

Fundamentals of Artificial Intelligence

Chapter 08: **First-Order Logic**

Roberto Sebastiani

DISI, Università di Trento, Italy – `roberto.sebastiani@unitn.it`

`https://disi.unitn.it/rseba/DIDATTICA/fai\_2025/`

Teaching assistant:

Paolo Morettin, `paolo.morettin@unitn.it`, `https://paolomorettin.github.io/`

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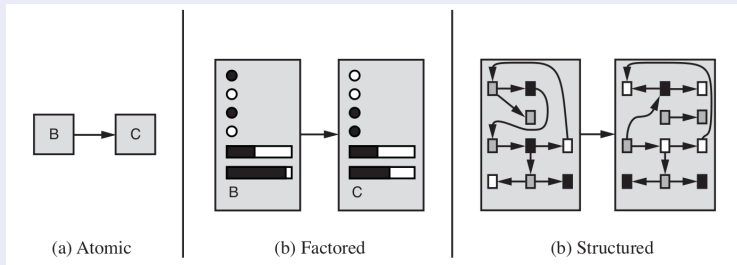
- 1 Generalities
- 2 Syntax and Semantics of FOL
 - Syntax
 - Semantics
 - Satisfiability, Validity, Entailment
- 3 Using FOL
 - FOL Agents
 - Example: The Wumpus World

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Recall: State Representations [Ch. 02]

Representations of states and transitions

- Three ways to represent states and transitions between them:
 - **atomic**: a state is a black box with no internal structure
 - **factored**: a state consists of a vector of attribute values
 - **structured**: a state includes objects, each of which may have attributes of its own as well as relationships to other objects
- increasing expressive power and computational complexity
- reality represented at different levels of abstraction



Pros of Propositional Logic

- PL language **is formal**
 - non-ambiguous semantics
 - unlike natural language, which is intrinsically ambiguous (ex “key”)
- PL **is declarative**
 - knowledge and inference are separate
 - inference is entirely domain independent
- PL **allows for partial/disjunctive/negated information**
 - unlike, e.g., data bases
- PL **is compositional**
 - the meaning of $(A \wedge B) \rightarrow C$ derives from the meaning of A,B,C
- The meaning of PL sentence is **context independent**
 - unlike with natural language, where meaning depends on context

Cons of Propositional Logic

- Is “**Atomic**”: based on **atomic events** which cannot be decomposed
- Assumes the world contains **facts** in the world that are either true or false, nothing else
 - ex: **Man_Socrates**, **Man_Plato**, **Man_Aristotle**, ... distinct atoms

⇒ PL has **has very limited expressive power**

- unlike natural language
- cannot concisely describe an environment with many objects
- e.g., cannot say “**pits cause breezes in adjacent squares**”
(need writing one sentence for each square)

Logics

- A logic is a triple $\langle \mathcal{L}, \mathcal{S}, \mathcal{R} \rangle$ where
 - $\mathcal{L}$, the logic's **language**: a class of sentences described by a formal grammar
 - $\mathcal{S}$, the logic's **semantics**: a formal specification of how to assign meaning in the “real world” to the elements of $\mathcal{L}$
 - $\mathcal{R}$, the logic's **inference system**: is a set of formal derivation rules over $\mathcal{L}$
- There are several logics:
 - **propositional** logic (PL)
 - **first-order** logic (FOL)
 - **modal** logics (MLs)
 - **description** logics (DLs)
 - **temporal** logics (TLs)
 - (fuzzy logics, probabilistic logics, ...)
 - ...

First-Order Logic (FOL)

- Is **structured**: a world/state includes objects, each of which may have attributes of its own as well as relationships to other objects
- Assumes the world contains:
 - **Objects**:
e.g., people, houses, numbers, theories, Jim Morrison, colors, basketball games, wars, centuries, ...
 - **Relations**:
e.g., red, round, bogus, prime, tall ...,
brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
 - **Functions**:
e.g., father of, best friend, one more than, end of, ...
- Allows to **quantify** on objects
 - ex: “All man are equal”, “some persons are left-handed”, ...

Outline

- 1 Generalities
- 2 **Syntax and Semantics of FOL**
 - Syntax
 - Semantics
 - Satisfiability, Validity, Entailment
- 3 Using FOL
 - FOL Agents
 - Example: The Wumpus World

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Syntax of FOL: Basic Elements

- **Constant symbols:** KingJohn, 2, UniversityofTrento,...
- **Predicate symbols:** Man(.), Brother(.,.), ($. > .$), AllDifferent(...),...
 - may have different arities (1,2,3,...)
 - may be **prefix** (e.g. Brother(.,.)) or **infix** (e.g. ($. > .$))
- **Function symbols:** Sqrt, LeftLeg, MotherOf
 - may have different arities (1,2,3,...)
 - may be **prefix** (e.g. Sqrt(.)) or **infix** (e.g. ($. + .$))
- **Variable symbols:** x, y, a, b, ...
- **Propositional Connectives:** $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$
- **Equality:** “=” (also “ $\neq$ ” s.t. “ $a \neq b$ ” shortcut for “ $\neg(a = b)$ ”)
- **Quantifiers:** “ $\forall$ ” (“forall”), “ $\exists$ ” (“exists”, aka “for some”)
- **Punctuation Symbols:** “,”, “(”, “)”

- Constants symbols are 0-ary function symbols
- Propositions are 0-ary predicates $\implies$ PL subcase of FOL
- **Signature:** the set of predicate, function & constant symbols

