Introduction to Formal Methods
Chapter 05: Symbolic CTL Model Checking

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Outline

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The Main Problem of CTL M.C. State Space Explosion

- The bottleneck:
  - Exhaustive analysis may require to store all the states of the Kripke structure, and to explore them one-by-one
  - The state space may be exponential in the number of components and variables
    (E.g., 300 Boolean vars $\Longrightarrow$ up to $2^{300} \approx 10^{100}$ states!)
  - State Space Explosion:
    - too much memory required
    - too much CPU time required to explore each state
- A solution: Symbolic Model Checking

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Symbolic Model Checking

Symbolic representation:

- manipulation of sets of states (rather than single states);
- sets of states represented by formulae in propositional logic;
  - set cardinality not directly correlated to size
- expansion of sets of transitions (rather than single transitions);

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Symbolic Model Checking [cont.]

- two main symbolic techniques:
  - Binary Decision Diagrams (BDDs)
  - Propositional Satisfiability Checkers (SAT solvers)
- Different model checking algorithms:
  - Fix-point Model Checking (historically, for CTL)
  - Fix-point Model Checking for LTL (conversion to fair CTL MC)
  - Bounded Model Checking (historically, for LTL)
  - Invariant Checking
  - ...

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Ordered Binary Decision Diagrams (OBDDs) [Bryant, '85]

Canonical representation of Boolean formulas

- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top, \bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

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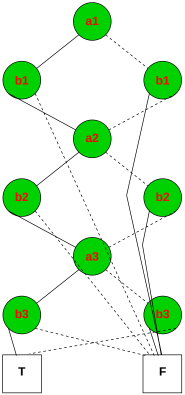
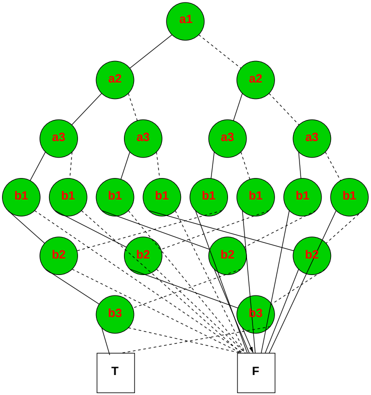
OBDD - Examples

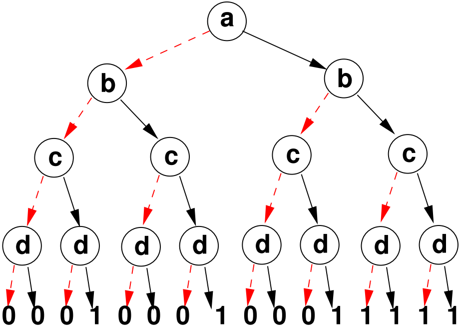




OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- Ordered Decision Tree: from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi = (a \wedge b) \vee (c \wedge d)$



From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following reductions:
  - share subnodes: point to the same occurrence of a subtree (via hash consing)
  - remove redundancies: nodes with same left and right children can be eliminated ("if A then B else B" $\Longrightarrow$ "B")

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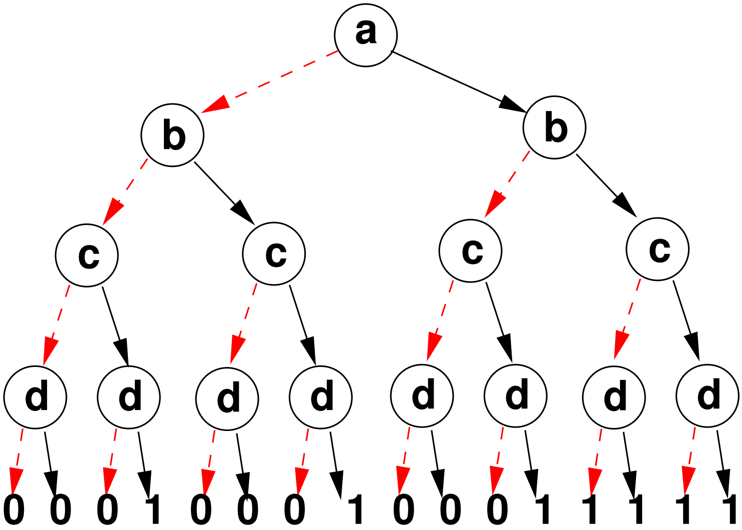
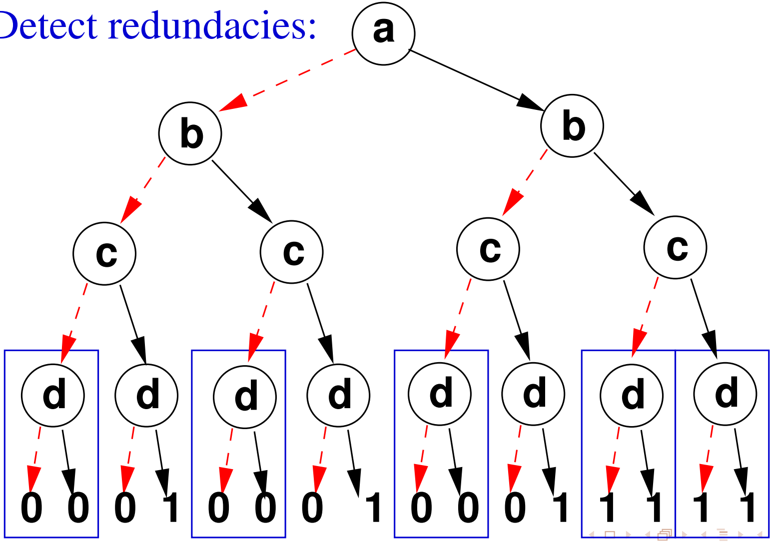
Reduction: example



Reduction: example

Detect redundacies:



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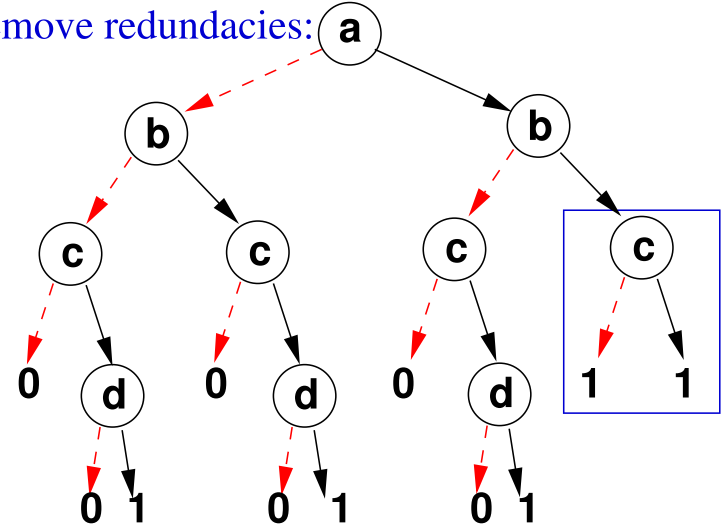

