Introduction to Formal Methods
Chapter 08: Automata-theoretic LTL Model Checking

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Outline

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System's computations

- The behaviors (computations) of a system can be seen as sequences of assignments to propositions.

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MODULE main
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ASSIGN
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   next(done):= case
        !done: {0,1};
        done: done;
     esac;
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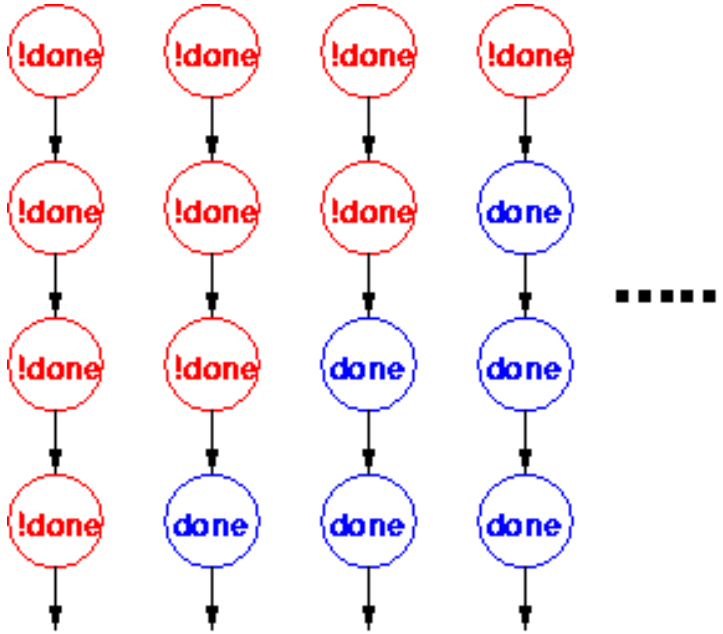

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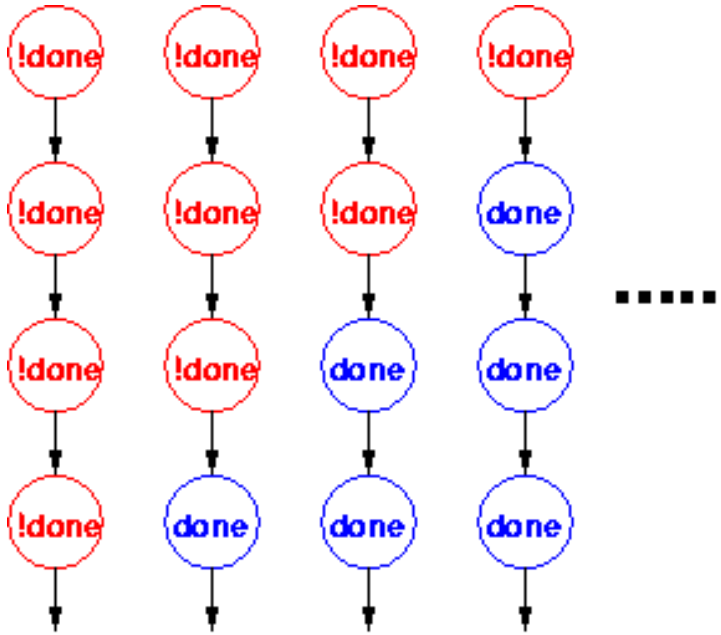

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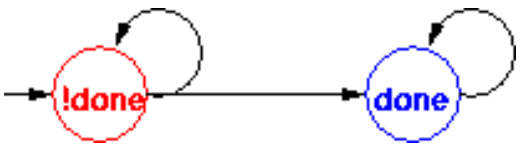

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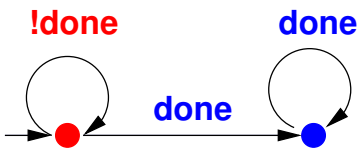
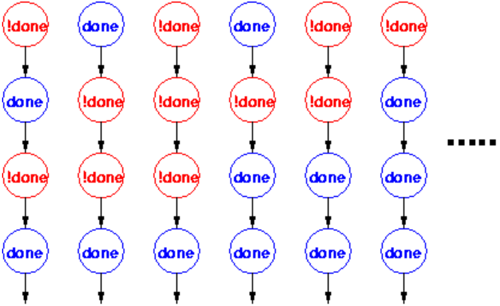
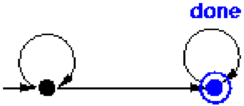


Correct computations

- Some computations are correct and others are not acceptable.
- We can build an automaton for the set of all acceptable computations.
- Example: eventually, done will be true forever (**FG**done).





Language Containment Problem

- Solution to the verification problem

  $\Longrightarrow$ Check if language of the system automaton is contained in the language accepted by the property automaton.

- The language containment problem is the problem of deciding if a language is a subset of another language.

$$\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2) \Longleftrightarrow \mathcal{L}(A_1) \cap \overline{\mathcal{L}(A_2)} = \{\}$$

- In order to solve the language containment problem, we need to know:

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Finite Word Languages

- An Alphabet Σ is a collection of symbols (letters).
  E.g. $\Sigma = \{a, b\}$.
- A finite word is a finite sequence of letters. (E.g. *aabb*.)
  The set of all finite words is denoted by Σ^*.
- A language U is a set of words, i.e. $U \subseteq \Sigma^*$.
  Example: Words over $\Sigma = \{a, b\}$ with equal number of a's and b's. (E.g. *aabb* or *abba*.)
- Language recognition problem: determine whether a word belongs to a language.
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Finite-State Automata

- Basic model of computational systems with finite memory.
- Widely applicable
  - Embedded System Controllers.
    Languages: Ester-el, Lustre, Verilog.
  - Synchronous Circuits.
  - Regular Expression Pattern Matching
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  - Protocols
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Notation

$a, b \in \Sigma$ finite alphabet.

$u, v, w \in \Sigma^*$ finite words.

- ϵ empty word.
- $u.v$ concatenation.
- $u^i = u.u.\ .u$ repeated i-times.

$U, V \subseteq \Sigma^*$ Finite word languages.

Finite-State Automata Definition

Definition

A Nondeterministic Finite-State Automaton (NFA) is $(Q, \Sigma, \delta, I, F)$ s.t.
Q Finite set of states.
Σ is a finite alphabet
$I \subseteq Q$ set of initial states.
$F \subseteq Q$ set of final states.
$\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).

We use $q \xrightarrow{a} q'$ to denote $(q, a, q') \in \delta$.

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Regular Languages

- A run of NFA A on $u = a_0, a_1, \ldots, a_{n-1}$ is a finite sequence of states $q_0, q_1, \ldots, q_n$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i < n$.
- An accepting run is one where $q_n \in F$.
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Finite-State Automata: examples

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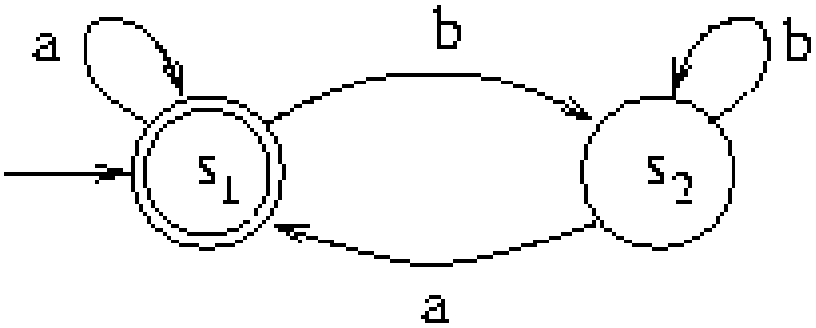

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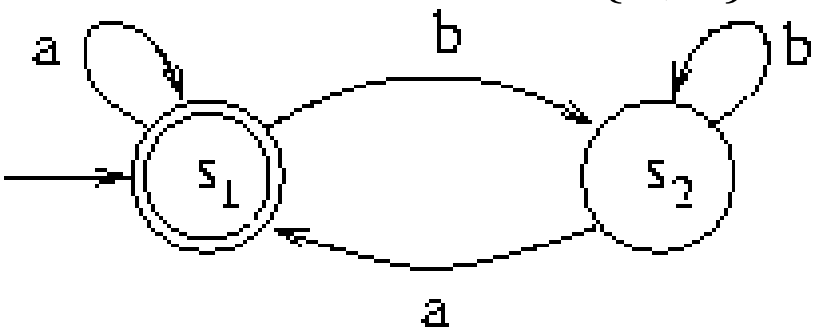

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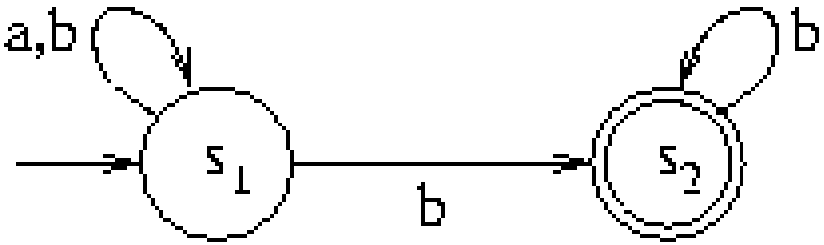

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Theorem (determinisation)

Given a NFA A we can construct a DFA A' s.t. $\mathcal{L}(A) = \mathcal{L}(A')$.
Size: $|A'| = 2^{O(|A|)}$.

- Each state of A' corresponds to a set $\{s_1, ..., s_j\}$ of states in A ($Q' \subseteq 2^Q$), with the intended meaning that :
  - A' is in the state $\{s_1, .., s_j\}$ if A is in one of the states $s_1, ..., s_j$
- The (unique) initial state is $I' =_{def} \{s_i \mid s_i \in I\}$
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Determinisation [cont.]

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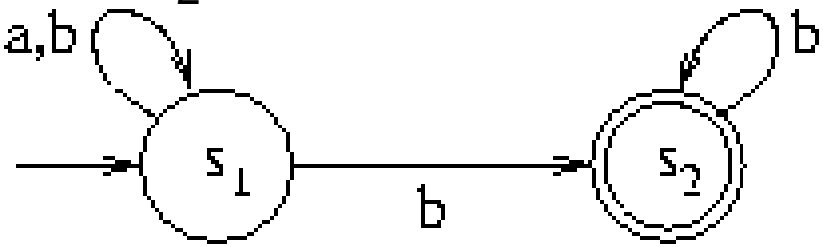

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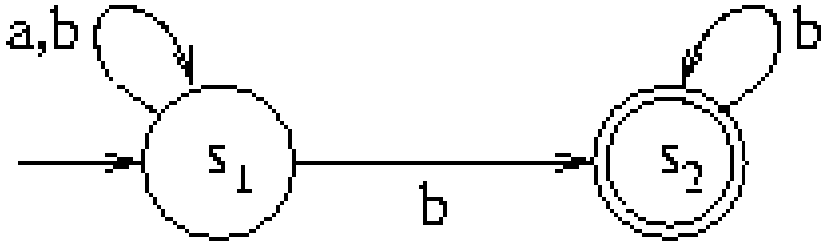

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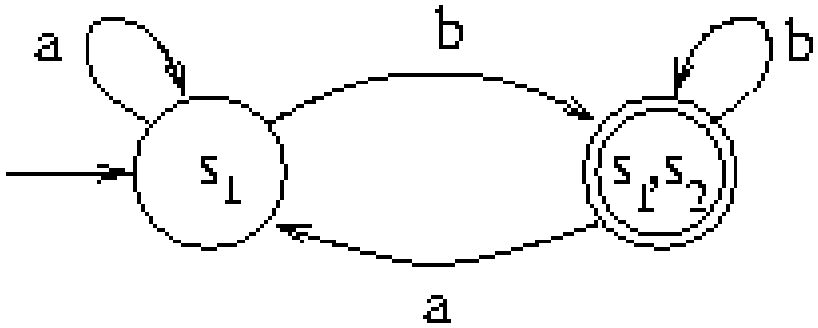

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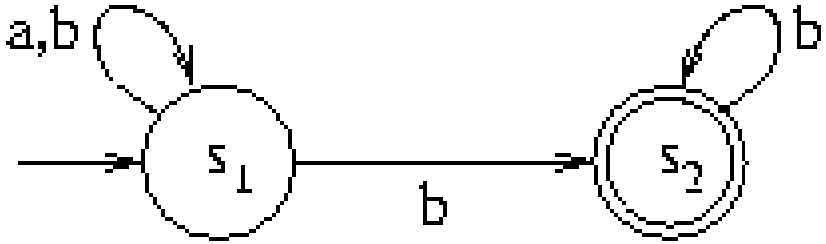

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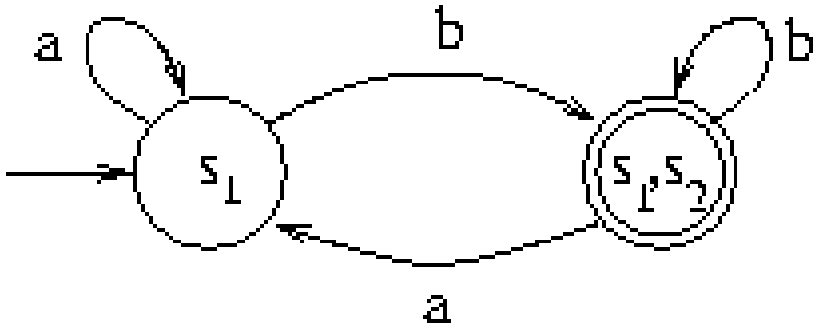

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Closure Properties

Theorem (Boolean closure)

Given NFA A_1, A_2 over Σ we can construct NFA A over Σ s.t.

- $\mathcal{L}(A) = \overline{\mathcal{L}(A_1)}$ (Complement). $|A| = 2^{O(|A_1|)}$.
- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$ (union). $|A| = |A_1| + |A_2|$.
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Theorem (Boolean closure)

Given NFA A_1, A_2 over Σ we can construct NFA A over Σ s.t.

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A NFA $A = (Q, \Sigma, \delta, I, F)$ is complemented by:

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Union of two NFAs

Definition: union of NFAs

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- $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
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Note

A is an automaton which just runs nondeterministically either A_1 or A_2

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Definition: product of NFAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
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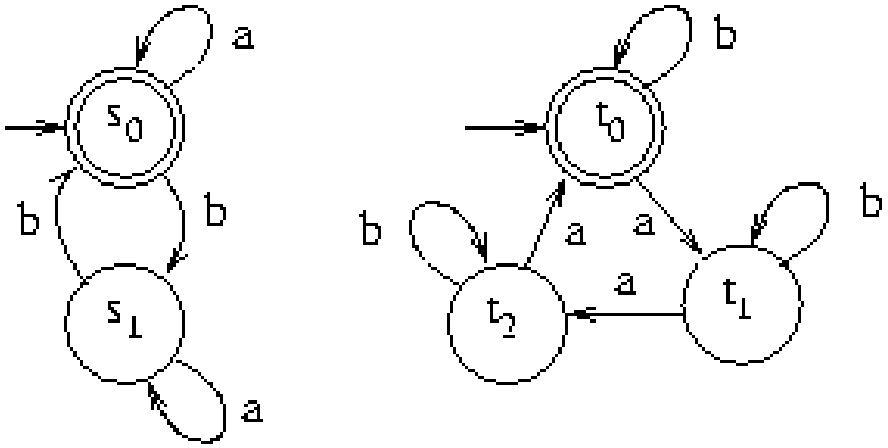

- A_1 recognizes words with an even number of b's.
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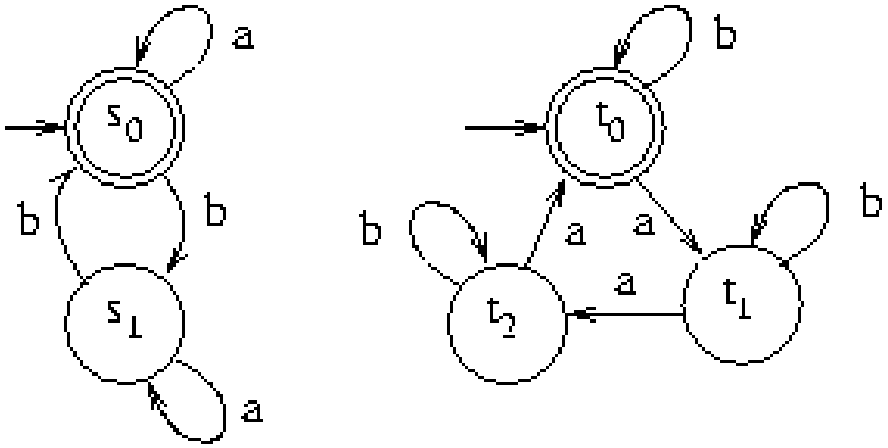

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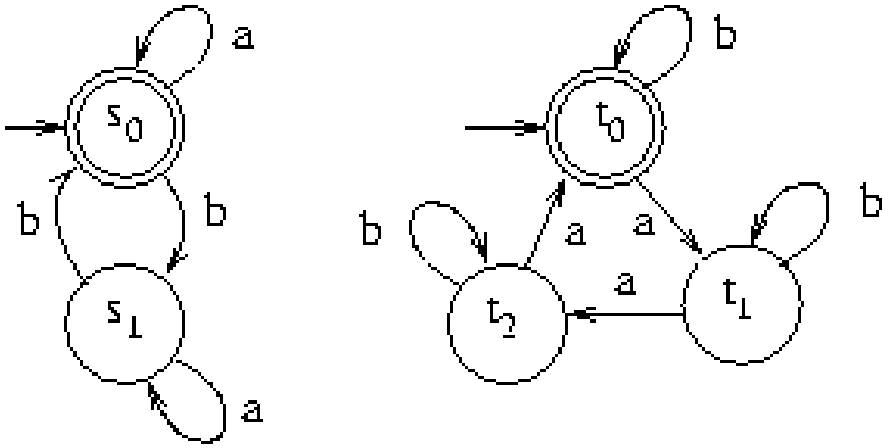

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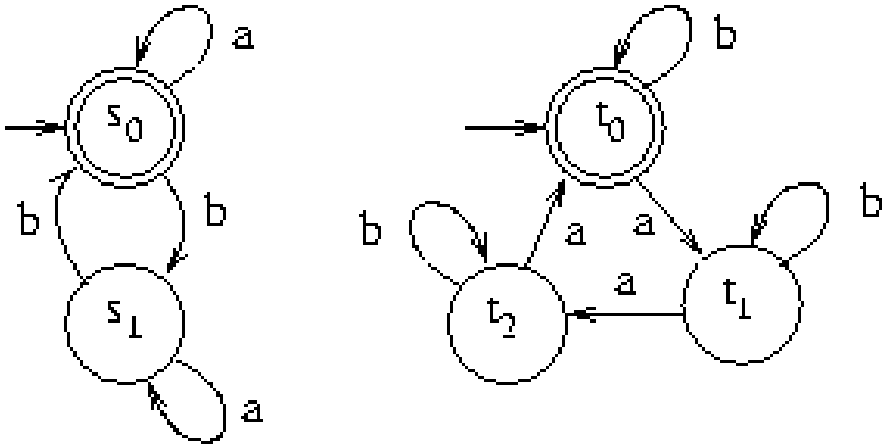

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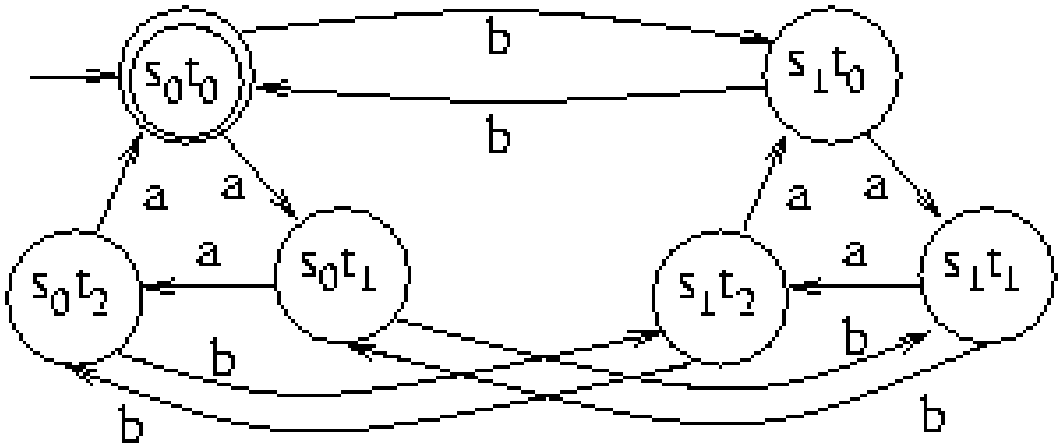

Regular Expressions

- Syntax: $\emptyset \mid \epsilon \mid a \mid reg_1.reg_2 \mid reg_1|reg_2 \mid reg^*$.
- Every regular expression reg denotes a language $\mathcal{L}(reg)$.
- Example: $a^*.(b|bb).a^*$. The words with either 1 b or 2 consecutive b's.

Theorem

For every regular expression reg we can construct a language equivalent NFA of size $O(|reg|)$.

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Modeling infinite computations of reactive systems.

- An ω-word α over Σ is an infinite sequence

  $$a_0,\ a_1,\ a_2 \ldots.$$

  Formally, $\alpha : \mathbb{N} \to \Sigma$.
  The set of all infinite words is denoted by Σ^ω.
- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.

  Example All words over $\{a, b\}$ with infinitely many a's.

Notation:
omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.
omega-languages $L, L_1 \subseteq \Sigma^\omega$
For $u \in \Sigma^+$, let $u^\omega = u.u.u \ldots$

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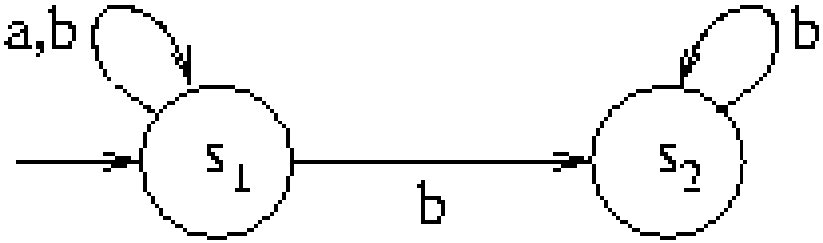
Omega-Automata

- We consider automaton running over infinite words.



