Fundamentals of Artificial Intelligence
Chapter 14: **Probabilistic Reasoning**

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Outline

Outline

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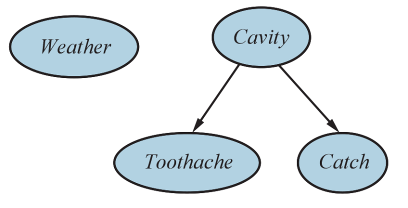
Bayesian Networks

Bayesian Networks (aka Belief Networks):

- Syntax: a directed acyclic graph (DAG):
  - each node represents a random variable (discrete or continuous)
  - directed arcs connect pairs of nodes: $X \rightarrow Y$ (X is a parent of Y)
  - a conditional distribution $\mathbf{P}(X_i | Parents(X_i))$ for each node X_i
- Conditional distribution represented as a conditional probability table (CPT)
  - distribution over X_i for each combination of parent values
- Allow for compact specification of full joint distributions
- Represent explicit conditional dependencies among variables:
  an arc from X to Y means that X has a direct influence on Y
- Topology encodes conditional independence assertions:
  - Toothache, Catch conditionally independent given Cavity
  - Tootchache, Catch depend on Cavity
  - Weather independent from others
- No arc $\Longleftrightarrow$ independence



Example (from Judea Pearl, UCLA)

"*The burglary alarm goes off very likely on burglary and occasionally on earthquakes. John and Mary are neighbors who agreed to call when the alarm goes off. Their reliability is different ...*"

- Variables: Burglary, Earthquake, Alarm, JohnCalls, MaryCalls
- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call
- CPTs:
  - alarm setoff if bunglar in 94% of cases
  - alarm setoff if hearthquake in 29% of cases
  - false alarm setoff in 0.1% of cases

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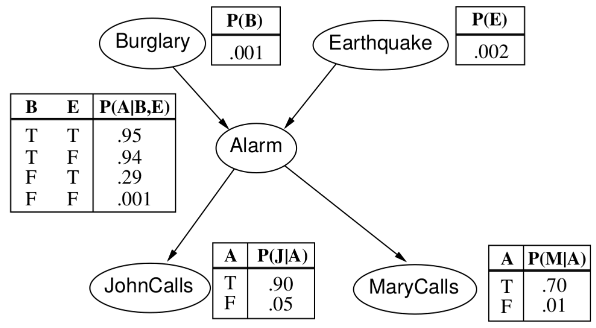


- Notice: in CPTs like $\mathbf{P}(A|B)$, only $P(a|B)$ are reported, because $P(\neg a|B) = 1 - P(a|B)$

Outline

Compactness of Bayesian Networks

- In most domains, it is reasonable to suppose that each random variable X_i is directly influenced by only a small number k_i of other variables, called parents of X_i (parents(X_i))
- A CPT for Boolean X_i with k_i Boolean parents has
  - 2^{k_i} rows for the combinations of parent values
  - each row requires one number p for $P(X_i = true)$ ($P(X_i = false) = 1 - P(X_i = true)$)

$\Longrightarrow$ If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

  - a full joint distribution requires $2^n - 1$ numbers
  - linear vs. exponential!
- Ex: for burglary example:
  - $1 + 1 + 4 + 2 + 2 = 10$ numbers vs. $2^5 - 1 = 31$

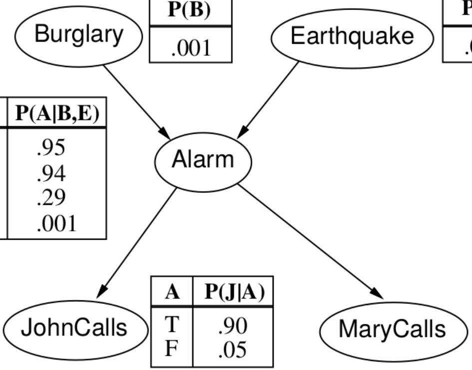
Global Semantics of Bayesian Networks

- Global semantics defines the full joint distribution as the product of the local conditional distributions:
  $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
  - if X_i has no parent, then conditional distributions reduce to prior probability $\mathbf{P}(X_i)$
- Intuition: order $X_1, ..., X_n$ s.t. $parents(X_i) \prec X_i$ for each i:
  $\mathbf{P}(X_1, ..., X_n)$
  $= \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, ..., X_{i-1})$ // chain rule
  $= \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$ // conditional independence

