Fundamentals of Artificial Intelligence
Chapter 09: **Inference in First-Order Logic**

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M.S. Course "Artificial Intelligence Systems", academic year 2024-2025

Last update: Thursday 5th September, 2024, 18:59

Outline

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Term/Subformula Substitutions

Notation

- Substitution: "Subst($\{e_1/e_2\}, e$)" or "$e\{e_1/e_2\}$":
  the expression obtained by simultaneously substituting every occurrence of e_1 with e_2 in e
  - e_1, e_2 either both terms (term substitution)
    or both subformulas (subformula substitution)
  - e is either a term or a formula (only term for term substitution)
- Examples:
  - (t. sub.): $(y + 1 = 1 + y)\{y/S(x)\} \Longrightarrow (S(x) + 1 = 1 + S(x))$
  - (s.f. sub.): $(Even(x) \lor Odd(x))\{Even(x)/Odd(S(x))\} \Longrightarrow ((Odd(S(x)) \lor Odd(x))$
- Multiple substitution: apply simulteneously all substitutions in a list: $e\{e_1/e_2, e_3/e_4\}$
  - ex: $(P(x, y) \rightarrow Q(x, y))\{x/1, y/2\} \Longrightarrow (P(1, 2) \rightarrow Q(1, 2))$
  - multiple substitutions are simultaneous:
    ex: $P(x) \lor Q(y)\{x/y, y/f(b)\} = P(y) \lor Q(f(b)$ (not $P(f(b)) \lor Q(f(b))$)
- If θ is a substitution list and e an expression (formula/term),
  then we denote the result of a substitution as $e\theta$

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Substitution with equivalent terms

Equal-term substitution rule

$$\frac{\Gamma \wedge (t_1 = t_2) \wedge \alpha}{\Gamma \wedge (t_1 = t_2) \wedge \alpha \wedge \alpha\{t_1/t_2\}}$$

- Ex: $(S(x) = x + 1) \wedge (0 \neq S(x)) \Longrightarrow (S(x) = x + 1) \wedge (0 \neq S(x)) \wedge (0 \neq x + 1)$
- Preserves validity: $M(\Gamma \wedge (t_1 = t_2) \wedge \alpha \wedge \alpha\{t_1/t_2\}) = M(\Gamma \wedge (t_1 = t_2) \wedge \alpha)$
- α can be safely dropped from the result

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Substitution with equivalent formulas

Equivalent-subformula substitution rule

$$\frac{\Gamma \wedge (\beta_1 \leftrightarrow \beta_2) \wedge \alpha}{\Gamma \wedge (\beta_1 \leftrightarrow \beta_2) \wedge \alpha \wedge \alpha\{\beta_1/\beta_2\}}$$

- Ex: $(Even(x) \leftrightarrow Odd(S(x))) \wedge (Even(x) \vee Odd(x)) \Longrightarrow$ $(Even(x) \leftrightarrow Odd(S(x))) \wedge (Even(x) \vee Odd(x)) \wedge (Odd(S(x)) \vee Odd(x))$
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Universal Instantiation (UI)

- Every instantiation of a universally quantified-sentence is entailed by it:

$$\frac{\Gamma \land \forall x.\alpha}{\Gamma \land \forall x.\alpha \land \alpha\{x/t\}}$$

  for every variable x and term t
- Ex: $\forall x.((King(x) \land Greedy(x)) \rightarrow Evil(x))$
  - $(King(John) \land Greedy(John)) \rightarrow Evil(John)$
  - $(King(Richard) \land Greedy(Richard)) \rightarrow Evil(Richard)$
  - $(King(Father(John)) \land Greedy(Father(John))) \rightarrow Evil(Father(John))$
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Existential Instantiation (EI)

- An existentially quantified-sentence can be substituted by one of its instantiation with a fresh constant:

$$\frac{\Gamma \wedge \exists x.\alpha}{\Gamma \wedge \alpha\{x/C\}}$$

  for every variable x and for a "fresh" constant C, i.e. a constant which does not appear in $\Gamma \wedge \exists x.\alpha$
- C is a Skolem constant, EI subcase of Skolemization (see later)
- Intuition: if there is an object satisfying some condition, then we give a (new) name to it
- Ex: $\exists x.(Crown(x) \wedge OnHead(x, John))$
  - $(Crown(C) \wedge OnHead(C, John))$
  - given "There is a crown on John's head", I call "C" such crown
- Preserves satisfiability (aka preserves inferential equivalence)
  $M(\Gamma \wedge \alpha\{x/C\}) \neq \emptyset$ *iff* $M(\Gamma \wedge \exists x.\alpha) \neq \emptyset$
  (i.e.. $(\Gamma \wedge \alpha\{x/C\}) \models \beta$ iff $(\Gamma \wedge \exists x.\alpha) \models \beta$, for every β)
- Example from math: $\exists x.(\frac{d(x^y)}{dy} = x^y)$, we call it "e" $\Longrightarrow (\frac{d(e^y)}{dy} = e^y)$

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Remarks

- About Universal Instantiation:
  - UI can be applied several times to add new sentences;
  - the new Γ is logically equivalent to the old Γ
- About Existential Instantiation:
  - EI can be applied once to replace the existential sentence;
  - the new Γ is not equivalent to the old,
  - but is (un)satisfiable iff the old Γ is (un)satisfiable

  $\Longrightarrow$ the new Γ can infer β iff the old Γ can infer β

Before applying UI or EI, sentences must be rewritten s.t. negations (even when implicit) must be pushed inside the quantifications:

- $\neg\forall x.\alpha \Longrightarrow \exists x.\neg\alpha$
- $\neg\exists x.\alpha \Longrightarrow \forall x.\neg\alpha$
- ex: $\forall x.P(x) \rightarrow \neg\exists y.Q(y)$

