

# Fundamentals of Artificial Intelligence

## Chapter 05: Adversarial Search and Games

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- 1 Games
- 2 Optimal Decisions in Games
  - Min-Max Search
  - Alpha-Beta Pruning
- 3 Adversarial Search with Resource Limits
- 4 Stochastic Games

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# Games and AI

- Games are a form of **multi-agent environment**
  - Q.: **What do other agents do and how do they affect our success?**
  - recall: cooperative vs. competitive multi-agent environments
  - competitive multi-agent environments give rise to **adversarial problems** (aka **games**)
- Q.: **Why study games in AI?**
  - lots of fun, historically entertaining
  - **easy to represent**: agents restricted to **small number of actions** with **precise rules**
  - interesting also because **computationally very hard**  
(ex: **chess has  $b \approx 35$ , #nodes  $\approx 10^{40}$** )
  - metaphor for important application domains  
(e.g. competitive markets, life sciences, sport, politics, warfare, ...)

# Search and Games

- Search (with no adversary)
  - solution is a (heuristic) method for finding a goal
  - heuristics techniques can find optimal solutions
  - evaluation function: estimate of cost from start to goal through given node
  - examples: path planning, scheduling activities, ...
- Games (with adversary), aka adversarial search
  - solution is a strategy: specifies a move for every possible opponent reply
  - evaluation function (utility): evaluate “goodness” of game position
  - examples: tic-tac-toe, chess, checkers, Othello, backgammon, ...
  - often computationally very hard  $\implies$  time limits force an approximate solution

# Types of Games

- Many different kinds of games
- Relevant features:
  - deterministic vs. stochastic (with chance)
  - one, two, or more players
  - zero-sum vs. general games
  - perfect information (can you see the state?) vs. imperfect
- Most common: deterministic, turn-taking, two-player, zero-sum games, perfect information
- Want algorithms for calculating a strategy (aka policy):
  - recommends a move from each state:  $policy : S \mapsto A$

	deterministic	chance
perfect information	chess, checkers, go, othello	backgammon monopoly
imperfect information	battleships, blind tictactoe	bridge, poker, scrabble nuclear war

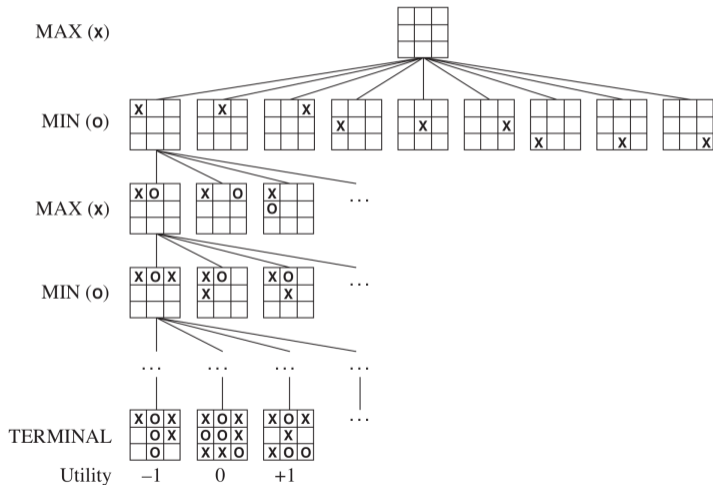
(\*) "blind tictactoe": a version of tic-tac-toe where the players don't get to see each others' moves.

# Games: Main Concepts

- We first consider games with **two players**: “MAX” and “MIN”
  - MAX moves first;
  - they take turns moving until the game is over
  - at the end of the game, points are awarded to the winner and penalties are given to the loser
- **A game is a kind of search problem:**
  - **Initial state  $S_0$** : specifies how the game is set up at the start
  - **$Player(s)$** : defines which player has the move in a state
  - **$Actions(s)$** : returns the set of legal moves in a state
  - **$Result(s, a)$** : the **transition model**, defines the result of a move
  - **$TerminalTest(s)$** : true iff the game is over (if so,  $s$  **terminal state**)
  - **$Utility(s, p)$** : (aka **objective function** or **payoff function**):  
defines the final numeric value for a game ending in state  $s$  for player  $p$ 
    - ex: **chess**: 1 (win), 0 (loss),  $\frac{1}{2}$  (draw)
    - ex: **tic-tac-toe**: 1 (win), -1 (loss), 0 (draw)
- $S_0$ ,  $Actions(s)$  and  $Result(s, a)$  recursively define the **game tree**
  - nodes are states, arcs are actions
  - ex: **tic-tac-toe**:  $\approx 10^5$  nodes, **chess**:  $\approx 10^{40}$  nodes, ...

# Game Tree: Example

Partial game tree for tic-tac-toe (2-player, deterministic, turn-taking)





# Zero-Sum Games vs. General Games

- General Games

- agents have independent utilities
- cooperation, indifference, competition, and more are all possible

- Zero-Sum Games: the total payoff to all players is the same for each game instance

- adversarial, pure competition
- agents have opposite utilities (values on outcomes)

⇒ Idea: With two-player zero-sum games, we can use one single utility value

- one agent maximizes it, the other minimizes it

⇒ optimal adversarial search as min-max search

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# Adversarial Search as Min-Max Search

- Assume MAX and MIN are very smart and always play optimally
- MAX must find a contingent strategy specifying:
  - MAX's move in the initial state
  - MAX's moves in the states resulting from every possible response by MIN,
  - MAX's moves in the states resulting from every possible response by MIN to those moves,
  - ...