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  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
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- Acceptance Conditions: Büchi (Muller, Rabin, Street):
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- Notation Let $\rho \in Q^\omega$. Then,
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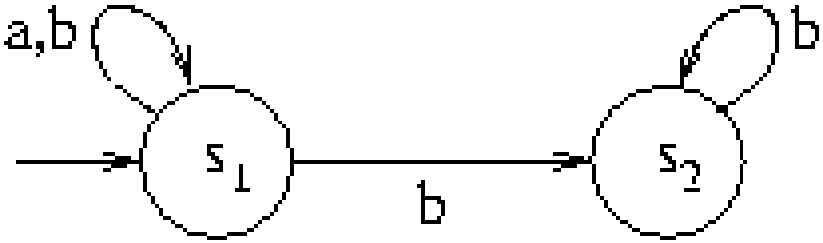

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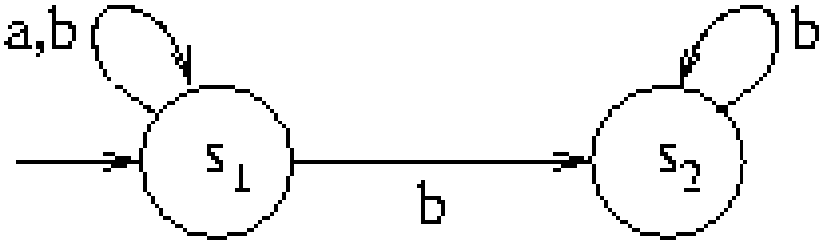

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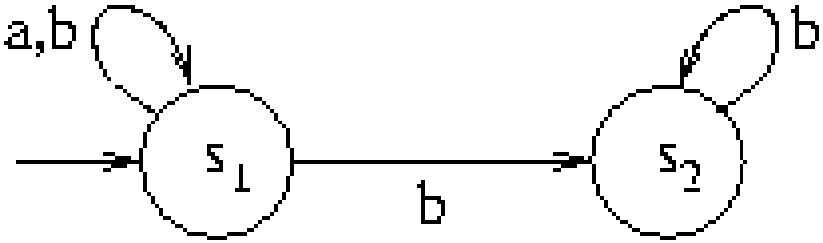

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Nondeterministic Büchi Automaton

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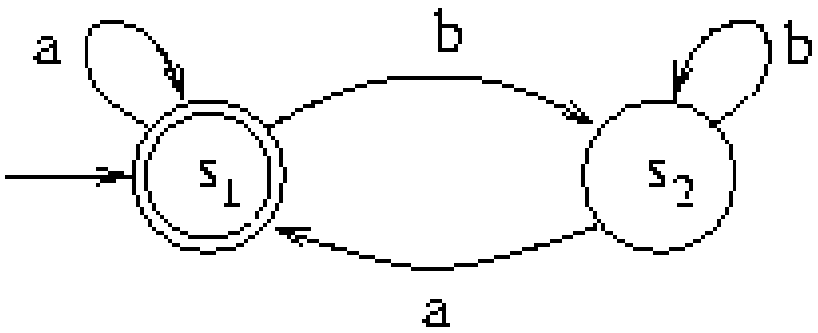

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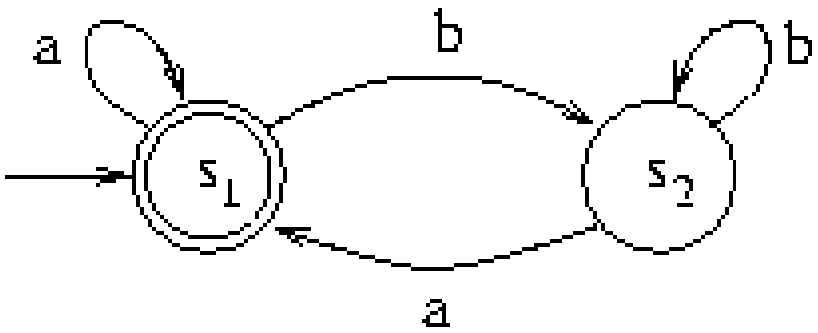

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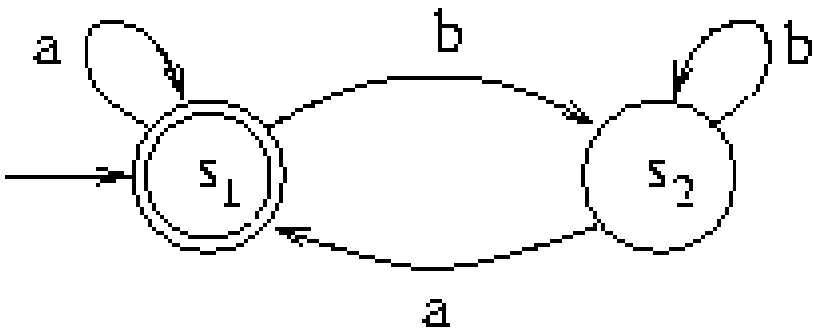
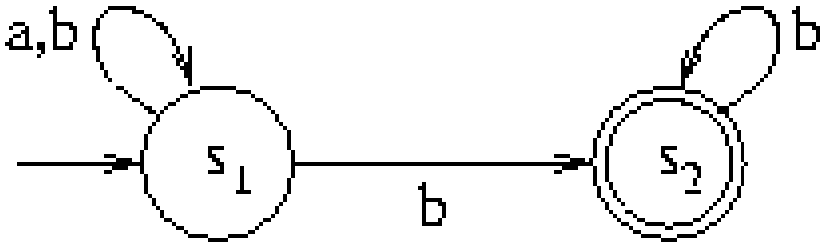


- With $F = \{s_1\}$ the automaton recognizes words with infinitely many a's.
- With $F = \{s_2\}$ the automaton recognizes words with infinitely many b's.

Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

Theorem

*DBA*s are strictly less powerful than *NBA*s.

The subset construction does not work:
let DA_2 be

- DA_2 is not equivalent to A_2
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- There is no DBA equivalent to A_2

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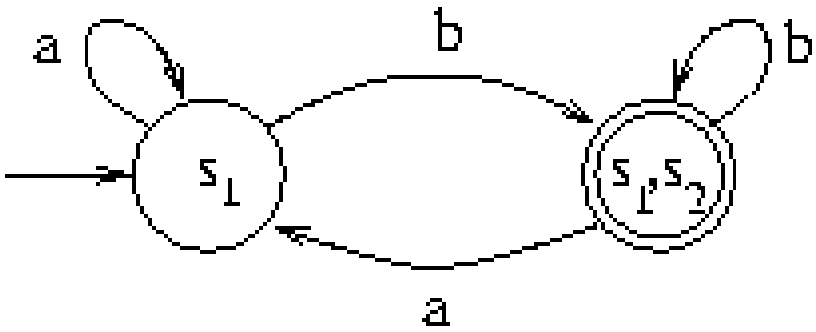

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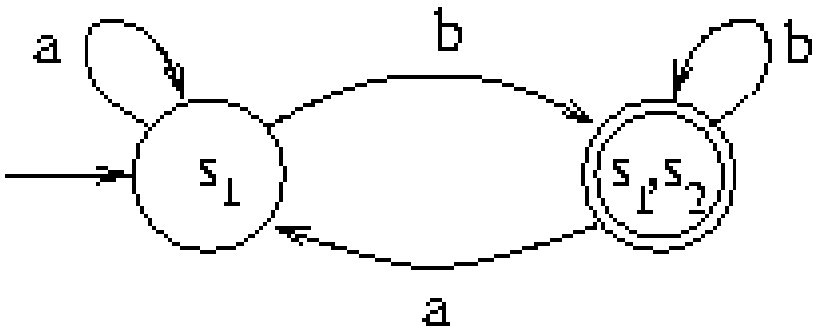

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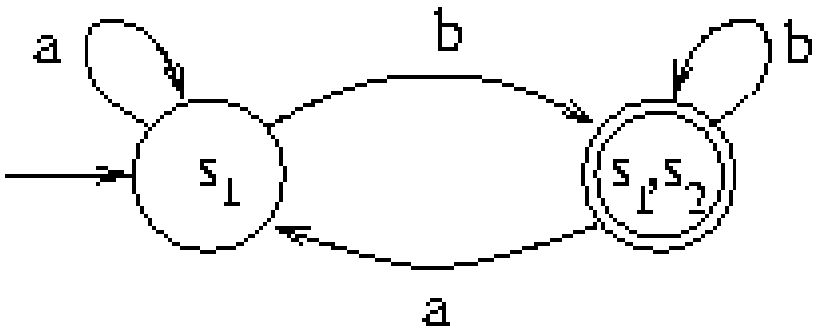

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Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
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Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
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Product of NBAs: Intuition

- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track
  $\Longrightarrow$ in order to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2
- Important subcase: If $F_2 = Q_2$, then

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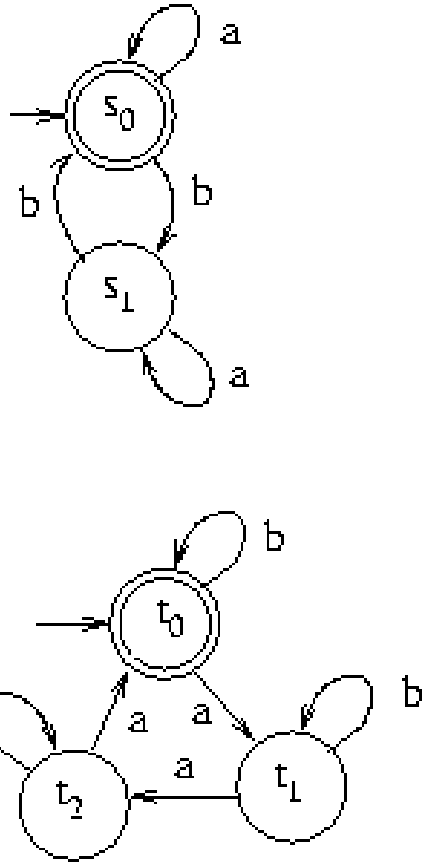

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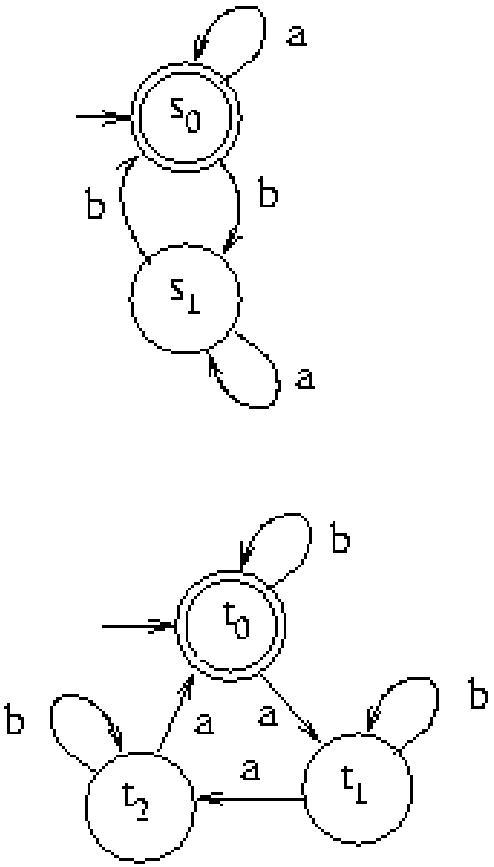

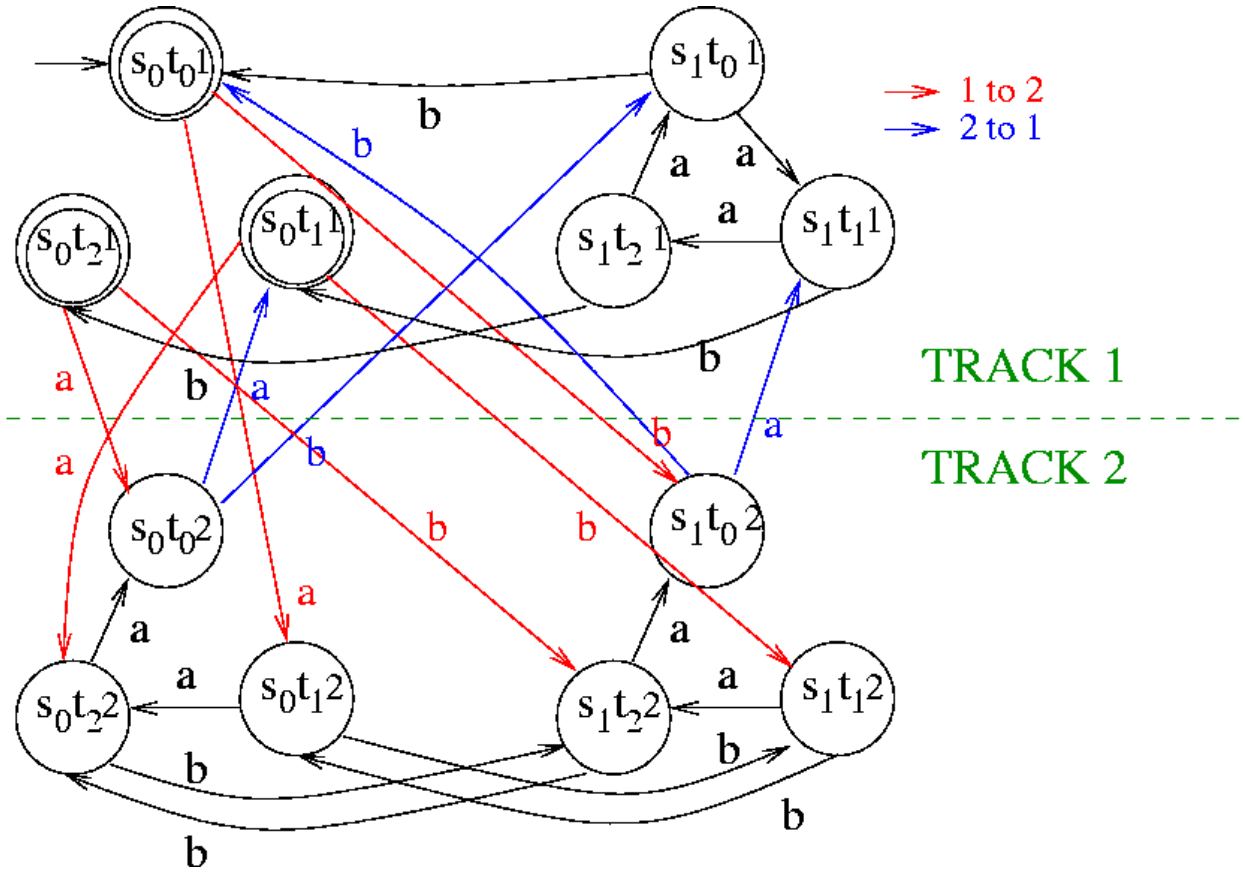

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Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
$|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

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Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
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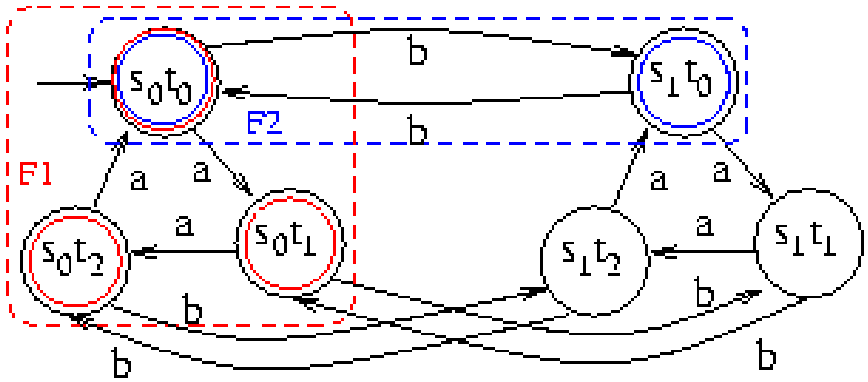

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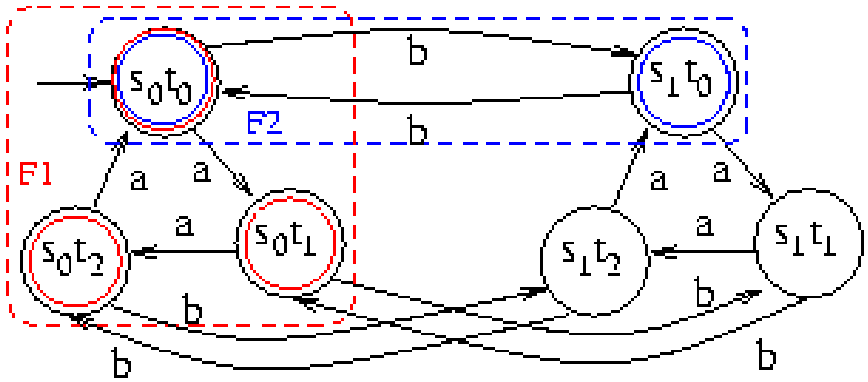

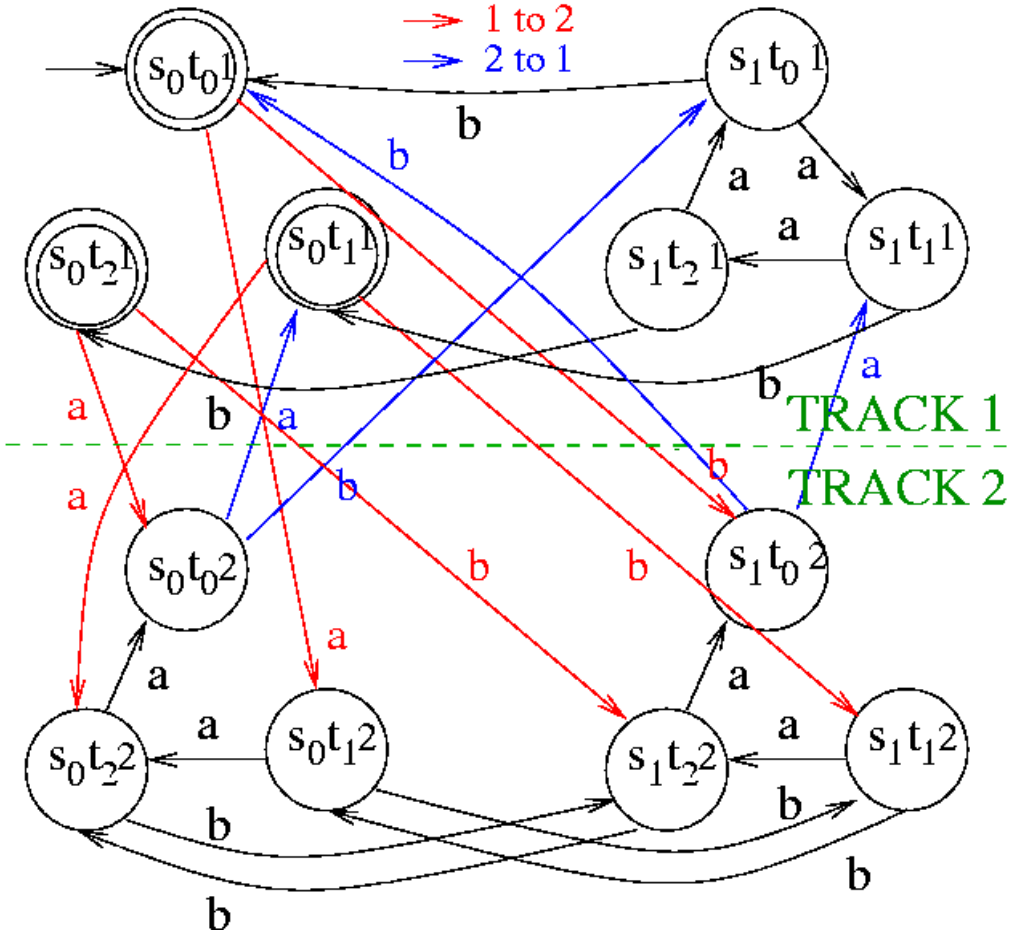

Omega-regular Expressions

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A language is called ω-regular if it has the form $\cup_{i=1}^{n} \; U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

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Outline

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- Equivalent of finding a final state reachable from an initial state.
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- Hereafter, assume w.l.o.g. that there is only one initial state.

NFA emptiness checking

- Equivalent of finding a final state reachable from an initial state.
- It can be solved with a DFS or a BFS.
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NFA Emptiness Checking (cont.)

// returns True if empty language, false otherwise

```
Bool DFS(NFA A) {
    stack S=I;
    Hashtable T=I;
    while S!=∅ {
        v=top(S);
        if v∈F return False
        if ∃w s.t. w∈δ(v) && T(w)==0 {
            hash(w,T);
            push(w,S);
        } else
            pop(S);
    }
    return True;
}
```

NBA emptiness checking

- Equivalent of finding an accepting cycle reachable from an initial state.
- A naive algorithm:
  (i) a DFS finds the final states f reachable from an initial state;
  (ii) for each f, a second DFS finds if it can reach f (i.e., if there exists a loop)
  - Complexity: $O(n^2)$.
- SCC-based algorithm:
  (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
  (ii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.
  - Complexity: $O(n)$.
  - Drawbacks: it stores too much information and does not find directly a counterexample.

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Double Nested DFS algorithm

- Double Nested DFS [Courcoubetis, Vardi, Wolper, Yannakakis, CAV'90]
  - two Hash tables:
    - `T1`: reachable states
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  - two stacks:
    - `S1`: current branch of states reachable
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  - two nested DFS's:
    - `DFS1` looks for a path from an initial state to a cycle starting from an accepting state
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  - It stops as soon as it finds a counterexample.
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Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
    stack S1=I; stack S2=∅;
    Hashtable T1=I; Hashtable T2=∅;
    while S1!=∅ {
        v=top(S1);
        if ∃w s.t. w∈δ(v) && T1(w)==0 {
            hash(w,T1);
            push(w,S1);
        } else {
            pop(S1);
            if (v∈F && !DFS2(v,S2,T2,A))
                return False;
    }     }
    return True;
}
```

Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
    hash(f,T);
    S = {f}
    while S!=∅ {
        v=top(S);
        if f∈δ(v) return False;
        if ∃w s.t. w∈δ(v) && T(w)==0 {
            hash(w);
            push(w);
        } else pop(S);
    }
    return True;
}
```

Remark: `T` passed by reference, is not reset at each call of `DFS2` !

Double nested DFS: intuition

DFS1 invokes DFS2 on each $f_1, ..., f_n$ only after popping it (postorder):

- suppose $DFS2$ is invoked on f_j before than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during $DFS2(f_i, ...)$ it is encountered a state S which has already been explored by $DFS2(f_j, ...)$ for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

$\Longrightarrow$ it is safe to backtrack.

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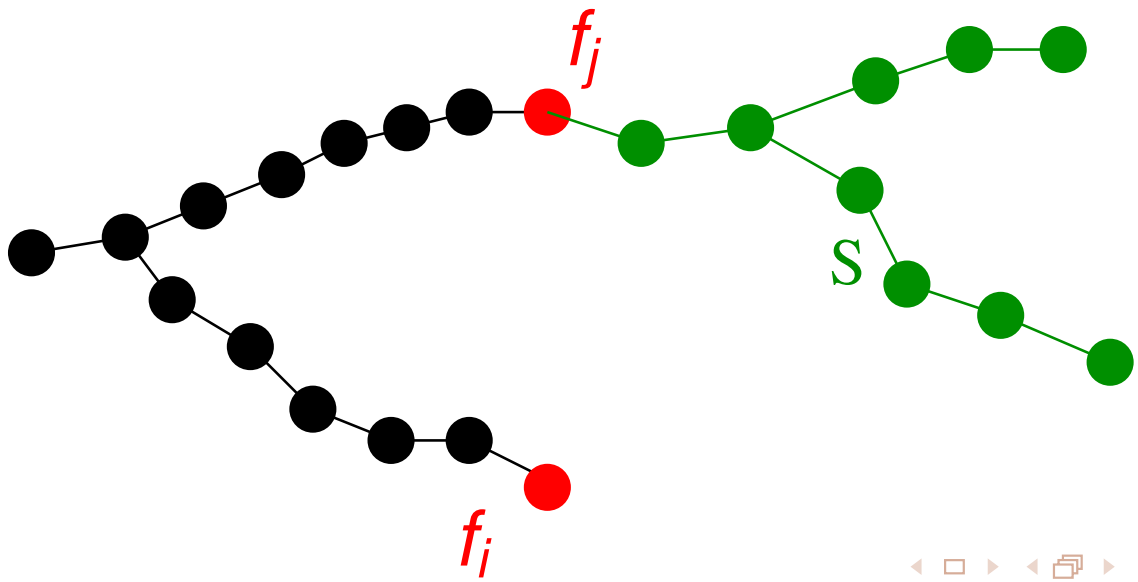

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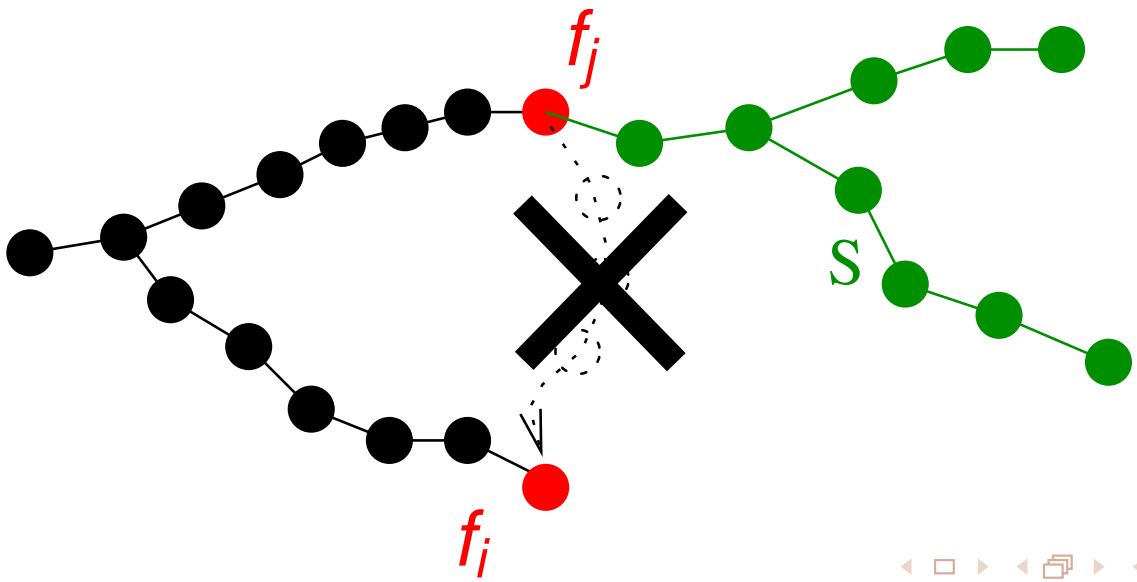