  $\Longrightarrow \neg\forall x.P(x) \vee \neg\exists y.Q(y)$

  $\Longrightarrow \exists x.\neg P(x) \vee \forall y.\neg Q(y)$

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   - Substitutions & Instantiations
   - From Propositional to First-Order Reasoning
   - Unification and Lifting
2. Handling Definite FOL KBs & Datalog
   - Forward Chaining (hints)
   - Backward Chaining (hints)
3. Resolution for General FOL KBs
   - CNF-Ization
   - Resolution
   - A Complete Example

Reduction to Propositional Inference (aka propositionalization))

- Idea: Given a FOL closed KB Γ and query α, Convert $(\Gamma \wedge \neg\alpha)$ to PL
  $\Longrightarrow$ use a PL SAT solver to check PL (un)satisfiability
- Trick:
  - replace variables with ground terms, creating all possible instantiations of quantified sentences
  - convert atomic sentences into propositional symbols

    e.g. "King(John)" $\Longrightarrow$ "King_John",
    e.g. "Brother(John,Richard)" $\Longrightarrow$ "Brother_John-Richard",
- Theorem: (Herbrand, 1930)
  If a ground sentence α is entailed by an FOL KB Γ,
  then it is entailed by a finite subset of the propositionalized KB Γ

  $\Longrightarrow$ Every FOL KB Γ can be propositionalized s.t. to preserve entailment
- The vice-versa does not hold
  $\Longrightarrow$ works if α is entailed, loops if α is not entailed

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    e.g. "King(John)" $\Longrightarrow$ "King_John",
    e.g. "Brother(John,Richard)" $\Longrightarrow$ "Brother_John-Richard",
- Theorem: (Herbrand, 1930)
  If a ground sentence α is entailed by an FOL KB Γ,
  then it is entailed by a finite subset of the propositionalized KB Γ

  $\Longrightarrow$ Every FOL KB Γ can be propositionalized s.t. to preserve entailment
- The vice-versa does not hold
  $\Longrightarrow$ works if α is entailed, loops if α is not entailed

Reduction to Propositional Inference: Example

- Suppose Γ contains only:

  $\forall x.((King(x) \land Greedy(x)) \rightarrow Evil(x))$
  $King(John)$
  $Greedy(John)$
  $Brother(Richard, John)$

- Instantiating the universal sentence in all possible ways:

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- The new Γ is propositionalized:

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- Propositionalization generates lots of irrelevant sentences

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Problems with Propositionalization [cont.]

- Problem: nested function applications
  - e.g. Father(John), Father(Father(John)), Father(Father(Father(John))), ...

$\Longrightarrow$ infinite instantiations

- Actual Trick: for k = 0 to ∞, use terms of function nesting depth k
  - create propositionalized Γ by instantiating depth-k terms
  - if $\Gamma \models \alpha$, then will find a contradiction for some finite k
  - if $\Gamma \not\models \alpha$, may find a loop forever
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Outline

Generalized Modus Ponens (GMP)

- "Lifted inference": Combine PL inference with UI/EI
- Aristotle's "Modus Ponens" syllogism:
  "All men are mortal; Socrates is a man; thus Socrates is mortal."

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- Ex:
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- $\{\forall x.\alpha, \forall x.\beta\} \Longleftrightarrow \{\forall x_1.\alpha\{x/x_1\}, \forall x_2.\beta\{x/x_2\}\}$, s.t. x_1, x_2 new

Most-General Unifier (MGU)

- Unifiers are not unique
  - ex: $Unify(Knows(John, x), Knows(y, z))$
    could return $\{y/John, x/z\}$ or $\{y/John, x/John, z/John\}$
- Given α, β, the unifier θ_1 is more general than the unifier θ_2 for α, β if exists θ_3 s.t. $\theta_2 = \theta_1\theta_3$
  - ex: $\{y/John, x/z\}$ more general than $\{y/John, x/John, z/John\}$:
    $\{y/John, x/John, z/John\} = \{y/John, x/z\}\{z/John\}$
- Theorem: If exists an unifier for α, β, then exists a most general unifier (MGU) θ for α, β
  - Ex: $\{y/John, x/z\}$ MGU for $Knows(John, x), Knows(y, z)$
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The Procedure Unify

function UNIFY(x, y, θ) **returns** a substitution to make x and y identical
inputs: x, a variable, constant, list, or compound expression
y, a variable, constant, list, or compound expression
θ, the substitution built up so far (optional, defaults to empty)

if θ = failure **then return** failure
else if $x = y$ **then return** θ
else if VARIABLE?(x) **then return** UNIFY-VAR(x, y, θ)
else if VARIABLE?(y) **then return** UNIFY-VAR(y, x, θ)
else if COMPOUND?(x) **and** COMPOUND?(y) **then**
return UNIFY(x.ARGS, y.ARGS, UNIFY(x.OP, y.OP, θ))
else if LIST?(x) **and** LIST?(y) **then**
return UNIFY(x.REST, y.REST, UNIFY(x.FIRST, y.FIRST, θ))
else return failure

---

function UNIFY-VAR(var, x, θ) **returns** a substitution

if $\{var/val\} \in \theta$ **then return** UNIFY(val, x, θ)
else if $\{x/val\} \in \theta$ **then return** UNIFY(var, val, θ)
else if OCCUR-CHECK?(var, x) **then return** failure
else return add $\{var/x\}$ to θ