$\Longrightarrow$ A Bayesian network is a distributed representation of the full joint distribution

Global Semantics: Example

- $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
- Ex: "Prob. that both John and Mary call, the alarm sets off but no burglary nor earthquake"
  $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) =$
  $P(j | m \wedge a \wedge \neg b \wedge \neg e) P(m | a \wedge \neg b \wedge \neg e) P(a | \neg b \wedge \neg e) P(\neg b | \neg e) P(\neg e) =$
  $P(j|a)\ P(m|a)\ P(a|\neg b \wedge \neg e)\ P(\neg b)\ P(\neg e) = 0.9 \cdot 0.7 \cdot 0.001 \cdot 0.999 \cdot 0.998$
  ≈ 0.00063



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

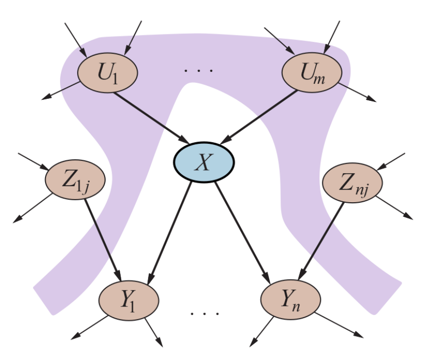
Exercises

- Compute:
  - The probability that John calls and Mary does not, the alarm is not set off with a burglar entering during an earthquake
  - The probability that John calls and Mary does not, given a burglar entering the house
  - The probability of an earthquake given the fact that John has called
  - ...

Outline

Local Semantics

- Local Semantics: each node is conditionally independent of its nondescendants given its parents: $\mathbf{P}(X|U_1,..,U_m,Z_{1j},...,Z_{nj}) = \mathbf{P}(X|U_1,..,U_m)$, for each X
  - “nondecendants” include ancestors
- Theorem: Local semantics holds iff global semantics holds: $\mathbf{P}(X_1,...,X_N) = \prod_{i=1} \mathbf{P}(X_i|parents(X_i))$



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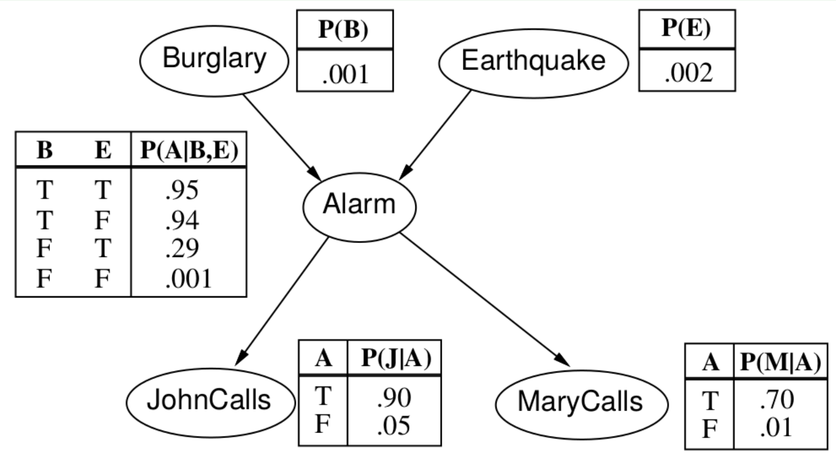
Local Semantics: Example

Ex: JohnCalls is independent of Burglary, Earthquake, and MaryCalls given the value of Alarm

$$\mathbf{P}(\mathit{JohnCalls}|\mathit{Alarm}, \mathit{Burglary}, \mathit{Earthquake}, \mathit{MaryCalls}) = \mathbf{P}(\mathit{JohnCalls}|\mathit{Alarm})$$



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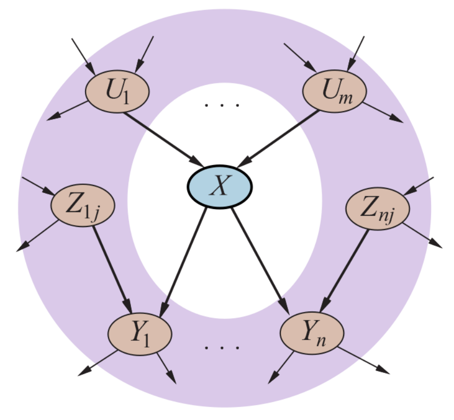
Independence Property: Markov Blanket