FOL: Syntax

- **Terms:**

- constant or variable or $function(term_1, \dots, term_n)$
- ex: KingJohn, x, LeftLeg(Richard), $(z * \log(2))$
- denote objects in the real world (aka domain)

- **Atomic sentences (aka atomic formulas):**

- $\top, \perp$
- $proposition$ or $predicate(term_1, \dots, term_n)$ or $term_1 = term_2$
- $(Length(LeftLeg(Richard)) > Length(LeftLeg(KingJohn)))$
- denote facts

- **Non-atomic sentences/formulas:**

- $\neg \alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta, \alpha \leftrightarrow \beta, \alpha \oplus \beta,$
 $\forall x. \alpha, \exists x. \alpha$ s.t. x (typically) occurs in α
- Ex: $\forall y. (Italian(y) \rightarrow President(Mattarella, y))$
 $\exists x \forall y. President(x, y) \rightarrow \forall y \exists x. President(x, y)$
 $\forall x. (P(x) \wedge Q(x)) \leftrightarrow ((\forall x. P(x)) \wedge (\forall x. Q(x)))$
 $\forall x. (((x \geq 0) \wedge (x \leq \pi)) \rightarrow (sin(x) \geq 0))$
- denote (complex) facts

FOL: Ground and Closed Formulas

- A term/formula is **ground** iff no variable occurs in it (ex: $2 \geq 1$)
- A formula is **closed** iff all variables occurring in it (if any) are quantified (ex: $\forall x \exists y. (x > y)$)

$\Rightarrow$ Ground formulas are closed, but not vice versa.

FOL: Syntax (BNF)

$\langle \text{Sentence} \rangle$	$::=$	$\langle \text{AtomicSentence} \rangle \mid \langle \text{ComplexSentence} \rangle$
$\langle \text{AtomicSentence} \rangle$	$::=$	$\top \mid \perp \mid$ $\langle \text{PredicateSymbol} \rangle(\langle \text{Term} \rangle, \dots) \mid$ $\langle \text{Term} \rangle = \langle \text{Term} \rangle$
$\langle \text{ComplexSentence} \rangle$	$::=$	$\neg \langle \text{Sentence} \rangle \mid$ $\langle \text{Sentence} \rangle \langle \text{Connective} \rangle \langle \text{Sentence} \rangle \mid$ $\langle \text{Quantifier} \rangle \langle \text{Sentence} \rangle$
$\langle \text{Term} \rangle$	$::=$	$\langle \text{ConstantSymbol} \rangle \mid \langle \text{Variable} \rangle \mid$ $\langle \text{FunctionSymbol} \rangle(\langle \text{Term} \rangle, \dots)$
$\langle \text{Connective} \rangle$	$::=$	$\wedge \mid \vee \mid \rightarrow \mid \leftarrow \mid \leftrightarrow \mid \oplus$
$\langle \text{Quantifier} \rangle$	$::=$	$\forall \langle \text{Variable} \rangle. \mid \exists \langle \text{Variable} \rangle.$
$\langle \text{Variable} \rangle$	$::=$	$a \mid b \mid \dots \mid x \mid y \mid \dots$
$\langle \text{ConstantSymbol} \rangle$	$::=$	$A \mid B \mid \dots \mid \textit{John} \mid 0 \mid 1 \mid \dots \mid \pi \mid \dots$
$\langle \text{FunctionSymbol} \rangle$	$::=$	$F \mid G \mid \dots \mid \textit{Cos} \mid \textit{FatherOf} \mid + \mid \dots$
$\langle \text{PredicateSymbol} \rangle$	$::=$	$P \mid Q \mid \dots \mid \textit{Red} \mid \textit{Brother} \mid > \mid \dots$

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- **Positive/negative occurrences**

- φ occurs positively in φ ;
- if $\neg\varphi_1$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ
- if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ ,
then φ_1 and φ_2 occur positively [negatively] in φ ;
- if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ ;
- if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ ,
then φ_1 and φ_2 occur positively and negatively in φ ;
- if $\forall x.\varphi_1$ or $\exists x.\varphi_1$ occurs positively [negatively] in φ ,
then φ_1 occurs positively [negatively] in φ

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Truth in FOL: Intuitions

- Sentences are true with respect to a **model**
 - containing a **domain** and an **interpretation**
- The **domain** contains ≥ 1 objects (**domain elements**) and relations and functions over them
- An **interpretation** specifies referents for
 - **variables** $\rightarrow$ objects
 - **constant symbols** $\rightarrow$ objects
 - **predicate symbols** $\rightarrow$ relations
 - **function symbols** $\rightarrow$ functional relations
- An atomic sentence $P(t_1, \dots, t_n)$ is true in an interpretation iff the objects referred to by $t_1, \dots, t_n$ are in the relation referred to by P

FOL: Semantics

FOL Models (aka possible worlds)

- A model $\mathcal{M}$ is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ ($\langle \text{domain}, \text{interpretation} \rangle$)
 - Domain $\mathcal{D}$: a **non-empty** set of objects (aka **domain elements**)
 - Interpretation $\mathcal{I}$: a (non-injective) map on elements of the signature
 - **constant symbols** $\mapsto$ **domain elements**:
a constant symbol C is mapped into a particular object $[C]^{\mathcal{I}}$ in $\mathcal{D}$
 - **predicate symbols** $\mapsto$ **domain relations**:
a k -ary predicate $P(\dots)$ is mapped into a subset $[P]^{\mathcal{I}}$ of $\mathcal{D}^k$
(i.e., the set of object tuples satisfying the predicate in this world)
 - **functions symbols** $\mapsto$ **domain functions**:
a k -ary function f is mapped into a domain function $[f]^{\mathcal{I}} : \mathcal{D}^k \mapsto \mathcal{D}$ ($[f]^{\mathcal{I}}$ must be total)
- (we denote by $[\cdot]^{\mathcal{I}}$ the result of the interpretation $\mathcal{I}$)

An **Interpretation** $\mathcal{I}$ is extended to assign domain values to variables, domain values to terms and truth values to formulas.