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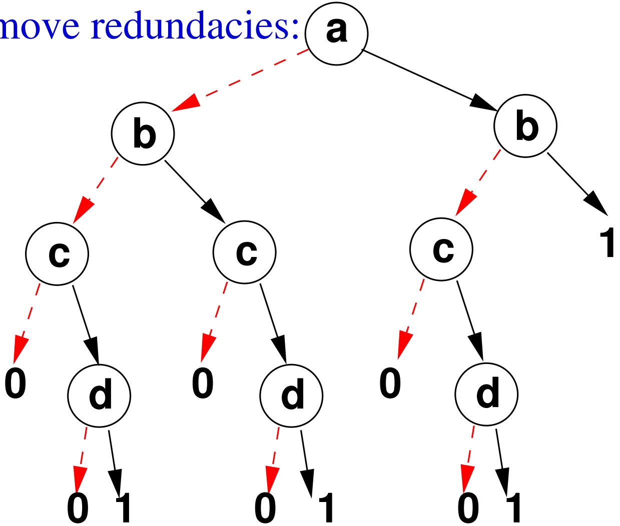

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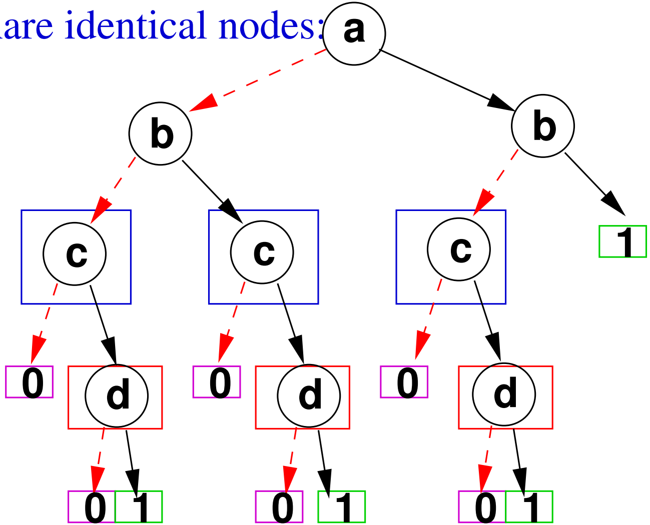

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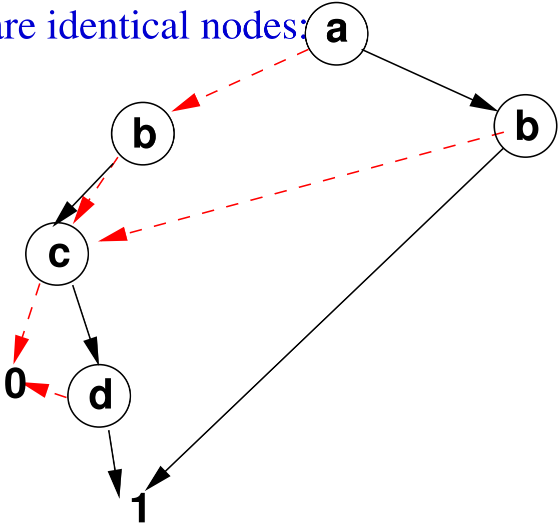

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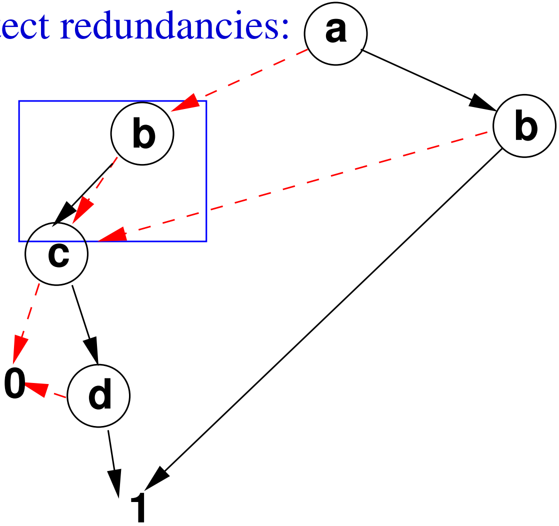

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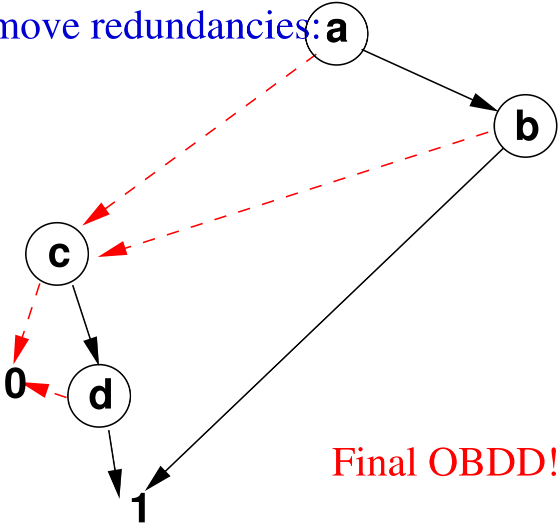

Final OBDD!

Recursive structure of an OBDD

Assume the variable ordering $A_1, A_2, ..., A_n$:

$$
\begin{array}{rcl}
OBDD(\top, \{A_1, A_2, ..., A_n\}) & = & 1 \\
OBDD(\bot, \{A_1, A_2, ..., A_n\}) & = & 0 \\
OBDD(\varphi, \{A_1, A_2, ..., A_n\}) & = & \textit{if } A_1 \\
 & & \textit{then } OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\}) \\
 & & \textit{else } OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\})
\end{array}
$$

Incrementally building an OBDD

- $obdd\_build(\top, \{...\}) := 1$,
- $obdd\_build(\bot, \{...\}) := 0$,
- $obdd\_build(A_i, \{...\}) := ite(A_i, 1, 0)$,
- $obdd\_build((\neg\varphi), \{A_1, ..., A_n\}) :=$
  $apply(\neg, obdd\_build(\varphi, \{A_1, ..., A_n\}))$
- $obdd\_build((\varphi_1\ op\ \varphi_2), \{A_1, ..., A_n\}) :=$
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"$ite(A_i, \varphi_i^\top, \varphi_i^\bot)$" is "If A_i Then $\varphi_i^\top$ Else $\varphi_i^\bot$"

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Incrementally building an OBDD (cont.)