In an Bayesian Network, each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents:

$\mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj},W_1,...,W_k) = \mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj})$, for each X



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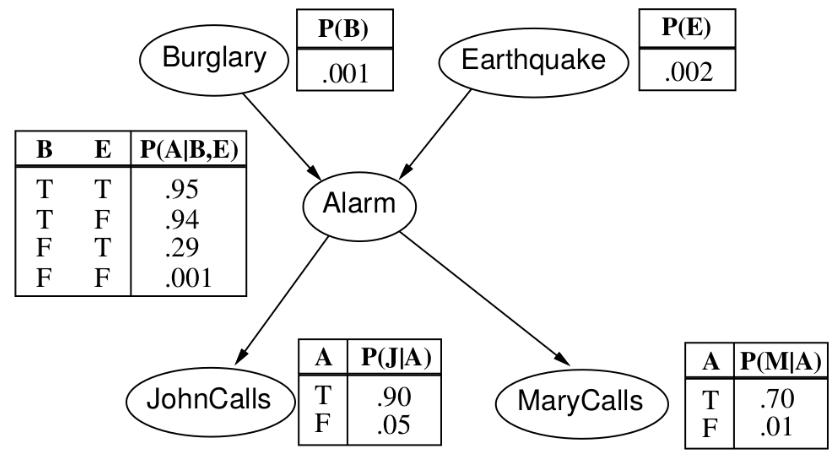
Markov Blanket: Example

Ex: Burglary is independent of JohnCalls and MaryCalls, given Alarm and Earthquake

$$\mathbf{P}(Burglary|Alarm, Earthquake, JohnCalls, MaryCalls) = \mathbf{P}(Burglary|Alarm, Earthquake)$$



Exercise

Verify numerically the two previous examples:

- Local Semantics
- Markov Blanket

Outline

Constructing Bayesian Networks

Building the graph

Given a set of random variables

1. Choose an ordering $\{X_1, ..., X_n\}$
   - in principle, any ordering will work (but some may cause blowups)
   - general rule: follow causality, $X \prec Y$ if $X \in causes(Y)$
2. For i=1 to n do
   1. add X_i to the network
   2. as $Parents(X_i)$, choose a subset of $\{X_1, ..., X_{i-1}\}$ s.t. $P(X_i | X_1, ..., X_{i-1}) = P(X_i | Parents(X_i))$

$\Longrightarrow$ Guarantees the global semantics by construction

$$\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$$

Constructing Bayesian Networks: Example

Suppose we choose the ordering $\{M, J, A, B, E\}$ (non-causal ordering):

$P(J|M) = P(J)$?

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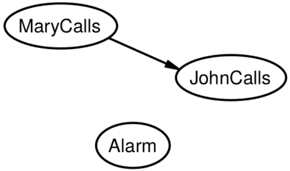
Constructing Bayesian Networks: Example

Suppose we choose the ordering $\{M, J, A, B, E\}$ (non-causal ordering):



$P(J|M) = P(J)$? No

$P(A|J, M) = P(A|J)$? $P(A|J, M) = P(A)$?

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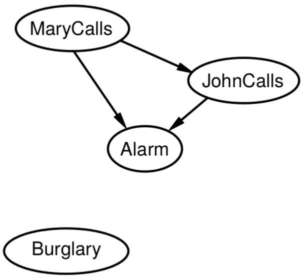
Constructing Bayesian Networks: Example

Suppose we choose the ordering $\{M, J, A, B, E\}$ (non-causal ordering):



$P(J|M) = P(J)$? No
$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$? No
$P(B|A,J,M) = P(B|A)$?
$P(B|A,J,M) = P(B)$?

