

# Fundamentals of Artificial Intelligence

## Chapter 13: Quantifying Uncertainty

**Roberto Sebastiani**

DISI, Università di Trento, Italy – [roberto.sebastiani@unitn.it](mailto:roberto.sebastiani@unitn.it)

[https://disi.unitn.it/rseba/DIDATTICA/fai\\_2023/](https://disi.unitn.it/rseba/DIDATTICA/fai_2023/)

Teaching assistants:

**Mauro Dragoni**, [dragoni@fbk.eu](mailto:dragoni@fbk.eu), <https://www.maurodragoni.com/teaching/fai/>

**Paolo Morettin**, [paolo.morettin@unitn.it](mailto:paolo.morettin@unitn.it), <https://paolomorettin.github.io/>

M.S. Course “Artificial Intelligence Systems”, academic year 2023-2024

Last update: Saturday 9<sup>th</sup> December, 2023, 16:40

Copyright notice: Most examples and images displayed in the slides of this course are taken from [Russell & Norvig, “Artificial Intelligence, a Modern Approach”, 3<sup>rd</sup> ed., Pearson], including explicitly figures from the above-mentioned book, so that their copyright is detained by the authors. A few other material (text, figures, examples) is authored by (in alphabetical order): Pieter Abbeel, Bonnie J. Dorr, Anca Dragan, Dan Klein, Nikita Kitaev, Tom Lenaerts, Michela Milano, Dana Nau, Maria Simi, who detain its copyright.

These slides cannot be displayed in public without the permission of the author.

# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited

# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited

# Acting Under Uncertainty

- Agents often make decisions based on incomplete information
  - partial observability
  - nondeterministic actions
- Partial solution (see previous chapters): maintain **belief states**
  - represent the set of all **possible world states** the agent might be in
  - generating a **contingency plan** handling every possible eventuality
- Several drawbacks:
  - must consider every possible explanation for the observation (even very-unlikely ones)  
⇒ impossibly complex belief-states
  - contingent plans handling every eventuality grow arbitrarily large
  - sometimes there is no plan that is **guaranteed** to achieve the goal
- Agent's knowledge cannot guarantee a successful outcome ...  
... but can provide some **degree of belief (likelihood)** on it
- **A rational decision depends on both the relative importance of (sub)goals and the likelihood that they will be achieved**
- **Probability theory** offers a clean way to quantify likelihood

# Acting Under Uncertainty: Example

## Automated taxi to Airport

- Goal: deliver a passenger to the airport on time
- Action  $A_t$ : leave for airport  $t$  minutes before flight
  - How can we be sure that  $A_{90}$  will succeed?
- Too many sources of uncertainty:
  - partial observability (ex: road state, other drivers' plans, etc.)
  - uncertainty in action outcome (ex: flat tire, etc.)
  - noisy sensors (ex: unreliable traffic reports)
  - complexity of modelling and predicting traffic

⇒ With purely-logical approach it is difficult to anticipate everything that can go wrong

- risks falsehood: " $A_{25}$  will get me there on time" or
- leads to conclusions that are too weak for decision making:  
" $A_{25}$  will get me there on time if there's no accident on the bridge , and it doesn't rain and my tires remain intact , and..."
- Over-cautious choices are not rational solutions either
  - ex:  $A_{1440}$  causes staying overnight at the airport

# Acting Under Uncertainty: Example (2)

## A medical diagnosis

- Given the symptoms (toothache) infer the cause (cavity)
- How to encode this relation in logic?
  - diagnostic rules:
    - $Toothache \rightarrow Cavity$  (wrong)
    - $Toothache \rightarrow (Cavity \vee GumProblem \vee Abscess \vee \dots)$   
(too many possible causes, some very unlikely)
  - causal rules:
    - $Cavity \rightarrow Toothache$  (wrong)
    - $(Cavity \wedge \dots) \rightarrow Toothache$  (many possible (con)causes)
- Problems in specifying the correct logical rules:
  - **Complexity**: too many possible antecedents or consequents
  - **Theoretical ignorance**: no complete theory for the domain
  - **Practical ignorance**: no complete knowledge of the patient