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  $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\perp))$
- $apply\ (op,\ ite(A_i, \varphi_i^\top, \varphi_i^\perp),\ ite(A_j, \varphi_j^\top, \varphi_j^\perp)) :=$
  if $(A_i = A_j)$ **then** $ite(A_i,\ apply\ (op, \varphi_i^\top, \varphi_j^\top),\ apply\ (op, \varphi_i^\perp, \varphi_j^\perp))$
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Incrementally building an OBDD (cont.)

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$
\begin{aligned}
& apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\
= \;& ite(A_1,\ apply(\vee, \top, ite(A_1, \top, \bot)),\ apply(\vee, \bot, ite(A_2, \top, \bot))\ ) \\
= \;& ite(A_1,\ \top,\ ite(A_2, \top, \bot)\ )
\end{aligned}
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- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

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= \;& ite(A_1,\ apply(\wedge, \top, \top),\ apply(\wedge,\ ite(A_2, \top, \bot),\ ite(A_2, \bot, \top)) \\
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OBBD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

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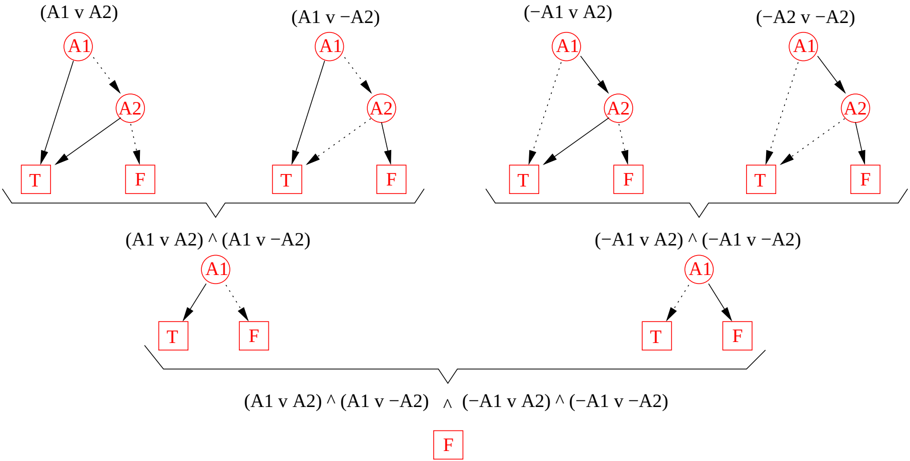

Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$

Linear size

Exponential size

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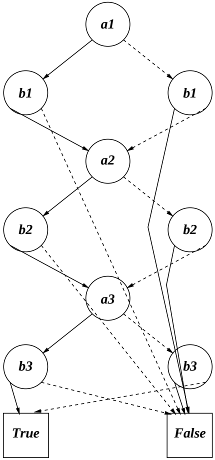

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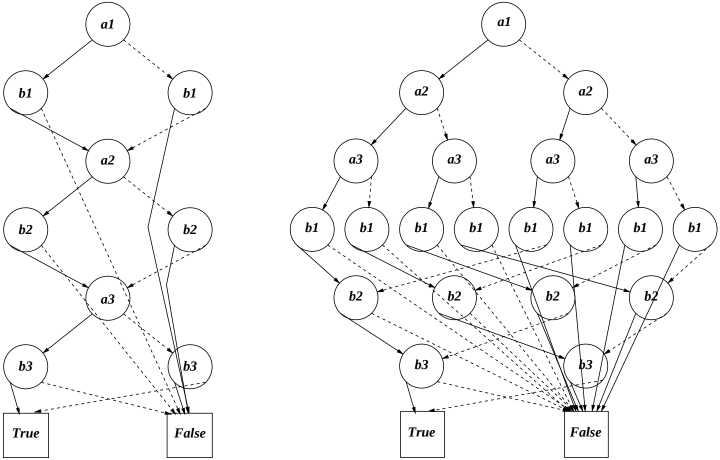

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OBDD's as canonical representation of Boolean formulas

- An OBDD is a canonical representation of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff OBDD(\varphi_1) = OBDD(\varphi_2)$$

- equivalence check requires constant time!
  $\Longrightarrow$validity check requires constant time! ($\varphi \leftrightarrow \top$)
  $\Longrightarrow$(un)satisfiability check requires constant time! ($\varphi \leftrightarrow \bot$)
- the set of the paths from the root to 1 represent all the models of the formula
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Exponentiality of OBDD's

- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless P = co-NP)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

Note

The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

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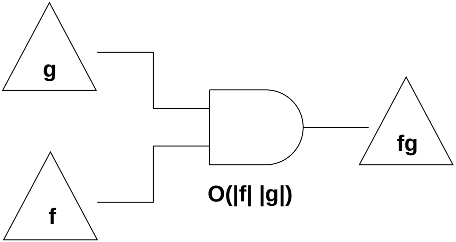

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Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then
  $$\exists v.f \quad := \quad f|_{v=0} \vee f|_{v=1}$$
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- v does no more occur in $\exists v.f$ and $\forall v.f$!!
- Multi-variable quantification: $\exists(w_1, \ldots, w_n).f \ := \ \exists w_1 \ldots \exists w_n.f$

- Intuition:
  - $\mu \models \exists v.f$ iff exists *tvalue* $\in \{\top, \bot\}$ s.t. $\mu \cup \{v := tvalue\} \models f$
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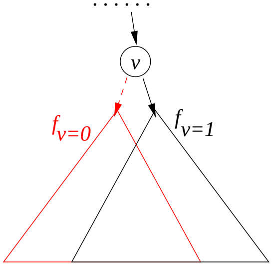
OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if f is a sub-OBDD labeled by variable v, then $f|_{v=1}$ and $f|_{v=0}$ are the "then" and "else" branches of f



$\Longrightarrow$ lots of sharing of subformulae!

OBDD – summary

- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
- Canonical representation of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents all models and counter-models of the formula.
- Require exponential space in worst-case
- Very efficient for some practical problems (circuits, symbolic model checking).