FOL: Semantics [cont.]

Interpretation of terms

$\mathcal{I}$ maps terms into domain elements

- Variables are assigned domain values
 - **variables** $\mapsto$ **domain elements**:
a variable x is mapped into a particular object $[x]^{\mathcal{I}}$ in $\mathcal{D}$
- A term $f(t_1, \dots, t_k)$ is mapped by $\mathcal{I}$ into the value $[f(t_1, \dots, t_k)]^{\mathcal{I}}$ returned by applying the domain function $[f]^{\mathcal{I}}$, into which f is mapped, to the values $[t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}}$ obtained by applying recursively $\mathcal{I}$ to the terms $t_1, \dots, t_k$:
 - $[f(t_1, \dots, t_k)]^{\mathcal{I}} = [f]^{\mathcal{I}}([t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}})$
 - Ex: if “**Me**, **Mother**, **Father**” are interpreted as usual, then “**Mother(Father(Me))**” is interpreted as my (paternal) grandmother
 - Ex: if “**+**, **-**, **.**, **0**, **1**, **2**, **3**, **4**” are interpreted as usual, then “ **$(3 - 1) \cdot (0 + 2)$** ” is interpreted as **4**

FOL: Semantics [cont.]

Interpretation of formulas

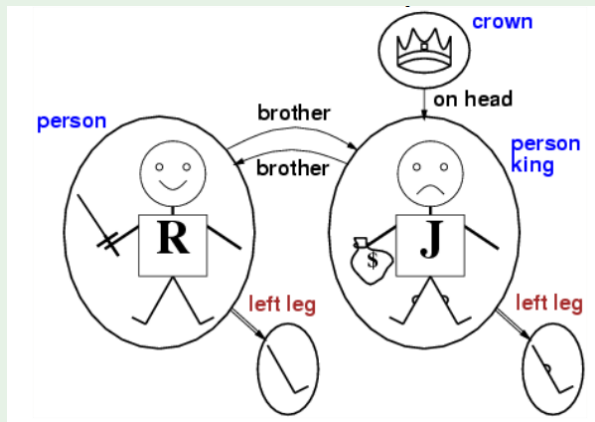
$\mathcal{I}$ maps formulas into truth values

- An atomic formula $P(t_1, \dots, t_k)$ is true in $\mathcal{I}$ iff the objects into which the terms $t_1, \dots, t_k$ are mapped by $\mathcal{I}$ comply to the relation into which P is mapped
 - $[P(t_1, \dots, t_k)]^{\mathcal{I}}$ is true iff $\langle [t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}} \rangle \in [P]^{\mathcal{I}}$
 - Ex: if “Me, Mother, Father, Married” are interpreted as tradition, then “Married(Mother(Me), Father(Me))” is interpreted as true
 - Ex: if “+, −, >, 0, 1, 2, 3, 4” are interpreted as usual, then “(4 − 0) > (1 + 2)” is interpreted as true
- An atomic formula $t_1 = t_2$ is true in $\mathcal{I}$ iff the terms t_1, t_2 are mapped by $\mathcal{I}$ into the same domain element
 - $[t_1 = t_2]^{\mathcal{I}}$ is true iff $[t_1]^{\mathcal{I}}$ same as $[t_2]^{\mathcal{I}}$
 - Ex: if “Mother” is interpreted as usual, Richard, John are brothers, then “Mother(Richard)=Mother(John)” is interpreted as true
 - Ex: if “+, −, 0, 1, 2, 3, 4” are interpreted as usual, then “(4 − 1) = (1 + 2)” is interpreted as true
- $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$ interpreted by $\mathcal{I}$ as in PL

Models for FOL: Example

Richard Lionheart and John Lackland

- $\mathcal{D}$: domain at right
- $\mathcal{I}$: s.t.
 - $[\text{Richard}]^{\mathcal{I}}$: Richard the Lionheart
 - $[\text{John}]^{\mathcal{I}}$: evil King John
 - $[\text{Brother}]^{\mathcal{I}}$: brotherhood
- $[\text{Brother}(\text{Richard}, \text{John})]^{\mathcal{I}}$ is true
- $[\text{LeftLeg}]^{\mathcal{I}}$ maps any individual to his left leg
- ...



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Models for FOL: Remark

- $[f]^{\mathcal{I}}$ total: must provide an output for every input
- e.g.: $[LeftLeg(crown)]^{\mathcal{I}}$?
- possible solution: assume “null” object ($[LeftLeg(crown) = null]^{\mathcal{I}}$)
(other solution, sorts, not considered here)

Universal Quantification

- $\forall x. \alpha(x, \dots)$ (x variable, typically occurs in x)
 - ex: $\forall x. (\text{King}(x) \rightarrow \text{Person}(x))$ (“all kings are persons”)
- $\forall x. \alpha(x, \dots)$ true in $\mathcal{M}$ iff
 α is true in $\mathcal{M}$ for every possible domain value x is mapped to
- Roughly speaking, can be seen as **a conjunction over all (typically infinite) possible instantiations of x in α**

$(\text{King}(\text{John}))$	$\rightarrow \text{Person}(\text{John})$	) $\wedge$
$(\text{King}(\text{Richard}))$	$\rightarrow \text{Person}(\text{Richard})$	) $\wedge$
$(\text{King}(\text{crown}))$	$\rightarrow \text{Person}(\text{crown})$	) $\wedge$
$(\text{King}(\text{LeftLeg}(\text{John})))$	$\rightarrow \text{Person}(\text{LeftLeg}(\text{John}))$	) $\wedge$
$(\text{King}(\text{LeftLeg}(\text{LeftLeg}(\text{John}))))$	$\rightarrow \text{Person}(\text{LeftLeg}(\text{LeftLeg}(\text{John})))$	) $\wedge$
...	...	