(a single-agent move is called half-move or ply)

- Analogous to the AND-OR search algorithm
  - MAX playing the role of OR
  - MIN playing the role of AND
- Optimal strategy: for which  $Minimax(s)$  returns the highest value

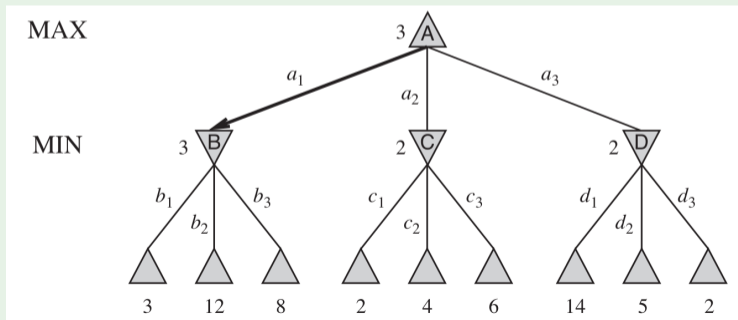
$$Minimax(s) \stackrel{\text{def}}{=} \begin{cases} Utility(s) & \text{if } TerminalTest(s) \\ \max_{a \in Actions(s)} Minimax(Result(s, a)) & \text{if } Player(s) = MAX \\ \min_{a \in Actions(s)} Minimax(Result(s, a)) & \text{if } Player(s) = MIN \end{cases}$$

# Min-Max Search: Example

## A two-ply game tree

- $\Delta$  nodes are “MAX nodes”,  $\nabla$  nodes are “MIN nodes”,
  - terminal nodes show the utility values for MAX
  - the other nodes are labeled with their minimax value
- Minimax maximizes the worst-case outcome for MAX

⇒ MAX's root best move is  $a_1$



# The Minimax Algorithm

## Depth-First Search Minimax Algorithm

```
function MINIMAX-DECISION(state) returns an action  
  return  $\arg \max_{a \in \text{ACTIONS}(s)} \text{MIN-VALUE}(\text{RESULT}(state, a))$ 
```

---

```
function MAX-VALUE(state) returns a utility value  
  if TERMINAL-TEST(state) then return UTILITY(state)  
   $v \leftarrow -\infty$   
  for each a in ACTIONS(state) do  
     $v \leftarrow \text{MAX}(v, \text{MIN-VALUE}(\text{RESULT}(s, a)))$   
  return v
```

---

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```

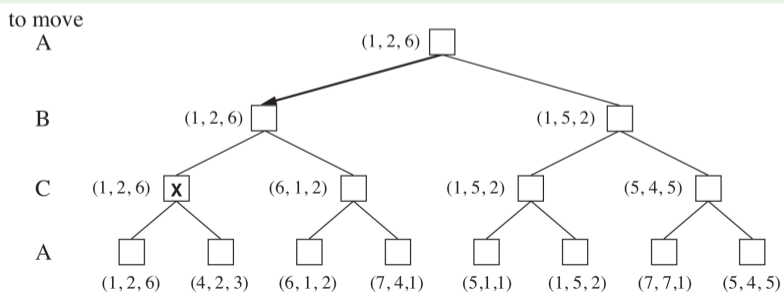
# Multi-Player Games: Optimal Decisions

- Replace the single value for each node with a **vector of values**
  - each value represent score from each player's viewpoint
  - terminal states: utility for each agent
  - agents, in turn, choose the action with best value for themselves
- Alliances are possible!
  - e.g., if one agent is in dominant position, the other can ally

# Multiplayer Min-Max Search: Example

## The first three plies of a game tree with three players (A, B, C)

- Each node labeled with values from each player's viewpoint
- Agents choose the action with best value for themselves
  - ⇒ A chooses the left move (1, 2, 6) (bad for A and B, good for C), or  
A chooses the left move (1, 5, 2) (equivalently bad for A, good for B, bad for C)
- If A and B are allied, then they may agree that B and then A choose (5,4,5) instead of (1,5,2)
  - ⇒ **benefit for both**





## Exercise

- Consider the Multiplayer Min-Max Search example of previous slide
  - Redo it with choice order A-C-B
  - Redo it with choice order C-A-B
  - Redo it with choice order C-B-A
  - Redo it with choice order B-A-C
  - Redo it with choice order B-C-A
- Do they have all the same outcome?
- For each case, try to define the best moves in case of alliance between the top two players

# The Minimax Algorithm: Properties

- Complete? Yes, if tree is finite
- Optimal? Yes, against an optimal opponent
  - What about non-optimal opponent?  
⇒ even better, but non optimal in this case
- Time complexity?  $O(b^m)$
- Space complexity?  $O(bm)$  (DFS)

For chess,  $b \approx 35$ ,  $m \approx 100 \implies 35^{100} = 10^{154}$  (!)

We need to prune the tree!

# Remark

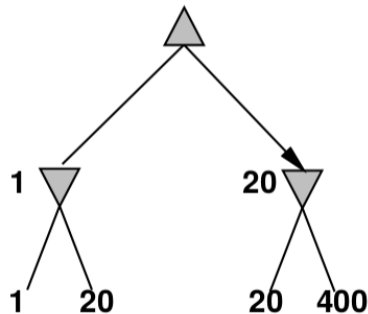
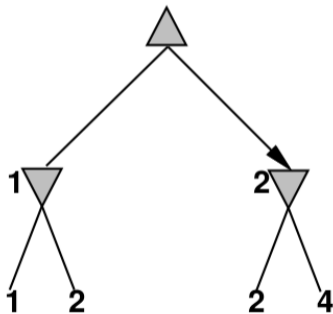
Exact values don't matter!