Exercises

- Find the MGU of the following formulas by the Unify() procedure, or say there is none. (If needed, standardize apart them beforehand.)
  - $Knows(John, x),\ Knows(y, Mother(y))$
  - $Knows(John, x),\ Knows(x, OJ)$
  - $R(f(x), z),\ R(f(g(B)), y)$
  - $P(f(x)),\ P(g(f(y)))$
  - $P(h(x), B),\ P(A, y)$
- Invent arbitrary pairs of (lists of) atomic FOL formulas and apply Unify() to them

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Outline

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First-Order Definite Clauses & Datalog

- We assume no function symbol and no $\exists$ under the scope of $\forall$ (see later for general case)
- FOL Definite Clauses: clauses with exactly one positive literal
  - we omit universal quantifiers

  $\Longrightarrow$ variables are (implicitly) universally quantified
  - we remove existential quantifiers by EI

  $\Longrightarrow$ existentially-quantified variables are substituted by fresh constants
- Represent implications of atomic formulas
  - Ex: $\forall x.((King(x) \wedge Greedy(x)) \rightarrow Evil(x))$

  $\Longrightarrow$ $(\neg King(x) \vee \neg Greedy(x) \vee Evil(x))$
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Example (Datalog)

KB:

The law says that it is a crime for an American to sell weapons to hostile nations.
The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Goal:

Prove that Colonel West is a criminal.

Example (Datalog) [cont.]

- it is a crime for an American to sell weapons to hostile nations:
  $\forall x, y, z.((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow Criminal(x))$

$\Longrightarrow \neg American(x) \vee \neg Weapon(y) \vee \neg Hostile(z) \vee \neg Sells(x, y, z) \vee Criminal(x)$

- Nono ... has some missiles
  $\exists x.(Owns(Nono, x) \wedge Missile(x)) \Longrightarrow Owns(Nono, M_1) \wedge Missile(M_1)$
- All of its missiles were sold to it by Colonel West
  $\forall x.((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

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  $\forall x.(Missile(x) \rightarrow Weapon(x)) \Longrightarrow \neg Missile(x) \vee Weapon(x)$
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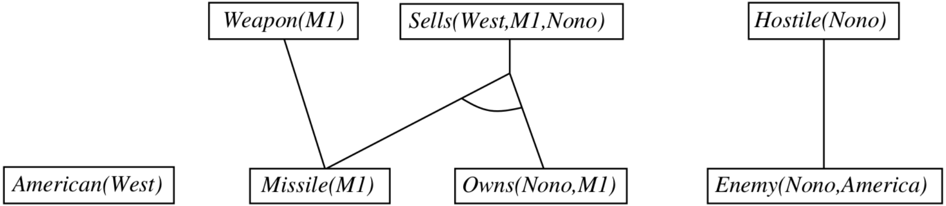

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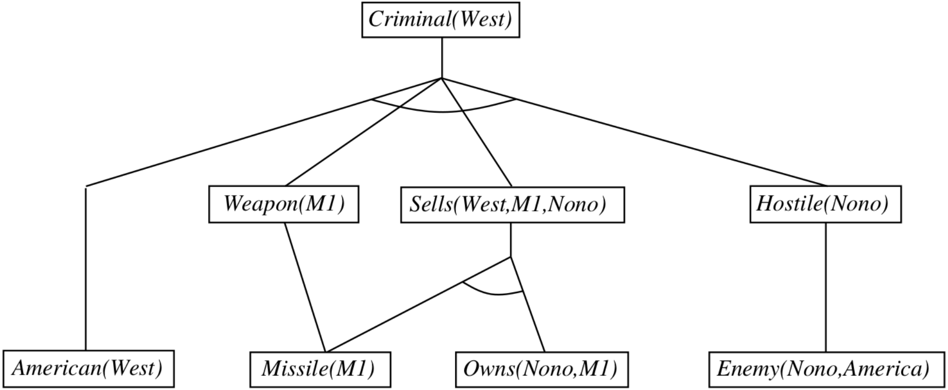

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Properties of Forward Chaining

Intuition: at every loop, add all new atomic sentences you can infer by GMP, checking them against the goal

- Sound: every inference is just an application of GMP
- Complete (for definite KBs): answers every query entailed by KB
- if $KB \models \alpha$, it always terminates
- if $KB \not\models \alpha$, may not terminate (Semi-decidable)
- Solves always Datalog queries in time: $O(p \cdot n^k)$, s.t. $p = \#predicates$, $n = \#number\ constants$, $k = maximum\ arity$
- Improvement: match a rule on iteration k only if a premise was added on iteration k-1
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Intuition: at every loop, add all new atomic sentences you can infer by GMP, checking them against the goal

- Sound: every inference is just an application of GMP
- Complete (for definite KBs): answers every query entailed by KB
- if $KB \models \alpha$, it always terminates
- if $KB \not\models \alpha$, may not terminate (Semi-decidable)
- Solves always Datalog queries in time: $O(p \cdot n^k)$, s.t. $p = \#predicates$, $n = \#number\ constants$, $k = maximum\ arity$
- Improvement: match a rule on iteration k only if a premise was added on iteration k-1
  $\Longrightarrow$ match each rule whose premise contains a newly added literal
- Matching can be expensive
  - matching conjunctive premises against known facts is NP-hard
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Outline

1. Basic First-Order Reasoning
   - Substitutions & Instantiations
   - From Propositional to First-Order Reasoning
   - Unification and Lifting
2. Handling Definite FOL KBs & Datalog
   - Forward Chaining (hints)
   - Backward Chaining (hints)
3. Resolution for General FOL KBs
   - CNF-Ization
   - Resolution
   - A Complete Example