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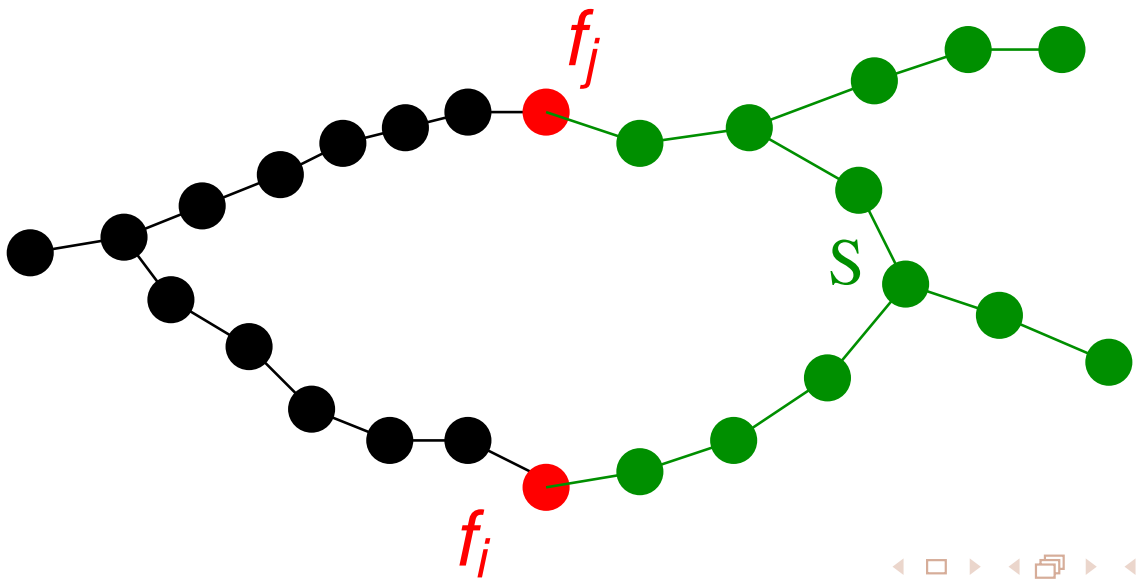

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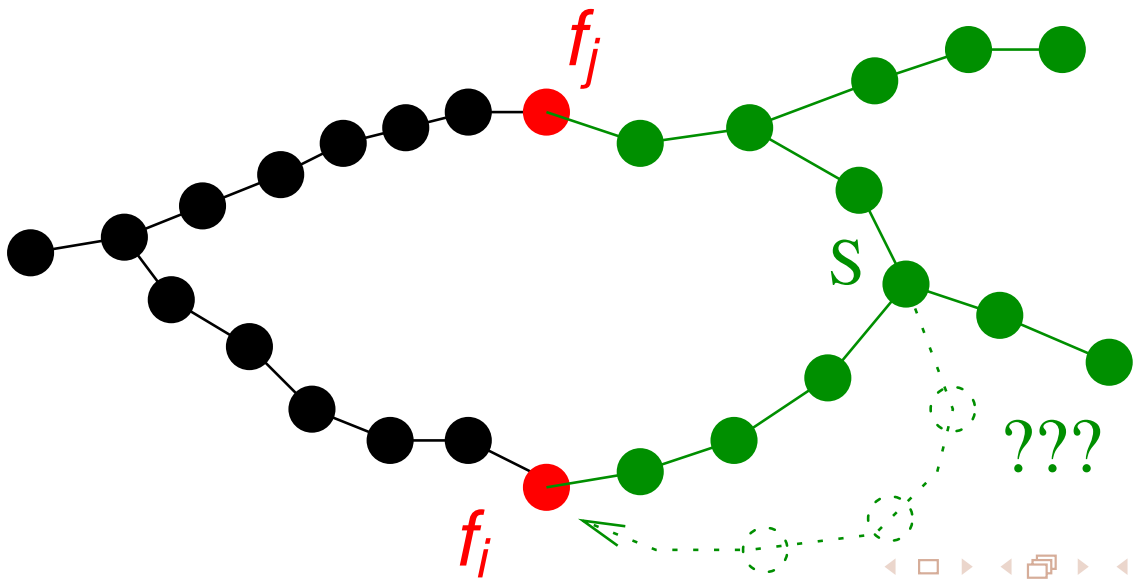

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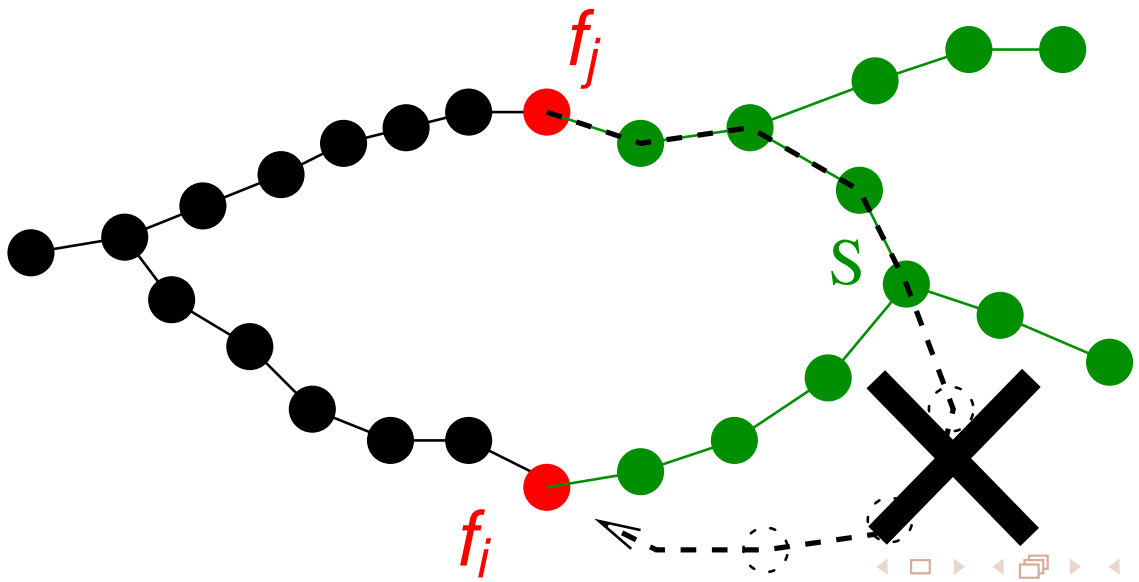

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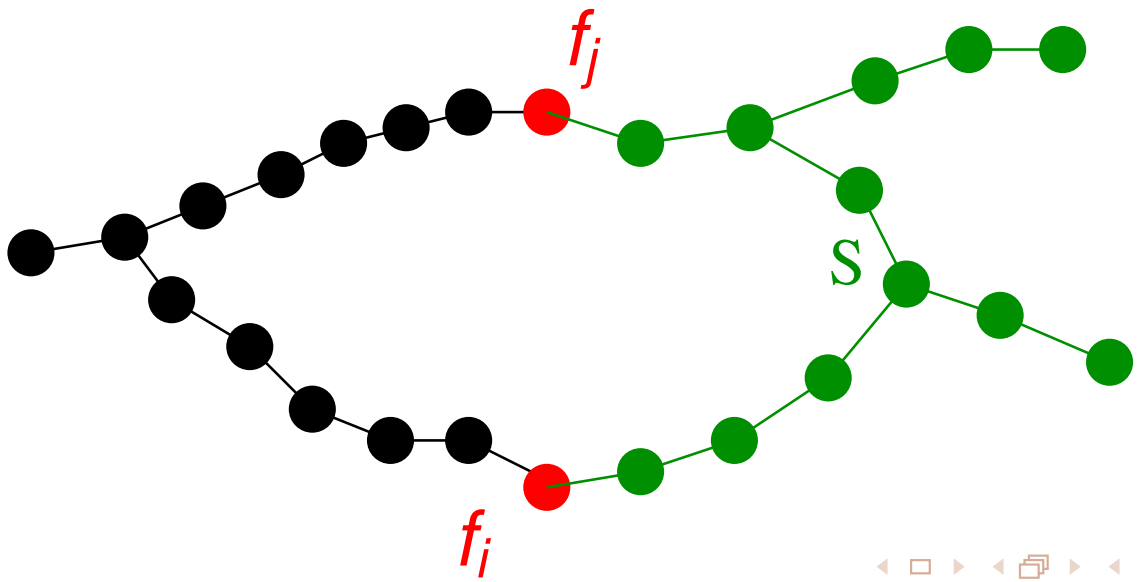

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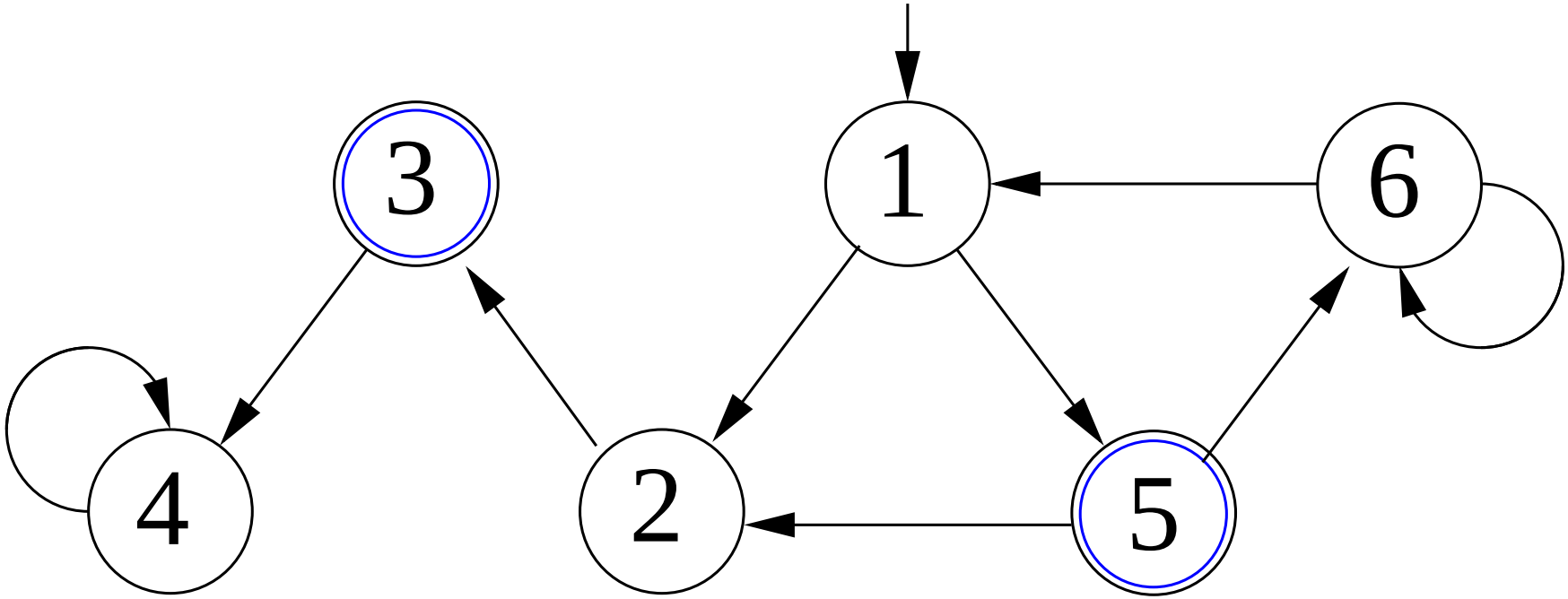

T1 T2

S1 S2

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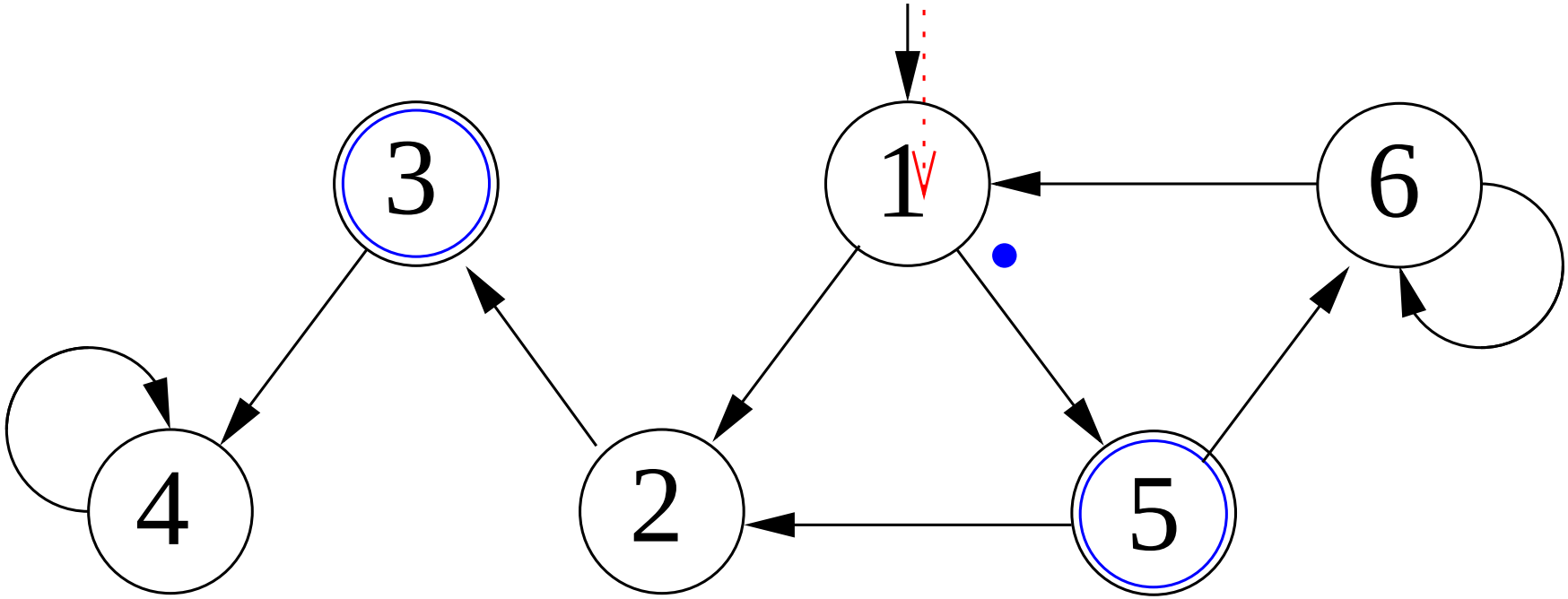


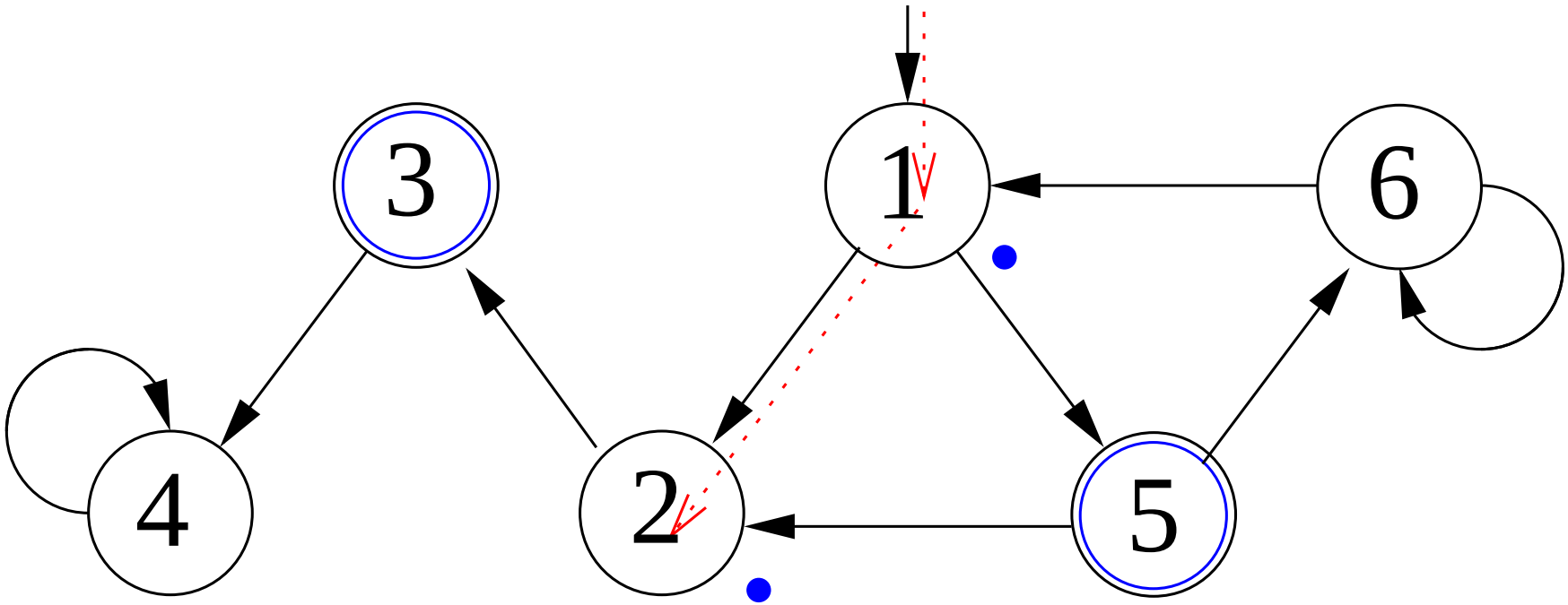
T1 1

T2

S1 1

S2

Double Nested DFS: example



T1 1 2

T2

S1 1 2

S2

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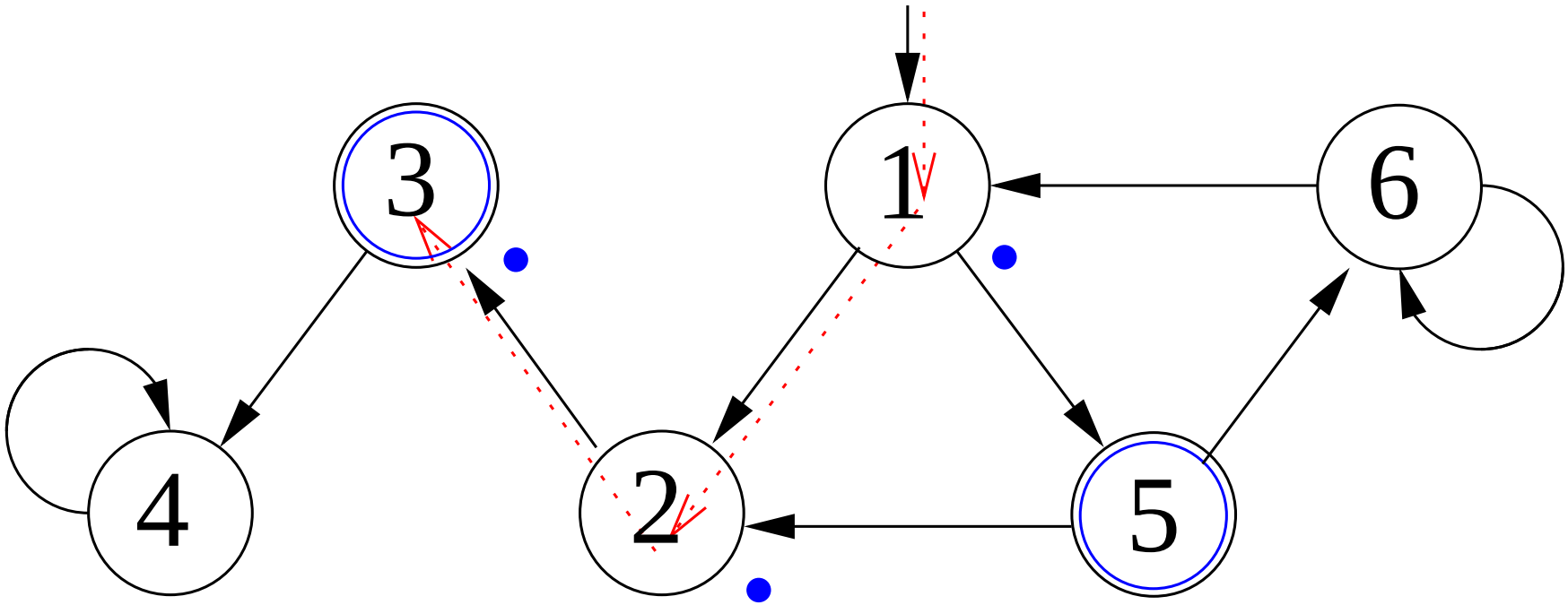

T1 1 2 3

T2

S1 1 2 3

S2

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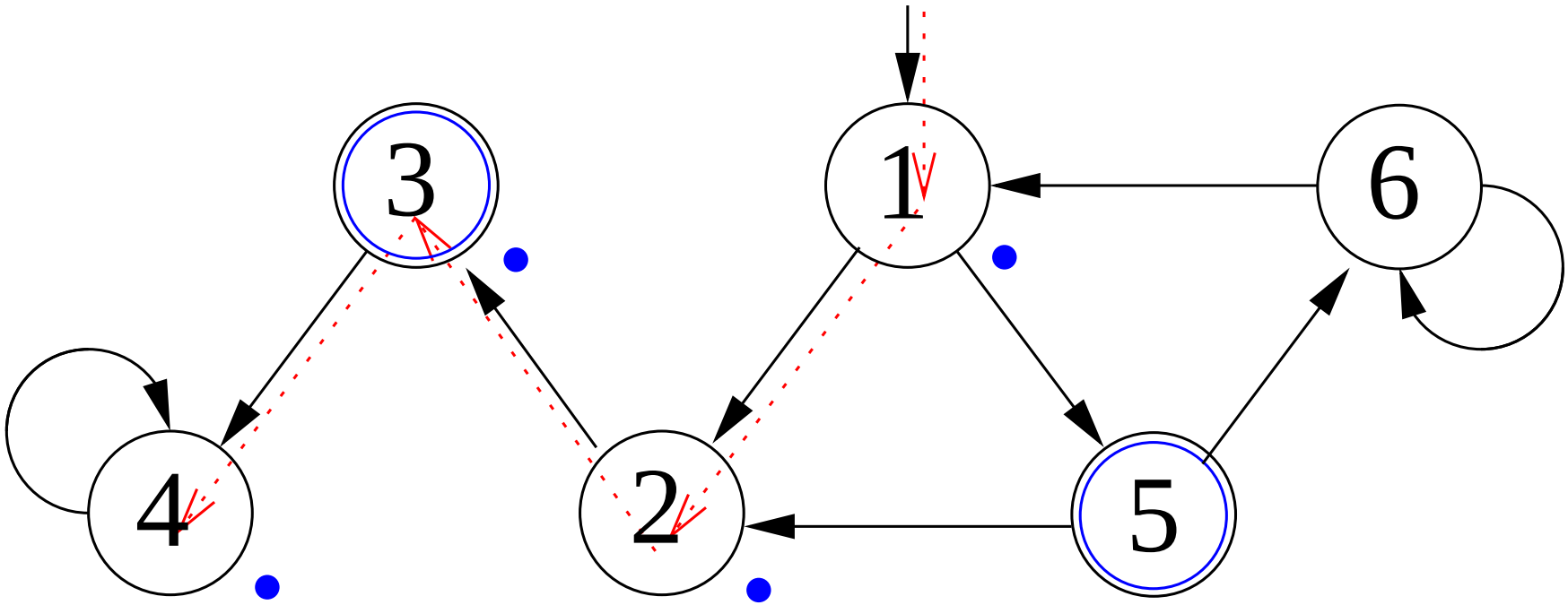