Behaviour preserved under any monotonic transformation of  $Eval()$

- Only the order matters!

MAX

MIN

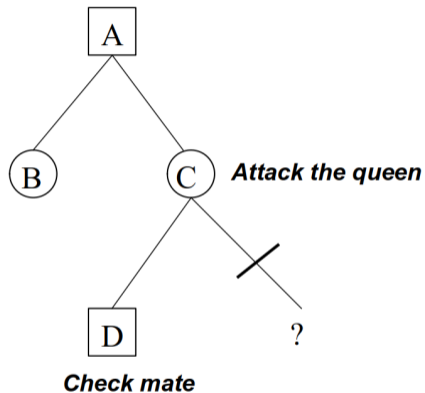


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# Outline

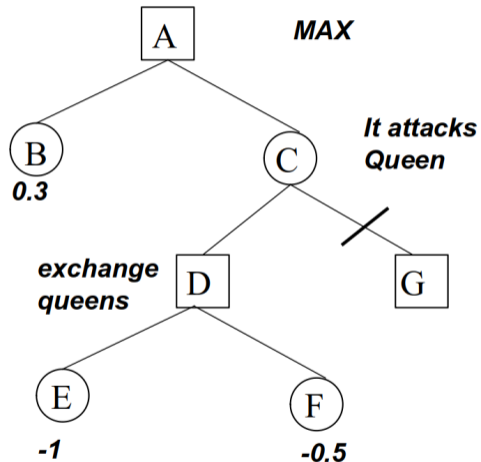
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## Example: Chess (1)



- No matter which is the evaluation of the other children of C (I realize that I should never move to C).

## Example: Chess (2)



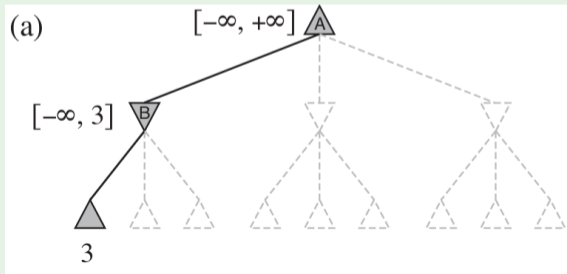
- Max in A avoids C because B is better. At most max gets from C a - 0.5 so 0.3 is better
- The subtree in G can be cut as soon as I receive the value of D.  
Indeed:  $C = \min(-0.5, G)$ ;  
 $A = \max(0.3, \min(-0.5, G)) = 0.3$

Since A is independent of G, the tree under G can be cut.

# Pruning Min-Max Search: Example

- Consider the Min-Max example, let  $[\alpha, \beta]$  track the currently-known bounds:  
( $\alpha$  (resp  $\beta$ ): best value for MAX (resp MIN) so far at any choice point along the path)
- (a): B labeled with  $[-\infty, 3]$  (MIN will not choose values  $\geq 3$  for B)
- (c): B labeled with  $[3, 3]$  (MIN cannot find values  $\leq 3$  for B)  $\implies$  A labeled with  $[3, +\infty]$
- (d): Is it necessary to evaluate the remaining leaves of C?  
NO! They cannot produce an upper bound  $\geq 2$   
 $\implies$  MAX cannot update the  $\alpha = 3$  bound due to C
- (e): MAX updates the upper bound to 14 (D is last subtree)
- (f): D labeled  $[2, 2] \implies$  MAX updates the upper bound to 3

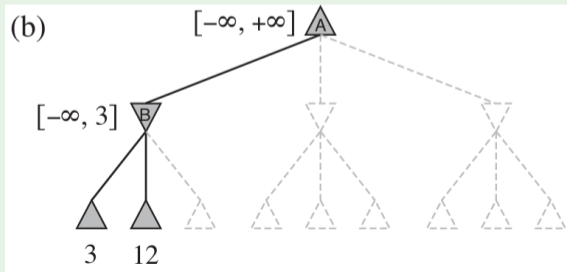
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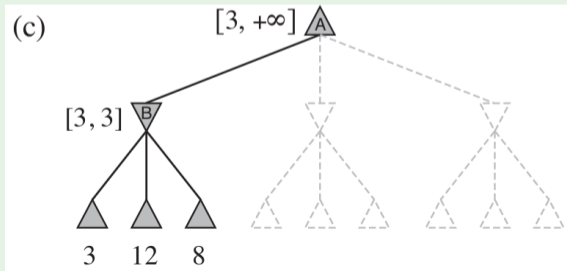