# Summarizing Uncertainty

- Probability allows to summarize the uncertainty on effects of
  - laziness: failure to enumerate exceptions, qualifications, etc.
  - ignorance: lack of relevant facts, initial conditions, etc.
- Probability can be derived from
  - statistical data (ex: 80% of toothache patients so far had cavities)
  - some knowledge (ex: 80% of toothache patients has cavities)
  - their combination thereof
- Probability statements are made with respect to a state of knowledge (aka evidence), not with respect to the real world
  - e.g., “The probability that the patient has a cavity, given that she has a toothache, is 0.8”:  
 $P(\text{HasCavity}(\text{patient}) \mid \text{hasToothAche}(\text{patient})) = 0.8$
- Probabilities of propositions change with new evidence:
  - “The probability that the patient has a cavity, given that she has a toothache and a history of gum disease, is 0.4”:  
 $P(\text{HasCavity}(\text{patient}) \mid \text{hasToothAche}(\text{patient}) \wedge \text{HistoryOfGum}(\text{patient})) = 0.4$

# Making Decisions Under Uncertainty

- Ex: Suppose I believe:

$$P(A_{25} \text{ gets me there on time} \mid \dots) = 0.04$$

$$P(A_{90} \text{ gets me there on time} \mid \dots) = 0.70$$

$$P(A_{120} \text{ gets me there on time} \mid \dots) = 0.95$$

$$P(A_{1440} \text{ gets me there on time} \mid \dots) = 0.9999$$

Which action to choose?

⇒ Depends on tradeoffs among preferences:

- missing flight vs. costs (airport cuisine, sleep overnight in airport)
- When there are conflicting goals the agent may express preferences among them by means of a **utility function**.
- Utilities are combined with probabilities in the **general theory of rational decisions**, aka **decision theory**:  
**Decision theory = Probability theory + Utility theory**
- **Maximum Expected Utility (MEU)**: an agent is rational if and only if it chooses the action that yields the maximum expected utility, averaged over all the possible outcomes of the action.



# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability**
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited

# Probabilities Basics: an AI-sh Introduction

- **Probabilistic assertions**: state how likely possible worlds are
- **Sample space  $\Omega$** : the set of all possible worlds
  - $\omega \in \Omega$  is a **possible world** (aka **sample point** or **atomic event**)
  - ex: **the dice roll (1,4)**
  - the possible worlds are **mutually exclusive** and **exhaustive**
  - ex: **the 36 possible outcomes of rolling two dice: (1,1), (1,2), ...**
- **A probability model (aka probability space)** is a sample space with an assignment  $P(\omega)$  for every  $\omega \in \Omega$  s.t.
  - $0 \leq P(\omega) \leq 1$ , for every  $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} P(\omega) = 1$
- Ex: **1-die roll:  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$**
- An **Event A** is any subset of  $\Omega$ , s.t.  $P(A) = \sum_{\omega \in A} P(\omega)$ 
  - events can be described by **propositions** in some formal language
  - ex:  $P(\text{Total} = 11) = P(5, 6) + P(6, 5) = 1/36 + 1/36 = 1/18$
  - ex:  $P(\text{doubles}) = P(1, 1) + P(2, 2) + \dots + P(6, 6) = 6/36 = 1/6$

# Random Variables

- Factored representation of possible worlds: sets of  $\langle \text{variable}, \text{value} \rangle$  pairs
- Variables in probability theory: **Random variables**
  - **domain**: the set of possible values a variable can take on  
ex: **Die**:  $\{1, 2, 3, 4, 5, 6\}$ , **Weather**:  $\{\text{sunny}, \text{rain}, \text{cloudy}, \text{snow}\}$ , **Odd**:  $\{\text{true}, \text{false}\}$
  - a r.v. can be seen as a function from sample points to the domain:  
ex:  $\text{Die}(\omega)$ ,  $\text{Weather}(\omega)$ ,... (“ $\omega$ ” typically omitted)
- **Probability Distribution** gives the probabilities of all the possible values of a random variable

$$X: P(X = x_i) \stackrel{\text{def}}{=} \sum_{\omega \in X(x_i)} P(\omega)$$

- ex:  $P(\text{Odd} = \text{true}) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2$

# Propositions and Probabilities

- We think a proposition  $a$  as the event  $A$  (set of sample points) where the proposition is true
  - $odd$  is a propositional random variable of range  $\{true, false\}$
  - notation:  $a \iff "A = true"$  (e.g.,  $odd \iff "Odd = true"$ )
- Given Boolean random variables  $A$  and  $B$ :
  - $a$ : set of sample points where  $A(\omega) = true$
  - $\neg a$ : set of sample points where  $A(\omega) = false$
  - $a \wedge b$ : set of sample points where  $A(\omega) = true, B(\omega) = true$