Universal Quantification [cont.]

- One may want to restrict the domain of universal quantification to elements of some kind P
 - ex “forall kings ...”, “forall integer numbers...”
- Idea: use an implication, with restrictive predicate as implicant:
 $\forall x.(P(x) \rightarrow \alpha(x, \dots))$
 - ex “ $\forall x.(King(x) \rightarrow \dots)$ ”, “ $\forall x.(Integer(x) \rightarrow \dots)$ ”,
- Beware of typical mistake: do not use “ $\wedge$ ” instead of “ $\rightarrow$ ”
 - ex: “ $\forall x.(King(x) \wedge Person(x))$ ” means “everything/one is a King and is a Person”
 - ex: “ $\forall x.(King(x) \rightarrow Person(x))$ ” means “everything/one who is a King is a Person” (i.e. “every king is a person”)
- “ $\forall$ ” distributes with “ $\wedge$ ”, but not with “ $\vee$ ”
 - $\forall x.(P(x) \wedge Q(x))$ equivalent to $(\forall x.P(x)) \wedge (\forall x.Q(x))$
 - “Everybody is a king and is a person” same as “Everybody is a king and everybody is a person”
 - $\forall x.(P(x) \vee Q(x))$ not equivalent to $(\forall x.P(x)) \vee (\forall x.Q(x))$:
 - “Everybody is a king or is a peasant” much weaker than “Everybody is a king or everybody is a peasant”
 $(\forall x.P(x)) \vee (\forall x.Q(x)) \models \forall x.(P(x) \vee Q(x))$,
 $\forall x.(P(x) \vee Q(x)) \not\models (\forall x.P(x)) \vee (\forall x.Q(x))$

Existential Quantification

- $\exists x.\alpha(x, \dots)$ (x variable, typically occurs in x)
 - ex: $\exists x.(King(x) \wedge Evil(x))$ (“there is an evil king”)
 - pronounced “exists x s.t. ...” or “for some x ...”
- $\exists x.\alpha(x, \dots)$ true in $\mathcal{M}$ iff
 α is true in $\mathcal{M}$ for some possible domain value x is mapped to
- Roughly speaking, can be seen as a disjunction over all (typically infinite) possible instantiations of x in α

$(King(Richard))$	$\wedge Evil(Richard)$	$)\vee$
$(King(John))$	$\wedge Evil(John)$	$)\vee$
$(King(crown))$	$\wedge Evil(crown)$	$)\vee$
$(King(LeftLeg(John)))$	$\wedge Evil(LeftLeg(John))$	$)\vee$
$(King(LeftLeg(LeftLeg(John))))$	$\wedge Evil(LeftLeg(LeftLeg(John)))$	$)\vee$
...	...	

Existential Quantification [cont.]

- One may want to restrict the domain of existential quantification to elements of some kind P
 - ex “exists a king s.t. ...”, “for some integer numbers...”
- Idea: **use a conjunction with restrictive predicate:**
 $\exists x.(P(x) \wedge \alpha(x, \dots))$
 - ex “ $\exists x.(King(x) \wedge \dots)$ ”, “ $\exists x.(Integer(x) \wedge \dots)$ ”,
- Beware of typical mistake: **do not use “ $\rightarrow$ ” instead of “ $\wedge$ ”**
- ex: “ $\exists x.(King(x) \rightarrow Evil(x))$ ” means “Someone is not a king or is evil”
 - ex: “ $\exists x.(King(x) \wedge Evil(x))$ ” means “Someone is king and is evil”
(i.e., “Some king is evil”)
- “ $\exists$ ” **distributes with “ $\vee$ ”, but not with “ $\wedge$ ”**
- $\exists x.(P(x) \vee Q(x))$ equivalent to $(\exists x.P(x)) \vee (\exists x.Q(x))$
 - “Somebody is a king or is a knight” same as
“Somebody is a king or somebody is a knight”
 - $\exists x.(P(x) \wedge Q(x))$ not equivalent to $(\exists x.P(x)) \wedge (\exists x.Q(x))$
 - “Somebody is a king and is evil” much stronger than
“Somebody is a king and somebody is evil”
 $\exists x.(P(x) \wedge Q(x)) \models (\exists x.P(x)) \wedge (\exists x.Q(x))$
 $(\exists x.P(x)) \wedge (\exists x.Q(x)) \not\models \exists x.(P(x) \wedge Q(x))$