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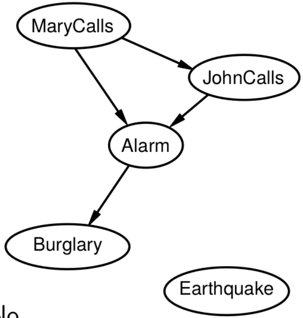
Constructing Bayesian Networks: Example

Suppose we choose the ordering $\{M, J, A, B, E\}$ (non-causal ordering):



$P(J|M) = P(J)$? No

$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$? No

$P(B|A,J,M) = P(B|A)$? Yes

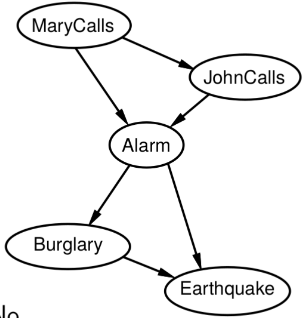
$P(B|A,J,M) = P(B)$? No

$P(E|B,A,J,M) = P(E|A)$?

$P(E|B,A,J,M) = P(E|A,B)$?

Constructing Bayesian Networks: Example

Suppose we choose the ordering $\{M, J, A, B, E\}$ (non-causal ordering):



$P(J|M) = P(J)$? No

$P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$? No

$P(B|A,J,M) = P(B|A)$? Yes

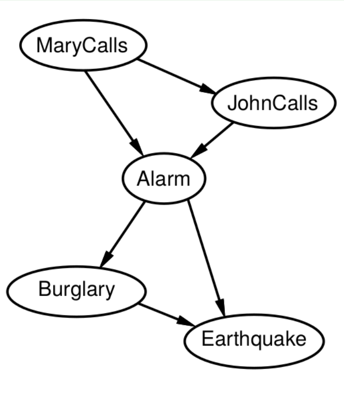
$P(B|A,J,M) = P(B)$? No

$P(E|B,A,J,M) = P(E|A)$? No

$P(E|B,A,J,M) = P(E|A,B)$? Yes

Constructing Bayesian Networks: Example [cont.]

- In non-causal directions
  - deciding conditional independence is hard
  - assessing conditional probabilities is hard
  - typically networks less compact
- Ex: $1+2+4+2+4=13$ numbers needed (rather than 10)
- Can be much worse
  - ex: try $\{M, J, E, B, A\}$ (see AIMA)
  - ex: try $\{J, M, E, B, A\}$
  - ex: try $\{J, M, E, A, B\}$
- Much better with causal orderings
  - ex: try either
    $\{B, E, A, J, M\}$
    $\{E, B, A, J, M\}$
    $\{B, E, A, M, J\}$
    $\{E, B, A, M, J\}$
  - i.e. $\{B, E\} \prec A \prec \{J, M\}$
    (both B and E cause A, A causes both M and J)



Building Conditional Probability Tables, CPTs

- Problem: CPT grow exponentially with number of parents
- If the causes don't interact: use a Noisy-OR distribution (generalization of logical or)
  - assume parents $U_1, ..., U_k$ include all causes (can add leak node representing "other causes"): $P(\neg X|\neg U_1 \ldots \neg U_k) = 1.0$
  - assume independent failure probability $q_i \stackrel{\text{def}}{=} P(\neg X|U_i \wedge \bigwedge_{j \neq i} \neg U_j)$ for each cause U_i: $P(\neg X|U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = \prod_{i=1}^{j} q_i$

  $\Longrightarrow$ number of parameters linear in number of parents!
- Ex: $q_{Cold} = 0.6,\ q_{Flu} = 0.2,\ q_{Malaria} = 0.1$:

$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	**0.0**	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

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Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network

Outline

Exact inference in Bayesian Networks

- Given:
  - X: the query variable (we assume one for simplicity)
  - **E**/**e**: the set of evidence variables $\{E_1, ..., E_m\}$ and of evidence values $\{e_1, ..., e_m\}$
  - **Y**/**y**: the set of unknown variables (aka hidden variables) $\{Y_1, ..., Y_l\}$ and unknown values $\{y_1, ..., y_l\}$

  $\Longrightarrow$ $\mathbf{X} = X \cup \mathbf{E} \cup \mathbf{Y}$

  A typical query asks for the posterior probability distribution:
  P(X | **E**=**e**) (also written **P**(X | **e**))
- Ex: $\mathbf{P}(Burglar|JohnCalls = true, MaryCalls = true)$
  - query: *Burglar*
  - evidence variables: $\mathbf{E} = \{JohnCalls, MaryCalls\}$
  - hidden variables: $\mathbf{Y} = \{Earthquake, Alarm\}$

Outline

Inference by Enumeration

- We defined a procedure for the task as: $\mathbf{P}(X|\mathbf{e}) = \alpha \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$