Outline

Symbolic Representation of Kripke Structures

- Symbolic representation:
  - sets of states as their characteristic function (Boolean formula)
  - provide logical representation and transformations of characteristic functions
- Example:
  - three state variables x_1, x_2, x_3:
    $\{ 000, 001, 010, 011 \}$ represented as "first bit false": $\neg x_1$
  - with five state variables x_1, x_2, x_3, x_4, x_5:
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Kripke Structures in Propositional Logic

- Let $M = (S, I, R, L, AF)$ be a Kripke structure
- States $s \in S$ are described by means of an array V of Boolean state variables.
- A state is a truth assignment to each atomic proposition in V.
  - 0100 is represented by the formula $(\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4)$
  - we call $\xi(s)$ the formula representing the state $s \in S$ (Intuition: $\xi(s)$ holds iff the system is in the state s)
- A set of states $Q \subseteq S$ can be represented by (any formula which is logically equivalent to) the formula $\xi(Q)$:

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One-to-one correspondence between sets and Boolean operators

- Set of all the states: $\xi(S) := \top$
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- Union represented by disjunction: $\xi(P \cup Q) := \xi(P) \vee \xi(Q)$
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- The transition relation R is a set of pairs of states: $R \subseteq S \times S$
- A transition is a pair of states (s, s')
- A new vector of variables V' (the next state vector) represents the value of variables after the transition has occurred
- $\xi(s, s')$ defined as $\xi(s) \wedge \xi(s')$ (Intuition: $\xi(s, s')$ holds iff the system is in the state s and moves to state s' in next step)
- The transition relation R can be (naively) represented by

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Each formula equivalent to $\xi(R)$ is a representation of R
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- $\xi(s, s')$ defined as $\xi(s) \wedge \xi(s')$ (Intuition: $\xi(s, s')$ holds iff the system is in the state s and moves to state s' in next step)
- The transition relation R can be (naively) represented by

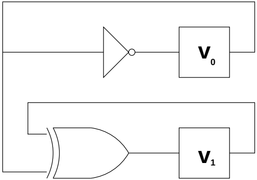
$$\bigvee_{(s,s')\in R} \xi(s, s') = \bigvee_{(s,s')\in R} (\xi(s) \wedge \xi(s'))$$

Note

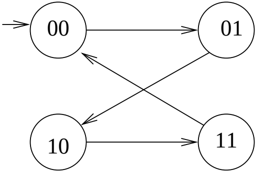
Each formula equivalent to $\xi(R)$ is a representation of R
$\Longrightarrow$ Typically R can be encoded by a much smaller formula than $\bigvee_{(s,s')\in R} \xi(s) \wedge \xi(s')$!

Example: a simple counter

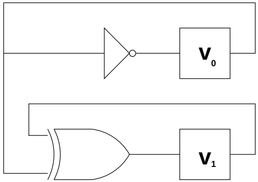
```
MODULE main
 VAR
   v0     : boolean;
   v1     : boolean;
   out    : 0..3;

 ASSIGN
   init(v0)  := 0;
   next(v0)  := !v0;

   init(v1)  := 0;
   next(v1)  := (v0 xor v1);

   out := toint(v0) + 2*toint(v1);
```



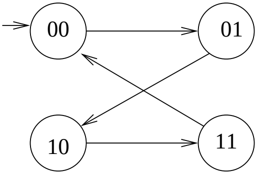
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



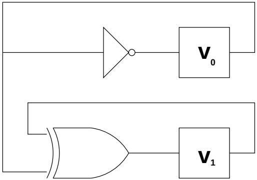
Example: a simple counter [cont.]



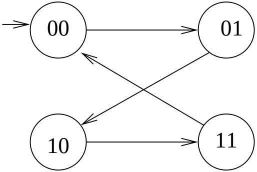
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



Example: a simple counter [cont.]

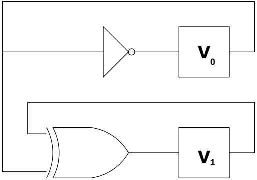


v_1	v_0	v'_1	v'_0
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

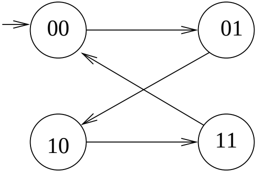


$$\xi(R) \quad = \quad (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \bigoplus v_1)$$

Example: a simple counter [cont.]



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

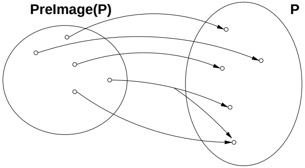


$$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$$

$$\begin{aligned} \bigvee_{(s,s')\in R} \xi(s) \wedge \xi(s') = \; & (\neg v_1 \wedge \neg v_0 \wedge \neg v_1' \wedge v_0') \vee \\ & (\neg v_1 \wedge v_0 \wedge v_1' \wedge \neg v_0') \vee \\ & (v_1 \wedge \neg v_0 \wedge v_1' \wedge v_0') \vee \\ & (v_1 \wedge v_0 \wedge \neg v_1' \wedge \neg v_0') \end{aligned}$$

Pre-Image

- (Backward) pre-image of a set:



Evaluate one-shot all transitions ending in the states of the set

- Set theoretic view:
  $PreImage(P, R) := \{s \mid \text{for some } s' \in P, (s, s') \in R\}$
- Logical view: $\xi(PreImage(P, R)) \ := \ \exists V'.(\xi(P)[V'] \wedge \xi(R)[V, V'])$
- μ over V is s.t $\mu \models \exists V'.(\xi(P)[V'] \wedge \xi(R)[V, V'])$ iff,
  for some μ' over V', we have: $\mu \cup \mu' \models (\xi(P)[V'] \wedge \xi(R)[V, V'])$,
  i.e., $\mu' \models \xi(P)[V']$ and $\mu \cup \mu' \models \xi(R)[V, V'])$
  - Intuition: $\mu \Longleftrightarrow s$, $\mu' \Longleftrightarrow s'$, $\mu \cup \mu' \Longleftrightarrow \langle s, s' \rangle$

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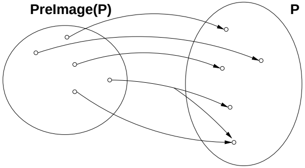

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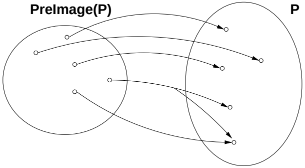

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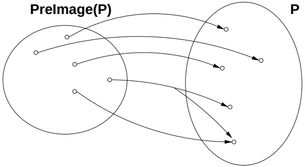

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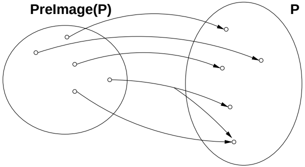

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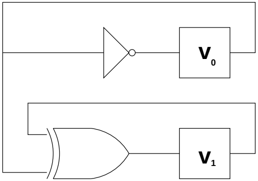
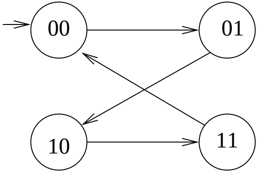
Example: simple counter



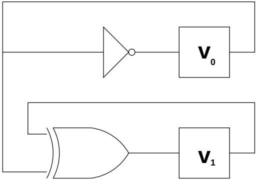
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