$\implies$  with Boolean random variables, sample points are PL models

- Proposition: disjunction of the sample points in which it is true

- ex:  $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

$\implies P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

- Some derived facts:

- $P(\neg a) = 1 - P(a)$

- $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$

# Probability Distributions

- **Probability Distribution** gives the probabilities of all the possible values of a random variable

- ex: **Weather**: {*sunny*, *rain*, *cloudy*, *snow*}

$$\Rightarrow \mathbf{P}(\text{Weather}) = (0.6, 0.1, 0.29, 0.01) \iff \left\{ \begin{array}{l} P(\text{Weather} = \textit{sunny}) = 0.6 \\ P(\text{Weather} = \textit{rain}) = 0.1 \\ P(\text{Weather} = \textit{cloudy}) = 0.29 \\ P(\text{Weather} = \textit{snow}) = 0.01 \end{array} \right\}$$

- normalized: their sum is 1

- **Joint Probability Distribution** for multiple variables

- gives the probability of every sample point

- ex:  $\mathbf{P}(\text{Weather}, \text{Cavity}) =$

<i>Weather</i> =	<i>sunny</i>	<i>rain</i>	<i>cloudy</i>	<i>snow</i>
<i>Cavity</i> = <i>true</i>	0.144	0.02	0.016	0.02
<i>Cavity</i> = <i>false</i>	0.576	0.08	0.064	0.08

- Every event is a sum of sample points,

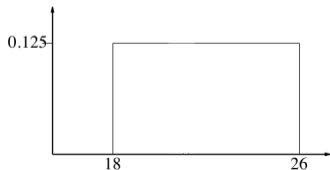
$\Rightarrow$  its probability is determined by the joint distribution

# Probability for Continuous Variables

- Express continuous probability distributions:
  - density functions  $f(x) \in [0, 1]$  s.t.  $\int_{-\infty}^{+\infty} f(x)dx = 1$
- $P(x \in [a, b]) = \int_a^b f(x) dx$ 
  - $\Rightarrow P(x \in [val, val]) = 0, P(x \in [-\infty, +\infty]) = 1$
  - ex:  $P(x \in [20, 22]) = \int_{20}^{22} 0.125 dx = 0.25$
- Density:  $P(x) = P(X = x) \stackrel{\text{def}}{=} \lim_{dx \rightarrow 0} P(X \in [x, x + dx])/dx$ 
  - ex:  $P(20.1) = \lim_{dx \rightarrow 0} P(X \in [20.1, 20.1 + dx])/dx = 0.125$
  - note:  $P(v) \neq P(x \in [v, v]) = 0$

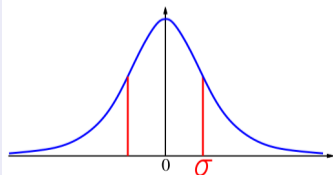
Uniform density between 18 and 26

$$f(x) = U[18, 26](x)$$



Gaussian density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



# Conditional Probabilities

- **Unconditional** or **prior probabilities** refer to degrees of belief in propositions **in the absence of any other information (evidence)**
  - ex:  $P(\text{cavity}) = 0.2$ ,  $P(\text{Total} = 11) = 1/18$ ,  $P(\text{double}) = 1/6$
- **Conditional** or **posterior probabilities** refer to degrees of belief in proposition **a given some evidence b:  $P(a|b)$** 
  - **evidence**: information already revealed
  - ex:  $P(\text{cavity}|\text{toothache}) = 0.6$ : p. of a cavity given a toothache (assuming no other information is provided!)
  - ex:  $P(\text{Total} = 11 | \text{die}_1 = 5) = 1/6$ : p. of total 11 given first die is 5

⇒ restricts the set of possible worlds to those where the first die is 5
- Note:  $P(a|\dots \wedge a) = 1$ ,  $P(a|\dots \wedge \neg a) = 0$ 
  - ex:  $P(\text{cavity}|\text{toothache} \wedge \text{cavity}) = 1$ ,  $P(\text{cavity}|\text{toothache} \wedge \neg \text{cavity}) = 0$
- Less specific belief still valid after more evidence arrives
  - ex:  $P(\text{cavity}) = 0.2$  holds even if  $P(\text{cavity}|\text{toothache}) = 0.6$
- New evidence may be irrelevant, allowing for simplification
  - ex:  $P(\text{cavity}|\text{toothache}, 49\text{ersWin}) = P(\text{cavity}|\text{toothache}) = 0.8$