$\Longrightarrow$ $\mathbf{P}(X, \mathbf{e}, \mathbf{y})$ can be rewritten as product of prior and conditional probabilities according to the Bayesian Network

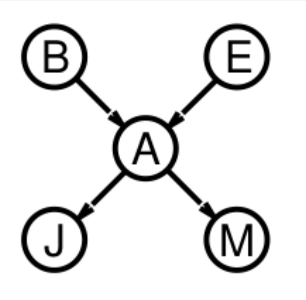
  - then apply factorization and simplify algebraically when possible

- Ex:

$$\mathbf{P}(B|j,m) =$$
$$\alpha \sum_e \sum_a \mathbf{P}(B,e,a,j,m) =$$
$$\alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a) =$$
$$\alpha \mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a)$$

$\Longrightarrow$ $P(b|j,m) =$

$$\alpha P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(m|a)$$



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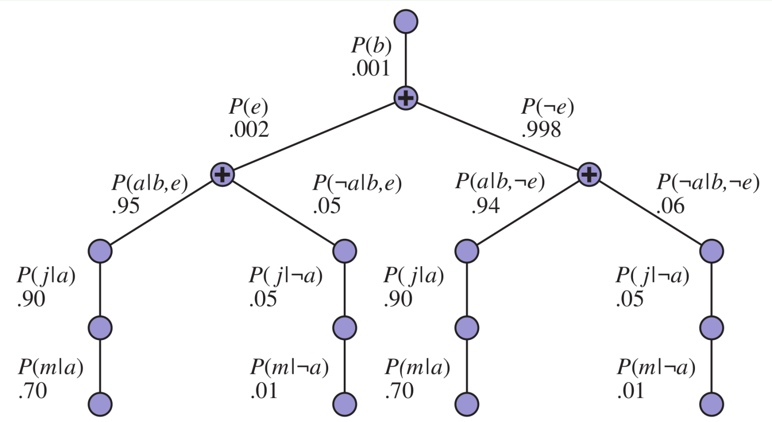
- Recursive depth-first enumeration: $O(n)$ space, $O(2^n)$ time with n propositional variables
- Enumeration can be inefficient: repeated computation

Inference by Enumeration: Example

$P(\ \ b|j,m) = \alpha P(\ \ b) \sum_e P(e) \sum_a P(a|\ \ b,e) P(j|a) P(m|a) = \alpha \cdot 0.00059224$

$P(\neg b|j,m) = \alpha P(\neg b) \sum_e P(e) \sum_a P(a|\neg b,e) P(j|a) P(m|a) = \alpha \cdot 0.0014919$

$\Longrightarrow \mathbf{P}(B|j,m) = \alpha \cdot \langle 0.00059224, 0.0014919 \rangle = [normal.] \approx \langle 0.284, 0.716 \rangle$



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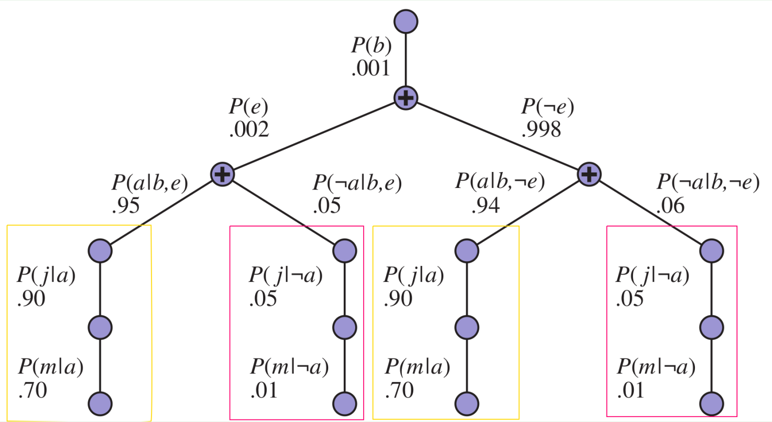
Repeated computation: $P(j|a)P(m|a)$ & $P(j|\neg a)P(m|\neg a)$ for each value of e