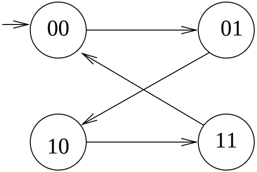
$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$
$\xi(P) := (v_0 \leftrightarrow v_1)$ (i.e., $P = \{00, 11\}$)

$$
\begin{aligned}
&\xi(PreImage(P,R)) &=\\
&\exists V'.(\xi(P)[V'] \wedge \xi(R)[V,V']) &=\\
&\exists v_0' v_1'.((v_0' \leftrightarrow v_1') \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) &=\\
&\underbrace{(\neg v_0 \wedge v_0 \bigoplus v_1)}_{v_0'=\top, v_1'=\top} \vee \underbrace{\bot}_{v_0'=\top, v_1'=\bot} \vee \underbrace{\bot}_{v_0'=\bot, v_1'=\top} \vee \underbrace{(v_0 \wedge \neg(v_0 \bigoplus v_1))}_{v_0'=\bot, v_1'=\bot} &=\\
&v_1 \quad (i.e., \{10, 11\})
\end{aligned}
$$

Example: simple counter



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



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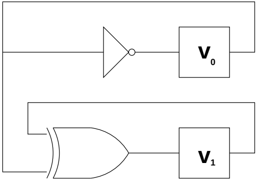
\underbrace{(\neg v_0 \wedge v_0 \bigoplus v_1)}_{v_0'=\top, v_1'=\top} \vee \underbrace{\perp}_{v_0'=\top, v_1'=\perp} \vee \underbrace{\perp}_{v_0'=\perp, v_1'=\top} \vee \underbrace{(v_0 \wedge \neg(v_0 \bigoplus v_1))}_{v_0'=\perp, v_1'=\perp} & = \\
v_1 \quad (i.e., \{10, 11\}) &
\end{array}
$$

Example: simple counter



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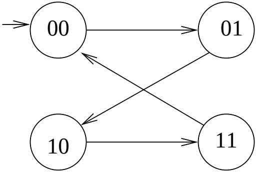

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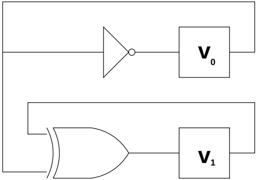
\end{aligned}$$

Example: simple counter



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0	0	0	1
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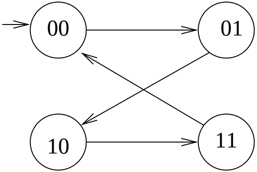

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&\underbrace{(\neg v_0 \land v_0 \bigoplus v_1)}_{v_0'=\top, v_1'=\top} \lor \underbrace{\bot}_{v_0'=\top, v_1'=\bot} \lor \underbrace{\bot}_{v_0'=\bot, v_1'=\top} \lor \underbrace{(v_0 \land \neg(v_0 \bigoplus v_1))}_{v_0'=\bot, v_1'=\bot} &=\\
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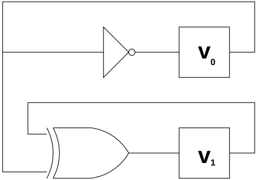
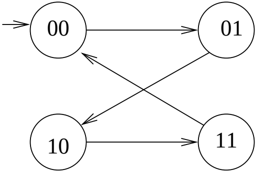


v_1	v_0	v'_1	v'_0
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



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\exists v'_0 v'_1.((v'_0 \leftrightarrow v'_1) \wedge (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \bigoplus v_1)) & = \\
\underbrace{(\neg v_0 \wedge v_0 \bigoplus v_1)}_{v'_0=\top, v'_1=\top} \vee \underbrace{\bot}_{v'_0=\top, v'_1=\bot} \vee \underbrace{\bot}_{v'_0=\bot, v'_1=\top} \vee \underbrace{(v_0 \wedge \neg(v_0 \bigoplus v_1))}_{v'_0=\bot, v'_1=\bot} & = \\
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\end{array}
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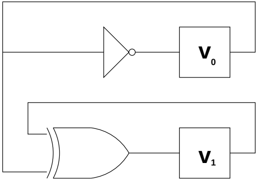

v_1	v_0	v_1'	v_0'
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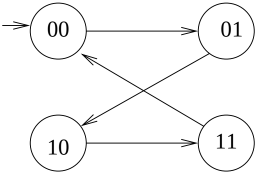

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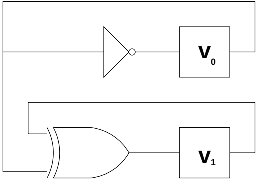

v_1	v_0	v_1'	v_0'
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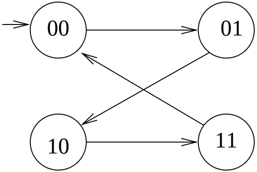

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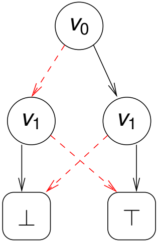
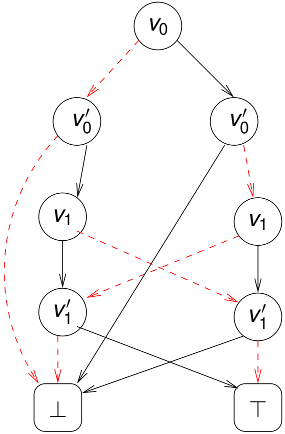
&v_1 \quad (i.e., \{10, 11\})
\end{aligned}
$$

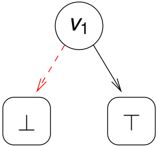
Pre-Image [cont.]



$\xi(P) = v_0 \leftrightarrow v_1$



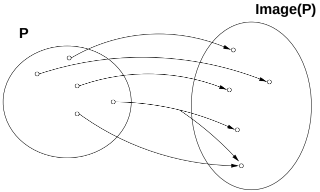
$\xi(R) = (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \bigoplus v_1)$



$\xi(PreImage(P, R)) =$
$\exists V'.((v'_0 \leftrightarrow v'_1) \wedge (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \bigoplus v$
v_1

Forward Image

- Forward image of a set:



Evaluate one-shot all transitions from the states of the set

- Set theoretic view:

$$Image(P, R) := \{s' \mid \text{for some } s \in P, (s, s') \in R\}$$

- Logical Characterization:

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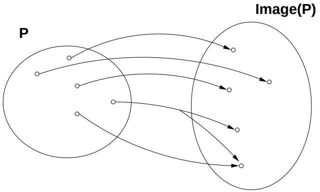

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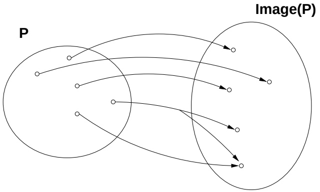

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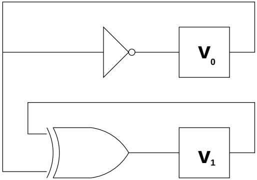
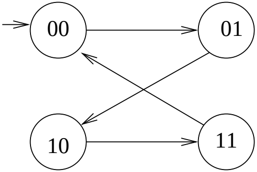
Example: simple counter



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)$
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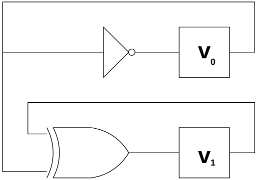