Backward Chaining: Example

$American(West)$, $Missile(M_1)$, $Owns(Nono, M_1)$, $Enemy(Nono, America)$
$\forall x, y, z.((American(x) \land Weapon(y) \land Hostile(z) \land Sells(x, y, z)) \rightarrow Criminal(x))$
$\forall x.(Missile(x) \rightarrow Weapon(x))$ $\forall x.((Missile(x) \land Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$
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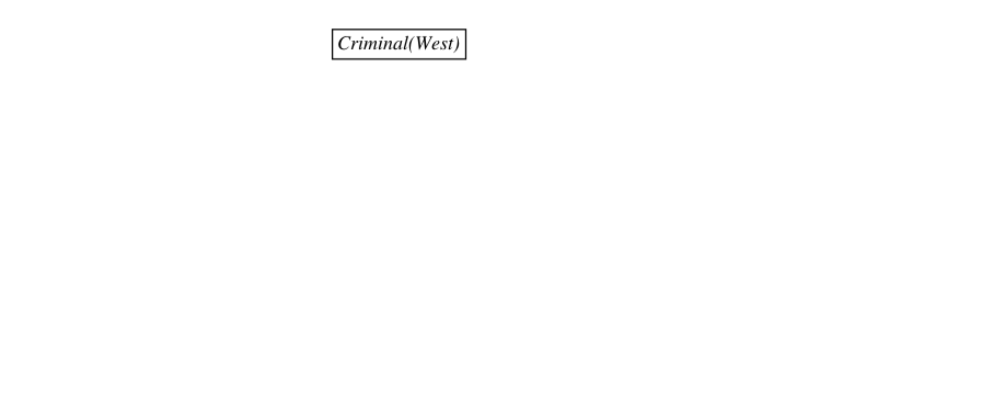

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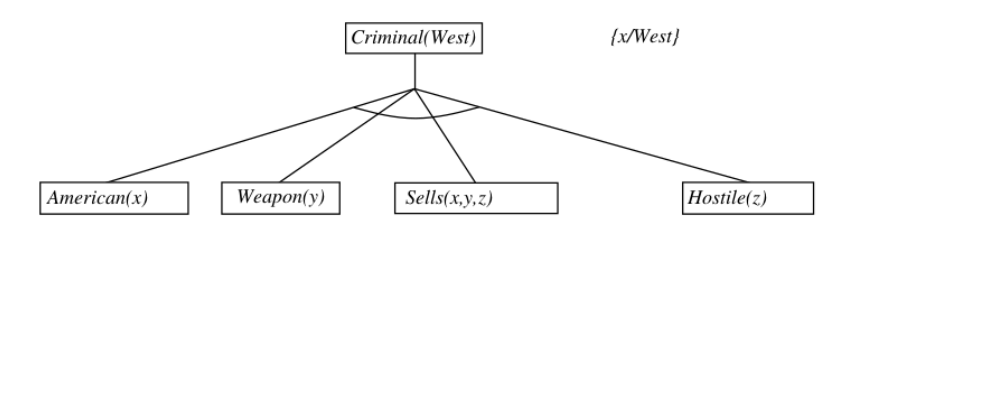

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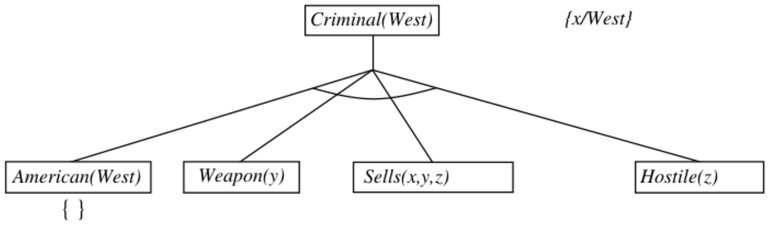

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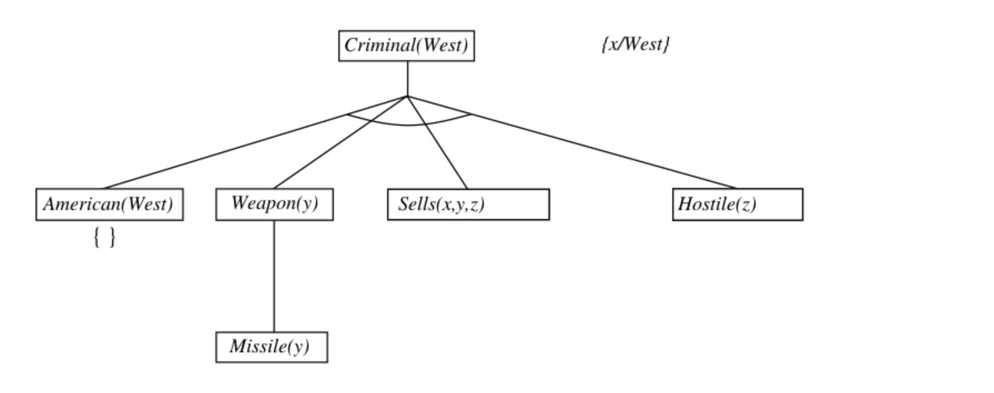

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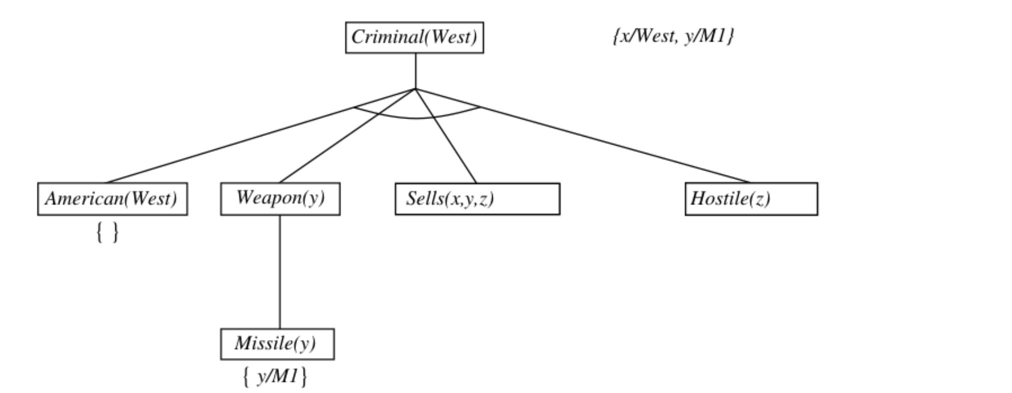