Inference by Enumeration: Example

$P(\ b|j,m) = \alpha P(\ b) \sum_e P(e) \sum_a P(a|\ b,e) P(j|a) P(m|a) = \alpha \cdot 0.00059224$
$P(\neg b|j,m) = \alpha P(\neg b) \sum_e P(e) \sum_a P(a|\neg b,e) P(j|a) P(m|a) = \alpha \cdot 0.0014919$
$\Longrightarrow \mathbf{P}(B|j,m) = \alpha \cdot \langle 0.00059224, 0.0014919 \rangle = [normal.] \approx \langle 0.284, 0.716 \rangle$



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Repeated computation: $P(j|a)P(m|a)$ & $P(j|\neg a)P(m|\neg a)$ for each value of e

Enumeration Algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X computes $\mathbf{P}(X \mid \mathbf{e})$
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* $\mathbf{Y}$ = *hidden variables* */

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$) computes P(xi, $\mathbf{Y}$, $\mathbf{e}$) (single probability value)
      where $\mathbf{e}_{x_i}$ is $\mathbf{e}$ extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$) X or evidence var
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$) hidden var
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network
   (b) Compute the query $P(P_{1,3}|b^*, p^*)$ via enumeration
   (c) Compare the result with that of the example in Ch. 13

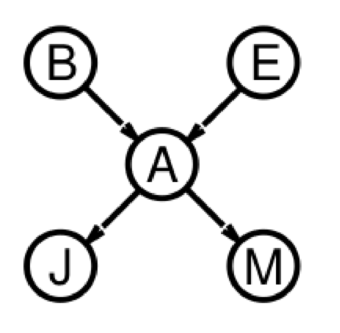
Outline

Inference by Variable Elimination

- Variable elimination:
  - carry out summations right-to-left (i.e., bottom-up in the tree)
  - store intermediate results (factors) to avoid recomputation
- Ex: $\mathbf{P}(B|j,m)$

$$
\begin{aligned}
&= \alpha \overbrace{\mathbf{P}(B)}^{\mathbf{f}_1(B)} \sum_e \overbrace{P(e)}^{\mathbf{f}_2(E)} \sum_a \overbrace{\mathbf{P}(a|B,e)}^{\mathbf{f}_3(A,B,E)} \overbrace{P(j|a)}^{\mathbf{f}_4(A)} \overbrace{P(m|a)}^{\mathbf{f}_5(A)} \\
&= \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) \\
&= \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B,E) \;\; (\textit{sum out } A) \\
&= \alpha \mathbf{f}_1(B) \times \mathbf{f}_7(B) \;\; (\textit{sum out } E) \\
&= \alpha \mathbf{f}_8(B)
\end{aligned}
$$

  - $\mathbf{f}_5(A) \stackrel{\text{def}}{=} \begin{bmatrix} P(m|\ \ a) \\ P(m|\neg a) \end{bmatrix}$, $\mathbf{f}_4(A) \stackrel{\text{def}}{=} \begin{bmatrix} P(j|\ \ a) \\ P(j|\neg a) \end{bmatrix}$, ...
  - "$+$" standard matric sum; "$\times$" pointwise product (see later)
  - $\mathbf{f}_6(B,E) = \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) =$ $(\mathbf{f}_3(a,B,E) \times \mathbf{f}_4(a) \times \mathbf{f}_5(a)) + (\mathbf{f}_3(\neg a,B,E) \times \mathbf{f}_4(\neg a) \times \mathbf{f}_5(\neg a))$
  - $\mathbf{f}_7(B) = \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B,E) =$ $(\mathbf{f}_2(e) \times \mathbf{f}_6(B,e) + (\mathbf{f}_2(\neg e) \times \mathbf{f}_6(B,\neg e))$



Variable Elimination: Basic Operations

- Factor summation: $\mathbf{f}_3(X_1, ..., X_j) = \mathbf{f}_1(X_1, ..., X_j) + \mathbf{f}_2(X_1, ..., X_j)$
  - standard matrix summation:

$$\begin{bmatrix} a_{11} & a_{21} & ... \\ ... & ... & ... \\ a_{n1} & a_{n1} & ... \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & ... \\ ... & ... & ... \\ b_{n1} & b_{n1} & ... \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{21} + b_{21} & ... \\ ... & ... & ... \\ a_{n1} + b_{n1} & a_{n1} + b_{n1} & ... \end{bmatrix}$$