T1 1 2 3 4

T2

S1 1 2 3 4

S2

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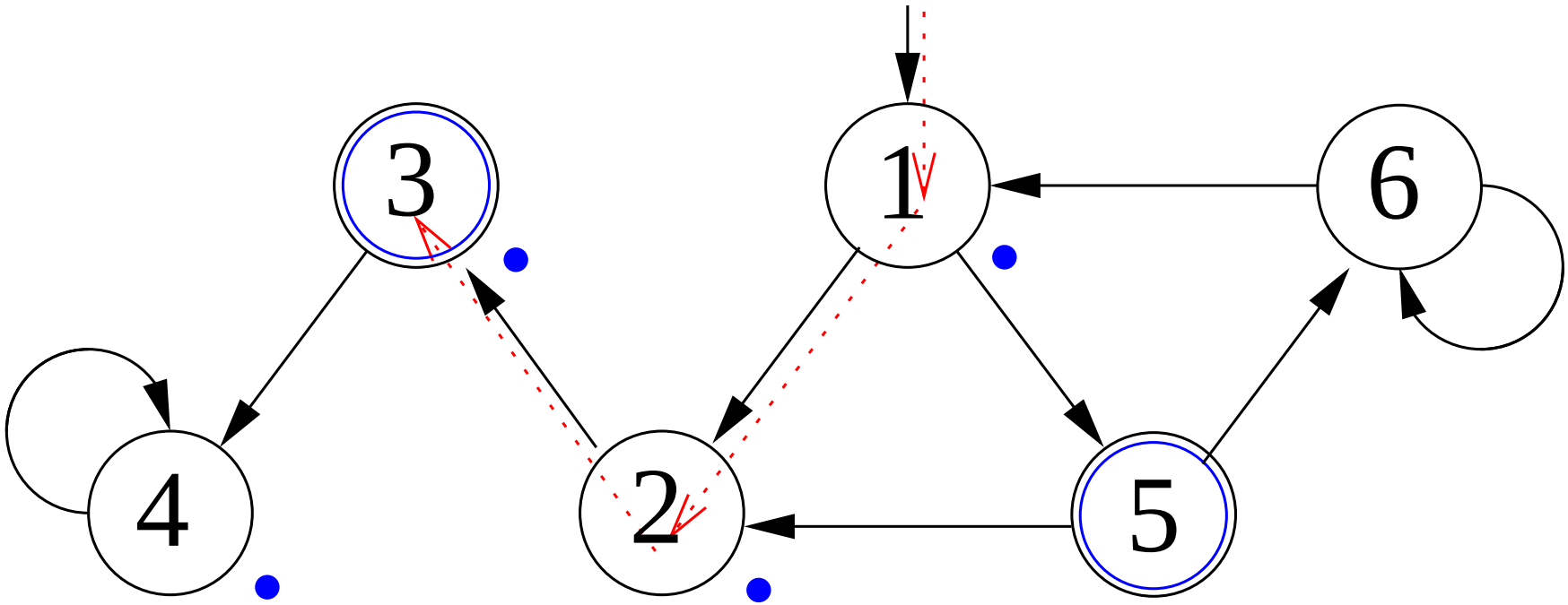

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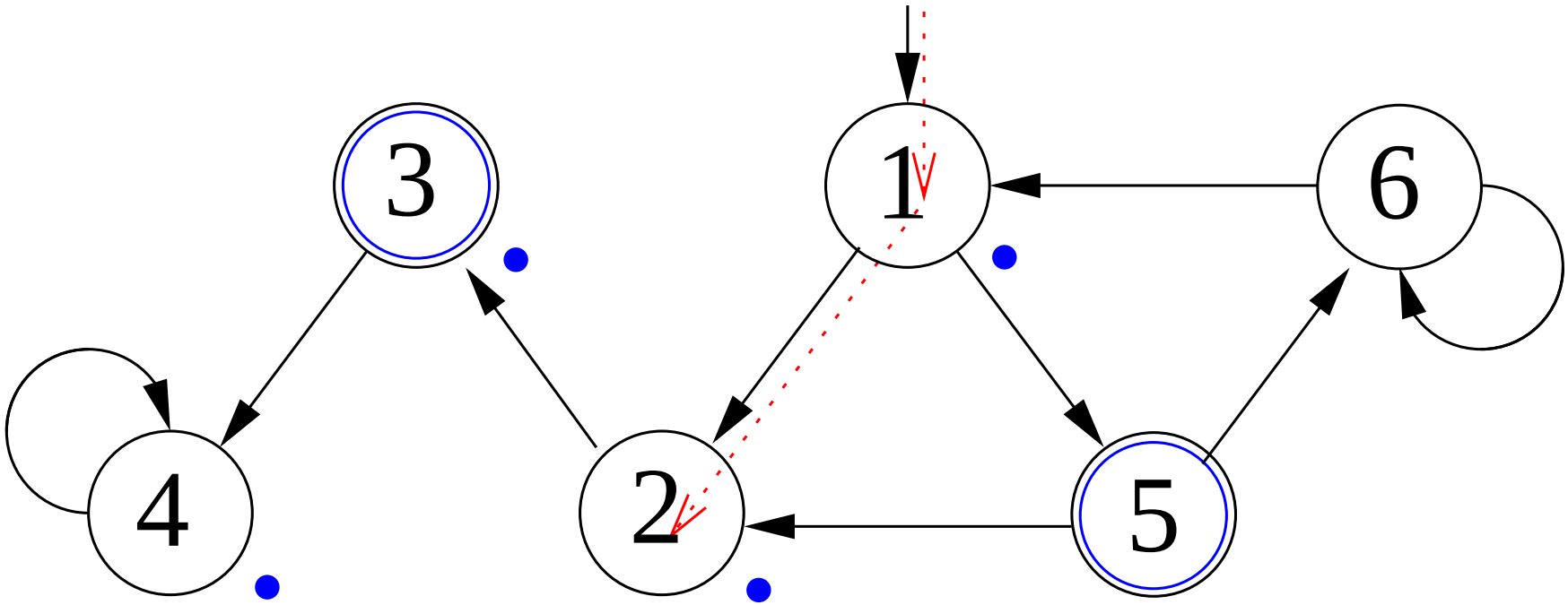

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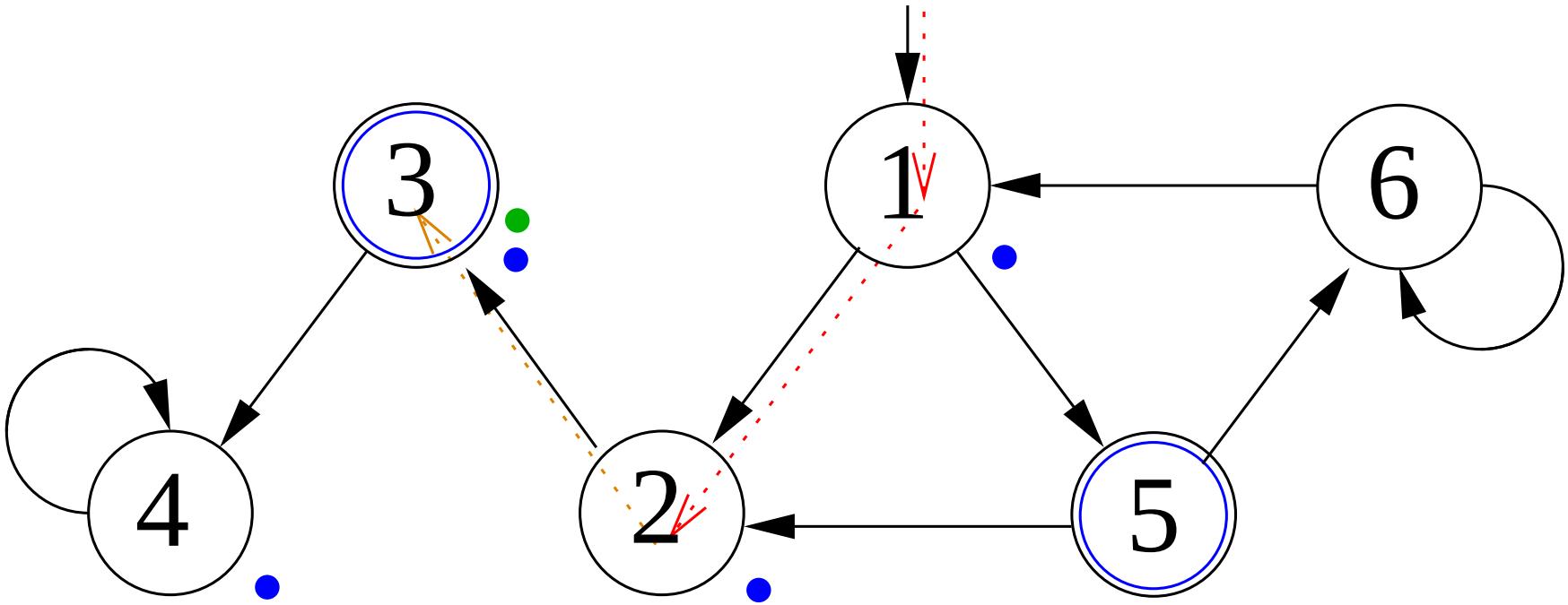

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T2 3

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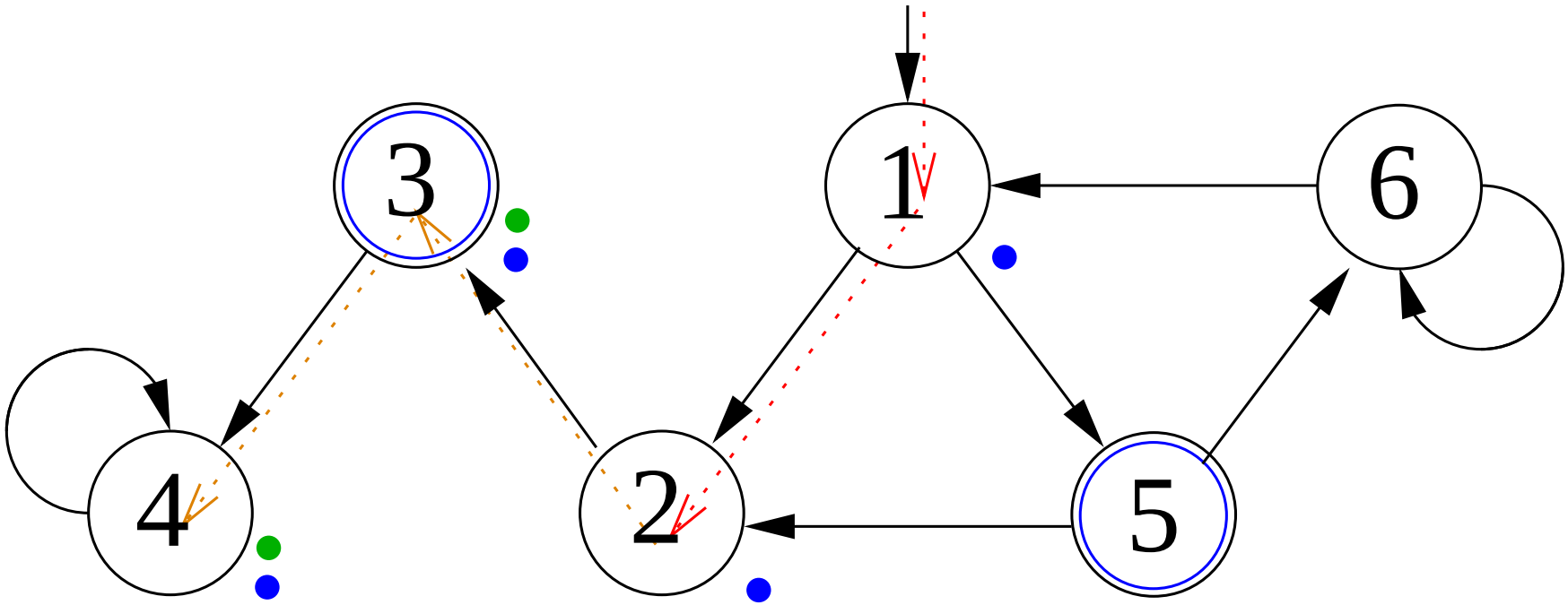

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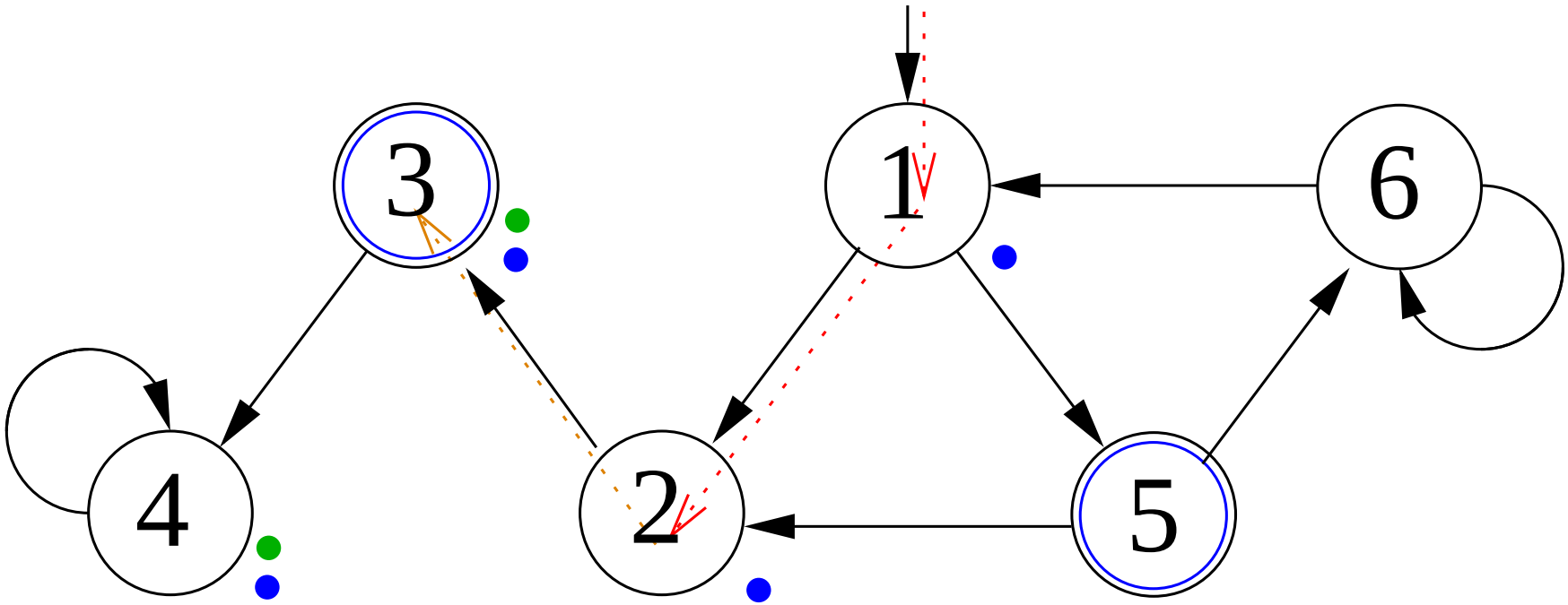

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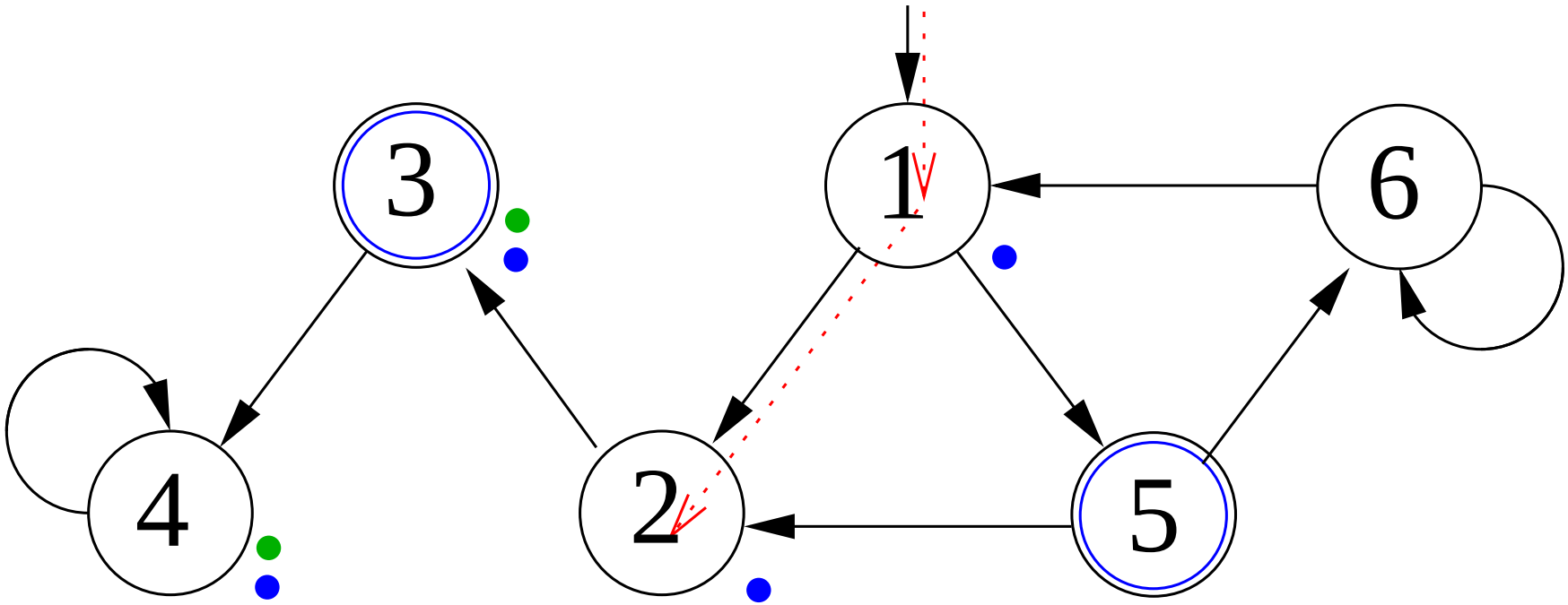
S2 3