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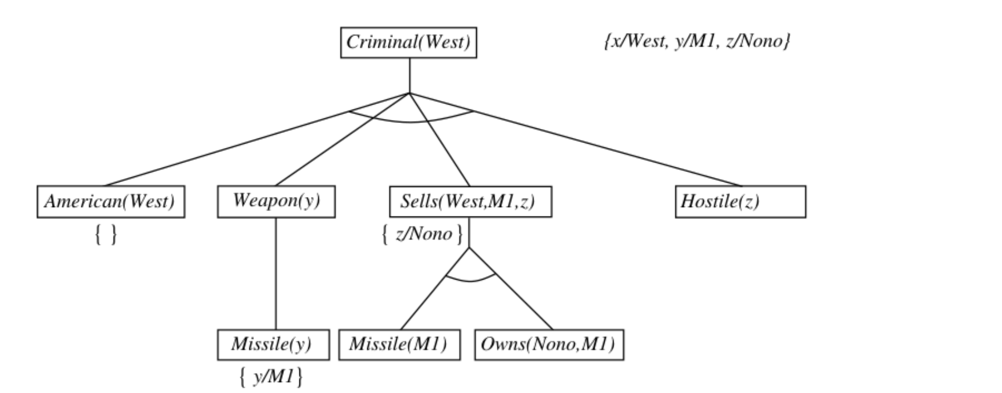

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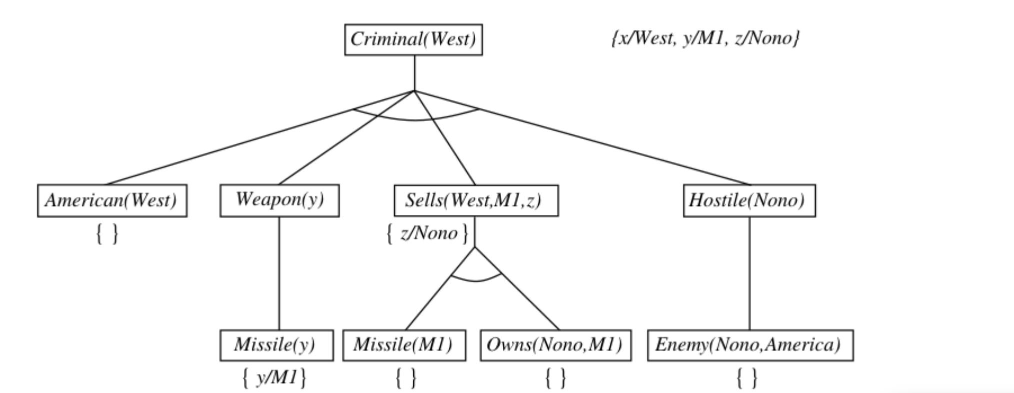

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Properties of Backward Chaining

Intuition: at every loop, pick a goal and one implication and apply GMP backwards, inferring the list of (unified) premises as sub-goals

- Depth-first recursive proof search: space is linear in size of proof
- Incomplete due to infinite loops
  - e.g., $P(x) \rightarrow P(x) \implies P(c), P(c), P(c)$... (easy to fix)
  - e.g., $Q(f(x)) \rightarrow Q(x) \implies Q(c), Q(f(c)), Q(f(f(c))), \ldots$
- Inefficient due to repeated subgoals
  - fix using caching of previous results $\implies$ need extra space!
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Conjunctive Normal Form (CNF)

- A FOL formula φ is in Conjunctive normal form iff it is a conjunction of disjunctions of quantifier-free literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

  - the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
  - every literal is a quantifier-free atom or its negation
  - free variables implicitly universally quantified
- Easier to handle: list of lists of literals.
  $\Longrightarrow$ no reasoning on the recursive structure of the formula
- Ex: $\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$

FOL CNF Conversion $CNF(\varphi)$

Convert into NNF

Every FOL formula φ can be reduced into CNF:

1. Eliminate implications and biconditionals:

$$\alpha \rightarrow \beta \quad \Longrightarrow \quad \neg\alpha \vee \beta$$
$$\alpha \leftrightarrow \beta \quad \Longrightarrow \quad (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

2. Push inwards negations recursively:

$$\neg(\alpha \wedge \beta) \quad \Longrightarrow \quad \neg\alpha \vee \neg\beta$$
$$\neg(\alpha \vee \beta) \quad \Longrightarrow \quad \neg\alpha \wedge \neg\beta$$
$$\neg\neg\alpha \quad \Longrightarrow \quad \alpha$$
$$\neg\forall x.\alpha \quad \Longrightarrow \quad \exists x.\neg\alpha$$
$$\neg\exists x.\alpha \quad \Longrightarrow \quad \forall x.\neg\alpha$$

$\Longrightarrow$ Negation normal form: negations only in front of atomic formulae

$\Longrightarrow$ quantified subformulas occur only with positive polarity

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FOL CNF Conversion $CNF(\varphi)$ [cont.]