  - must have the same argument variables
- Pointwise product: Multiply the array elements for the same variable values
- Ex: $\mathbf{f}_4(A) \times \mathbf{f}_5(A) = \begin{bmatrix} P(j| \ a) \\ P(j|\neg a) \end{bmatrix} \times \begin{bmatrix} P(m| \ a) \\ P(m|\neg a) \end{bmatrix} = \begin{bmatrix} P(j| \ a)P(m| \ a) \\ P(j|\neg a)P(m|\neg a) \end{bmatrix}$
- General case:
  $\mathbf{f}_3(X_1, ..., X_j, Y_1, ..., Y_k, Z_1, ..., Z_l) = \mathbf{f}_1(X_1, ..., X_j, Y_1, ..., Y_k) \times \mathbf{f}_2(Y_1, ..., Y_k, Z_1, ..., Z_l)$
  - union of arguments
  - values: $\mathbf{f}_3(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{f}_1(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}_2(\mathbf{y}, \mathbf{z})$
  - matrix size: $\mathbf{f}_1 : 2^{j+k}$, $\mathbf{f}_2 : 2^{k+l}$, $\mathbf{f}_3 : 2^{j+k+l}$

Variable Elimination: Basic Operations

- $\mathbf{f}_3(A, B, C) = \mathbf{f}_1(A, B) \times \mathbf{f}_2(B, C)$
- Summing out one variable: $f(B, C) = \sum_a f_3(A, B, C) = f_3(a, B, C) + f_3(\neg a, B, C) =$
$$\begin{bmatrix} 0.06 & 0.24 \\ 0.42 & 0.28 \end{bmatrix} + \begin{bmatrix} 0.18 & 0.72 \\ 0.06 & 0.04 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.96 \\ 0.48 & 0.32 \end{bmatrix}$$

A	B	$\mathbf{f}_1(A, B)$	B	C	$\mathbf{f}_2(B, C)$	A	B	C	$\mathbf{f}_3(A, B, C)$
T	T	.3	T	T	.2	T	T	T	$.3 \times .2 = .06$
T	F	.7	T	F	.8	T	T	F	$.3 \times .8 = .24$
F	T	.9	F	T	.6	T	F	T	$.7 \times .6 = .42$
F	F	.1	F	F	.4	T	F	F	$.7 \times .4 = .28$
						F	T	T	$.9 \times .2 = .18$
						F	T	F	$.9 \times .8 = .72$
						F	F	T	$.1 \times .6 = .06$
						F	F	F	$.1 \times .4 = .04$

Variable Elimination Algorithm

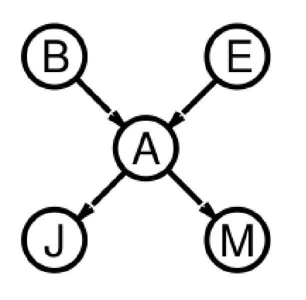
function ELIMINATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$
  for each var **in** ORDER(bn.VARS) **do**
    $factors \leftarrow$ [MAKE-FACTOR(var, $\mathbf{e}$)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

- Efficiency depends on variable ordering ORDER(...)
- Efficiency improvements:
  - factor out of summations factors not depending on sum variable
  - remove irrelevant variables

Factor Out Constant Factors

- If $f_1, ..., f_i$ do not depend on X,
  then move them out of a $\sum_x(...)$:
  $\sum_x f_1 \times \cdots \times f_k =$
  $f_1 \times \cdots \times f_i \sum_x (f_{i+1} \times \cdots \times f_k) =$
  $f_1 \times \cdots \times f_i \times f_{\bar{X}}$
- Ex: $\sum_a \mathbf{f}_1(A,B) \times \mathbf{f}_2(B,C)$
  $= \mathbf{f}_2(B,C) \times \sum_a \mathbf{F}_1(A,B)$
- Ex: $\mathbf{P}(JohnCalls|Burglary = true)$:
  $\mathbf{P}(J|b) = \alpha \sum_e \sum_a \sum_m \mathbf{P}(J,m,b,e,a) =$
  $\alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b,e)\mathbf{P}(J|a)P(m|a) =$
  $\alpha P(b) \sum_e P(e) \sum_a P(a|b,e)\mathbf{P}(J|a) \sum_m P(m|a)$