Examples

- Brothers are siblings
 - $\forall x, y. (Brothers(x, y) \rightarrow Siblings(x, y))$
- “Siblings” is symmetric
 - $\forall x, y. (Siblings(x, y) \leftrightarrow Siblings(y, x))$
- One’s mother is one’s female parent
 - $\forall x, y. (Mother(x, y) \leftrightarrow (Female(x) \wedge Parent(x, y)))$
- A first cousin is a child of a parent’s sibling
 - $\forall x_1, x_2. (FirstCousin(x_1, x_2) \leftrightarrow$
 $\exists p_1, p_2. (Siblings(p_1, p_2) \wedge Parent(p_1, x_1) \wedge Parent(p_2, x_2)))$
- Dogs are mammals
 - $\forall x. (Dog(x) \rightarrow Mammal(x))$

Equality

- Equality is a special predicate: $t_1 = t_2$ is true under a given interpretation if and only if t_1 and t_2 refer to the same object
 - Ex: $1 = 2$ and $x * x = x$ are satisfiable (!)
 - Ex: $2 = 2$ is valid
- Ex: definition of *Sibling* in terms of *Parent*
$$\forall x, y. (Siblings(x, y) \leftrightarrow [\neg(x = y) \wedge \exists p_1, p_2. (\neg(p_1 = p_2) \wedge$$

$$Parent(p_1, x) \wedge Parent(p_2, x) \wedge Parent(p_1, y) \wedge Parent(p_2, y))])$$

Example

- No one is his/her own sibling
 - $\forall x. \neg \text{Siblings}(x, x)$
- Sisters are female, brothers are male
 - $\forall x, y. ((\text{Sisters}(x, y) \rightarrow (\text{Female}(x) \wedge \text{Female}(y))) \wedge (\text{Brothers}(x, y) \rightarrow (\text{Male}(x) \wedge \text{Male}(y))))$
- Every married person has a spouse
 - $\forall x. ((\text{Person}(x) \wedge \text{Married}(x)) \rightarrow \exists y. \text{Spouse}(x, y))$
- Married people have spouses
 - $\forall x. ((\text{Person}(x) \wedge \text{Married}(x)) \rightarrow \exists y. \text{Spouse}(x, y))$
- Only married people have spouses
 - $\forall x, y. ((\text{Person}(x) \wedge \text{Person}(y) \wedge \text{Spouse}(x, y)) \rightarrow (\text{Married}(x) \wedge \text{Married}(y)))$
- People cannot be married to their siblings
 - $\forall x, y. (\text{Spouse}(x, y) \rightarrow \neg \text{Siblings}(x, y))$

Example (cont.)

- Not everybody has a spouse
 - $\neg \forall x. (Person(x) \rightarrow \exists y. Spouse(x, y))$ or
 - $\exists x. (Person(x) \wedge \neg \exists y. Spouse(x, y))$
- Everybody has a mother
 - $\forall x. (Person(x) \rightarrow \exists y. Mother(y, x))$
- Everybody has a mother and only one
 - $\forall x. Person(x) \rightarrow (\exists y. Mother(y, x) \wedge \neg \exists z. (\neg(y = z) \wedge Mother(z, x)))$

Properties of Quantifiers

Notation variants: $\forall x(\forall y.\alpha) \iff \forall x\forall y.\alpha \iff \forall x, y.\alpha \iff \forall xy.\alpha$

(same with $\exists$)

- if x does not occur in φ , $\forall x.\varphi$ equivalent to $\exists x.\varphi$ equivalent to φ
- $\forall xy.P(x, y)$ equivalent to $\forall yx.P(x, y)$
 - ex: $\forall xy.(x < y)$ same as $\forall yx.(x < y)$
- $\exists xy.P(x, y)$ equivalent to $\exists yx.P(x, y)$
 - ex: $\exists xy.Twins(x, y)$ same as $\exists yx.Twins(x, y)$
- $\exists x\forall y.P(x, y)$ not equivalent to $\forall y\exists x.P(x, y)$
 - ex: $\forall y\exists x.Father(x, y)$ much weaker than $\exists x\forall y.Father(x, y)$
“everybody has a father” vs. “exists a father of everybody”
 $\exists x\forall y.P(x, y) \models \forall y\exists x.P(x, y)$
 $\forall y\exists x.P(x, y) \not\models \exists x\forall y.P(x, y)$

Remark

- Variable names are irrelevant: e.g., $\forall x.P(x)$ is the same as $\forall y.P(y)$
- ... provided there are no name conflicts: e.g., $\forall x.\exists yP(x, y)$ is **not** the same as $\forall y.\exists yP(y, y)$!

Duality of Universal and Existential Quantification

- $\forall$ and $\exists$ are dual

- $\forall x.\alpha \iff \neg \exists x.\neg \alpha$
- $\neg \forall x.\alpha \iff \exists x.\neg \alpha$
- $\exists x.\alpha \iff \neg \forall x.\neg \alpha$
- $\neg \exists x.\alpha \iff \forall x.\neg \alpha$

- Examples

- $\forall x.Likes(x, Icecream)$ equivalent to $\neg \exists x.\neg Likes(x, Icecream)$
- $\exists x.Likes(x, Broccoli)$ equivalent to $\neg \forall x.\neg Likes(x, Broccoli)$

- Negated restricted quantifiers switch “ $\rightarrow$ ” with “ $\wedge$ ”

- $\forall x.(P(x) \rightarrow \alpha) \iff \neg \exists x.(P(x) \wedge \neg \alpha)$
- $\neg \forall x.(P(x) \rightarrow \alpha) \iff \exists x.(P(x) \wedge \neg \alpha)$
- ...

