formula into NNF guarantees that only positive **U**/**F**-subformulas and negative **R**-/**G**-subformulas are considered here

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The fact that we initially converted the formula into NNF guarantees that only positive **U**/**F**-subformulas and negative **R**-/**G**-subformulas are considered here

Dealing with **U**-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \Longleftrightarrow (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X} \varphi_1 \mathbf{U} \varphi_2))$ are a property, not a definition of **U**:
  $\Longrightarrow$ they implicitly admit a "weaker" semantics of $\varphi_1 \mathbf{U} \varphi_2$, in which $\varphi_1 \mathbf{U} \varphi_2$ always holds and φ_2 never holds
- It cannot happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.

$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ holds

- In LTL: $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ ("avoid bad loop")

Dealing with **U**-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \iff (\varphi_2 \lor (\varphi_1 \land \mathbf{X}\varphi_1 \mathbf{U} \varphi_2))$ are a property, not a definition of **U**:
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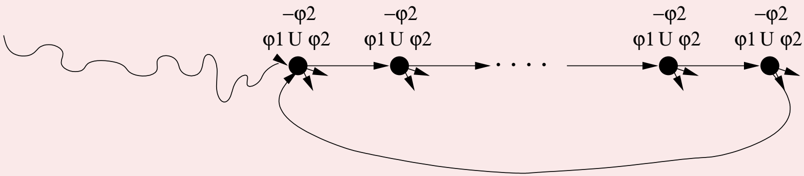

$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \lor \varphi_2)$ holds

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Dealing with U-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \iff (\varphi_2 \lor (\varphi_1 \land \mathbf{X} \varphi_1 \mathbf{U} \varphi_2))$ are a property, not a definition of **U**:
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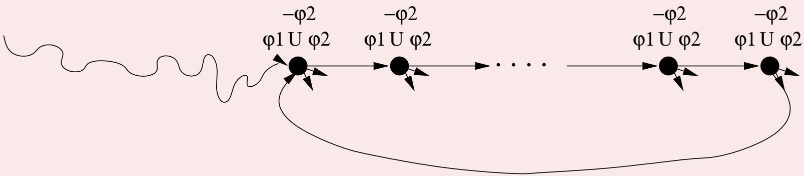

$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \lor \varphi_2$ holds

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Dealing with **U**-subformulas: Intuition

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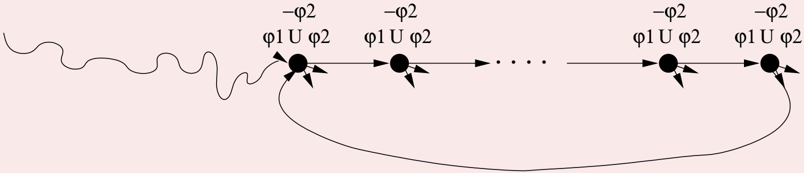

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On-the-fly Construction of A_ϕ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of ϕ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - Combines steps 1. and 2. of previous slides
  - Expand() defined recursively as follows

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On-the-fly Construction of A_ϕ - Expand

Given a set of formulas Φ to expand and a state s, we define the set of states $Expand(\Phi, s)$ recursively as follows:

- if $\Phi = \emptyset$, $Expand(\Phi, s) = \{s\}$
- if $\bot \in \Phi$, $Expand(\Phi, s) = \emptyset$
- if $\top \in \Phi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
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- if $l \in \Phi$ and $s = \langle\lambda, \chi, \sigma\rangle$, l propositional literal
  $Expand(\Phi, s) = Expand(\Phi\backslash\{l\}, \langle\lambda \cup \{l\}, \chi, \sigma \cup \{l\}\rangle)$
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On-the-fly Construction of A_ϕ - Expand

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  $$Expand(\Phi, s) = Expand(\Phi \cup \{\psi_1\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle) \cup Expand(\Phi \cup \{\psi_2\} \backslash \{\psi_1 \vee \psi_2\}, \langle \lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\} \rangle)$$
  (split *s* in two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
- if $\psi_1 \mathbf{U} \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
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  (split *s* in two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{U} \psi_2$ to σ)
- if $\psi_1 \mathbf{R} \psi_2 \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
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Two relevant subcases: $\mathbf{F}\psi \stackrel{\text{def}}{=} \top \mathbf{U} \psi$ and $\mathbf{G}\psi \stackrel{\text{def}}{=} \bot \mathbf{R} \psi$