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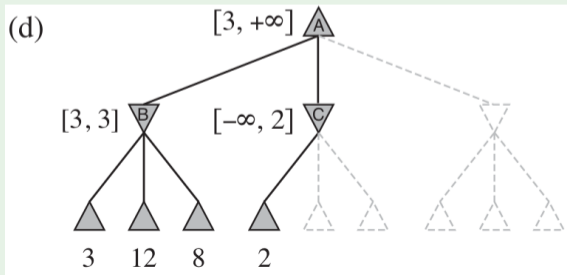
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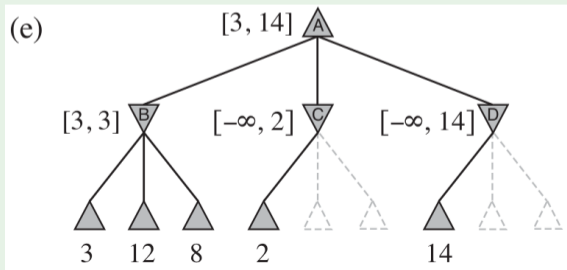
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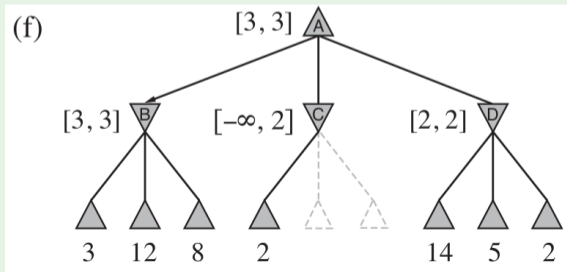
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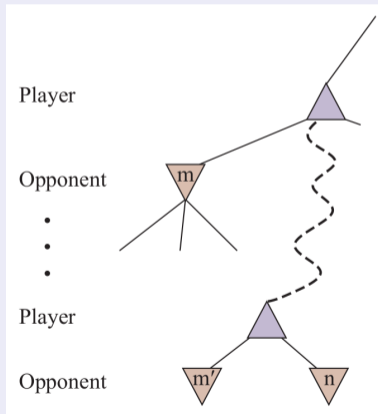
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# Alpha-Beta Pruning Technique for Min-Max Search

- Idea: consider a node  $n$  (terminal or intermediate) and its **current** value
    - If player has a better choice at the same level of  $n$  ( $m'$ ) or at any point higher up in the tree ( $m$ ), then  **$n$  will never be reached in actual play**
  - ⇒ as soon as we know enough of  $n$  to draw this conclusion, **we can prune  $n$**
  - **Alpha-Beta Pruning**: nodes labeled with  $[\alpha, \beta]$  s.t.:
    - $\alpha$  : **best value for MAX (highest) so far at any choice point along the path**
      - ⇒ lower bound for future values
    - $\beta$  : **best value for MIN (lowest) so far at any choice point along the path**
      - ⇒ upper bound for future values
- ⇒ **Prune  $n$  if its value is worse (lower) than the current  $\alpha$  value for MAX (dual for  $\beta$ , MIN)**



# The Alpha-Beta Search Algorithm

```
function ALPHA-BETA-SEARCH(state) returns an action  
   $v \leftarrow \text{MAX-VALUE}(\textit{state}, -\infty, +\infty)$   
  return the action in  $\text{ACTIONS}(\textit{state})$  with value  $v$ 
```

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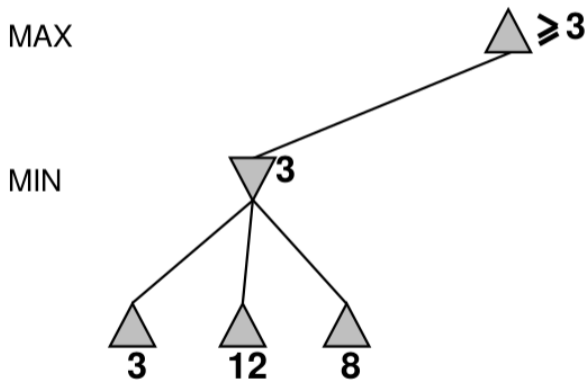
```
function MAX-VALUE(state,  $\alpha$ ,  $\beta$ ) returns a utility value  
if  $\text{TERMINAL-TEST}(\textit{state})$  then return  $\text{UTILITY}(\textit{state})$   
   $v \leftarrow -\infty$   
  for each  $a$  in  $\text{ACTIONS}(\textit{state})$  do  
     $v \leftarrow \text{MAX}(v, \text{MIN-VALUE}(\text{RESULT}(s,a), \alpha, \beta))$   
    if  $v \geq \beta$  then return  $v$  // MIN will never choose a bigger value  
     $\alpha \leftarrow \text{MAX}(\alpha, v)$   
  return  $v$ 
```

---

```
function MIN-VALUE(state,  $\alpha$ ,  $\beta$ ) returns a utility value  
if  $\text{TERMINAL-TEST}(\textit{state})$  then return  $\text{UTILITY}(\textit{state})$   
   $v \leftarrow +\infty$   
  for each  $a$  in  $\text{ACTIONS}(\textit{state})$  do  
     $v \leftarrow \text{MIN}(v, \text{MAX-VALUE}(\text{RESULT}(s,a), \alpha, \beta))$   
    if  $v \leq \alpha$  then return  $v$  // MAX will never choose a smaller value  
     $\beta \leftarrow \text{MIN}(\beta, v)$   
  return  $v$ 
```

## Example revisited: Alpha-Beta Cuts

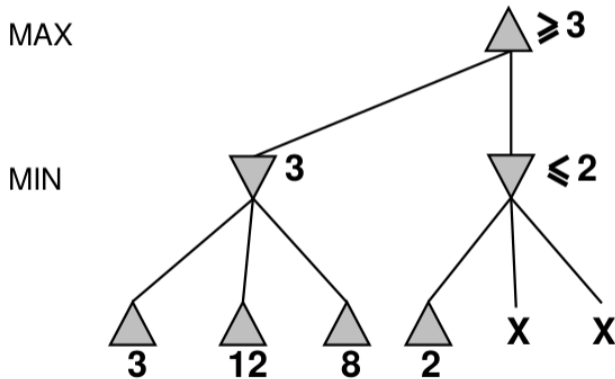
- Notation:  $\geq \alpha$ ;  $\leq \beta$ ;