- Ex: “not all kings are evil” same as “some king is not evil”

- $\neg \forall x.(King(x) \rightarrow Evil(x)) \iff \exists x.(King(x) \wedge \neg Evil(x))$

- Unsurprising, since $\langle \forall, \exists \rangle$ are $\langle \wedge, \vee \rangle$ over infinite instantiations

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Satisfiability, Validity, Entailment

- A model $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{D}, \mathcal{I} \rangle$ satisfies φ ($\mathcal{M} \models \varphi$) iff $[\varphi]^{\mathcal{I}}$ is true
- $M(\varphi) \stackrel{\text{def}}{=} \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$ (the set of models of φ)
- φ is **satisfiable** iff $\mathcal{M} \models \varphi$ for some $\mathcal{M}$ (i.e. $M(\varphi) \neq \emptyset$)
- α **entails** β ($\alpha \models \beta$) iff, for all $\mathcal{M}$, $\mathcal{M} \models \alpha \implies \mathcal{M} \models \beta$ (i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is **valid** ($\models \varphi$) iff $\mathcal{M} \models \varphi$ for all $\mathcal{M}$ s (i.e., $\mathcal{M} \in M(\varphi)$ for all $\mathcal{M}$ s)
- α, β are **equivalent** iff $\alpha \models \beta$ and $\beta \models \alpha$ (i.e. $M(\alpha) = M(\beta)$)

Sets of formulas as conjunctions

Let $\Gamma \stackrel{\text{def}}{=} \{\varphi_1, \dots, \varphi_n\}$. Then:

- Γ **satisfiable** iff $\bigwedge_{i=1}^n \varphi_i$ **satisfiable**
- $\Gamma \models \phi$ iff $\bigwedge_{i=1}^n \varphi_i \models \phi$
- Γ **valid** iff $\bigwedge_{i=1}^n \varphi_i$ **valid**

Properties & Results

Property

φ is valid iff $\neg\varphi$ is unsatisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is unsatisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

Examples

- $P(x), \forall x.(x \geq y), \{\forall x.(x \geq 0), \forall x.(x + 1 > x)\}$ satisfiable
- $P(x) \wedge \neg P(x), \neg(x = x), (\forall x, y. Q(x, y)) \rightarrow \neg Q(a, b)$ unsatisfiable
- $\forall x.P(x) \rightarrow \exists x.P(x)$ valid
- $\forall x.P(x) \models \exists x.P(x)$
- $\neg(\forall x.P(x)) \rightarrow \exists x.P(x)$ unsatisfiable
- $\forall x.P(x) \wedge \neg \exists x.P(x)$ unsatisfiable

$(1 > 2)$ is satisfiable. Why?

Exercises

- Is $\forall x.P(x)$ equivalent to $\forall y.P(y)$?
- Is $\forall xy.P(x, y)$ equivalent to $\forall yx.P(y, x)$?
- $\forall x.\exists x.P(x)$ is equivalent to:
 - $\exists x.P(x)$
 - $\forall x.P(x)$
 - neither
- $\exists x.\forall x.P(x)$ is equivalent to:
 - $\exists x.P(x)$
 - $\forall x.P(x)$
 - neither

Enumeration of Models?

- We *can enumerate* the models for a given FOL sentence:

- For each number of universe elements n from 1 to ∞

- For each k -ary predicate P_k in the sentence

- For each possible k -ary relation on n objects

- For each constant symbol C in the sentence

- For each one of n objects C is mapped to

- ...

- $\implies$ Enumerating models is not going to be easy!

Semi-decidability of FOL

Theorem

Entailment (validity, unsatisfiability) in FOL is only **semi-decidable**:

- if $\Gamma \models \alpha$, this can be checked in finite time
- if $\Gamma \not\models \alpha$, no algorithm is guaranteed to check it in finite time



Outline

- 1 Generalities
- 2 Syntax and Semantics of FOL
 - Syntax
 - Semantics
 - Satisfiability, Validity, Entailment
- 3 Using FOL
 - FOL Agents
 - Example: The Wumpus World

Outline

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FOL Knowledge-Based Agent

- We can assert FOL sentences (**assertions**) into the KB: **Tell[...]**

- ex: **Tell[KB, King(John)]**
- ex: **Tell[KB, Person(Richard)]**
- ex: **Tell[KB, $\forall x.(King(x) \rightarrow Person(x))$]**

- We can ask **queries** (aka **goals**) to the KB: **Ask[...]**

- ex: **Ask[KB, King(John)]**
- ex: **Ask[KB, Person(John)]**
- ex: **Ask[KB, $\exists x.Person(x)$]**

$\Rightarrow$ **Ask[KB, α]** returns true only if $KB \models \alpha$

- Other queries: **AskVars[...]**, asking for variable values

$\Rightarrow$ returns one (or more) **binding lists** (aka **substitutions**) $\{var/term; var/term, \dots\}$

- ex: **AskVars[KB, $\exists x.Person(x)$]** $\Rightarrow \{x/John\}; \{x/Richard\}$
- typical for Horn clauses
(e.g. with **King(John) $\vee$ King(Richard)**,
the query **AskVars[KB, $\exists x.King(x)$]** would not cause a binding list)