# Conditional Probabilities [cont.]

- **Conditional probability:**  $P(a|b) \stackrel{\text{def}}{=} \frac{P(a \wedge b)}{P(b)}$ , s.t.  $P(b) > 0$ 
  - ex:  $P(\text{Total} = 11 | \text{die}_1 = 5) = \frac{P(\text{Total}=11 \wedge \text{die}_1=5)}{P(\text{die}_1=5)} = \frac{1/6 \cdot 1/6}{1/6} = 1/6$
  - observing  $b$  restricts the possible worlds to those where  $b$  is true
- **Production rule:**  $P(a \wedge b) = P(a|b) \cdot P(b) = P(b|a) \cdot P(a)$
- **Production rule for whole distributions:**  $\mathbf{P}(X, Y) = \mathbf{P}(X|Y) \cdot \mathbf{P}(Y)$ 
  - ex:  $\mathbf{P}(\text{Weather}, \text{Cavity}) = \mathbf{P}(\text{Weather}|\text{Cavity})\mathbf{P}(\text{Cavity})$ , that is:  
 $P(\text{sunny}, \text{cavity}) = P(\text{sunny}|\text{cavity})P(\text{cavity})$   
...  
 $P(\text{snow}, \neg\text{cavity}) = P(\text{snow}|\neg\text{cavity})P(\neg\text{cavity})$
  - a  $4 \times 2$  set of equations, not matrix multiplication!
- **Chain rule** is derived by successive application of product rule:  
$$\begin{aligned} & \mathbf{P}(X_1, \dots, X_n) \\ &= \mathbf{P}(X_1, \dots, X_{n-1})\mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \mathbf{P}(X_1, \dots, X_{n-2})\mathbf{P}(X_{n-1}|X_1, \dots, X_{n-2})\mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \dots \\ &= \prod_{i=1}^n \mathbf{P}(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$



# Logic vs. Probability

<i>Logic</i>	<i>Probability</i>
$a$	$P(a) = 1$
$\neg a$	$P(a) = 0$
$a \rightarrow b$	$P(b a) = 1$
$\frac{(a, a \rightarrow b)}{b}$	$\frac{P(a) = 1, P(b a) = 1}{P(b) = 1}$
$\frac{(\neg b, a \rightarrow b)}{\neg a}$	$\frac{P(b) = 0, P(b a) = 1}{P(a) = 0}$
$\frac{(a \rightarrow b, b \rightarrow c)}{a \rightarrow c}$	$\frac{P(b a) = 1, P(c b) = 1}{P(c a) = 1}$

(Courtesy of Maria Simi, UniPI)

# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration**
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited

# Probabilistic Inference via Enumeration

## Basic Ideas

- Start with the joint distribution
- For any proposition  $\varphi$ , sum the atomic events where  $\varphi$  is true:  $P(\varphi) = \sum_{\omega : \omega \models \varphi} P(\omega)$

# Probabilistic Inference via Enumeration: Example

## Example: Generic Inference

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- For any proposition  $\varphi$ , sum the atomic events where  $\varphi$  is true:  $P(\varphi) = \sum_{\omega : \omega \models \varphi} P(\omega)$ :
- Ex:  $P(\textit{cavity} \vee \textit{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

# Probabilistic Inference via Enumeration: Example

## Example: Generic Inference

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- For any proposition  $\varphi$ , sum the atomic events where  $\varphi$  is true:  $P(\varphi) = \sum_{\omega : \omega \models \varphi} P(\omega)$ :
- Ex:  $P(\textit{cavity} \vee \textit{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

# Marginalization

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- **Marginalization** (aka **summing out**):  
sum up the probabilities for each possible value of the other variables:

$$\mathbf{P}(\mathbf{Y}) = \sum_{\mathbf{z} \in \mathbf{Z}} \mathbf{P}(\mathbf{Y}, \mathbf{z})$$

$$\text{Ex: } \mathbf{P}(\textit{Toothache}) = \sum_{\mathbf{z} \in \{\textit{Catch}, \textit{Cavity}\}} \mathbf{P}(\textit{Toothache}, \mathbf{z})$$

- **Conditioning**: variant of marginalization, involving conditional probabilities instead of joint probabilities (using the product rule)

$$\mathbf{P}(\mathbf{Y}) = \sum_{\mathbf{z} \in \mathbf{Z}} \mathbf{P}(\mathbf{Y}|\mathbf{z})P(\mathbf{z})$$

$$\text{Ex: } \mathbf{P}(\textit{Toothache}) = \sum_{\mathbf{z} \in \{\textit{Catch}, \textit{Cavity}\}} \mathbf{P}(\textit{Toothache}|\mathbf{z})P(\mathbf{z})$$