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## Example revisited: Alpha-Beta Cuts

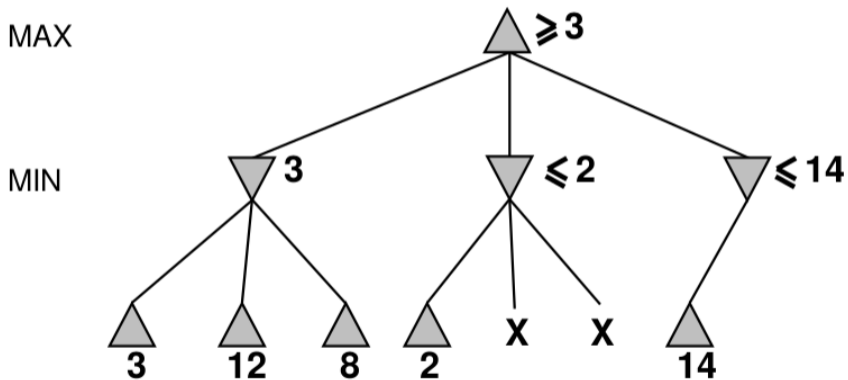
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## Example revisited: Alpha-Beta Cuts

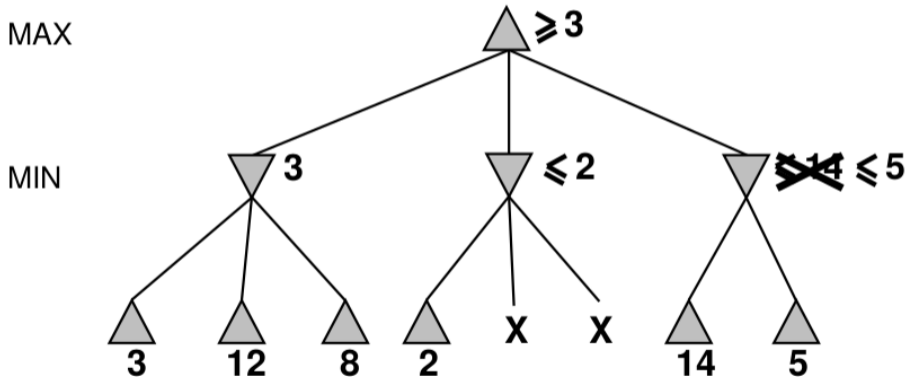
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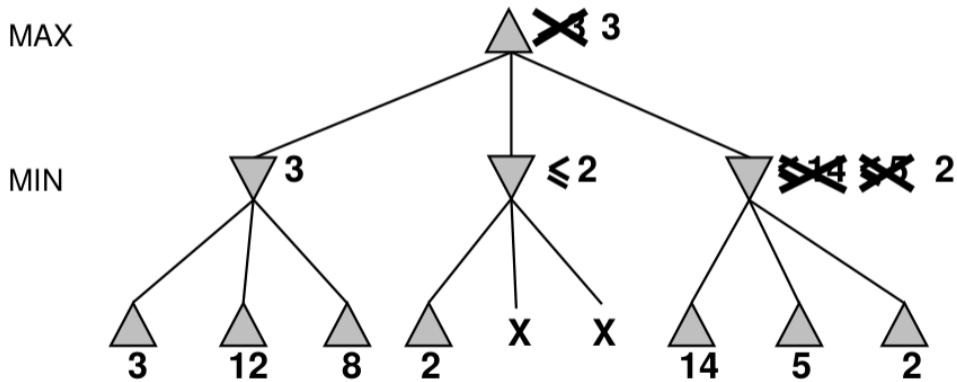
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# Example revisited: Alpha-Beta Cuts

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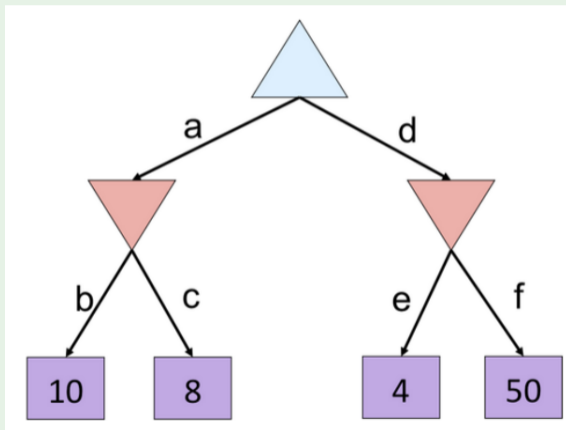


# Properties of Alpha-Beta Search

- Pruning does not affect the final result  $\implies$  **correctness preserved**
- Good move ordering improves effectiveness of pruning
  - Ex: if MIN expands 3<sup>rd</sup> child of D first (see 2<sup>nd</sup> last example), then the others are pruned $\implies$  try to examine first the successors that are likely to be best
- With “perfect” ordering, time complexity reduces to  $O(b^{m/2})$ 
  - aka “killer-move heuristic” $\implies$  doubles solvable depth!
- With “random” ordering, time complexity reduces to  $O(b^{3m/4})$
- “Graph-based” version further improves performances
  - track explored states via hash table