Remove quantifiers

3. Standardize variables: each quantifier should use a different var
   $(\forall x.\exists y.\alpha) \land \exists y.\beta \land \forall x.\gamma \implies (\forall x.\exists y.\alpha) \land \exists y_1.\beta\{y/y_1\} \land \forall x_1.\gamma\{x/x_1\}$
4. Skolemize (a generalization of EI):
   Each existential variable is replaced by a fresh Skolem function applied to the enclosing universally-quantified variables

$$\begin{array}{lcl}
\exists y.\alpha & \implies & \alpha\{y/c\} \\
\forall x.(...\exists y.\alpha...) & \implies & \forall x.(...\alpha\{y/F_1(x)\}...) \\
\forall x_1 x_2.(...\exists y.\alpha...) & \implies & \forall x_1 x_2.(...\alpha\{y/F_1(x_1,x_2)...)\} \\
\exists y_1 \forall x_1 x_2 \exists y_2 \forall x_3 \exists y_3.\alpha & \implies & \forall x_1 x_2 x_3.\alpha\{y_1/c, y_2/F_1(x_1,x_2), y_3/F_2(x_1,x_2,x_3)\}
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   Ex: $\forall x \exists y.Father(y,x) \implies \forall x.Father(s(x),x)$
   ($s(x)$ implictly means "father of x" although s() is a fresh function)
5. Drop universal quantifiers: $\forall x_1...x_k.\alpha \implies \alpha$
   $\implies$ free variables implicitly universally quantified

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6. CNF-ize propositionally (see previous chapters):
   either apply recursively the DeMorgan's Rule: $(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
   or rename subformulas and add definitions: $(\alpha \wedge \beta) \vee \gamma \implies (B \vee \gamma) \wedge CNF(B \leftrightarrow (\alpha \wedge \beta))$
7. Standardize Apart (again) (Personal suggestion, not in AIMA book):
   prevent the same (implicitly universally-quantified) variable to occur in distinct clauses
   (correct because $\forall x.(\alpha \wedge \beta)$ equivalent to $\forall x.\alpha \wedge \forall y.\beta$)

Properties of FOL CNF-ization

- Preserves satisfiability: $M(\varphi) \neq \emptyset$ iff $M(CNF(\varphi)) \neq \emptyset$
- $\Longrightarrow$ Preserves entailment: $\varphi \models \alpha$ iff $CNF(\varphi) \models \alpha$ (in fact, $\varphi \wedge \neg\alpha$ unsat iff $\varphi \wedge \neg CNF(\alpha)$ unsat)
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Conversion to CNF: Example

Consider: "Everyone who loves all animals is loved by someone"

$\forall x.([\forall y.(Animal(y) \to Loves(x,y))] \to [\exists y.Loves(y,x)])$

1. Eliminate implications and biconditionals:

   $\forall x.(\neg[\forall y.(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y.Loves(y,x)])$

2. Push inwards negations recursively (NNF)

   $\forall x.([\exists y.\neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y.Loves(y,x)])$

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3. Standardize variables:

   $\forall x.([\exists y.(Animal(y) \land \neg Loves(x,y))] \lor [\exists z.Loves(z,x)])$

4. Skolemize:

   $\forall x.([Animal(F(x)) \land \neg Loves(x,F(x))] \lor [Loves(G(x),x)])$

   (F(x): "an animal unloved by x"; G(x): "someone who loves x")

5. Drop universal quantifiers::

   $[Animal(F(x)) \land \neg Loves(x,F(x))] \lor [Loves(G(x),x)]$

6. CNF-ize propositionally (and standardize apart the result):

$(Animal(F(x)) \lor Loves(G(x),x)) \land (\neg Loves(x_1, F(x_1)) \lor Loves(G(x_1), x_1))$

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Conversion to CNF: Example

Consider: "Everyone who loves all animals is loved by someone"

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Remark about Skolemization

Common mistake to avoid

- Do <u>not</u>
  - apply Skolemization or
  - drop universal quantifiers

  before converting into NNF & standardize apart variables!
- Polarity of quantified subformulas affects Skolemization!

$\Longrightarrow$ NNF-ization may convert $\exists$'s into $\forall$'s, and vice versa

- Same-name quantified variable may cause errors

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$\forall x.([\forall y.(Animal(y) \rightarrow Loves(x,y))] \rightarrow [\exists y.Loves(y,x)])$

1. Too-early Skolemization & universal-quantifier dropping:
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"y" should be a Skolem function F(x) instead
because "$\forall y.(...)$" occurred negatively
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Exercise

Did Curiosity kill the cat?

Formalize and CNF-ize the following:

Everyone who loves all animals is loved by someone.
Anyone who kills an animal is loved by no one.
Jack loves all animals.
Either Jack or Curiosity killed the cat, who is named Tuna.
Did Curiosity kill the cat?

(See also AIMA book for FOL formalization and CNF-ization)

Outline

Resolution

- **FOL resolution rule**, let $\theta \stackrel{\text{def}}{=} mgu(l_i, \neg m_j)$, s.t. $l_i\theta = \neg m_j\theta$:

$$\frac{(l_1 \vee \ldots \vee l_i \vee \ldots \vee l_k) \qquad (m_1 \vee \ldots \vee m_j \vee \ldots \vee m_n)}{(l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n)\theta}$$

- Ex: $\frac{Man(Socrates) \qquad (\neg Man(x) \vee Mortal(x))}{Mortal(Socrates)}$ s.t. $\theta \stackrel{\text{def}}{=} \{x/Socrates\}$
- To prove that $\Gamma \models \alpha$ in FOL:
  - convert $\Gamma \wedge \neg\alpha$ to CNF
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    - the empty clause is generated $\Longrightarrow \Gamma \models \alpha$
    - no more resolution step is applicable $\Longrightarrow \Gamma \not\models \alpha$
    - resource (time, memory) exhausted $\Longrightarrow$??
  - Hint: apply resolution first to unit clauses (unit resolution)
  - Unit resolution alone complete for definite clauses
    - choose positive unit-clauses first (DFS) $\Longrightarrow$ Forward chaining
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- Refutation-Complete:
  - If there is a substitution θ such that $\Gamma \models \theta\alpha$, then it will return θ
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  - If there is a substitution θ such that $\Gamma \models \theta\alpha$, then it will return θ
  - If there is no such θ, then the procedure may not terminate
  - Many strategies and tools available