Double Nested DFS: example



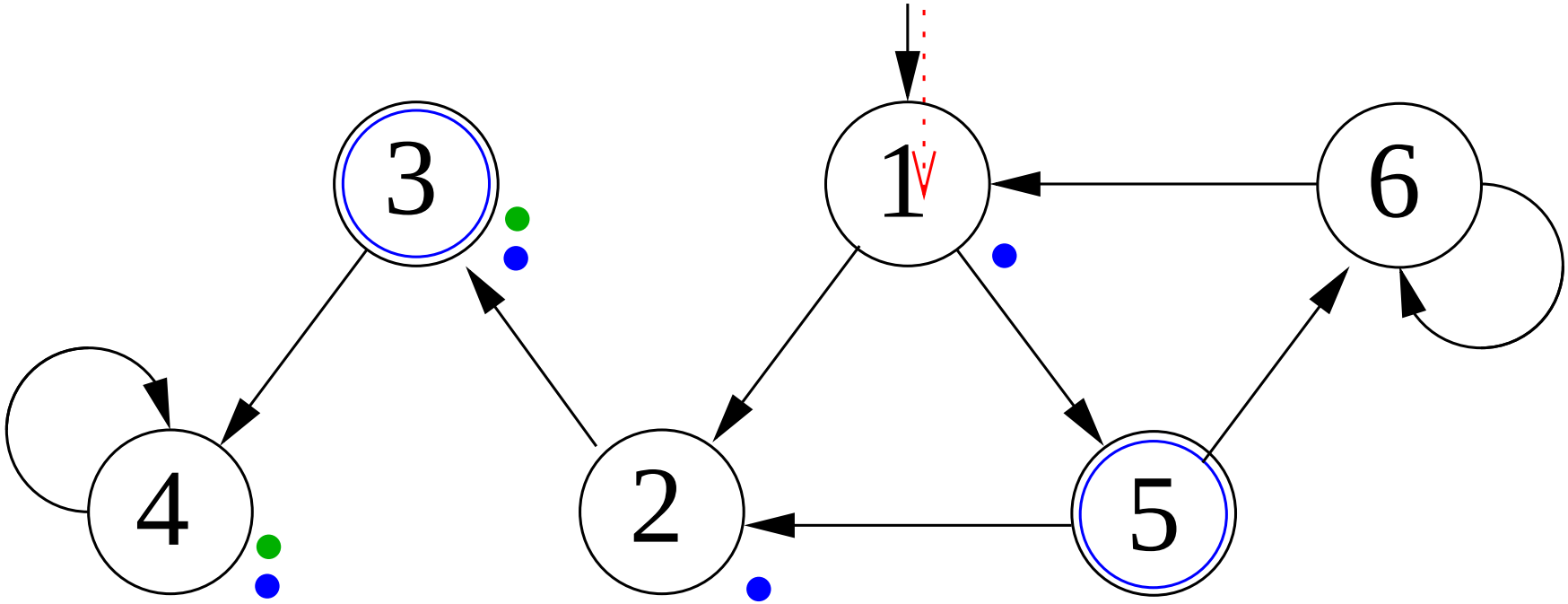
T1 1 2 3 4

T2 3 4

S1 1 2

S2

Double Nested DFS: example



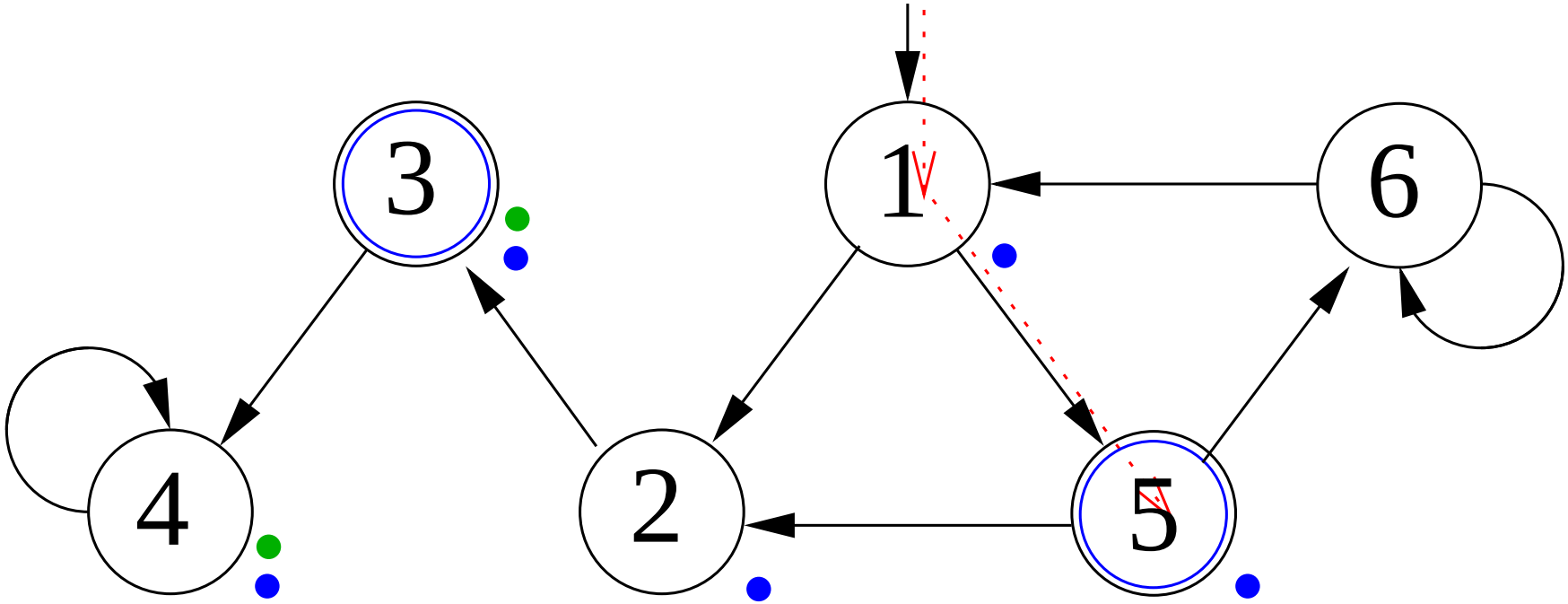
T1 1 2 3 4

T2 3 4

S1 1

S2

Double Nested DFS: example



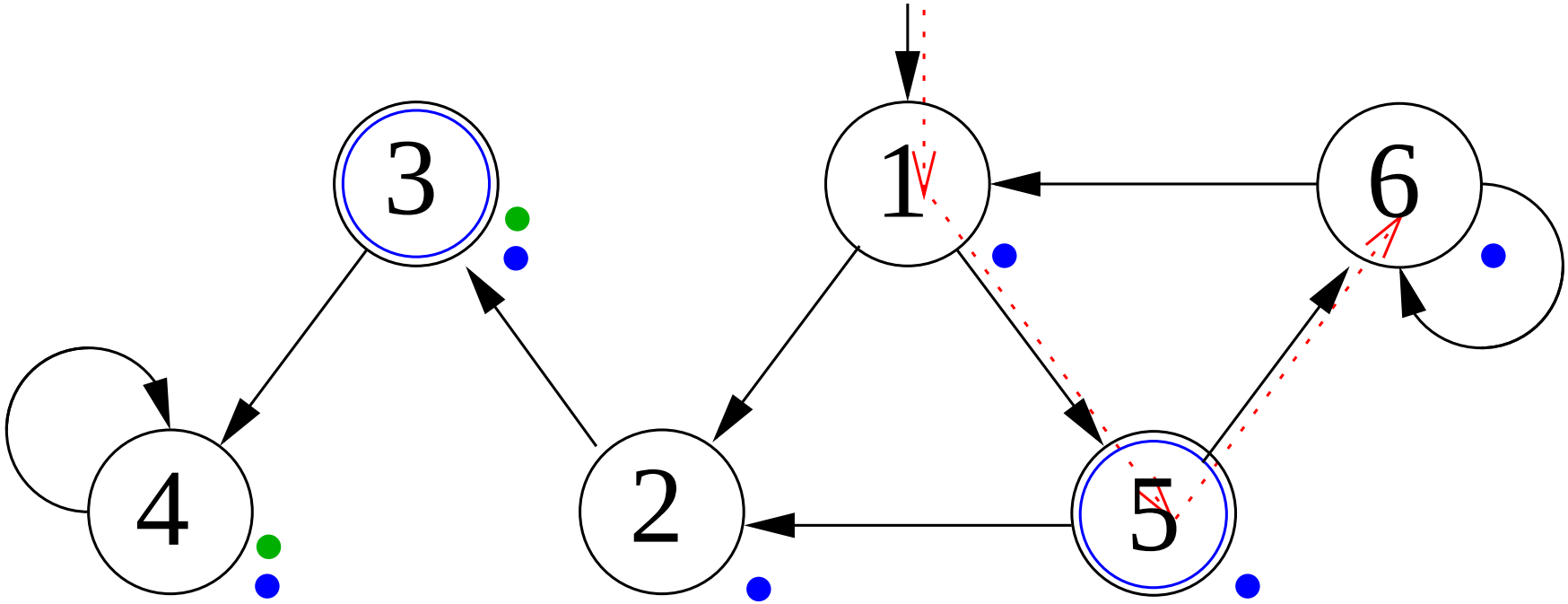
T1 1 2 3 4 5

T2 3 4

S1 1 5

S2

Double Nested DFS: example



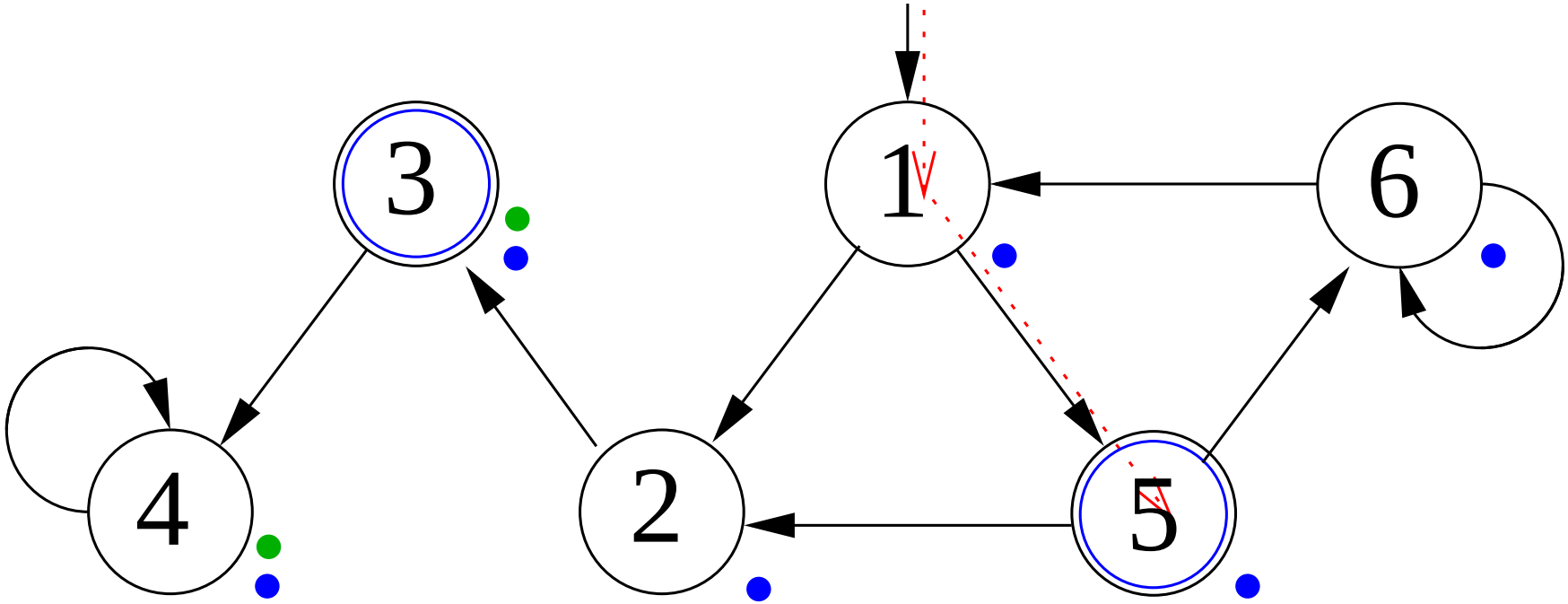
T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

Double Nested DFS: example



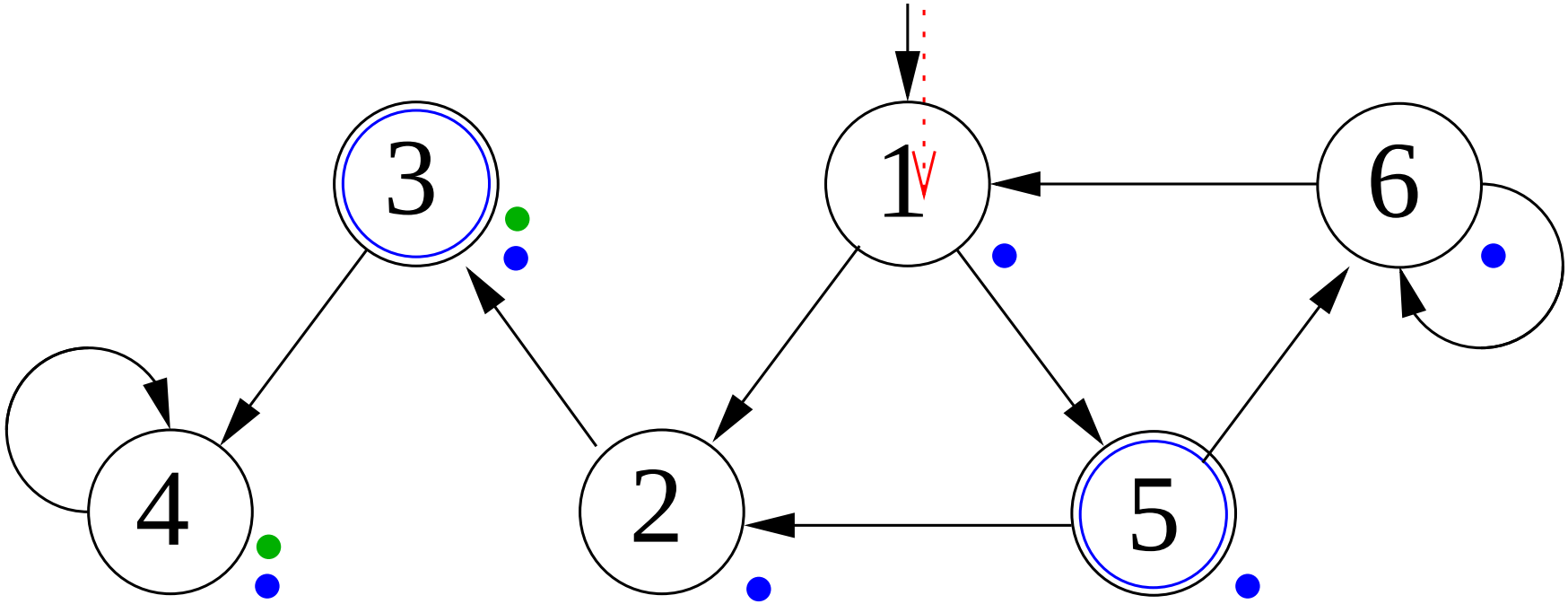
T1 1 2 3 4 5 6

T2 3 4

S1 1 5

S2

Double Nested DFS: example



T1 1 2 3 4 5 6

T2 3 4

S1 1

S2

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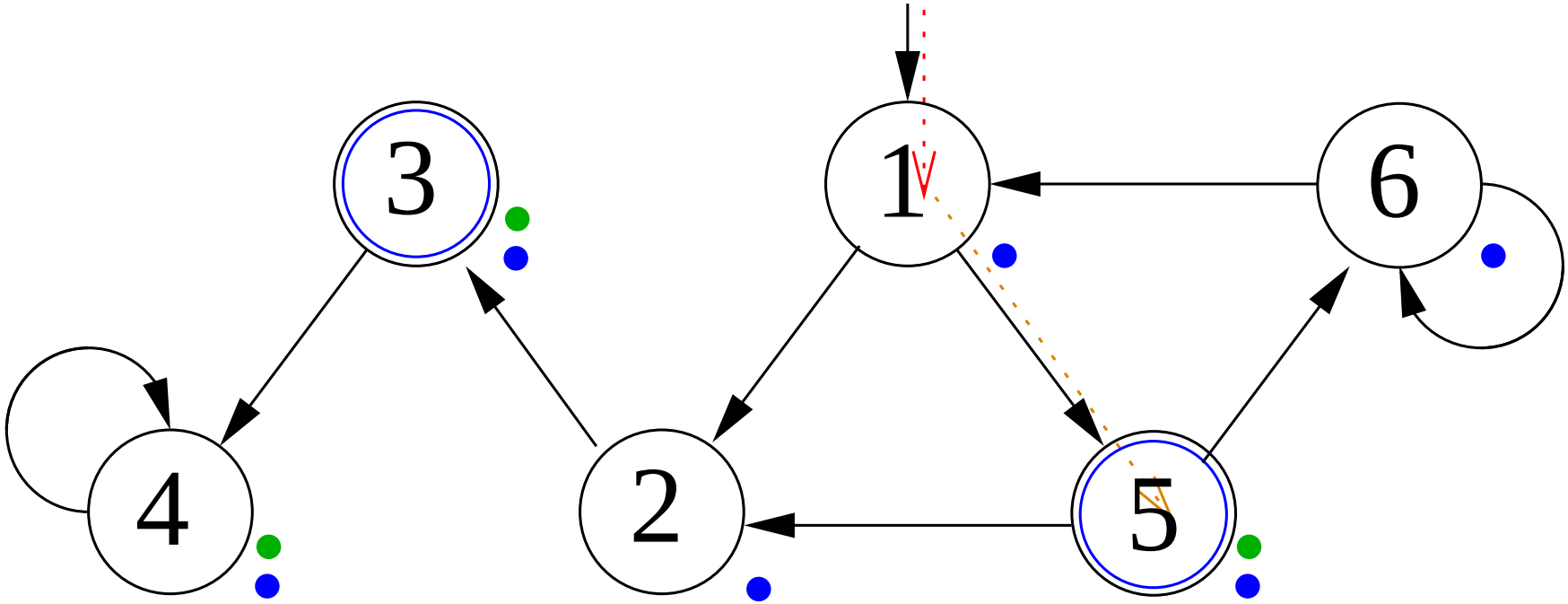

T1 1 2 3 4 5 6

T2 3 4 5

S1 1

S2 5

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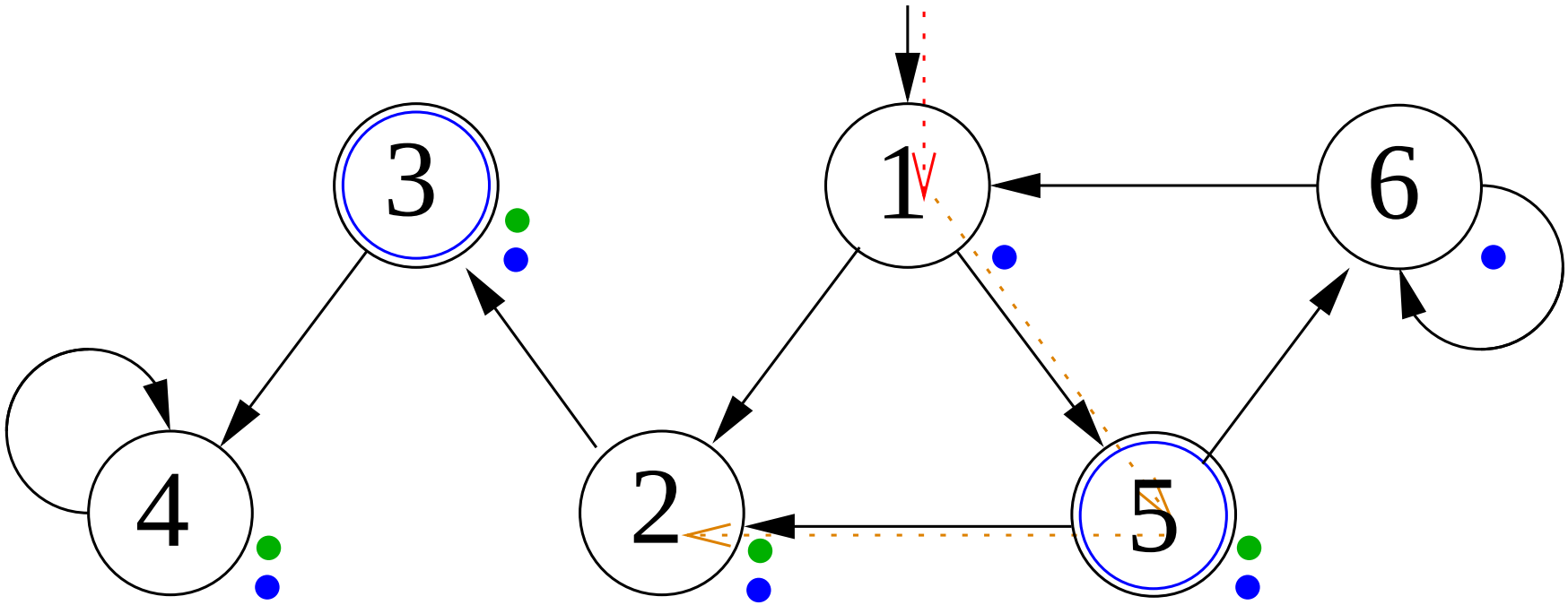


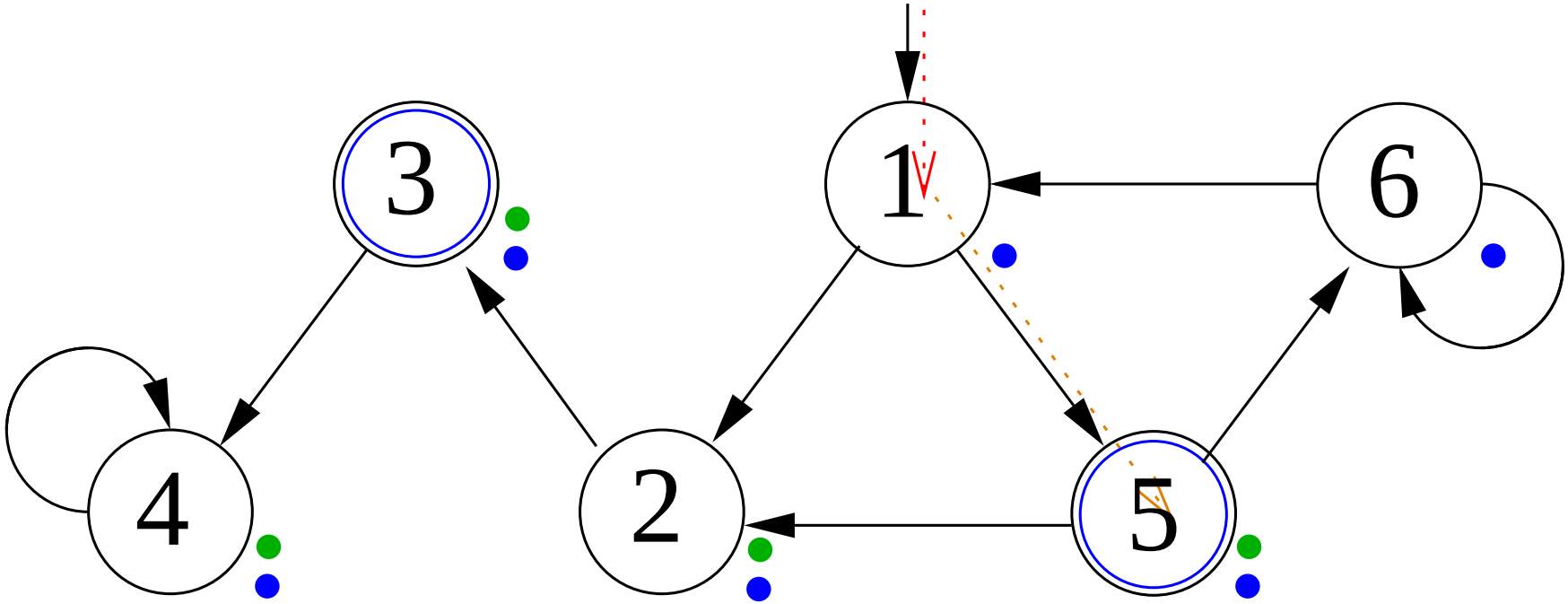
T1 1 2 3 4 5 6

T2 3 4 5 2

S1 1

S2 5 2

Double Nested DFS: example



T1 1 2 3 4 5 6

T2 3 4 5 2

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S2 5

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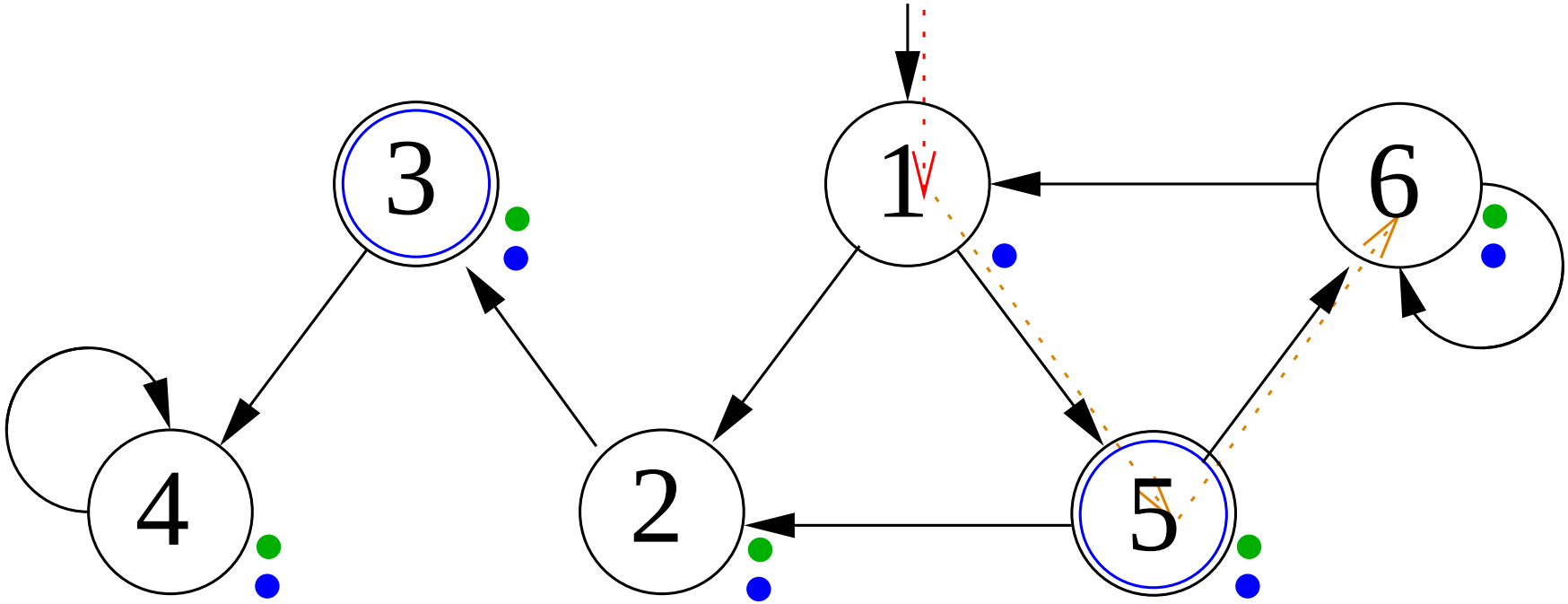

T1 1 2 3 4 5 6

T2 3 4 5 2 6

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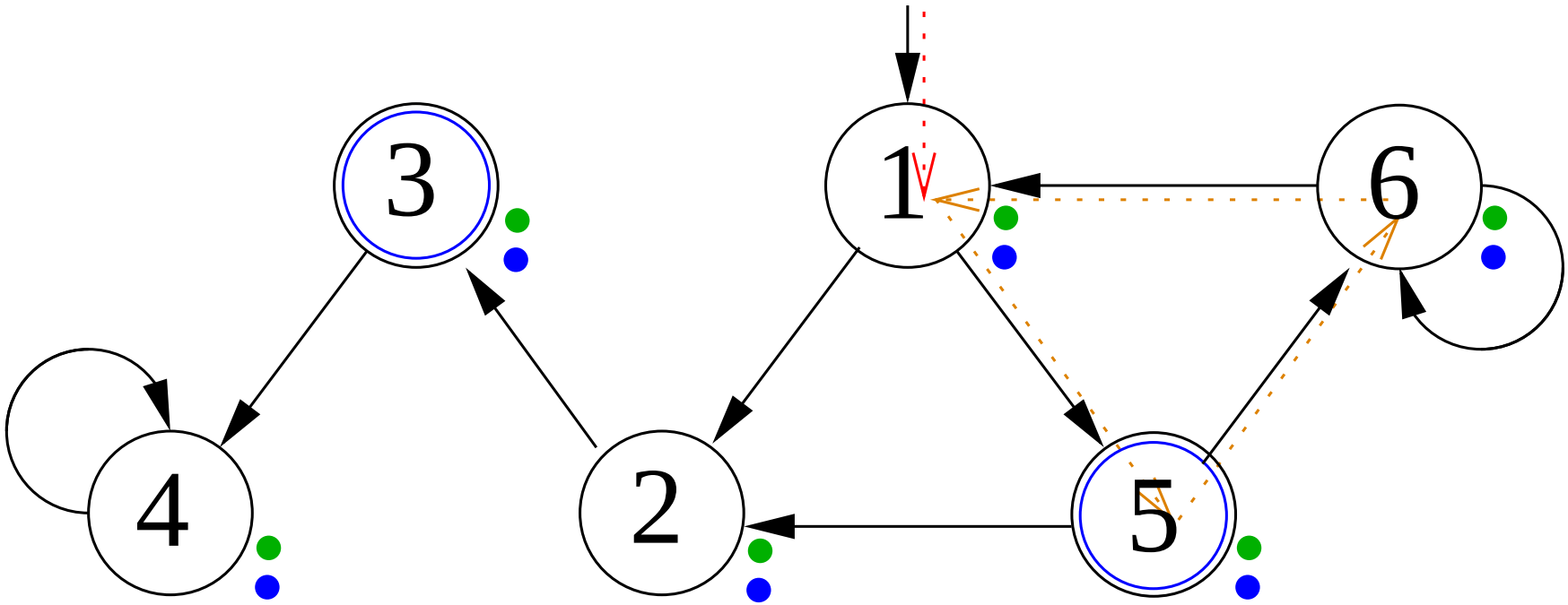

T1 1 2 3 4 5 6

T2 3 4 5 2 6 1

S1 1

S2 5 6 1

Outline

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Automata-Theoretic LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$
\begin{aligned}
& M \models \mathbf{A}\psi \text{ (CTL}^*\text{)} \\
\iff\ & M \models \psi \quad \text{(LTL)} \\
\iff\ & \mathcal{L}(M) \subseteq \mathcal{L}(\psi) \\
\iff\ & \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset \\
\iff\ & \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset \\
\iff\ & \mathcal{L}(A_M) \cap \mathcal{L}(A_{\neg\psi}) = \emptyset \\
\iff\ & \mathcal{L}(A_M \times A_{\neg\psi}) = \emptyset
\end{aligned}
$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\implies$ $A_M \times A_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ.

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Automata-Theoretic LTL M.C. (dual version)

- Let M be a Kripke model and $\varphi \stackrel{\text{def}}{=} \neg\psi$ be an LTL formula

$$
\begin{aligned}
& M \models \mathbf{E}\varphi \\
\iff\ & M \not\models \mathbf{A}\neg\varphi \\
\iff\ & \ldots \\
\iff\ & \mathcal{L}(A_M \times A_\varphi) \neq \emptyset
\end{aligned}
$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- A_φ is a Büchi Automaton which represents all and only the paths that satisfy φ

$\Longrightarrow$ $A_M \times A_\varphi$ represents all and only the paths appearing in both A_M and A_φ.