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Given a set of LTL formulas Ψ, we define
$Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle\emptyset, \emptyset, \emptyset\rangle)$.
For an LTL formula ϕ, we construct a Generalized NBA
$A_\phi = (Q, \Sigma, \delta, I, FT)$ as follows:

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Example: $\phi = \mathbf{FG}p$

- $$\begin{aligned}&\mathit{Cover}(\{\mathbf{FG}p\})\\ &= \mathit{Expand}(\{\mathbf{FG}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle) \cup \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\{\mathbf{FG}p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle)\\ &= \{\langle\emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\}\rangle, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle\}\end{aligned}$$
- $$\begin{aligned}\mathit{Cover}(\{\mathbf{G}p\}) &= \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\{p\}, \langle\emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\}\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle)\\ &= \{\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle\}\end{aligned}$$
- Optimization:
  merge $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\}\rangle$ and $\langle\{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\}\rangle$

Example: $\phi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
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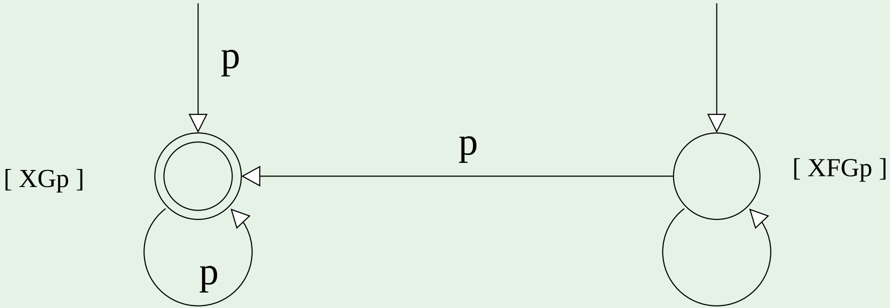

Example: $\phi = p\mathbf{U}q$

- $$\begin{aligned} & Cover(\{p\mathbf{U}q\}) \\ & = Expand(\{p\mathbf{U}q\}, \langle\emptyset, \emptyset, \emptyset\rangle) \\ & = Expand(\{p\}, \langle\emptyset, \{p\mathbf{U}q\}, \{p\mathbf{U}q\}\rangle) \cup Expand(\{q\}, \langle\emptyset, \emptyset, \{p\mathbf{U}q\}\rangle) \\ & = Expand(\emptyset, \langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle) \cup Expand(\emptyset, \langle\{q\}, \emptyset, \{p\mathbf{U}q, q\}\rangle) \\ & = \{\langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle\} \cup \{\langle\{q\}, \{\top\}, \{p\mathbf{U}q, q\}\rangle\} \end{aligned}$$
- $Cover(\{\top\}) = \{\langle\emptyset, \{\top\}, \{\top\}\rangle\}$

Example: $\phi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle$, $s_2 =_{def} \langle\{q\}, \{\top\}, \{p\mathbf{U}q, q\}\rangle$, $s_3 =_{def} \langle\emptyset, \{\top\}, \{\top\}\rangle$.
- $Q = \{s_1, s_2, s_3\}$,
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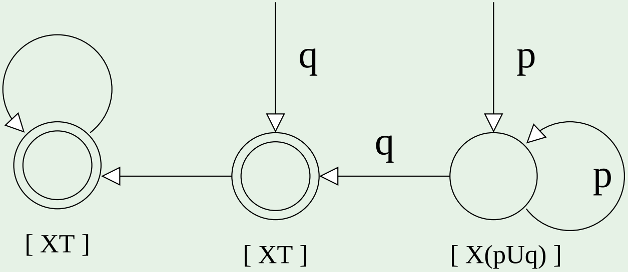

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$$
\begin{aligned}
&\mathit{Cover}(\{\mathbf{GF}p\})\\
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$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \iff \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: **GF**p