Example: The Kinship Domain

Domain of family relationships

- Binary predicate symbols (family relationships):
 - Parent, Sibling, Brother, Sister, Child, Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle
- function symbols:
 - Mother, Father
- Knowledge base KB:
 - 1 $\forall x, y. (x = \text{Mother}(y) \leftrightarrow (\text{Female}(x) \wedge \text{Parent}(x, y)))$
 - 2 $\forall x, y. (\text{Brother}(x, y) \leftrightarrow (\text{Male}(x) \wedge \text{Sibling}(x, y)))$
 - 3 $\forall x, y. (\text{Grandparent}(x, y) \leftrightarrow \exists z. (\text{Parent}(x, z) \wedge \text{Parent}(z, y)))$
 - 4 $\forall x, y. (\text{Sibling}(x, y) \leftrightarrow ((x \neq y) \wedge \exists p_1, p_2. ((p_1 \neq p_2) \wedge \text{Parent}(p_1, x) \wedge \text{Parent}(p_1, y) \wedge (\text{Parent}(p_2, x) \wedge \text{Parent}(p_2, y))))$
 - 5 ...
- Queries inferred from KB
 - ex: (4) $\models \forall x, y. (\text{Sibling}(x, y) \leftrightarrow \text{Sibling}(y, x))$

Notation: “ $t \neq s$ ” shortcut for “ $\neg(t = s)$ ”

Example: Integer Numbers

Peano Arithmetic

- Basic symbols
 - Unary predicate symbol: NatNum (natural number)
 - Unary function symbol: S (Successor)
 - Constant symbol: 0
- Defined symbols:
 - Binary function symbols: $+, *$ (infix)
 - Constant symbols: $1, 2, 3, 4, 5, 6, \dots$
- Knowledge base KB:
 - 1 $\text{NatNum}(0)$
 - 2 $\forall x. (\text{NatNum}(x) \rightarrow \text{NatNum}(S(x)))$
 - 3 $\forall x. (\text{NatNum}(x) \rightarrow (0 \neq S(x)))$
 - 4 $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
 - 5 $\forall x. (\text{NatNum}(x) \rightarrow (x = (0 + x)))$
 - 6 $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow (S(x) + y = S(x + y)))$
 - 7 $1 = S(0), 2 = S(1), 3 = S(2), \dots$
- Queries inferred from KB
 - ex: $(4) \models \forall x, y. ((\text{NatNum}(x) \wedge (\text{NatNum}(y))) \rightarrow ((x + y) = (y + x)))$

Exercises

About the Kinship domain

- Try to add the axioms defining other predicates or functions (e.g. Brother, Sister, Child, Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle, ...)
- Add some ground atom or its negation to the KB (ex: Brother(Steve,Mary), Mary=Mother(Paul),...)
- Try to solve some query by entailment (e.g. Uncle(Steve,Paul), $\exists x. \text{Uncle}(x, \text{Paul})$, ...)

About the Peano Arithmetic domain

- Try to add the axioms defining other predicate or functions (e.g. " $n \leq m$ " or " $m * n$ ", " n^m ")
- Add some ground atom or its negation to the KB (ex: $1 = S(0)$, $2 = S(1)$, ...)
- Try to solve some query by entailment (e.g. $3 + 2 = 5$, $2 * 3 = 6$, ...)

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Example: The Wumpus World

The FOL KB

- **Perception**: binary predicate $\text{Percept}([s, b, g, b, sc], t)$
 - (recall: perception is [Stench, Breeze, Glitter, Bump, Scream])
 - **Stench, Breeze, Glitter, Bump, Scream** constant symbols
 - time step t represented as integer
- Percepts imply facts about the current state.
 - $\forall t, s, g, m, c. (\text{Percept}([s, \text{Breeze}, g, m, c], t) \rightarrow \text{Breeze}(t))$
 - $\forall t, s, g, m, c. (\text{Percept}([s, \text{Null}, g, m, c], t) \rightarrow \neg \text{Breeze}(t))$
 - ...
- **Environment**:
 - **Square**: term (pair of integers): $[1, 2]$
 - **Adjacency**: binary predicate **Adjacent**:
$$\forall x, y, a, b. (\text{Adjacent}([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$$
 - **Position**: predicate $\text{At}(\text{Agent}, s, t)$, ex: $\text{At}(\text{Agent}, [1, 1], 1)$
 - Unique position: $\forall x, s_1, s_2, t. ((\text{At}(x, s_1, t) \wedge \text{At}(x, s_2, t)) \rightarrow s_1 = s_2)$
 - **Wumpus**: predicate $\text{Wumpus}(s)$, ex: $\text{Wumpus}([3, 1])$
 - **Pits**: predicate $\text{Pit}(s)$, ex: $\text{Pit}([3, 1])$

- For Wumpus, AIMA suggests;
 - **Wumpus**: constant, ex $\forall t. At(Wumpus, [2, 2], t)$
- Simplification: assume Wumpus status does not evolve with time
 - predicate $Wumpus(s)$, ex: $Wumpus([3, 1])$
 - ⇒ makes inference much easier
 - if we consider the case the Wumpus is killed by arrow, then we need reintroducing the “At” formalization

Example: The Wumpus World [cont.]

The FOL KB [cont.]

- Infer properties from percepts:
 - $\forall s, t. ((At(Agent, s, t) \wedge Breeze(t)) \rightarrow Breezy(s))$
 - $\forall s, t. ((At(Agent, s, t) \wedge \neg Breeze(t)) \rightarrow \neg Breezy(s))$
- Infer information about pits & Wumpus
 - $\forall s. (Breezy(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Pit(r)))$
 - $\forall s. (Stench(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Wumpus(r)))$
- Evolution on time: successor states:
 - $\forall t. (HaveArrow(t+1) \leftrightarrow (HaveArrow(t) \wedge \neg Action(Shoot, t)))$
- **Actions:** terms **Turn(Right), Turn(Left), Forward, Shoot, Grab, Climb**
 - simple reflex action: $\forall t. (Glitter(t) \rightarrow BestAction(Grab, t))$
 - Query: $AskVars[\exists a. BestAction(a, 5)] \Rightarrow \{a/Grab\}$