Automata-Theoretic LTL Model Checking

Four steps:

(i) Compute A_M

(ii) Compute A_φ

(iii) Compute the product $A_M \times A_\varphi$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

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Outline

Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke structure $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
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  - Transitions:

$$\delta : \quad q \xrightarrow{a} q' \text{ iff } (q, q') \in R \text{ and } L(q') = a$$
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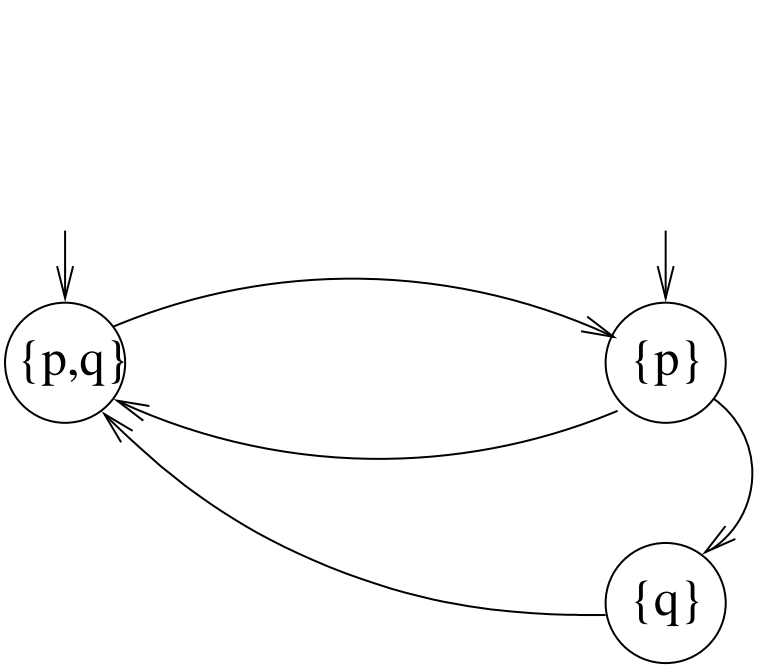
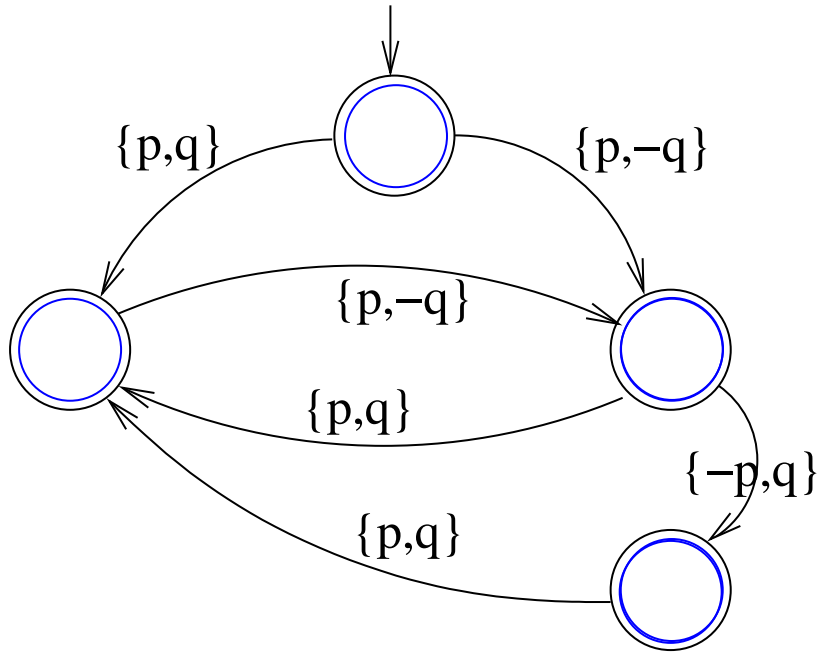


Kripke Structure

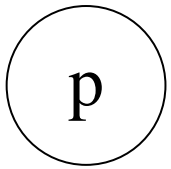


Buechi Automaton

$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



- in a Kripke Structure, it means that p is true and all other propositions are false;
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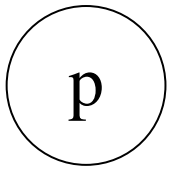

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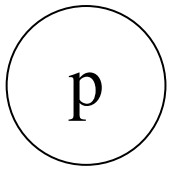

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- Transforming a fair K.S. $M = \langle S, S_0, R, L, AP, FT \rangle$, $FT = \{F_1, ..., F_n\}$, into a generalized NBA $A_M = \langle Q, \Sigma, \delta, I, FT' \rangle$ s.t.:
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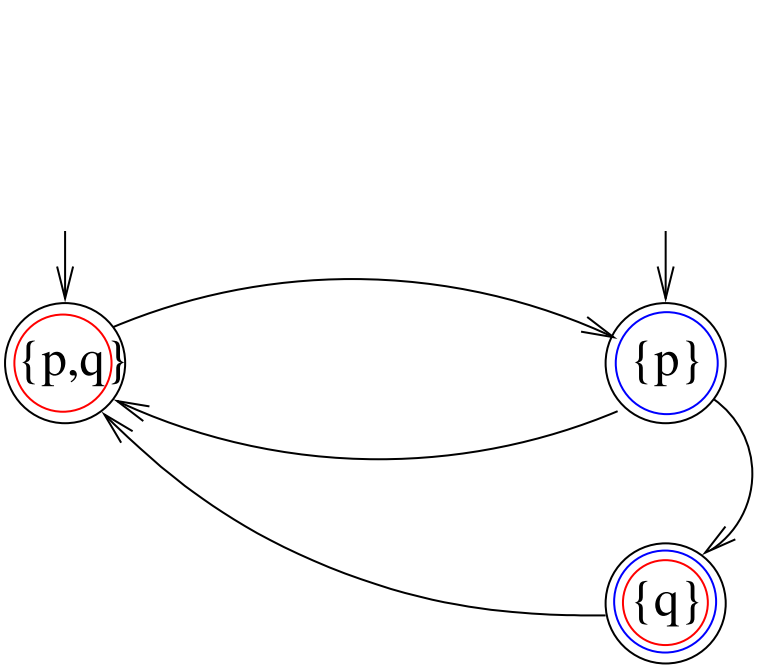
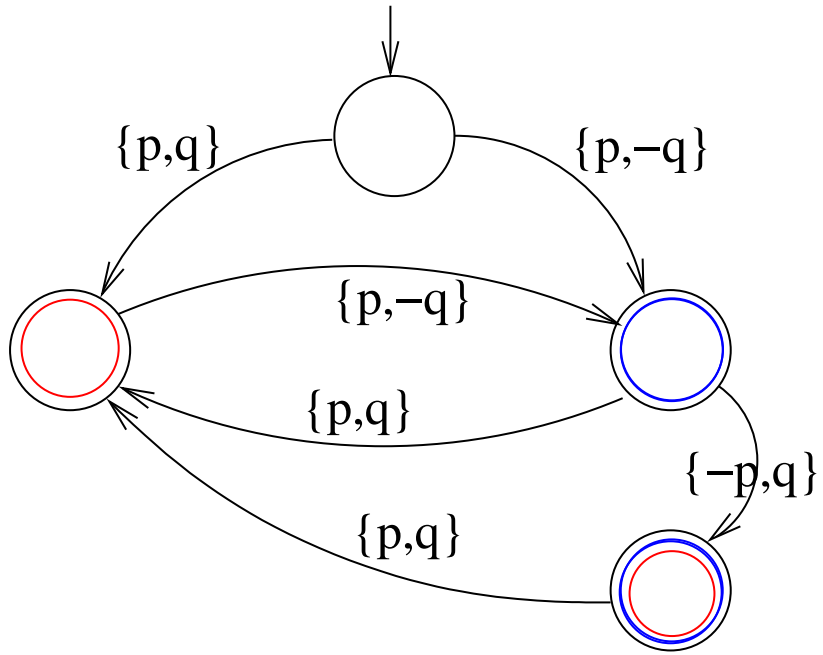
Computing a (Generalized) BA A_M from a Fair Kripke Structure M: Example



Fair Kripke Structure



Generalized Buechi Automaton

$\Longrightarrow$ Substantially, add one initial state, move labels from states to incoming edges, set fair states as accepting states

Outline

Translation problem

Problem

Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL model checking (in other words, every model checking algorithm that verifies the correctness of an LTL formula translates it in some sort of finite-state machine).
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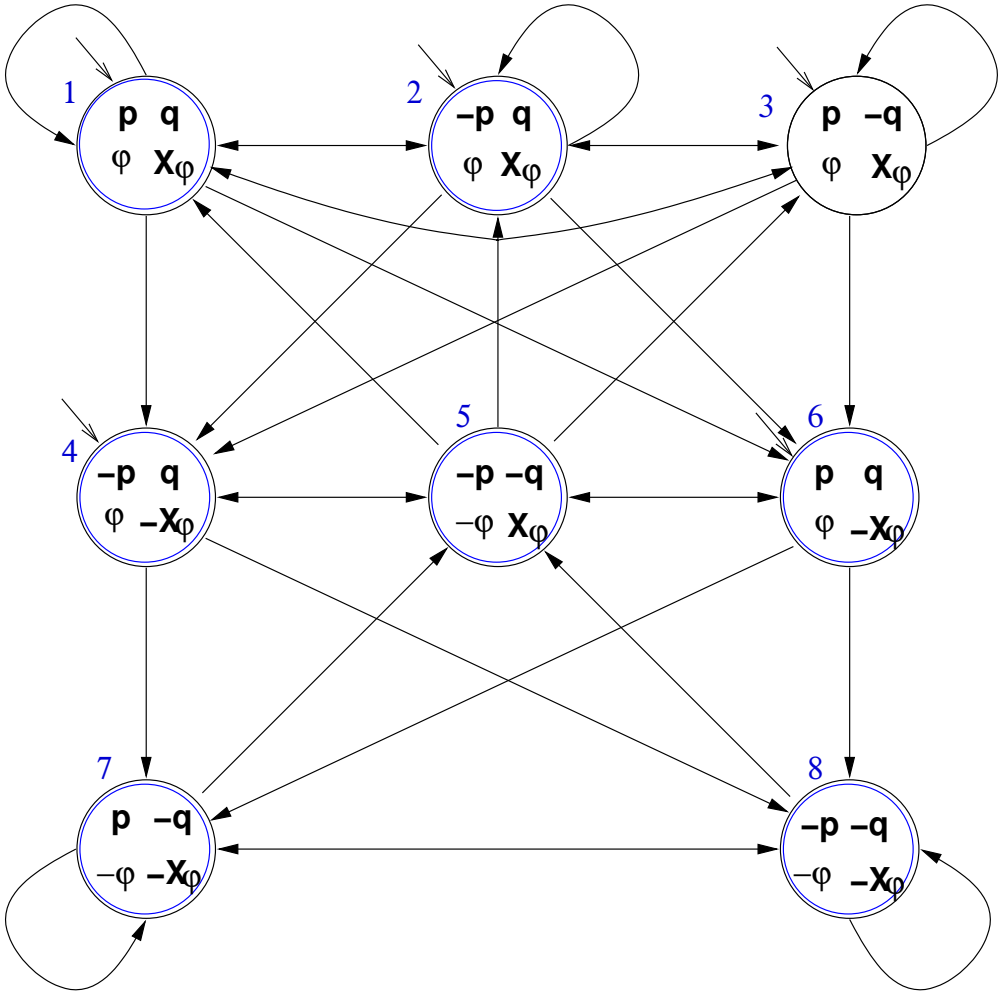

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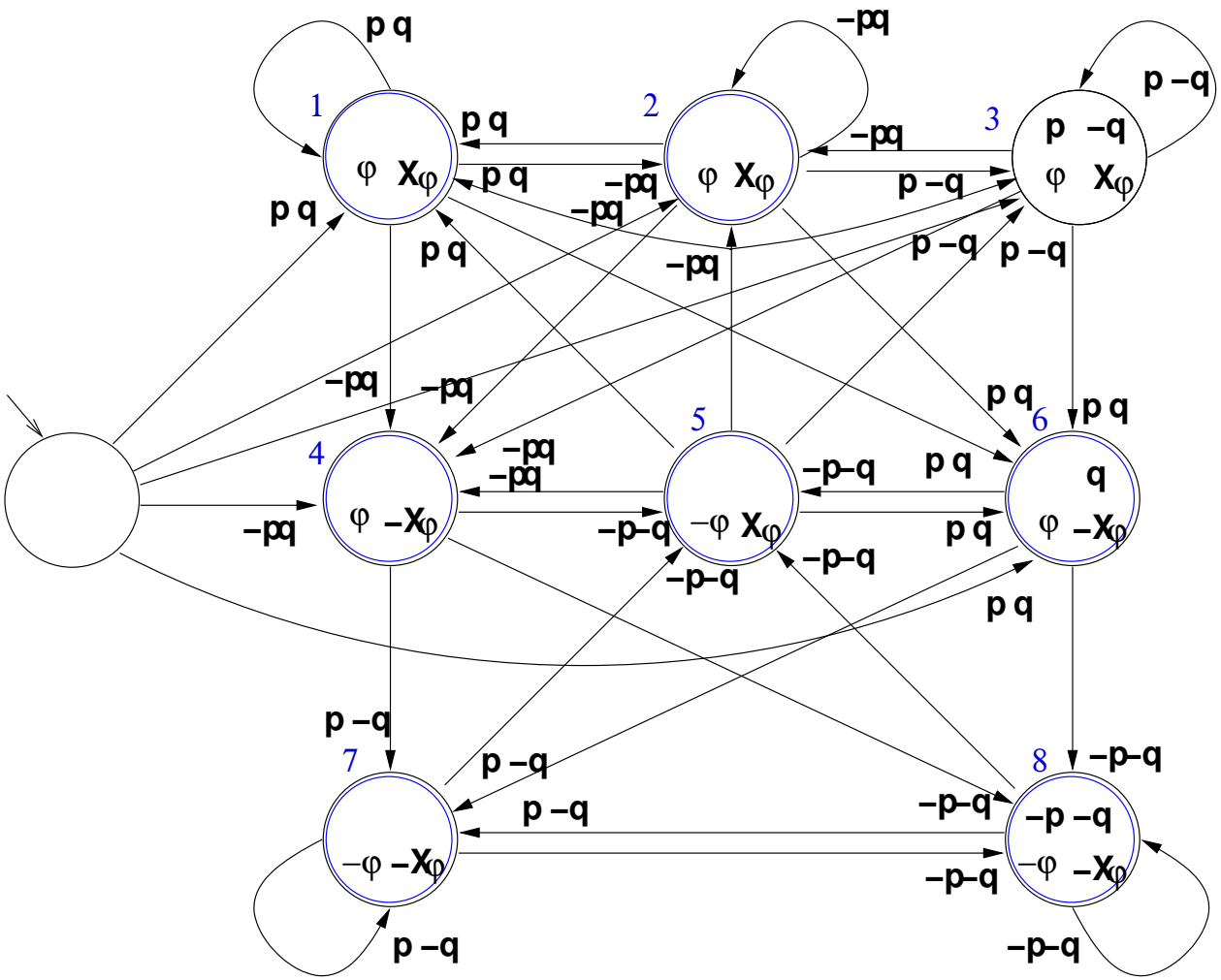

Outline

LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, **X**, **U**, **R** on propositional literals.
- Done by pushing negations down to literal level:

$$\neg(\varphi_1 \vee \varphi_2) \implies (\neg\varphi_1 \wedge \neg\varphi_2)$$
$$\neg(\varphi_1 \wedge \varphi_2) \implies (\neg\varphi_1 \vee \neg\varphi_2)$$
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  $\implies$ the resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).
  - encoding linear if a DAG representation is used
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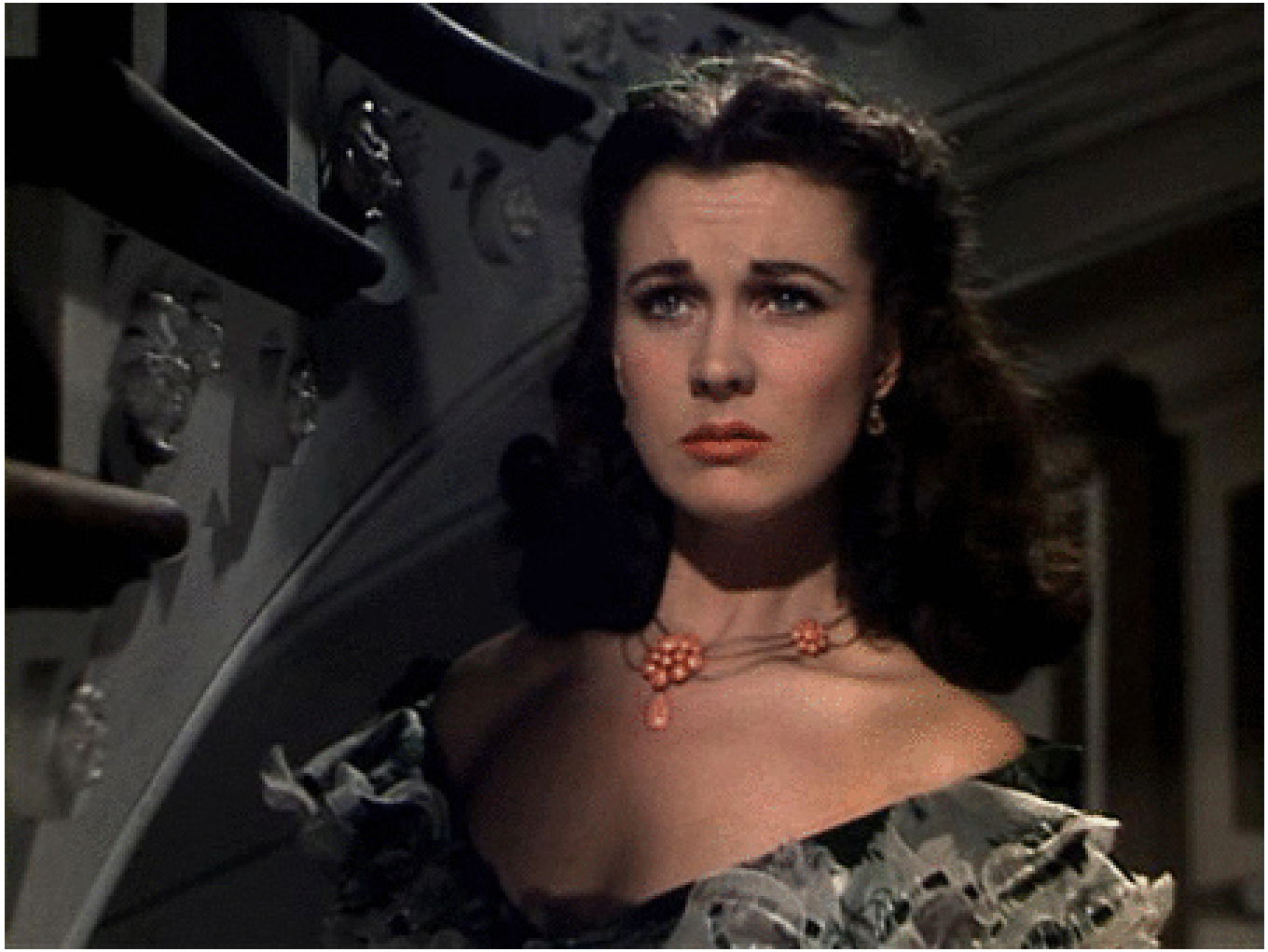
On-the-fly Construction of A_φ (Intuition)

Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ
$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$
$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$
until we get a Boolean combination of elementary subformulas of φ
(An elementary formula is a proposition or a **X**-formula.)

Tableaux rules: a quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, and then push the conjunctions inside the next:

$$\varphi \implies \bigvee_i \Big(\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X}\psi_{ik}\Big) \implies \bigvee_i \Big(\bigwedge_j l_{ij} \wedge \mathbf{X}\bigwedge_k \psi_{ik}\Big).$$

- Each disjunct $\Big(\overbrace{\bigwedge_j l_{ij}}^{labels} \wedge \overbrace{\mathbf{X}\bigwedge_k \psi_{ik}}^{next\ part}\Big)$ represents a state:
  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
  - $\mathbf{X}\bigwedge_k \psi_{ik}$ represents the next part of the state (obbligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

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- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
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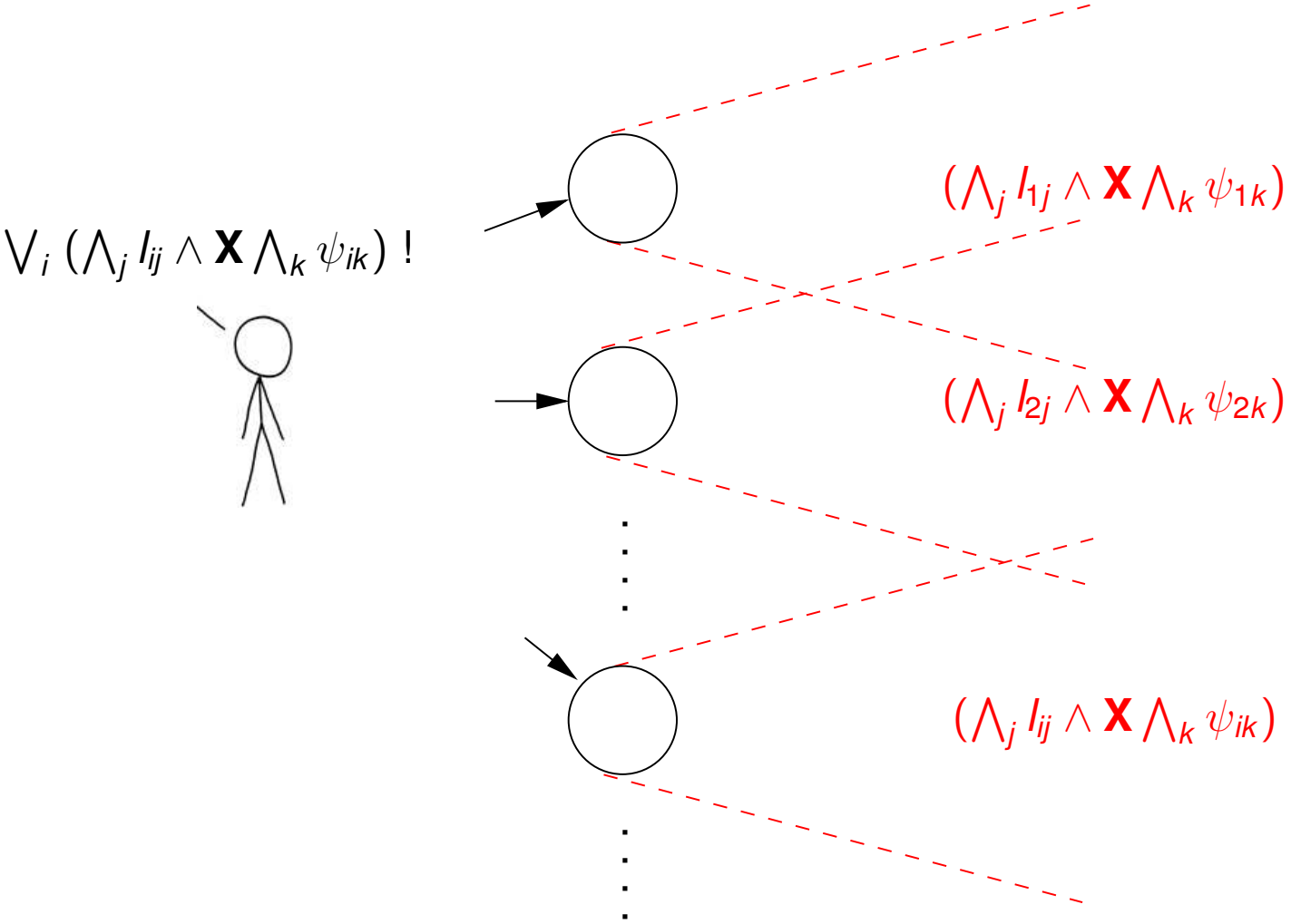
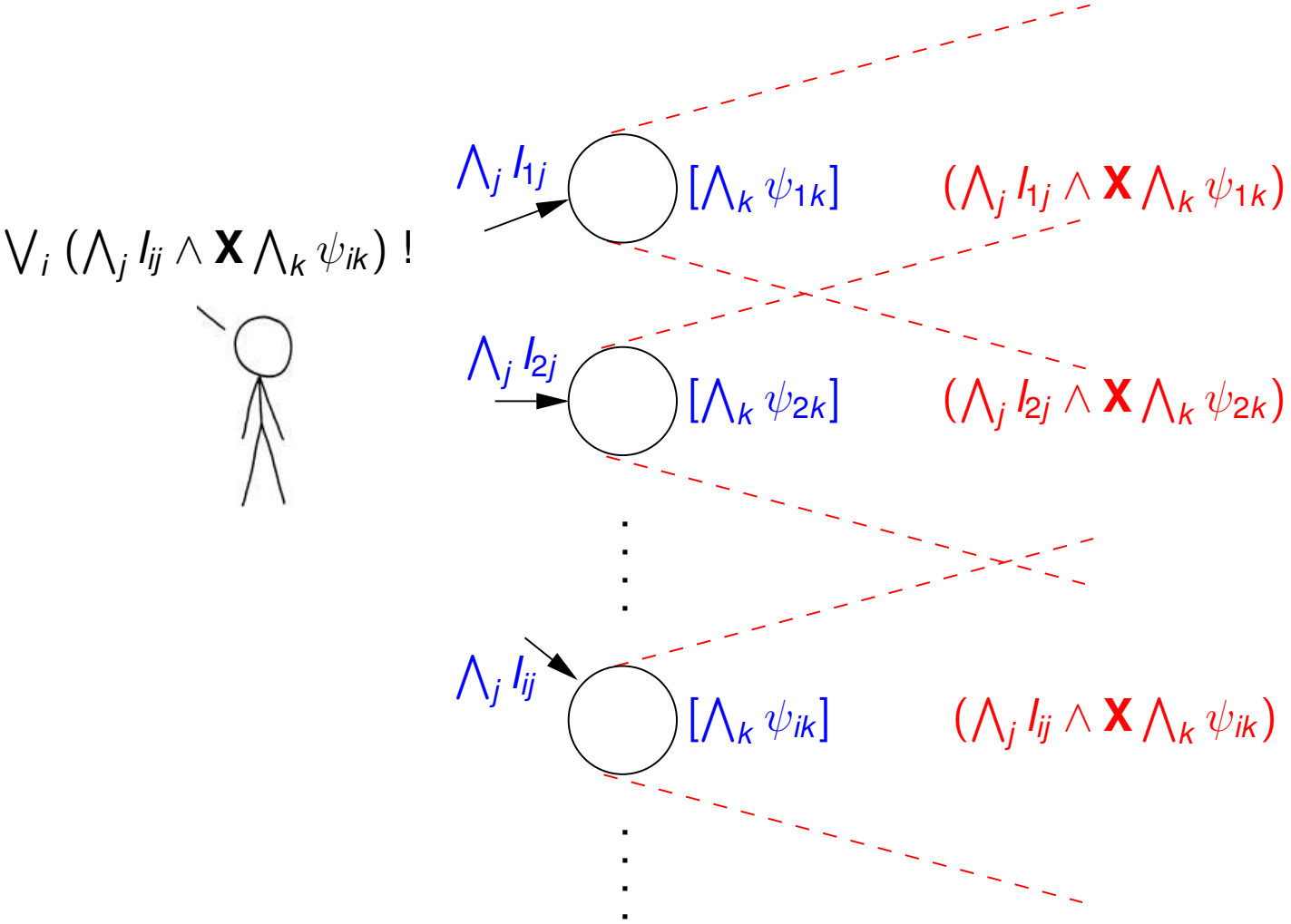


On-the-fly Construction of A_φ (Intuition) [cont.]