# Exercise I

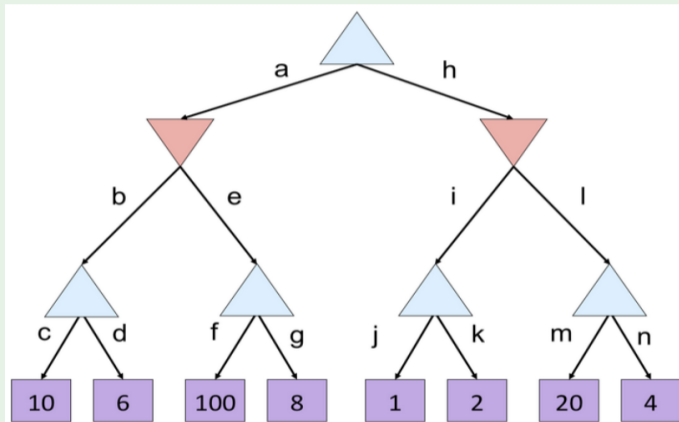
Apply alpha-beta search to the following tree



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## Exercise II

Apply alpha-beta search to the following tree



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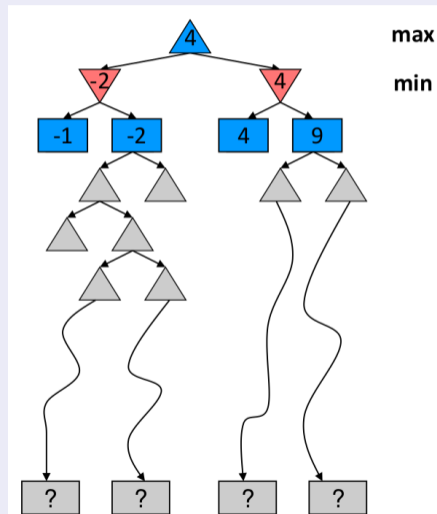
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# Adversarial Search with Resource Limits

Problem: In realistic games, full search is impractical!

- Complexity:  $b^d$  (ex. chess:  $\approx 35^{100}$ )
- Idea [Shannon, 1949]: **Depth-limited search**
  - cut off minimax search earlier, after limited depth
  - replace **terminal utility function** with **evaluation for non-terminal nodes**
- Ex (chess): depth  $d = 8$  (decent)  
 $\implies \alpha\text{-}\beta: 35^{8/2} \approx 10^5$  (feasible)





# Adversarial Search with Resource Limits [cont.]

- Idea:

- cut off the search earlier, at limited depths
- apply a heuristic evaluation function to states in the search

⇒ effectively turning nonterminal nodes into terminal leaves

- Modify *Minimax()* or Alpha-Beta search in two ways:

- replace the utility function *Utility(s)* by a heuristic evaluation function *Eval(s)*, which estimates the position's utility
- replace the terminal test *TerminalTest(s)* by a cutoff test *CutOffTest(s, d)*, that decides when to apply *Eval()*
- plus some bookkeeping to increase depth *d* at each recursive call

⇒ Heuristic variant of *Minimax()*:

$$H\text{-Minimax}(s, d) \stackrel{\text{def}}{=} \begin{cases} Eval(s) & \text{if } CutOffTest(s, d) \\ \max_{a \in Actions(s)} H\text{-Minimax}(Result(s, a), d + 1) & \text{if } Player(s) = MAX \\ \min_{a \in Actions(s)} H\text{-Minimax}(Result(s, a), d + 1) & \text{if } Player(s) = MIN \end{cases}$$

⇒ Heuristic variant of alpha-beta: substitute the terminal test with

**If *CutOffTest(s)* then return *Eval(s)***

# Evaluation Functions

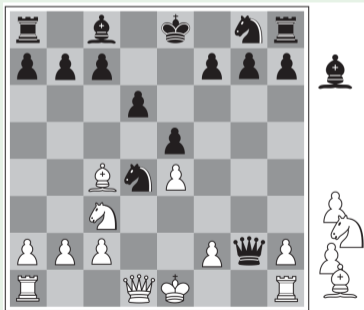
## *Eval(s)*

- Should be relatively cheap to compute
- Returns an estimate of the expected utility from a given position
  - Ideal function: returns the actual minimax value of the position
- Should order terminal states the same way as the utility function
  - e.g., wins > draws > losses
- For nonterminal states, should be strongly correlated with the actual chances of winning
- Defines equivalence classes of positions (same  $Eval(s)$  value)
  - e.g. returns a value reflecting the % of states with each outcome
- Typically weighted linear sum of features:  
$$Eval(s) = w_1 \cdot f_1(s) + w_2 \cdot f_2(s) + \dots + w_n \cdot f_n(s)$$
  - ex (chess):  $f_{pawns}(s) = \#white\ pawns - \#black\ pawns$ ,  
 $w_{pawns} = 1$ ;  $w_{bishops} = w_{knights} = 3$ ,  $w_{rooks} = 5$ ,  $w_{queens} = 9$
- May depend on depth
  - ex: knights more valuable with low depths, rooks more valuable with high depths
- May be very inaccurate for some positions

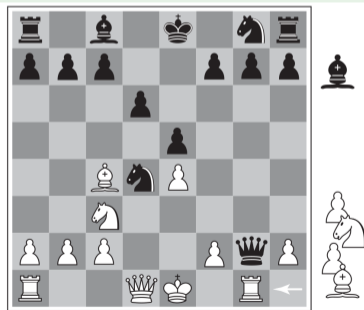
# Example

- Two same-score positions (White: -8, Black: -3)
  - (a) Black has an advantage of a knight and two pawns,  
⇒ should be enough to win the game
  - (b) White will capture the queen,  
⇒ give it an advantage that should be strong enough to win

(Personal note: only very-stupid black player would get into (b))