Personal remark

Simplified action axiomatization: “Move(...)” instead of “Turn(...), Forward”

Example: Exploring the Wumpus World

KB initially contains:

$$\forall x, y, a, b. (Adjacent([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$$
$$\forall t, s, g, m, c. (Percept([s, Null, g, m, c], t) \rightarrow \neg Breeze(t))$$
$$\forall t, b, g, m, c. (Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$$
$$\forall s, t. ((At(Agent, s, t) \wedge \neg Breeze(t)) \rightarrow \neg Breezy(s))$$
$$\forall s, t. ((At(Agent, s, t) \wedge \neg Stench(t)) \rightarrow \neg Stenchy(s))$$
$$\forall s. (Breezy(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Pit(r)))$$
$$\forall s. (Stench(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Wumpus(r)))$$
$$\forall s. (Ok(s) \leftrightarrow (\neg Pit(s) \wedge \neg Wumpus(s)))$$

- A is initially in 1,1: $At(A, [1, 1], 0)$

- Perceives no stench, no breeze:

$$Tell[KB, Percept([Null, Null, Null, Null, Null], 0)]$$
$$\Rightarrow \neg Breeze(0), \neg Stench(0),$$
$$\Rightarrow \neg Breezy([1, 1]), \neg Stenchy([1, 1]),$$
$$\Rightarrow \neg Pit([1, 2]), \neg Pit([2, 1]), \neg Wumpus([1, 2]), \neg Wumpus([2, 1]),$$
$$\Rightarrow Ok([1, 2]), Ok([2, 1])$$
$$AskVars[KB, \exists a. BestAction(a, 0)]$$
$$\Rightarrow \{a/Move([1, 2])\}, \{a/Move([2, 1])\}$$

OK			
OK A	OK		

Example: Exploring the Wumpus World

KB initially contains:

$\neg Pit([1, 1]), \neg Wumpus([1, 1]), \dots$

$\forall x, y, a, b. (Adjacent([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$

$\forall t, s, g, m, c. (Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$

$\forall t, b, g, m, c. (Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$

$\forall s, t. ((At(Agent, s, t) \wedge Breeze(t)) \rightarrow Breezy(s))$

$\forall s, t. ((At(Agent, s, t) \wedge \neg Stench(t)) \rightarrow \neg Stenchy(s))$

$\forall s. (Breezy(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Pit(r)))$

$\forall s. (Stench(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Wumpus(r)))$

- Agent moves to [2,1]: $At(A, [2, 1], 1)$

- Perceives a breeze and no stench:

$Tell[KB, Percept([Null, Breeze, Null, Null, Null], 1)]$

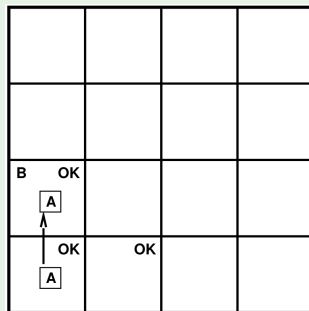
$\Rightarrow Breeze(1), \neg Stench(1),$

$\Rightarrow Breezy([2, 1]), \neg Stenchy([2, 1]),$

$\Rightarrow \exists r. (Adjacent(r, [2, 1]) \wedge Pit(r)),$
 $\neg Wumpus([3, 1]), \neg Wumpus([2, 2]),$

$\Rightarrow (Pit([3, 1]) \vee Pit([2, 2]))$

$AskVars[KB, \exists a. Action(a, 1)] \Rightarrow \{a / Move([1, 1])\}$



Example: Exploring the Wumpus World

KB initially contains:

$\neg Pit([1, 1]), \neg Wumpus([1, 1]), \dots$

$\forall x, y, a, b. (Adjacent([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$

$\forall t, s, g, m, c. (Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$

$\forall t, b, g, m, c. (Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$

$\forall s, t. ((At(Agent, s, t) \wedge Breeze(t)) \rightarrow Breezy(s))$

$\forall s, t. ((At(Agent, s, t) \wedge \neg Stench(t)) \rightarrow \neg Stenchy(s))$

$\forall s. (Breezy(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Pit(r)))$

$\forall s. (Stench(s) \leftrightarrow \exists r. (Adjacent(r, s) \wedge Wumpus(r)))$

- Agent moves to [2,1]: $At(A, [2, 1], 1)$

- Perceives a breeze and no stench:

$Tell[KB, Percept([Null, Breeze, Null, Null, Null], 1)]$

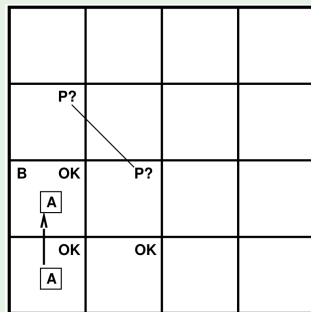
$\Rightarrow Breeze(1), \neg Stench(1),$

$\Rightarrow Breezy([2, 1]), \neg Stenchy([2, 1]),$

$\Rightarrow \exists r. (Adjacent(r, [2, 1]) \wedge Pit(r)),$
 $\neg Wumpus([3, 1]), \neg Wumpus([2, 2]),$

$\Rightarrow (Pit([3, 1]) \vee Pit([2, 2]))$

$AskVars[KB, \exists a. Action(a, 1)] \Rightarrow \{a / Move([1, 1])\}$



Exercise

Complete the example in the FOL case (see the PL case).