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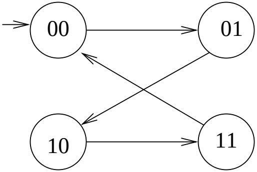

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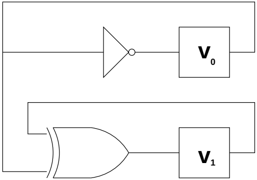

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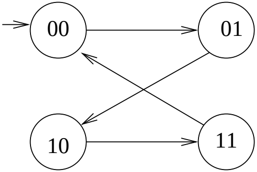

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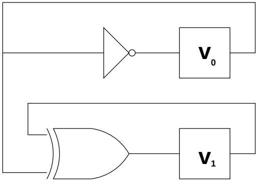

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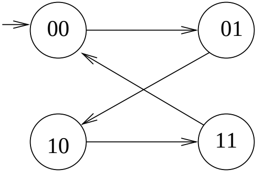

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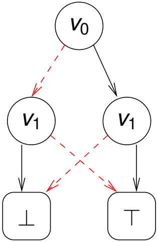
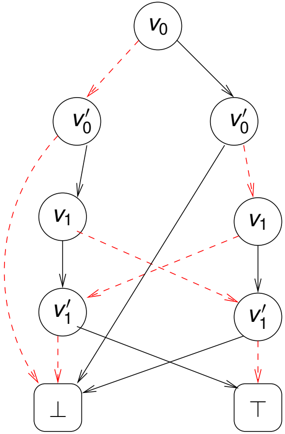
$$\begin{aligned} \xi(Image(P,R)) &= \exists V.(\xi(P)[V] \wedge \xi(R)[V, V']) \\ &= \exists V.((v_0 \leftrightarrow v_1) \wedge (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \bigoplus v_1)) \\ &= \ldots \\ &= \neg v_1' \quad (i.e., \ \{00, 01\}) \end{aligned}$$

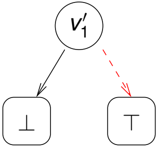
Forward Image [cont.]



$\xi(P) = v_0 \leftrightarrow v_1$



$\xi(R) = (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \bigoplus v_1)$



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Application of the Transition Relation

- Image and PreImage of a set of states S computed by means of quantified Boolean formulae
- The whole set of transitions can be fired (either forward or backward) in one logical operation
- The symbolic computation of PreImage and Image provide the primitives for symbolic search of the state space of FSM's

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Outline

Symbolic CTL model checking

- Problem: $M \models \varphi$?,
  - $M = \langle S, I, R, L, AP \rangle$ being a Kripke structure and
  - φ being a CTL formula
- Solution: represent I and R as Boolean formulas $\xi(I), \xi(R)$ and encode them as OBDDs, and
- Apply fix-point CTL M.C. algorithm:
  - using OBDDs to represent sets of states and relations,
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Assume φ written in terms of $\neg$, $\wedge$, **EX**, **EU**, **EG**

- A general M.C. algorithm (fix-point):
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  Subformulas $Sub(\varphi)$ of φ are checked bottom-up
- $\xi([\varphi_i])$ computed directly, without computing $[\varphi_i]$ explicitly!!!
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Symbolic Denotation of a CTL formula φ: $\xi([\varphi])$

$\xi([\varphi]) := \xi(\{s \in S : M, s \models \varphi\})$

Notation: if X_1 and X_2 are OBDDs and *op* is a Boolean operator, we write "X_1 op X_2" for "reduce(apply(op,X_1,X_2))"

Symbolic Denotation of a CTL formula φ: $\xi([\varphi])$

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$$
\begin{array}{lcl}
\xi([false]) & = & \bot \\
\xi([true]) & = & \top \\
\xi([p]) & = & p \\
\xi([\neg\varphi_1]) & = & \neg\xi([\varphi_1] \\
\xi([\varphi_1 \wedge \varphi_2]) & = & \xi([\varphi_1]) \wedge \xi([\varphi_2]) \\
\xi([\mathbf{EX}\varphi]) & = & \exists V'.(\ \xi([\varphi])[V'] \wedge \xi(R)[V, V']) \\
\xi([\mathbf{EG}\beta]) & = & \nu Z.(\ \xi([\beta]) \wedge \xi([\mathbf{EX}Z])\ ) \\
\xi([\mathbf{E}(\beta_1 \mathbf{U} \beta_2)]) & = & \mu Z.(\ \xi([\beta_2]) \vee (\xi([\beta_1]) \wedge \xi([\mathbf{EX}Z])\ )
\end{array}
$$

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General M.C. Procedure

OBDD Check(CTL_formula β) {
  if (*In_OBDD_Hash*(β))
    return *OBDD_Get_From_Hash*(β);
  case β **of**
  true: **return** *obdd_true*;
  false: **return** *obdd_false*;
  $\neg\beta_1$: **return** $\neg$ Check(β_1);
  $\beta_1 \wedge \beta_2$: **return** (Check(β_1) $\wedge$ Check(β_2));
  EXβ_1: **return** PreImage(Check(β_1));
  EGβ_1: **return** Check_EG(Check(β_1));
  E(β_1**U**β_2): **return** Check_EU(Check(β_1),Check(β_2));
}

PreImage

OBDD PreImage(**OBDD** X) $\{$
    return $\exists V'.(\ X[V'] \wedge \xi(R)[V, V']);$
$\}$

Check_EG

OBDD Check_EG(**OBDD** X) {

$\quad Y' := X;\ j := 1;$

$\quad$**repeat**

$\quad\quad Y := Y';\ j := j + 1;$

$\quad\quad Y' := Y \wedge PreImage(Y));$

$\quad$**until** $(Y' \leftrightarrow Y);$

return $Y;$

}

Check_EU

```
OBDD Check_EU(OBDD X_1,X_2) {
    Y' := X_2; j := 1;
    repeat
        Y := Y'; j := j + 1;
        Y' := Y ∨ (X_1 ∧ PreImage(Y));
    until (Y' ↔ Y);
return Y;
}
```

CTL Symbolic Model Checking – Summary

- Based on fixed point CTL M.C. algorithms
- Kripke structure encoded as Boolean formulas (OBDDs)
- All operations handled as (quantified) Boolean operations
- Avoids building the state graph explicitly
- reduces dramatically the state explosion problem
  $\Longrightarrow$ problems of up to 10^{120} states handled!!