(a) White to move



(b) White to move

# Cutting-off the Search

## *CutOffTest(state, depth)*

- Most straightforward approach: **set a fixed depth limit**
  - d chosen s.t. a move is selected within the allocated time
  - sometimes may produce very inaccurate outcomes (see previous example)
- More robust approach: **apply Iterative Deepening**
- More sophisticated: apply *Eval()* only to **quiescent** states
  - **quiescent**: unlikely to exhibit wild swings in value in the near future
  - e.g. positions with direct favorable captures are not quiescent  
(previous example (b))

⇒ **further expand non-quiescent states until quiescence is reached**

# Deterministic Games in Practice

- **Checkers:** (1994) **Chinook** ended 40-year-reign of world champion **Marion Tinsley**
  - used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board
  - a total of 443,748,401,247 positions
- **Chess:** (1997) **Deep Blue** defeated world champion **Gary Kasparov** in a six-game match
  - searches 200 million positions per second
  - uses very sophisticated evaluation, and undisclosed methods
- **Othello:**
  - Human champions refuse to compete against computers, which are too good
- **Go:** (2016) **AlphaGo** beats world champion **Lee Sedol**
  - number of possible positions > number of atoms in the universe

## AlphaGo beats GO world champion, Lee Sedol (2016)



# Outline

- 1 Games
- 2 Optimal Decisions in Games
  - Min-Max Search
  - Alpha-Beta Pruning
- 3 Adversarial Search with Resource Limits
- 4 Stochastic Games**

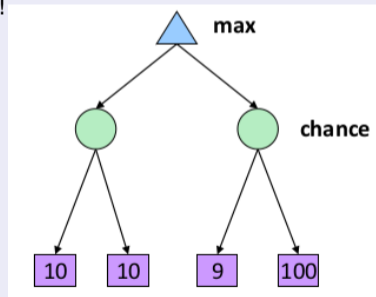
# Stochastic Games: Generalities

- In real life, **unpredictable external events** may occur
- **Stochastic Games** mirror unpredictability by **random steps**:
  - e.g. **dice throwing, card-shuffling, coin flipping, tile extraction, ...**
- Ex: **Backgammon**
- Cannot calculate definite minimax value, only **expected values**
- Uncertain outcomes controlled by **chance**, not an adversary!
  - adversarial  $\implies$  **worst case**
  - chance  $\implies$  **average case**
- Ex: **if chance is 0.5 each (coin)**:
  - **minimax: 10**
  - **average:  $(100+9)/2=54.5$**



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# An Example: Backgammon

- Rules

- 15 pieces each
- white moves clockwise to 25, black moves counterclockwise to 0
- a piece can move to a position unless  $\geq 2$  opponent pieces there
- if there is one opponent, it is captured and must start over
- termination: all whites in 25 or all blacks in 0

- Ex: Possible white moves (dice: 6,5):

(5-10,5-11)

(5-11,19-24)

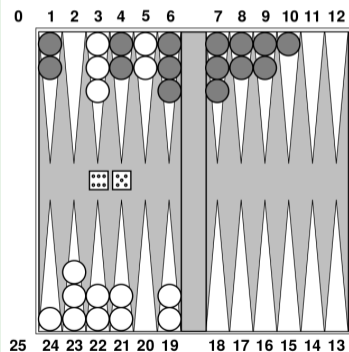
(5-10,10-16)

(5-11,11-16)

- Combines strategy with **luck**

⇒ **stochastic component** (dice)

- double rolls (1-1),..., (6-6)  
have 1/36 probability each
- other 15 distinct rolls  
have a 1/18 probability each

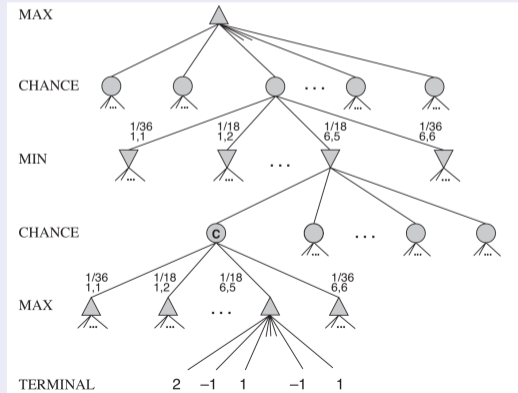


(© S. Russell & P. Norwig, AIMA)

# Stochastic Games Trees

Idea:

- A tree for a stochastic game includes chance nodes in addition to MAX and MIN nodes.
  - chance nodes above agent represent stochastic events for agent (e.g. dice roll)
  - outgoing arcs represent stochastic event outcomes
  - labeled with stochastic event and relative probability



# Algorithm for Stochastic Games: *ExpectMinimax()*

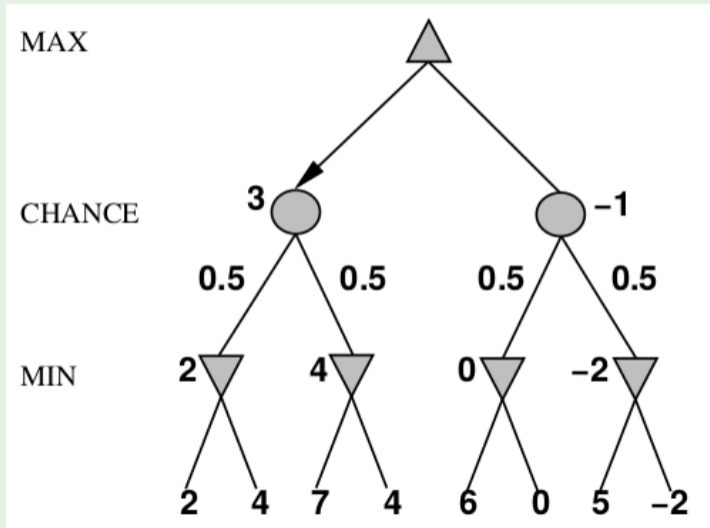
- Extension of *Minimax()*, handling also chance nodes:

$$ExpectMinimax(s) \stackrel{\text{def}}{=} \begin{cases} Utility(s) & \text{if } TerminalTest(s) \\ \max_{a \in Actions(s)} ExpectMinimax(Result(s, a)) & \text{if } Player(s) = MAX \\ \min_{a \in Actions(s)} ExpectMinimax(Result(s, a)) & \text{if } Player(s) = MIN \\ \sum_r P(r) \cdot ExpectMinimax(Result(s, r)) & \text{if } Player(s) = Chance \end{cases}$$

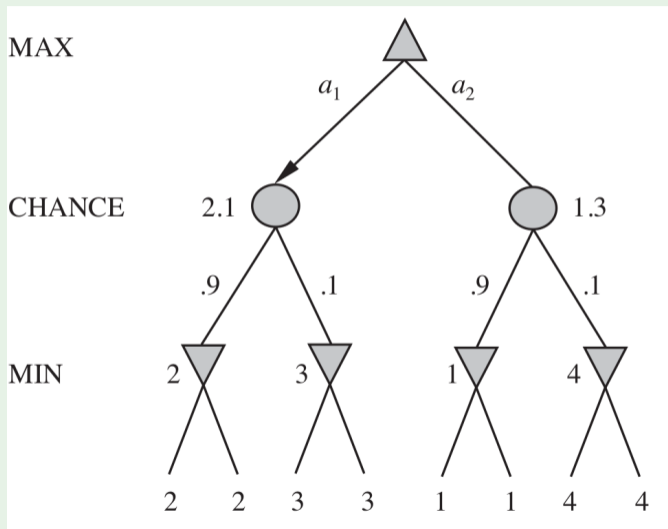
- $P(r)$ : probability of stochastic event outcome  $r$
- chance seen as an actor ("Chance")
- stochastic event outcomes  $r$  (e.g., dice values) seen as actions

⇒ Returns the weighted average of the minimax outcomes (recall that  $\sum_r P(r) = 1$ )

# Simple Example with Coin-Flipping



# Example (Non-uniform Probabilities)



## Remark (compare with deterministic case)

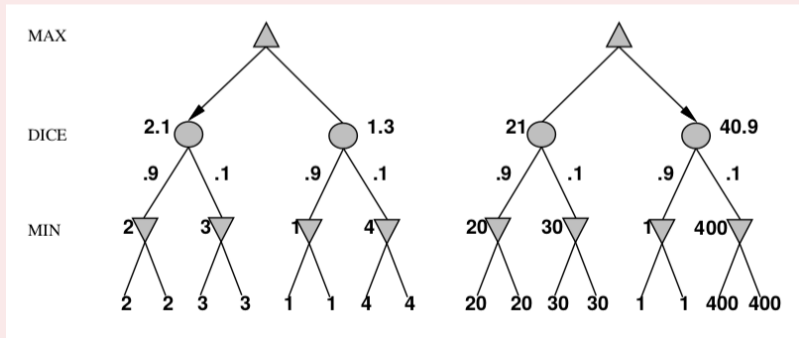
Exact values do matter!

Behaviour **not** preserved under monotonic transformations of  $Utility()$

- preserved only by **positive linear transformation** of  $Utility()$

- hint:  $p_1 v_1 \geq p_2 v_2 \implies p_1(av_1 + b) \geq p_2(av_2 + b)$  if  $a \geq 0$

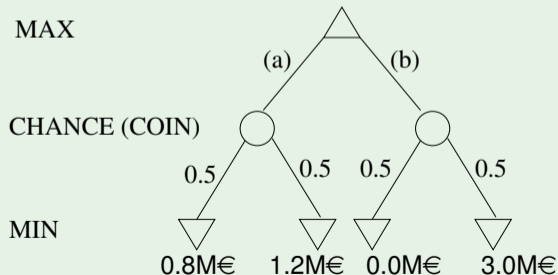
$\implies Utility()$  should be proportional to the expected payoff



# Example

## Beware of money as utility function!

- Ex: choose between two alternatives in a coin-toss tree:
  - (a) gain 0.8M€ (heads) vs. gain 1.2M€ (tails)
  - (b) gain 0.0M€ (heads) vs. gain 3.0M€ (tails).
- Which one will you choose? Why?
- If you choose (a), what is wrong with applying ExpectMinimax() here?





# Stochastic Games in Practice

- Dice rolls increase  $b$ : 21 possible rolls with 2 dice
    - ⇒  $O(b^m \cdot n^m)$ ,  $n$  being the number of distinct roll
  - Ex: Backgammon has  $\approx 20$  moves
    - ⇒ depth 4:  $20 \cdot (21 \cdot 20)^3 \approx 10^9$  (!)
  - Alpha-beta pruning much less effective than with deterministic games
- ⇒ Unrealistic to consider high depths in most stochastic games
- Heuristic variants of *ExpectMinimax()* effective, low cutoff depths
  - Ex: TD-GGAMMON uses depth-2 search + very-good *Eval()*
    - *Eval()* “learned” by running million training games
    - competitive with world champions