Resolution

- **FOL resolution rule**, let $\theta \stackrel{\text{def}}{=} mgu(l_i, \neg m_j)$, s.t. $l_i\theta = \neg m_j\theta$:

$$\frac{(l_1 \vee ... \vee l_i \vee ... \vee l_k) \qquad (m_1 \vee ... \vee m_j \vee ... \vee m_n)}{(l_1 \vee ... \vee l_{i-1} \vee l_{i+1} \vee ... \vee l_k \vee m_1 \vee ... \vee m_{j-1} \vee m_{j+1} \vee ... \vee m_n)\theta}$$

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Example: Resolution with Definite Clauses

KB:
The law says that it is a crime for an American to sell weapons to hostile nations.
The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.
Goal: Prove that Colonel West is a criminal.

Example: Resolution with Definite Clauses [cont.]

- it is a crime for an American to sell weapons to hostile nations:
  $\forall x, y, z.((American(x) \land Weapon(y) \land Hostile(z) \land Sells(x, y, z)) \rightarrow Criminal(x))$

$\Longrightarrow$ $\neg American(x) \lor \neg Weapon(y) \lor \neg Hostile(z) \lor \neg Sells(x, y, z) \lor Criminal(x)$

- Nono ... has some missiles
  $\exists x.(Owns(Nono, x) \land Missile(x)) \Longrightarrow Owns(Nono, M_1) \land Missile(M_1)$
- All of its missiles were sold to it by Colonel West
  $\forall x.((Missile(x) \land Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

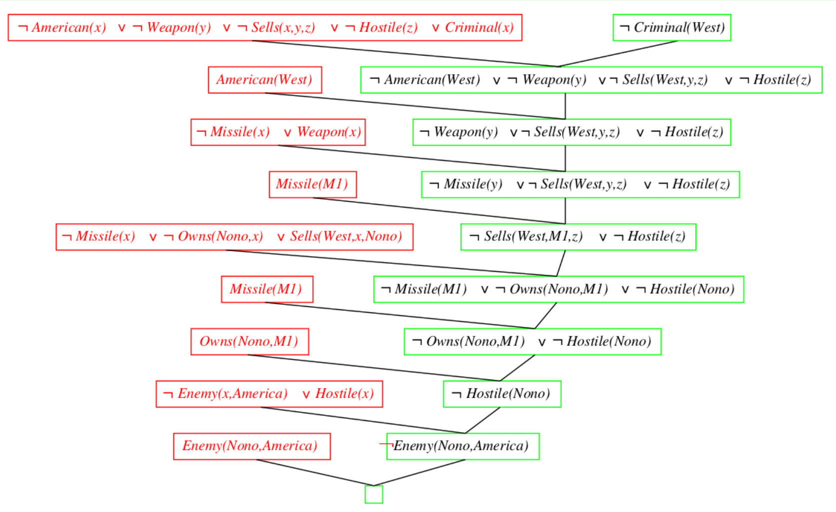
$\Longrightarrow$ $\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$

- Missiles are weapons:
  $\forall x.(Missile(x) \rightarrow Weapon(x)) \Longrightarrow \neg Missile(x) \lor Weapon(x)$
- An enemy of America counts as “hostile”: $\forall x.(Enemy(x, America) \rightarrow Hostile(x))$

$\Longrightarrow$ $\neg Enemy(x, America) \lor Hostile(x)$

- West, who is American ...: $American(West)$
- The country Nono, an enemy of America ...: $Enemy(Nono, America)$

Example: Resolution with Definite Clauses



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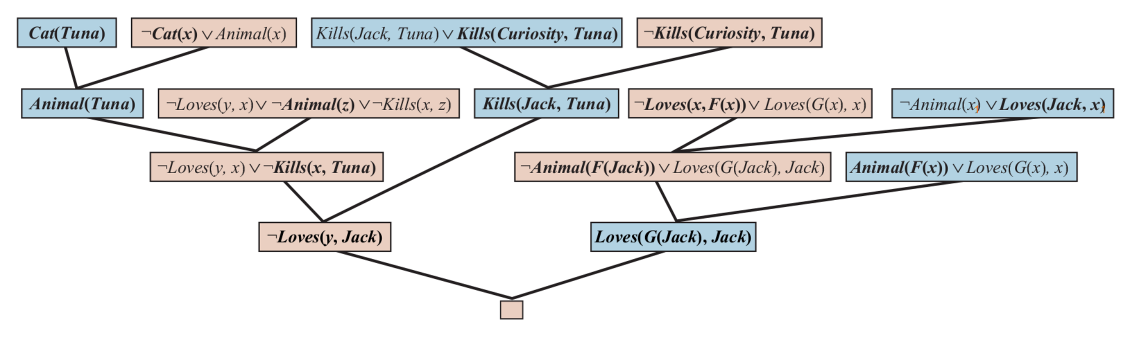
Exercise: Resolution with Definite Clauses

Resolve the problem of previous example:

1. selecting positive unit clauses first (DFS) $\Longrightarrow$ Forward chaining
2. selecting negative clauses first first (DFS) $\Longrightarrow$ Backward chaining
3. selecting unit-literals in any order first $\Longrightarrow$ Mixed chaining

Example: Resolution with General Clauses

Everyone who loves all animals is loved by someone.
Anyone who kills an animal is loved by no one.
Jack loves all animals.
Either Jack or Curiosity killed the cat, who is named Tuna.
Did Curiosity kill the cat?
(See previous exercise or AIMA book for FOL formalization and CNF-ization.)