- Let $s_1 =_{def} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$,
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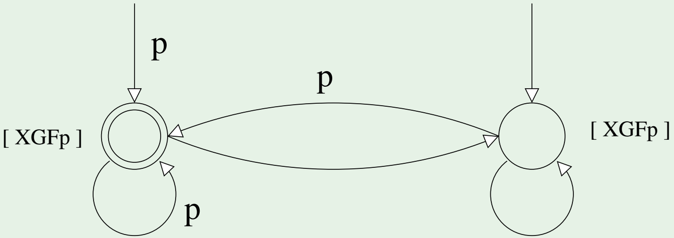

NBAs of disjunctions of formulas

Remark

If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \stackrel{\text{def}}{=} (\mathbf{GF}p \rightarrow \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.
Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ:

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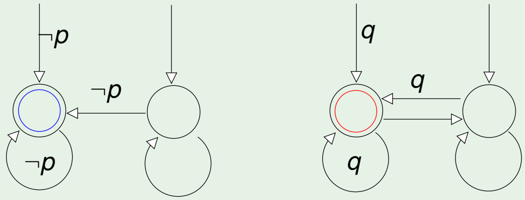

Suggested Exercises:

- Find an NBA encoding:
  - p
  - $(p \wedge q) \vee (\neg p \wedge \neg q)$
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{G}r$

Outline

Automata-Theoretic LTL Model Checking: Complexity

Four steps:

(i) Compute A_M:
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Final Remarks

- Büchi automata are in general more expressive than LTL!

$\Longrightarrow$ some tools (e.g., Spin) allow specifications to be expressed directly as NBAs

$\Longrightarrow$ complementation of NBA important!

- For every LTL formula, there are many possible equivalent NBAs

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Outline

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Write the product Büchi automaton $BA1 \times BA2$.

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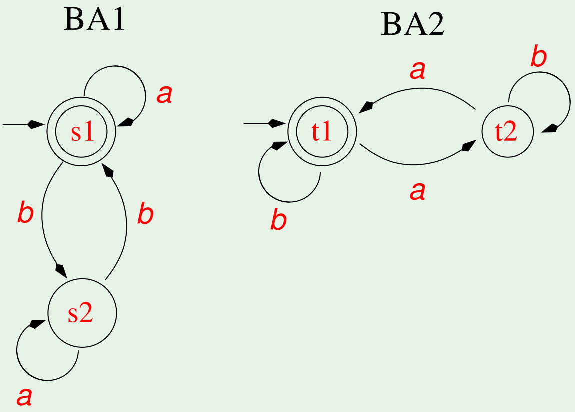

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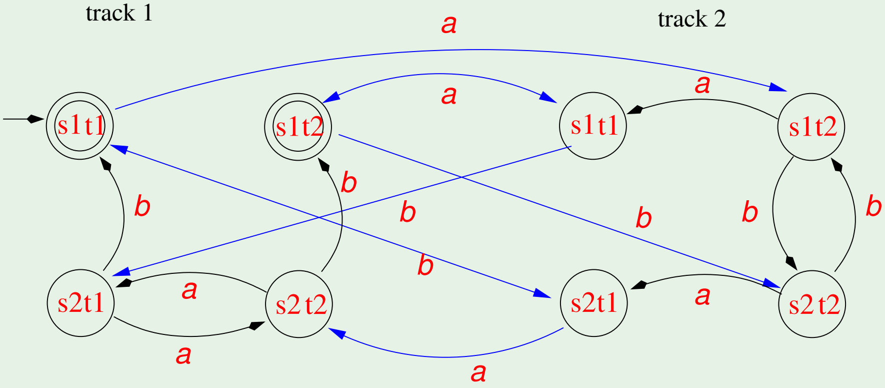

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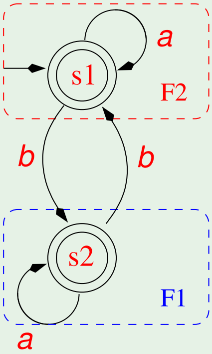
Ex: De-generalization of Büchi Automata

Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \stackrel{\text{def}}{=} \{s2\}$, $F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