Outline

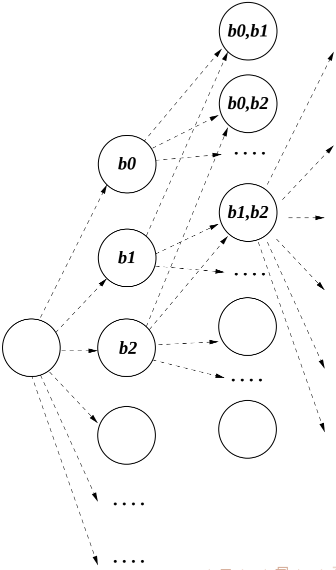
A simple example

```
MODULE main
VAR
  b0 : boolean;
  b1 : boolean;
  ...
ASSIGN
  init(b0) := 0;
  next(b0) := case
    b0  : 1;
    !b0 : {0,1};
  esac;
  init(b1) := 0;
  next(b1) := case
    b1  : 1;
    !b1 : {0,1};
  esac;
```

A simple example [cont.]

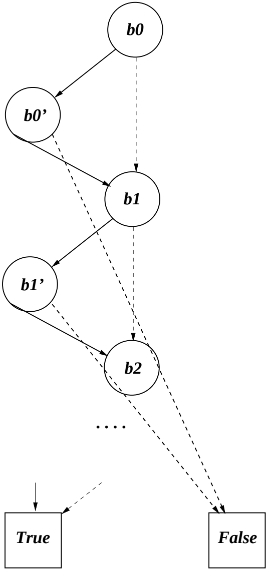
- N Boolean variables $b0, b1, \ldots$
- Initially, all variables set to 0
- Each variable can pass from 0 to 1, but not vice-versa
- 2^N states, all reachable
- (Simplified) model of a student career behaviour.

A simple example: FSM



(transitive trans. omitted)
2^N STATES
$O(2^N)$ TRANSITIONS

A simple example: $OBDD(\xi(R))$



$2N + 2$ NODES

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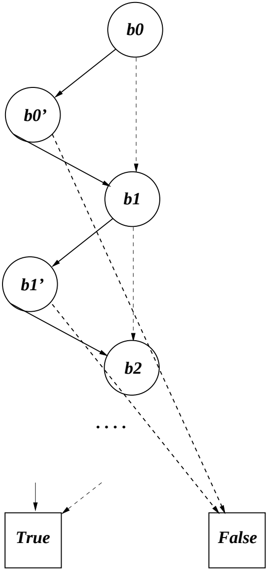

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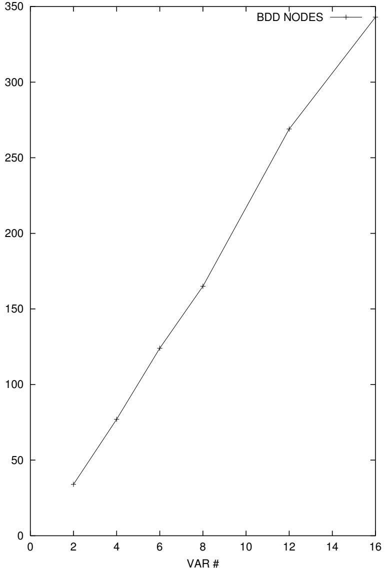
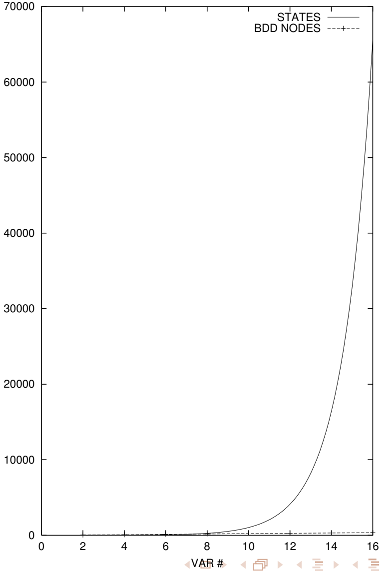
A simple example: states vs. OBDD nodes [NuSMV.2]

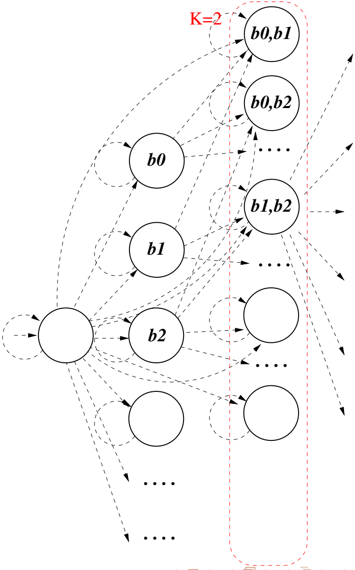




A simple example: reaching K bits true

- Property $\mathbf{EF}(b0 + b1 + ... + b(N-1) \geq K)$ ($K \leq N$)
  (it may be reached a state in which K bits are true)
- E.g.: “it is reachable a state where K exams are passed”

A simple example: FSM



$\binom{N}{K} + \binom{N}{K+1} + \ldots + \binom{N}{N}$

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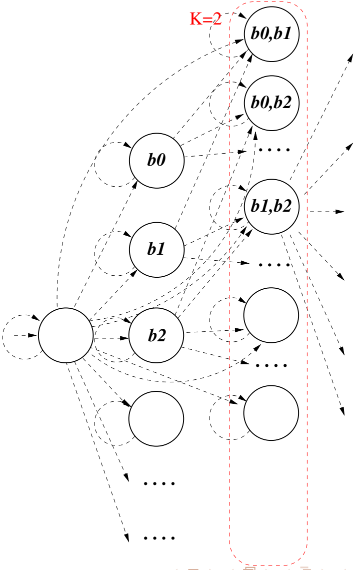

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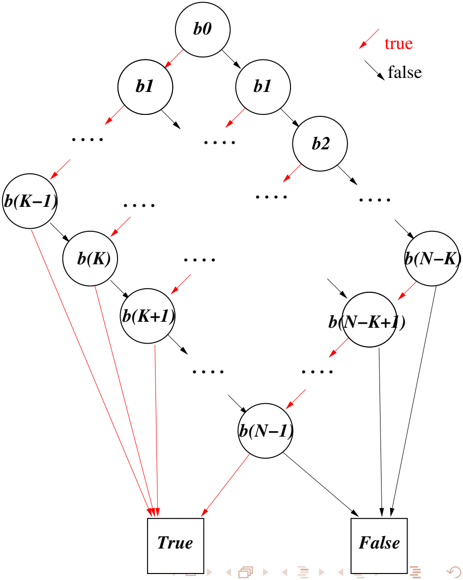

$(N-K+1)\cdot K+2$ NODES

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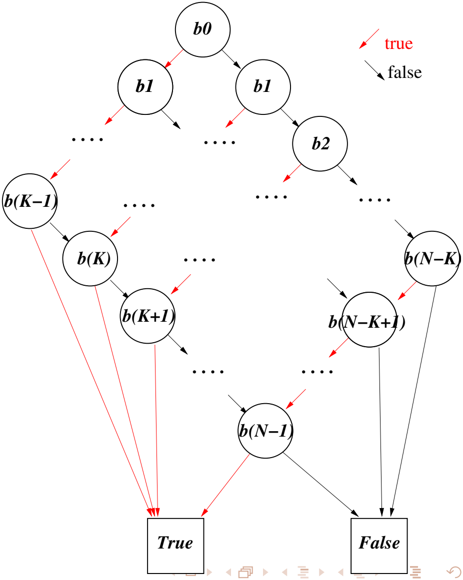