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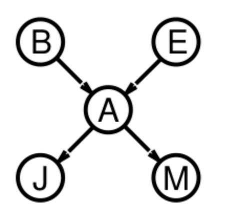
Remove Irrelevant Variables

- Sometimes we fave summations like $\sum_y P(y|...)$
  - $\sum_y P(y|...) = 1 \implies$ can be dropped
- Ex: $\mathbf{P}(JohnCalls|Burglary = true)$:
  $\mathbf{P}(J|b) = ... =$
  $\alpha P(b) \sum_e P(e) \sum_a P(a|b,e)\mathbf{P}(J|a)\overbrace{\sum_m P(m|a)}^{1} = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e)\mathbf{P}(J|a)$
- Theorem: For query X and evidence **E**,
  Y is irrelevant unless $Y \in Ancestors(X \cup \mathbf{E})$
- Ex: $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and $Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$

$\Longrightarrow$ *MaryCalls* is irrelevant

- Related to backward-chaining with Horn clauses



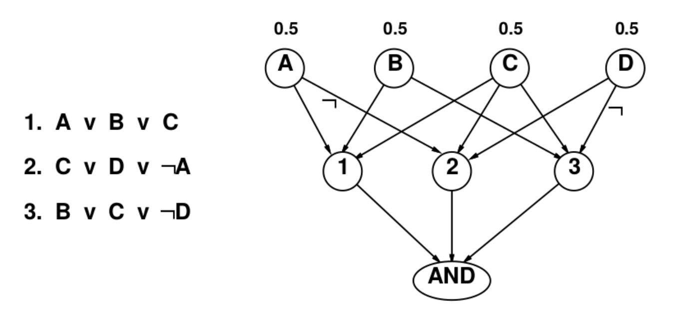
Exercises

1. Try to compute queries (your choice) on the burglary problem using variable elimination
2. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network
   (b) Compute the query $P(P_{1,3}|b^*, p^*)$ via variable elimination
   (c) Compare the result with that of the example in Ch. 13

Outline

Complexity of Exact Inference

- We can reduce SAT to exact inference in Bayesian Networks
  - $P(AND = \top) = \frac{|Models(\varphi)|}{2^{\#vars}}$
  - φ satisfiable $\iff P(AND = \top) > 0$
- Both $\mathbf{P}(clause_i|vars)$ and $\mathbf{P}(AND|clauses)$ $\{0,1\}$-CPTs (deterministic)



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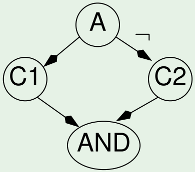
$\Longrightarrow$ Exact inference in Bayesian Networks is NP-Hard

Example: From SAT to BN Inference

$\varphi \stackrel{\text{def}}{=} (a \wedge \neg a)$



$$c1 \stackrel{\text{def}}{=} a \implies P(c1|a) = 1, P(c1|\neg a) = 0$$

$$c2 \stackrel{\text{def}}{=} \neg a \implies P(c2|a) = 0, P(c2|\neg a) = 1$$

$$and \stackrel{\text{def}}{=} c1 \wedge c2 \implies \begin{cases} P(and|c1c2) = 1 \\ P(and|c1\neg c2) = P(and|\neg c1 c2) = P(and|\neg c1 \neg c2) = 0 \end{cases}$$

$$\begin{array}{rcl} P(and) & = & \sum_{c1} \sum_{c2} \sum_{a} P(and|c1c2)P(c1|a)P(c2|a)P(a) \\ & = & \sum_{c1} \sum_{c2} P(and|c1c2) \sum_{a} P(c1|a)P(c2|a)P(a) \\ \implies & = & 1 \cdot \sum_{a} P(c1|a)P(c2|a)P(a) + 0 \cdot ... + 0 \cdot ... + 0 \cdot ... \\ & = & 1 \cdot (P(c1|a)P(c2|a)P(a) + P(c1|\neg a)P(c2|\neg a)P(a\neg)) \\ & = & 1 \cdot ((1 \cdot 0 \cdot 0.5) + (0 \cdot 1 \cdot 0.5)) = 0 \end{array}$$

$\implies$ UNSAT

Exercise

- For each of the following formulas φ, convert φ into a Bayesian network, and determine the number of its models
  - $(\neg A \vee B) \wedge (A \vee \neg B)$
  - $A \wedge (\neg A \vee B) \wedge \neg B$


"That's all Folks!"