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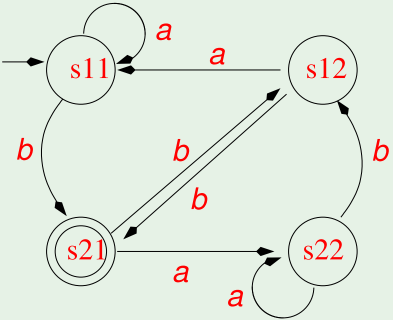

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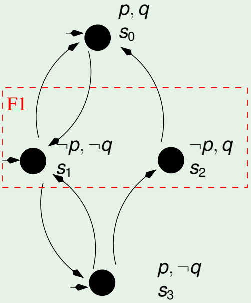
Ex: From Kripke models to Büchi automata

Given the following fair Kripke model M, convert it into an equivalent Buchi automaton.



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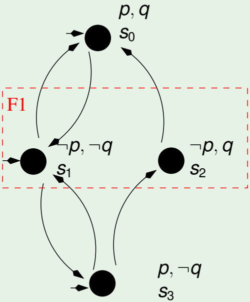

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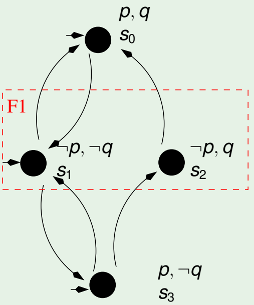

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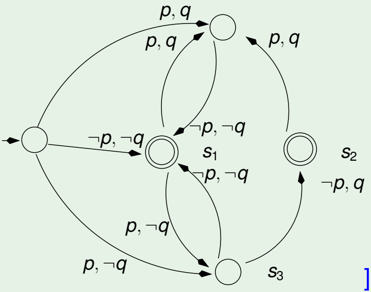

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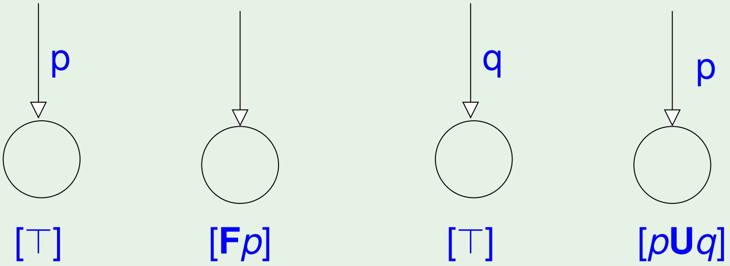

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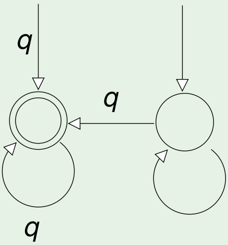

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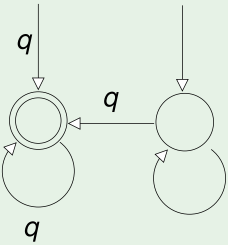

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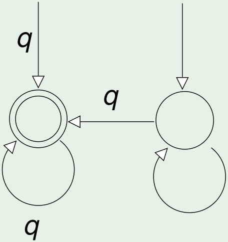

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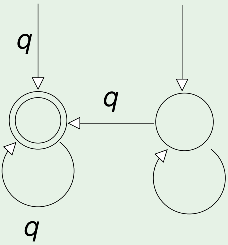

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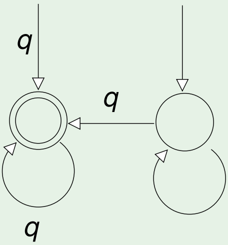

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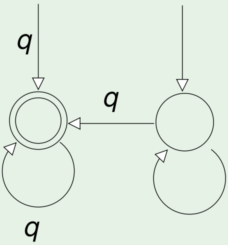

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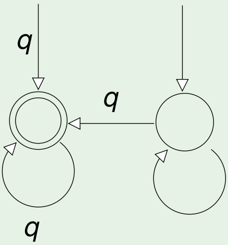

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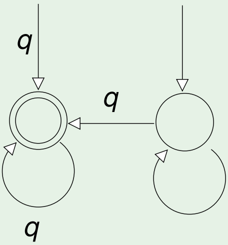

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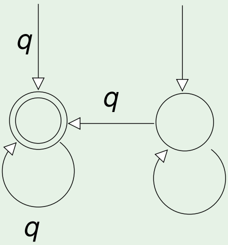

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