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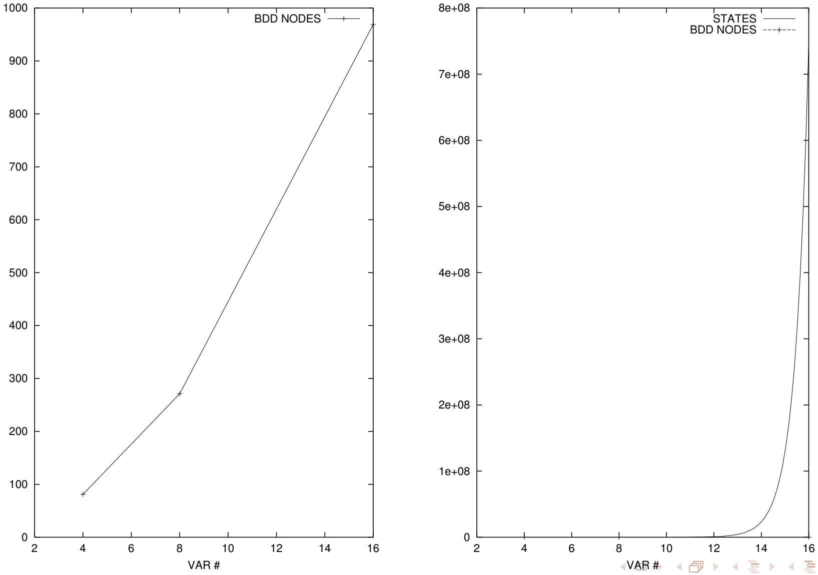

Outline

Back to OBDDs: Efficiency Issues

OBDD packages provides efficient basis for Symbolic Model Checking:

- unique representant for each OBDD via hash tables
- complement-based representation of negation
- memoizing partial computations
- garbage collection mechanisms
- variable reordering algorithms, dynamic activation
- specialized algorithms for relational products
  for Image/PreImage computations

Symbolic Model Checkers

- Most hardware design companies have their own Symbolic Model Checker(s)
  - Intel, IBM, Motorola, Siemens, ST, Cadence, ...
  - very advanced tools
  - proprietary technolgy!
- On the academic side
  - CMU SMV [McMillan]
  - VIS [Berkeley, Colorado]
  - Bwolen Yang's SMV [CMU]
  - NuSMV [CMU, IRST, UNITN, UNIGE]
  - ...

Outline

Ex: OBDDs

Let $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.\varphi$. Using the variable ordering "A, B, C", draw the OBDD corresponding to the formulas φ and φ'.

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[Solution:

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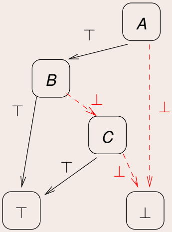
[Solution:



]

Ex: OBDDs (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

Ex: OBDDs (cont.)

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[Solution:

]

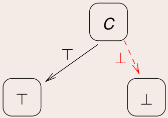
Ex: OBDDs (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

[Solution:

$$
\begin{array}{lllllll}
\varphi' & \stackrel{\text{def}}{=} & \exists A.\forall B.\varphi & & & & \\
 & = & \forall B.(A \wedge (B \vee C)))[A := \top] & & & \vee & (\forall B.(A \wedge (B \vee C)))[A := \bot] \\
 & = & \forall B.(B \vee C)) & & & \vee & \forall B.\bot \\
 & = & ((B \vee C)[B := \top] & \wedge & (B \vee C)[B := \bot]) & \vee & \bot \\
 & = & (\top & \wedge & C) & & \\
 & = & C & & & &
\end{array}
$$

which corresponds to the following OBDD:



]

Ex: Symbolic CTL Model Checking

Given the following finite state machine expressed in NuSMV input language:

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MODULE main
VAR v1 : boolean; v2 : boolean;
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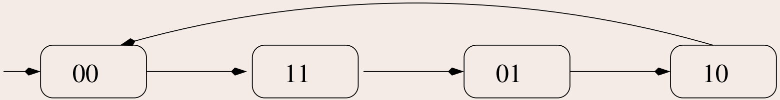

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Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula representing symbolically $\mathbf{EX}P$. [The formula must be computed symbolically, not simply inferred from the graph of the previous question!]

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  [Solution:

$$
\begin{array}{rcl}
\mathbf{EX}(P) & = & \exists v_1', v_2'.(T(v_1, v_2, v_1', v_2') \wedge P(v_1', v_2')) \\
 & = & \exists v_1', v_2'.((v_1' \leftrightarrow \neg v_1) \wedge (v_2' \leftrightarrow (v_1 \leftrightarrow v_2)) \wedge \underbrace{(v_1' \wedge v_2')}_{\Longrightarrow v_1'=\top, v_2'=\top}) \\
 & = & \overbrace{(\neg v_1 \wedge \neg v_2)}^{v_1'=\top, v_2'=\top} \vee \bot \vee \bot \vee \bot \\
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\end{array}
$$

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write:

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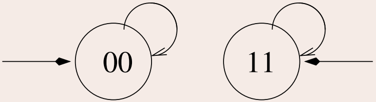
  [Solution: 

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Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula $R^1(v_1', v_2')$ representing the set of states which can be reached after exactly 1 step.
  NOTE: this must be computed symbolically, not simply deduced from the graph of question b).

Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula $R^1(v_1', v_2')$ representing the set of states which can be reached after exactly 1 step.
  NOTE: this must be computed symbolically, not simply deduced from the graph of question b).

  [Solution:

$$\begin{array}{rcl}
R^1(v_1', v_2') & = & \exists v_1, v_2.(I(v_1, v_2) \wedge T(v_1, v_2, v_1', v_2')) \\
& = & \exists v_1, v_2.((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1')) \\
& = & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1'))[v_1 = \bot, v_2 = \bot] \vee \\
& & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1'))[v_1 = \bot, v_2 = \top] \vee \\
& & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1'))[v_1 = \top, v_2 = \bot] \vee \\
& & ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v_2') \wedge (v_2 \leftrightarrow v_1'))[v_1 = \top, v_2 = \top] \\
& = & (\neg v_1' \wedge \neg v_2') \vee \bot \vee \bot \vee (v_1' \wedge v_2') \\
& = & (\neg v_1' \wedge \neg v_2') \vee (v_1' \wedge v_2') \\
& = & (v_1' \leftrightarrow v_2')
\end{array}$$

  .]