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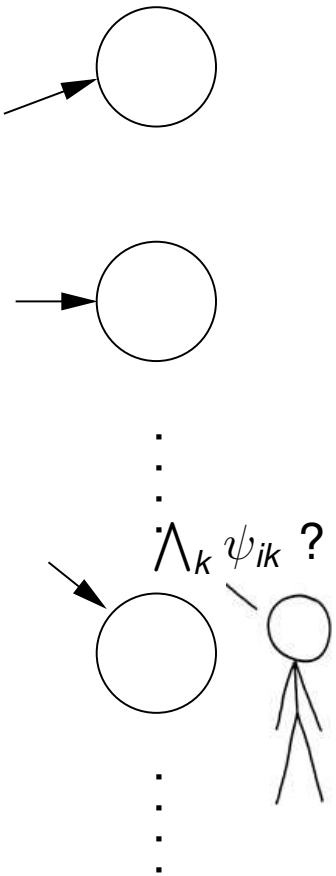

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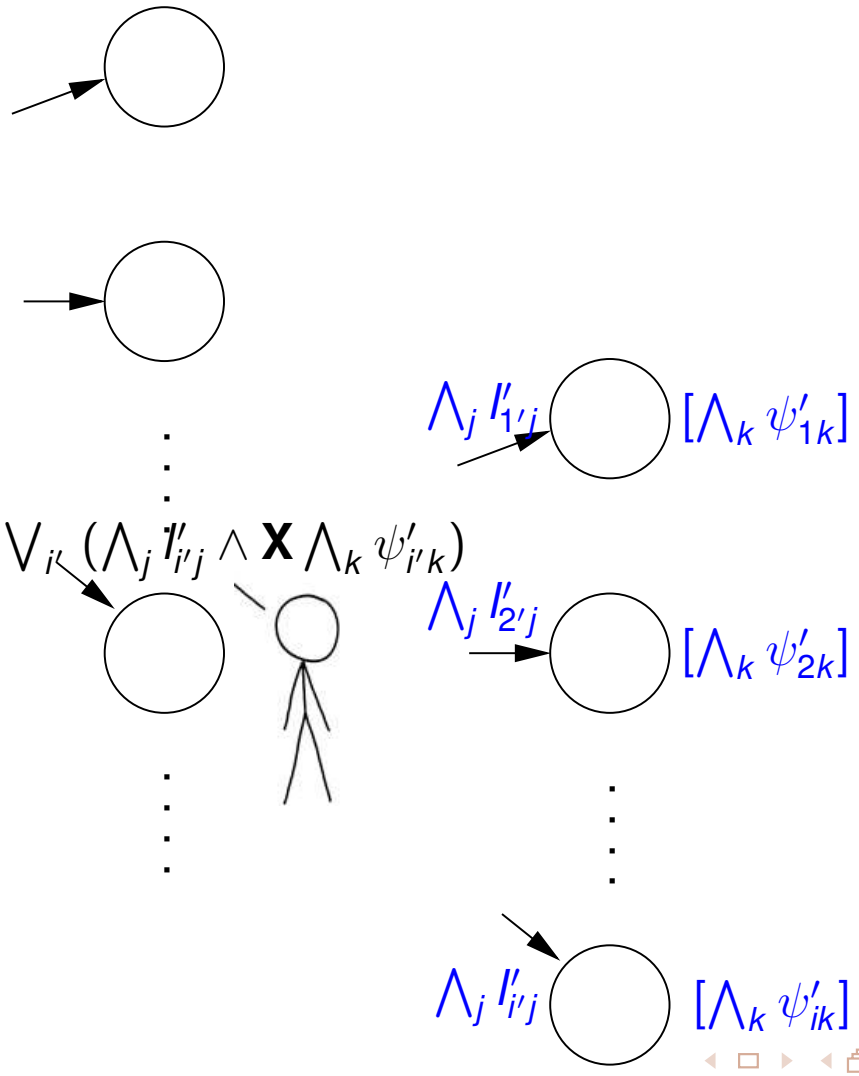

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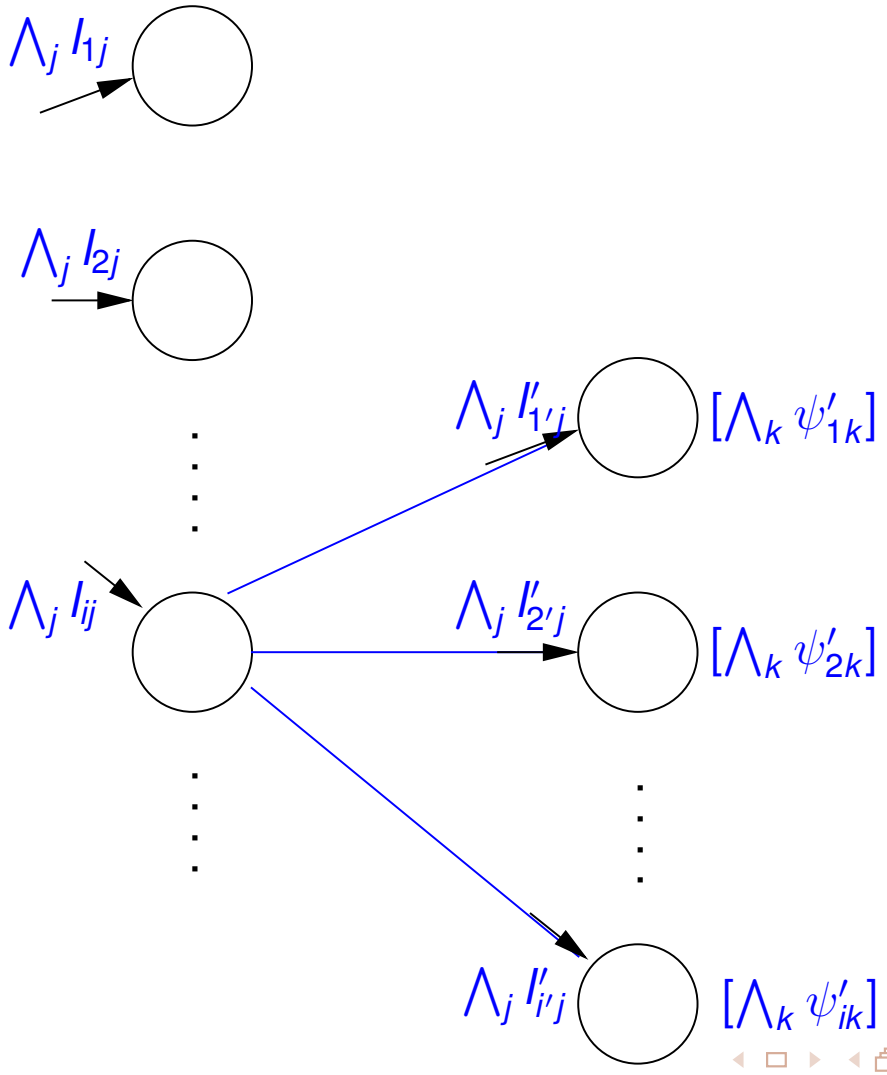

On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j, mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no U-subformulas, then mark all states with F_1 —i.e., $FT \stackrel{\text{def}}{=} \{Q\}$).

On-the-fly Construction of A_ϕ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of ϕ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $\mathit{Cover}(\Psi) \stackrel{\text{def}}{=} \mathit{Expand}(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - Combines steps 1. and 2. of previous slides
  - Expand() defined recursively as follows

On-the-fly Construction of A_ϕ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
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  - σ is the set of the subformulas of ϕ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \stackrel{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $\mathit{Cover}(\Psi) \stackrel{\text{def}}{=} \mathit{Expand}(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
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- if $\top \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
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Two relevant subcases: $\mathbf{F}\psi \stackrel{\text{def}}{=} \top\mathbf{U}\psi$ and $\mathbf{G}\psi \stackrel{\text{def}}{=} \bot\mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Phi$ and $s = \langle \lambda, \chi, \sigma \rangle$,
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Definition of A_ϕ

Given a set of LTL formulas Ψ, we define
$Cover(\Psi) \stackrel{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.
For an LTL formula ϕ, we construct a Generalized NBA $A_\phi = (Q, Q_0, \Sigma, L, T, FT)$ as follows:

- $\Sigma = 2^{vars(\phi)}$
- Q is the smallest set such that
  - $Cover(\{\phi\}) \subseteq Q$
  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
- $Q_0 = Cover(\{\phi\})$.
- $L(\langle \lambda, \chi, \sigma \rangle) = \{a \in \Sigma | a \models \lambda\}$
- $(s, s') \in T$ iff, $s = \langle \lambda, \chi, \sigma \rangle$ and $s' \in Cover(\chi)$
- $FT = \langle F_1, F_2, ..., F_k \rangle$ where, for all $(\psi_i \mathbf{U} \phi_i)$ occurring positively in ϕ, $F_i = \{\langle \lambda, \chi, \sigma \rangle \in Q \mid (\psi_i \mathbf{U} \phi_i) \notin \sigma \ \ or \ \ \phi_i \in \sigma\}$.
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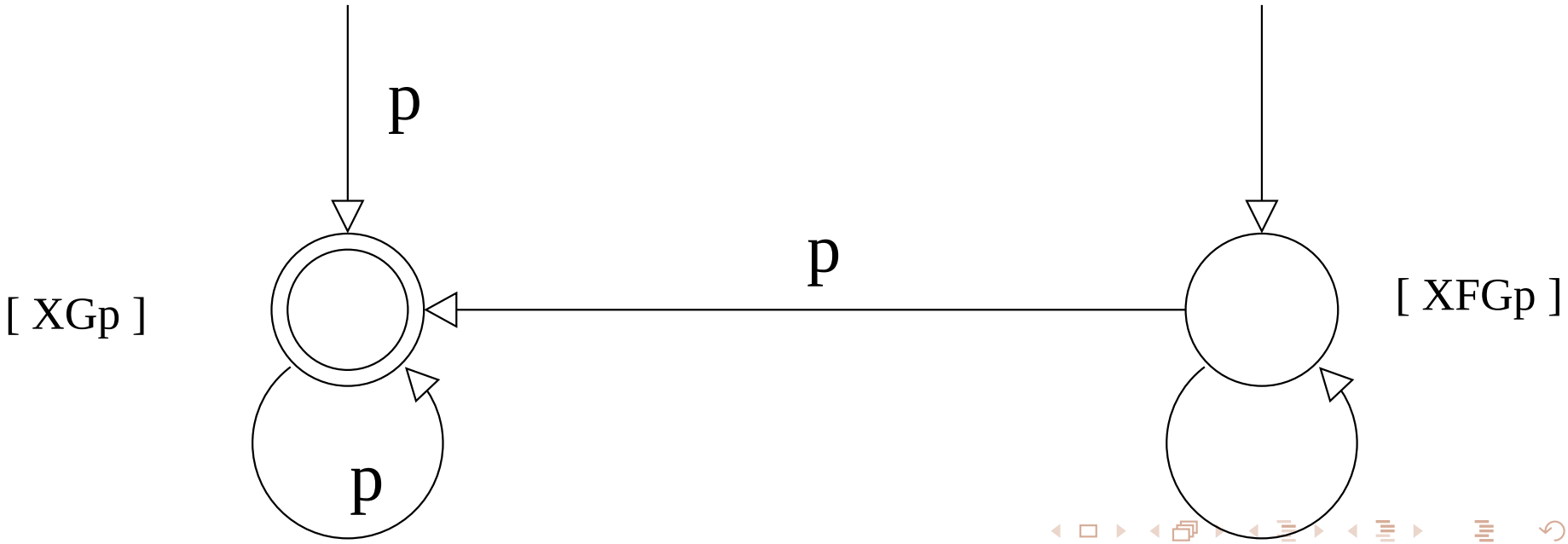
Example: $\phi = \mathbf{FG}p$

- $$\begin{aligned}
&\mathit{Cover}(\{\mathbf{FG}p\}) \\
&= \mathit{Expand}(\{\mathbf{FG}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\
&= \mathit{Expand}(\emptyset, \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle) \cup \mathit{Expand}(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \{\mathbf{FG}p\} \rangle) \\
&= \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup \mathit{Expand}(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p\} \rangle) \\
&= \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle\} \cup \mathit{Expand}(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle) \\
&= \{\langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle\}
\end{aligned}$$
- $$\begin{aligned}
\mathit{Cover}(\{\mathbf{G}p\}) &= \mathit{Expand}(\{\mathbf{G}p\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\
&= \mathit{Expand}(\{p\}, \langle \emptyset, \{\mathbf{G}p\}, \{\mathbf{G}p\} \rangle) \\
&= \mathit{Expand}(\emptyset, \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle) \\
&= \{\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle\}
\end{aligned}$$
- Optimization:
  merge $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$ and $\langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{G}p, p\} \rangle$

Example: $\phi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
- $Q_0 = \{s_1, s_2\}$.
- T : $s_1 \to \{s_1, s_2\}$, $s_2 \to \{s_2\}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2\}$.



Example: $\phi = p\mathbf{U}q$

- $$\begin{aligned} & \mathit{Cover}(\{p\mathbf{U}q\}) \\ = \; & \mathit{Expand}(\{p\mathbf{U}q\}, \langle \emptyset, \emptyset, \emptyset \rangle) \\ = \; & \mathit{Expand}(\{p\}, \langle \emptyset, \{p\mathbf{U}q\}, \{p\mathbf{U}q\} \rangle) \cup \mathit{Expand}(\{q\}, \langle \emptyset, \emptyset, \{p\mathbf{U}q\} \rangle) \\ = \; & \mathit{Expand}(\emptyset, \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle) \cup \mathit{Expand}(\emptyset, \langle \{q\}, \emptyset, \{p\mathbf{U}q, q \\ = \; & \{\langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle\} \cup \{\langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\} \rangle\} \end{aligned}$$
- $\mathit{Cover}(\{\top\}) = \{\langle \emptyset, \{\top\}, \{\top\} \rangle\}$

Example: $\phi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle$, $s_2 =_{def} \langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\} \rangle$, $s_3 =_{def} \langle \emptyset, \{\top\}, \{\top\} \rangle$.
- $Q = \{s_1, s_2, s_3\}$,
- $Q_0 = \{s_1, s_2\}$,
- T : $s_1 \rightarrow \{s_1, s_2\}$, $s_2 \rightarrow \{s_3\}$, $s_3 \rightarrow \{s_3\}$
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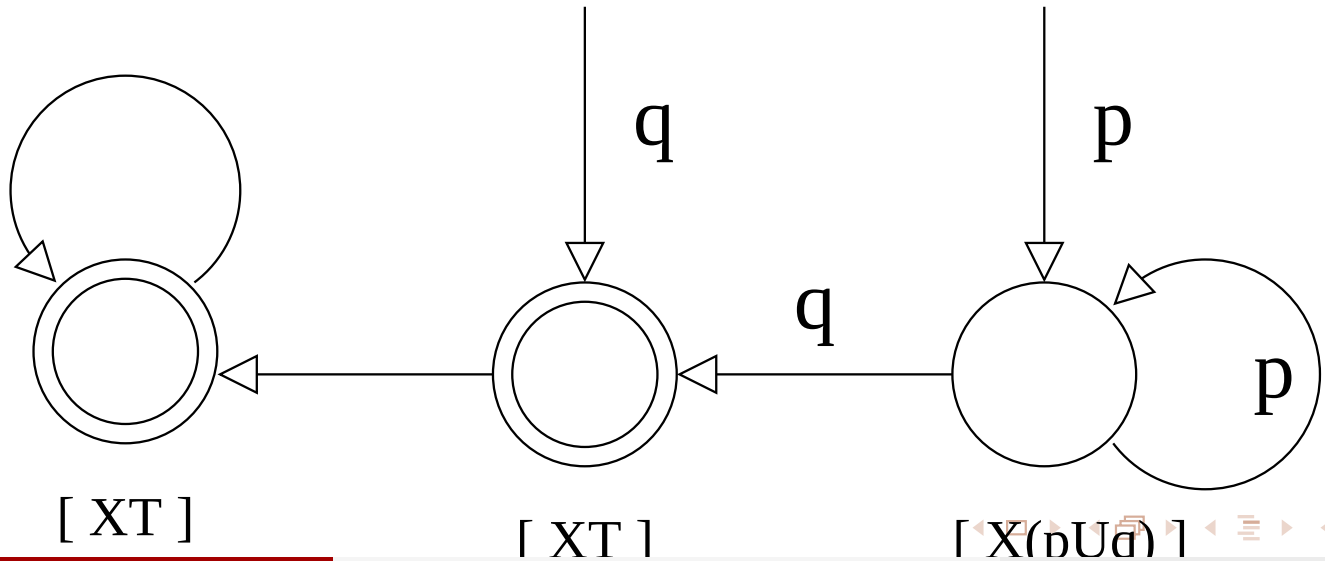

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$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \iff \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: **GF***p*

- Let $s_1 =_{def} \langle \{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\} \rangle$,
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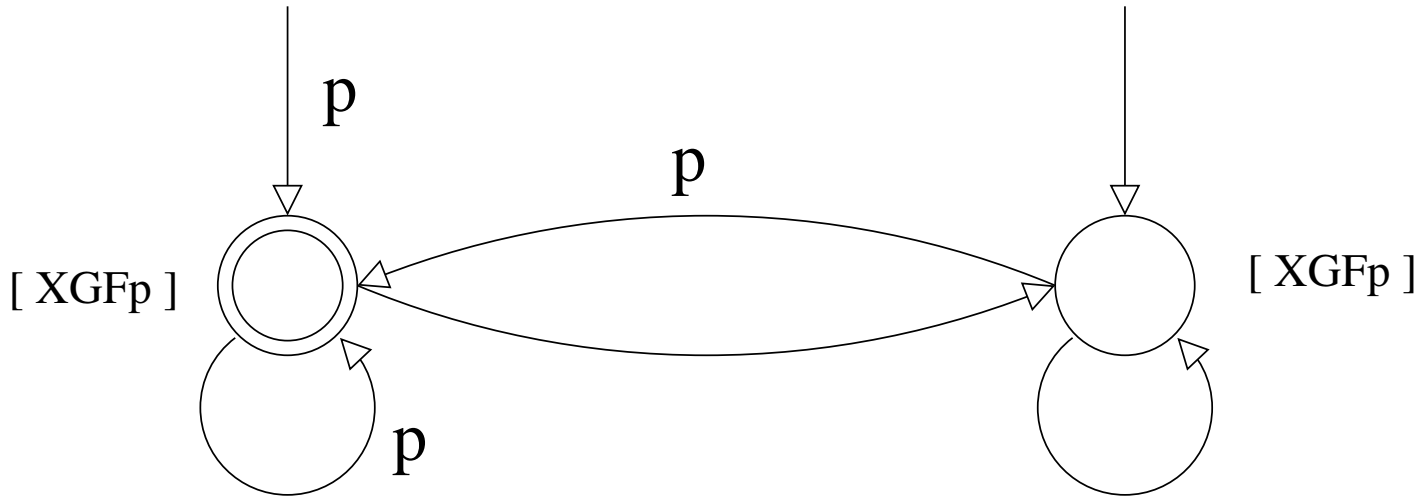

NBAs of disjunctions of formulas

Remark

If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

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Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ:

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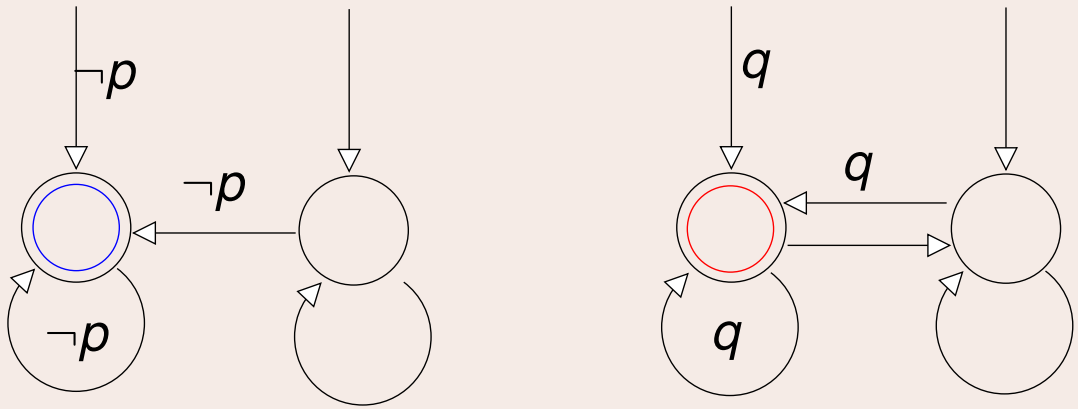

Suggested Exercises:

- Find an NBA encoding:
  - p
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{G}r$

Outline

Automata-Theoretic LTL Model Checking: complexity

Four steps:

(i) Compute A_M: $|A_M| = O(|M|)$

(ii) Compute A_φ: $|A_\varphi| = O(2^{|\varphi|})$

(iii) Compute the product $A_M \times A_\varphi$:
$|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$: $O(|A_M \times A_\varphi|) = O(|M| \cdot 2^{|\varphi|})$