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Resolution Strategies

Saturation Calculus:

- Given N_0 : set of (implicitly universally quantified) clauses.
- Derive N_0, N_1, N_2, N_3, ... s.t. $N_{i+1} = N_i \cup \{C\}$,
  - where C is the conclusion of a resolution step from premises in N_i
- (under reasonable restrictions) is refutationally complete :

$$N_0 \models \bot \implies \bot \in N_i \text{ for some i}$$

Problem

- The resolution rule is prolific.
  - it generates many useless intermediate results
  - it may generate the same clauses in many different ways
- This motivates the introduction of resolution restrictions.

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Resolution Restrictions

Ordered resolution

- define stable atom ordering;
- resolve only maximal literals

Hyper-Resolution

- Clauses are divided into
  - "nuclei": those with ≥ 1 negative literals
  - "electrons" : those with positive literals only
- Resolution can occur only among one nucleus and one electron
- $$Ex: \quad \frac{\dfrac{\neg P(x) \lor \neg Q(x) \lor R(x) \quad Q(A) \lor C}{\neg P(A) \lor R(A) \lor C} \qquad P(A) \lor D}{R(A) \lor C \lor D}$$
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Exercise

- Solve the example of Colonel West using Hyper-Resolution strategy
- Solve the example of Curiosity & Tuna using Hyper-Resolution Strategy

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Outline

Exercise

Problem

Consider the following FOL formula set Γ:

1. $\forall x.\{[\forall y.(\mathrm{Child}(y) \rightarrow \mathrm{Loves}(x, y))] \rightarrow [\exists y.\mathrm{Loves}(y, x)]\}$
2. $\forall x.[\mathrm{Child}(x) \rightarrow \mathrm{Loves}(\mathrm{Mark}, x)]$
3. $\mathrm{Beats}(\mathrm{Mark}, \mathrm{Paul}) \vee \mathrm{Beats}(\mathrm{John}, \mathrm{Paul})$
4. $\mathrm{Child}(\mathrm{Paul})$
5. $\forall x.\{[\exists z.(\mathrm{Child}(z) \wedge \mathrm{Beats}(x, z))] \rightarrow [\forall y.\neg \mathrm{Loves}(y, x)]\}$

(a) Compute the CNF-ization of Γ, Skolemize & standardize variables

(b) Write a FOL-resolution inference of the query $\mathrm{Beats}(\mathrm{John}, \mathrm{Paul})$ from the CNF-ized KB

Exercise solution

CNF-ization

(a) Compute the CNF-ization of Γ, Skolemize & standardize variables

1. $\forall x.\{[\forall y.(\text{Child}(y) \rightarrow \text{Loves}(x,y))] \rightarrow [\exists y.\text{Loves}(y,x)]\}$
   $\forall x.\{[\neg\forall y.(\text{Child}(y) \rightarrow \text{Loves}(x,y))] \vee [\exists y.\text{Loves}(y,x)]\}$
   $\forall x.\{[\exists y.(\text{Child}(y) \wedge \neg\text{Loves}(x,y))] \vee [\exists y.\text{Loves}(y,x)]\}$
   $\{[(\text{Child}(F(x)) \wedge \neg\text{Loves}(x,F(x)))] \vee [\text{Loves}(G(x),x)]\}$
   1. $\text{Child}(F(x)) \vee \text{Loves}(G(x),x)$
   2. $\neg\text{Loves}(y,F(y)) \vee \text{Loves}(G(y),y)$
2. $\neg\text{Child}(z) \vee \text{Loves}(\text{Mark},z)$
3. $\text{Beats}(\text{Mark},\text{Paul}) \vee \text{Beats}(\text{John},\text{Paul})$
4. $\text{Child}(\text{Paul})$
5. $\forall x.\{[\exists z.(\text{Child}(z) \wedge \text{Beats}(x,z))] \rightarrow [\forall y.\neg\text{Loves}(y,x)]\}$
   $\forall x.\{[\neg\exists z.(\text{Child}(z) \wedge \text{Beats}(x,z))] \vee [\forall y.\neg\text{Loves}(y,x)]\}$
   $\forall x.\{[\forall z.(\neg\text{Child}(z) \vee \neg\text{Beats}(x,z))] \vee [\forall y.\neg\text{Loves}(y,x)]\}$
   $\neg\text{Child}(z_2) \vee \neg\text{Beats}(x_2,z_2) \vee \neg\text{Loves}(y_2,x_2)$

where $F()$, $G()$ are Skolem unary functions.

Exercise solution [cont.]

Resolution

(b) Write a FOL-resolution inference of the query Beats(John, Paul) from the CNF-ized KB:

6. $[1.2,\ 2.] \Longrightarrow \neg \mathrm{Child}(F(\mathrm{Mark})) \lor \mathrm{Loves}(G(\mathrm{Mark}), \mathrm{Mark})$;
7. $[1.1,\ 6.] \Longrightarrow \mathrm{Loves}(G(\mathrm{Mark}), \mathrm{Mark})$;
8. $[4,\ 5.] \Longrightarrow \neg \mathrm{Beats}(x_2, \mathrm{Paul}) \lor \neg \mathrm{Loves}(y_2, x_2)$;
9. $[7,\ 8.] \Longrightarrow \neg \mathrm{Beats}(\mathrm{Mark}, \mathrm{Paul})$;
10. $[3,\ 9.] \Longrightarrow \mathrm{Beats}(\mathrm{John}, \mathrm{Paul})$;