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Final Remarks

- Büchi automata are in general more expressive than LTL!
  $\Longrightarrow$ Some tools (e.g., Spin) allow specifications to be expressed directly as NBAs
  $\Longrightarrow$ complementation of NBA important!
- for every LTL formula, there are many possible equivalent NBAs
  $\Longrightarrow$ lots of research for finding "the best" conversion algorithm
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Outline

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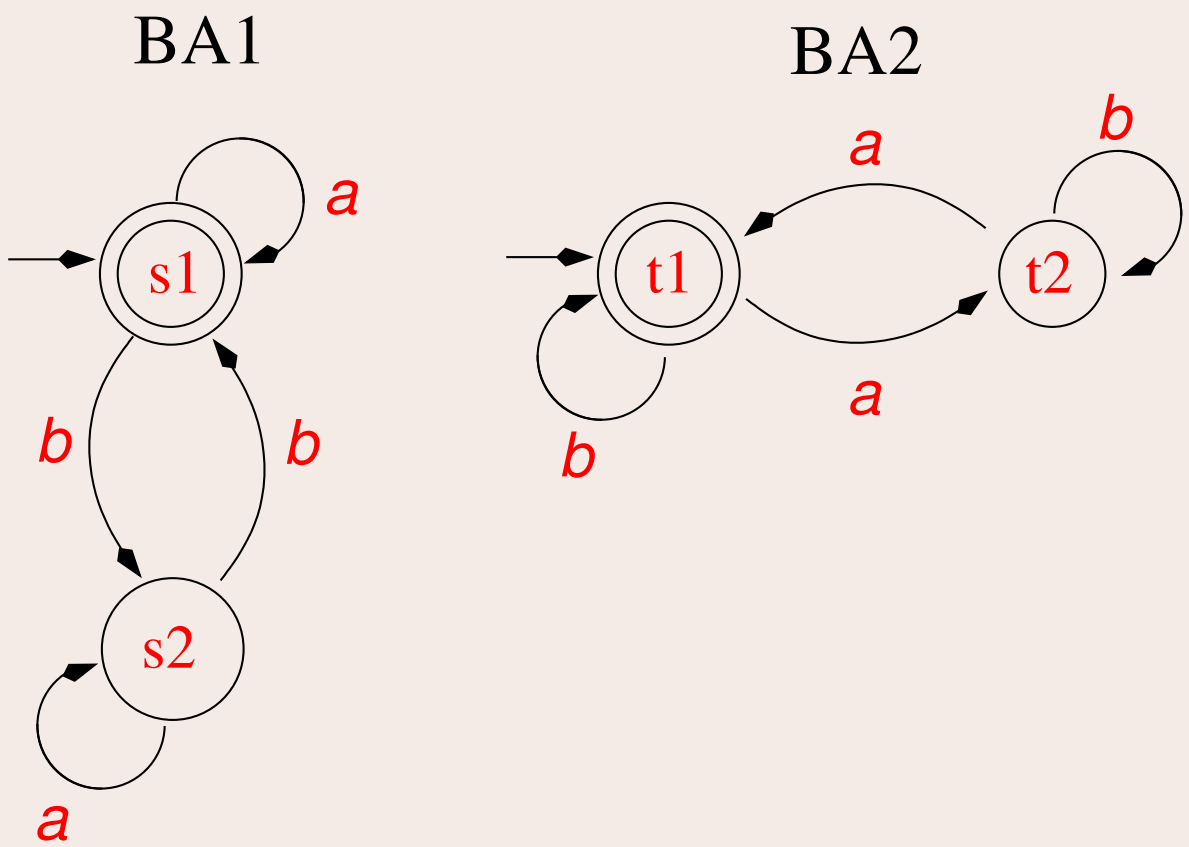

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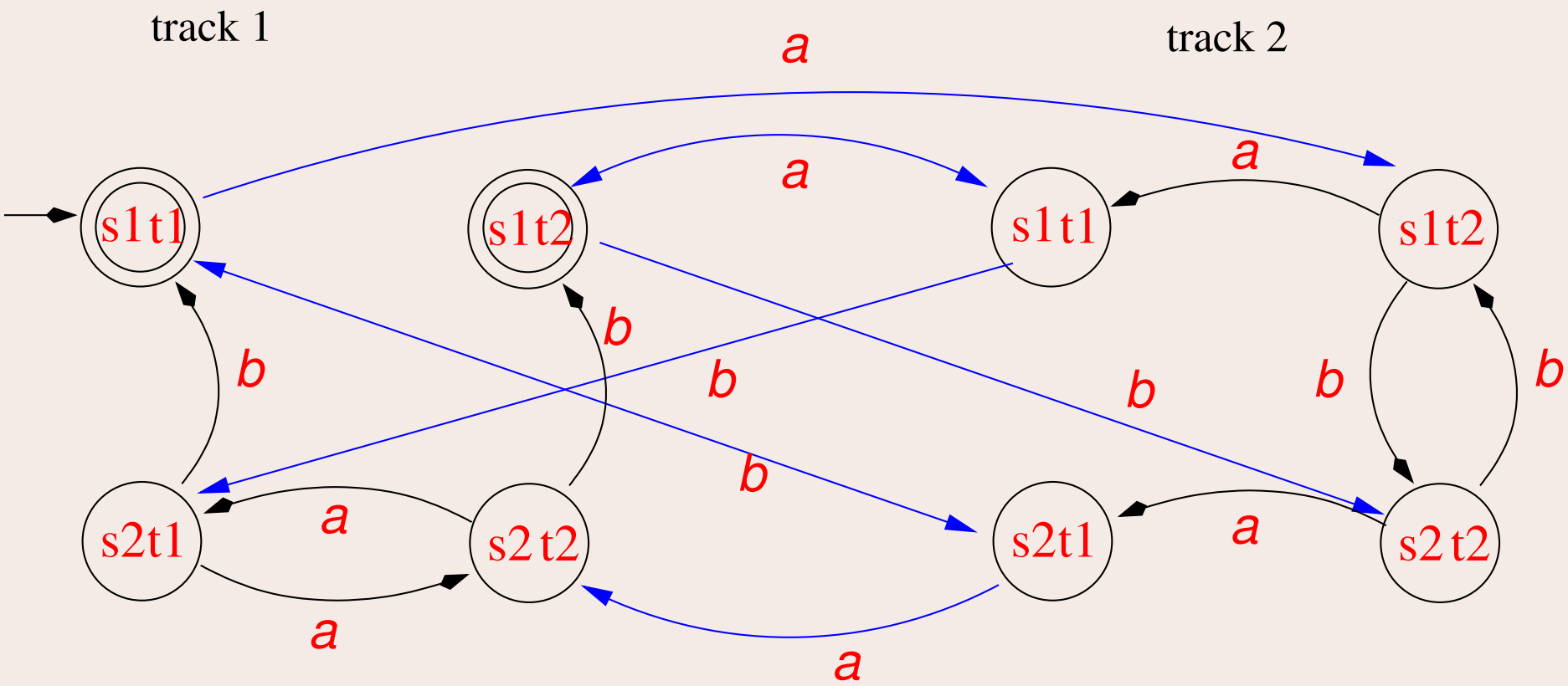

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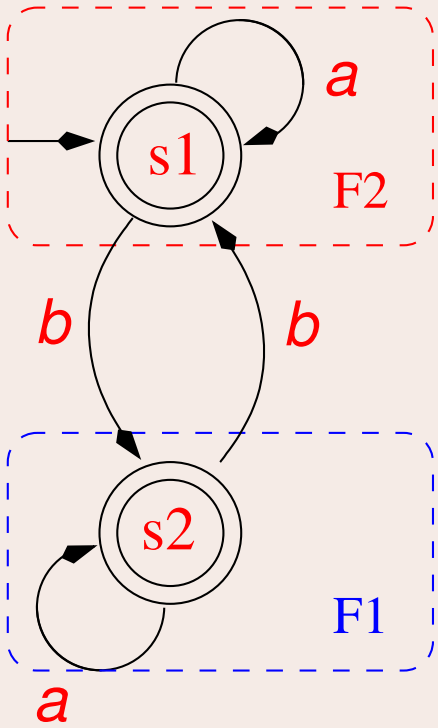
Ex: De-generalization of Büchi Automata

Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT \rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \stackrel{\text{def}}{=} \{s2\}, F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

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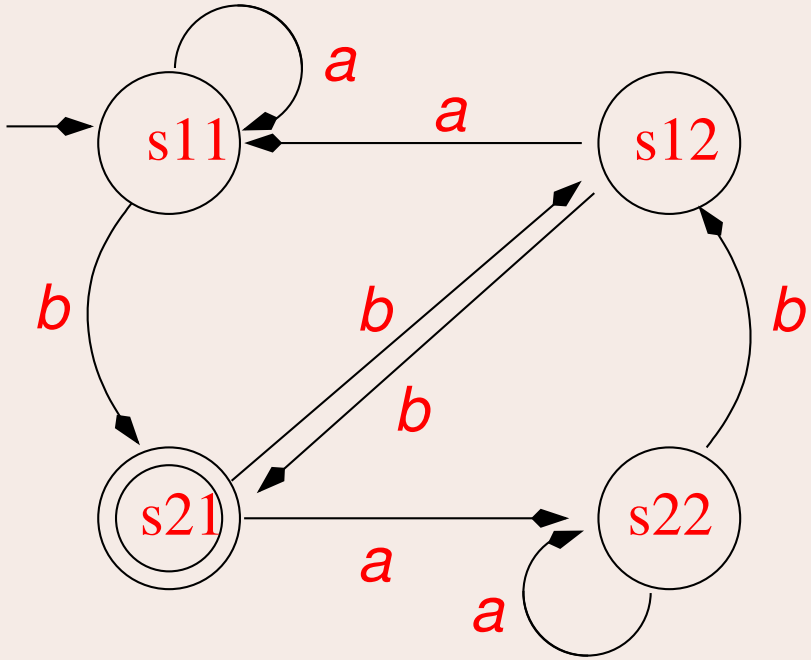

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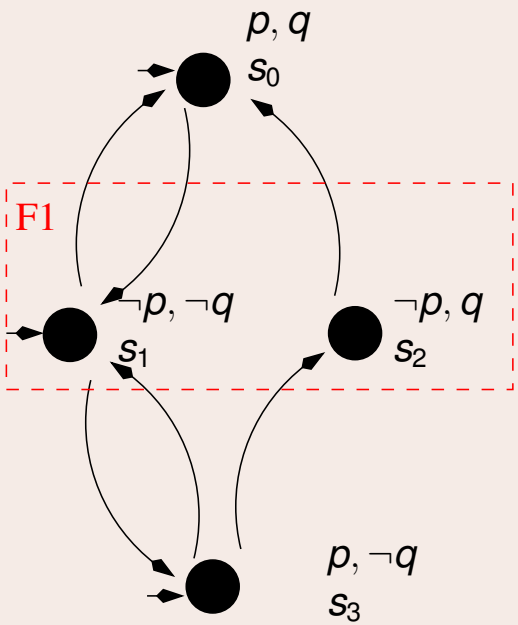
Given the following fair Kripke model M, convert it into an equivalent Buchi automaton.



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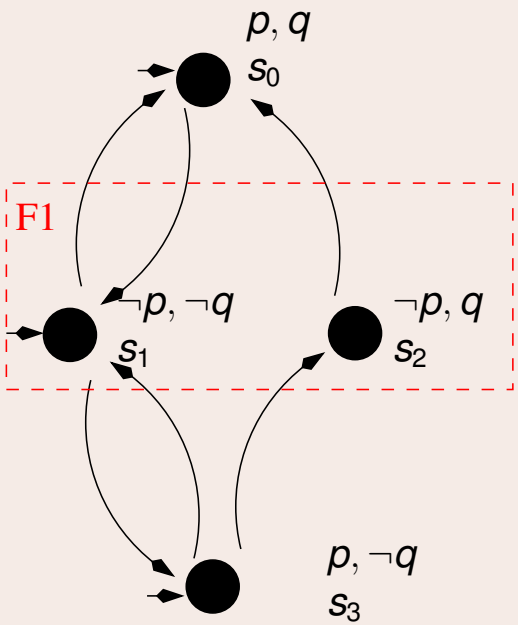

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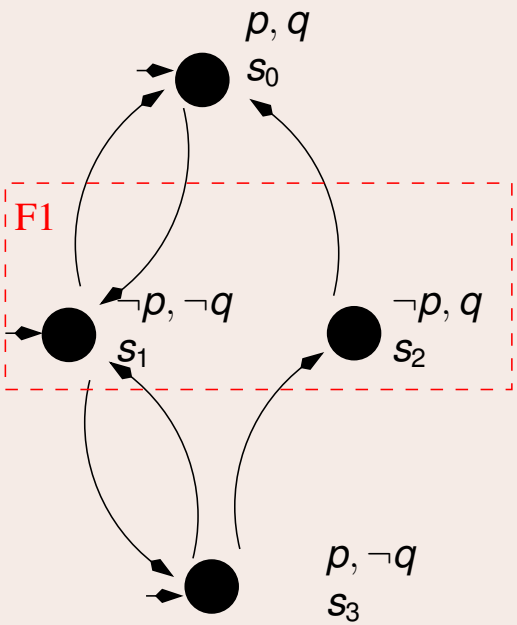

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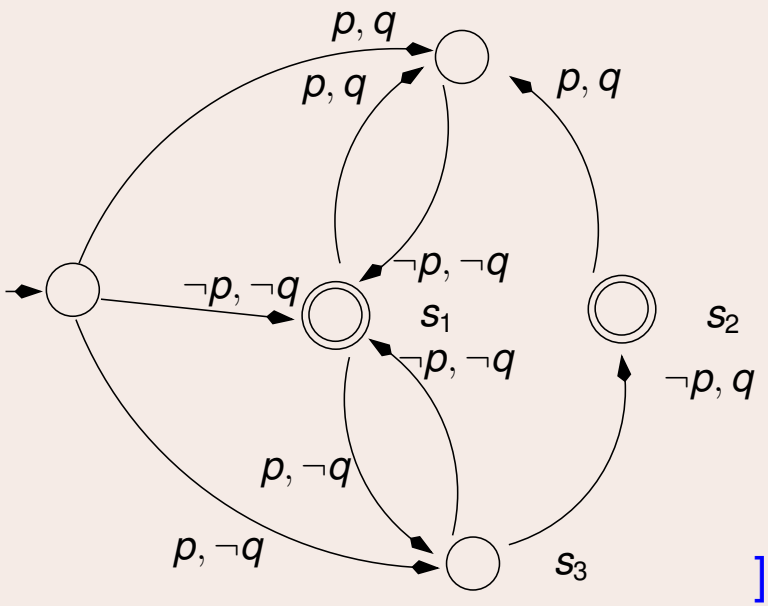

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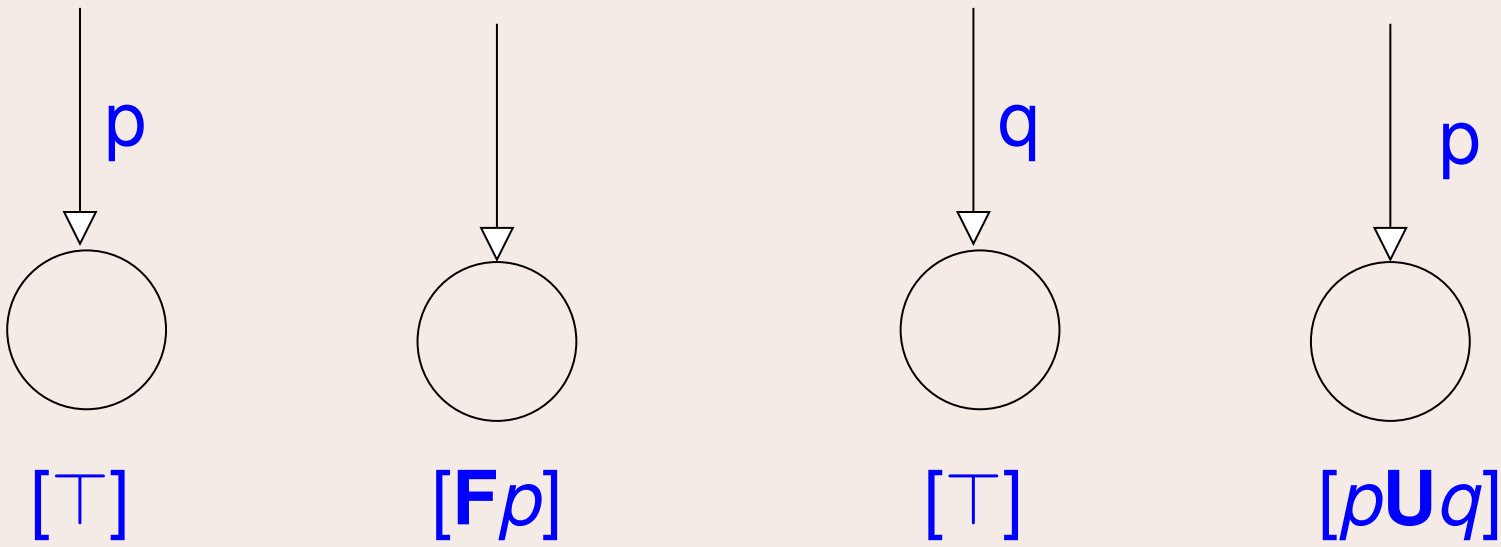

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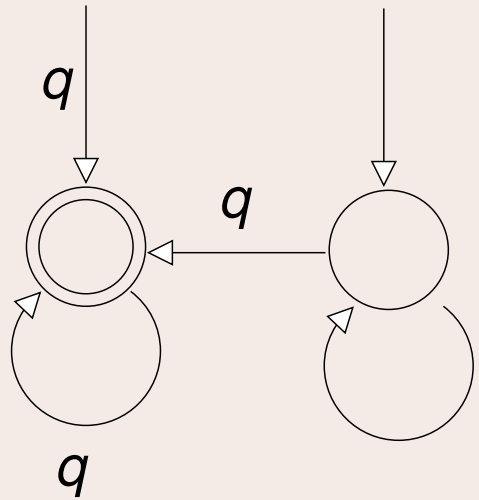

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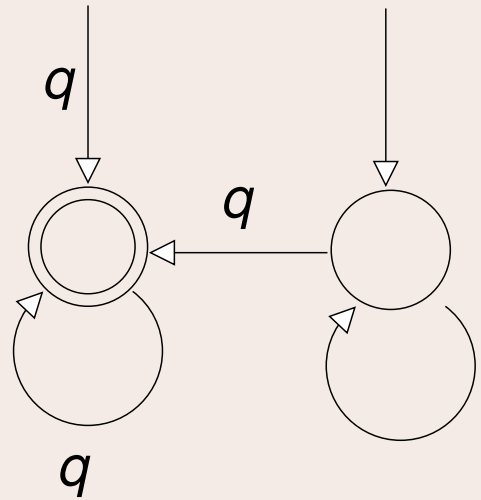

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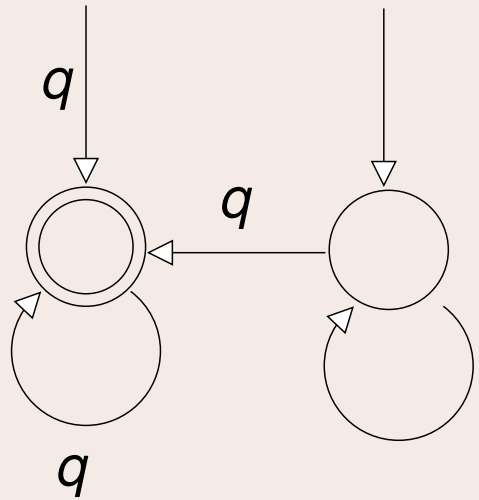

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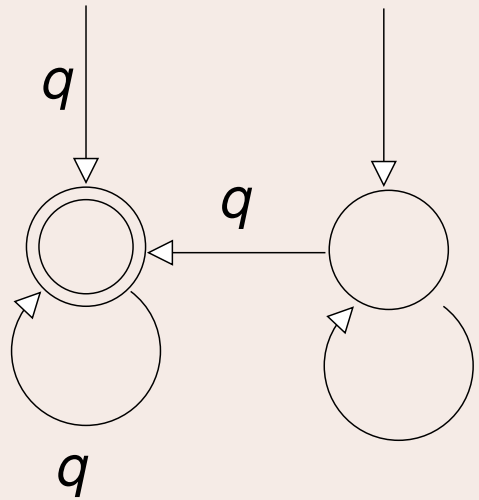

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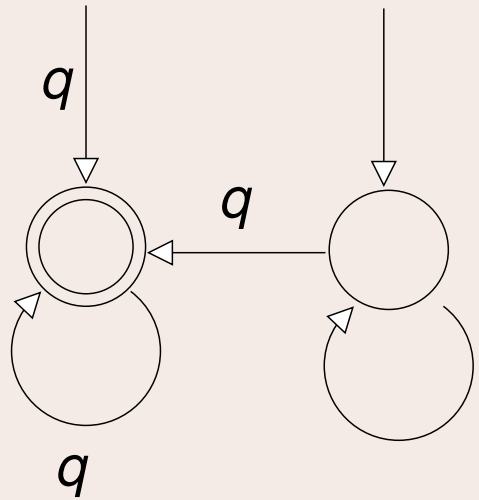

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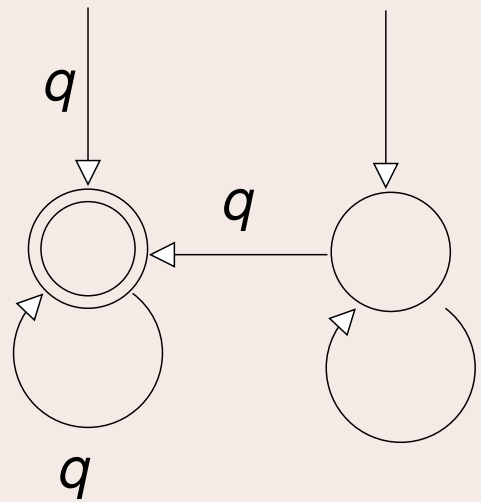

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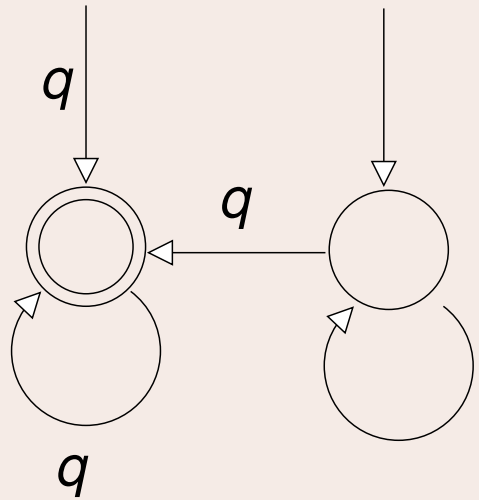

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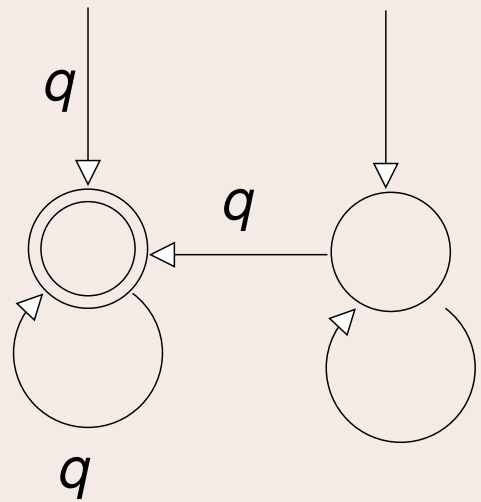

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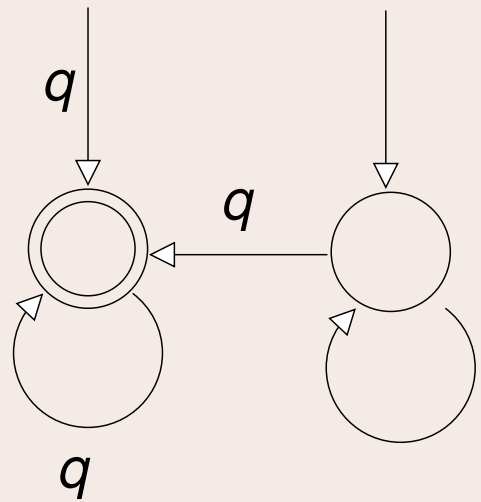

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