# Marginalization: Example

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- Marginalization (aka **summing out**):  
sum up the probabilities for each possible value of the other variables:

$$\mathbf{P}(\mathbf{Y}) = \sum_{\mathbf{z} \in \mathbf{Z}} \mathbf{P}(\mathbf{Y}, \mathbf{z})$$

$$\text{Ex: } \mathbf{P}(\textit{Toothache}) = \sum_{\mathbf{z} \in \{\textit{Catch}, \textit{Cavity}\}} \mathbf{P}(\textit{Toothache}, \mathbf{z})$$

$$P(\textit{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

$$P(\neg \textit{toothache}) = 1 - P(\textit{toothache}) = 1 - 0.2 = 0.8$$

$$\Rightarrow \mathbf{P}(\textit{Toothache}) = \langle 0.2, 0.8 \rangle$$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

# Marginalization: Example

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- Marginalization (aka summing out):  
sum up the probabilities for each possible value of the other variables:

$$\mathbf{P}(\mathbf{Y}) = \sum_{\mathbf{z} \in \mathbf{Z}} \mathbf{P}(\mathbf{Y}, \mathbf{z})$$

$$\text{Ex: } \mathbf{P}(\textit{Toothache}) = \sum_{\mathbf{z} \in \{\textit{Catch}, \textit{Cavity}\}} \mathbf{P}(\textit{Toothache}, \mathbf{z})$$

$$P(\textit{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

$$P(\neg \textit{toothache}) = 1 - P(\textit{toothache}) = 1 - 0.2 = 0.8$$

$$\Rightarrow \mathbf{P}(\textit{Toothache}) = \langle 0.2, 0.8 \rangle$$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>



# Conditional Probability via Enumeration: Example

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- Conditional Probability:

$$\text{Ex: } P(\neg \textit{cavity} | \textit{toothache}) = \frac{P(\neg \textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})}$$
$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

$$\text{Ex: } P(\textit{cavity} | \textit{toothache}) = \frac{P(\textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})} = \dots = 0.6$$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

# Conditional Probability via Enumeration: Example

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- Conditional Probability:

$$\text{Ex: } P(\neg \textit{cavity} | \textit{toothache}) = \frac{P(\neg \textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})}$$
$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

$$\text{Ex: } P(\textit{cavity} | \textit{toothache}) = \frac{P(\textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})} = \dots = 0.6$$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

# Conditional Probability via Enumeration: Example

- Start with the joint distribution  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$
- Conditional Probability:

$$\text{Ex: } P(\neg \textit{cavity} | \textit{toothache}) = \frac{P(\neg \textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})}$$
$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

$$\text{Ex: } P(\textit{cavity} | \textit{toothache}) = \frac{P(\textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})} = \dots = 0.6$$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

# Normalization

- Let  $\mathbf{X}$  be all the variables. Typically, we want  $\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e})$ :
  - the **conditional joint distribution** of the **query variables**  $\mathbf{Y}$
  - given specific values  $\mathbf{e}$  for the **evidence variables**  $\mathbf{E}$
  - let the **hidden variables** be  $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{X} \setminus (\mathbf{Y} \cup \mathbf{E})$
- The summation of joint entries is done by summing out the hidden variables:  
 $\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h} \in \mathbf{H}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$ 
  - where  $\alpha \stackrel{\text{def}}{=} 1/\mathbf{P}(\mathbf{E} = \mathbf{e})$  (Notice: **different  $\alpha$ 's for different values of  $\mathbf{e}$ !**)
- $\implies$  it is easy to compute  $\alpha$  by normalization
  - note: the terms in the summation are joint entries, because  $\mathbf{Y}, \mathbf{E}, \mathbf{H}$  together exhaust the set of random variables  $\mathbf{X}$
- Idea: compute **whole distribution** on **query variable** by:
  - fixing **evidence variables** and summing over **hidden variables**
  - **normalize** the final distribution, so that  $\sum \dots = 1$
- Complexity:  $O(2^n)$ ,  $n$  number of propositions  $\implies$  impractical for large  $n$ 's

Common practice: deal with non-normalized distributions, normalize at the end of the process (see e.g. “Wumpus world” example at the end of this chapter)

# Normalization: Example

- $\alpha \stackrel{\text{def}}{=} 1/P(\textit{toothache})$  (previous example) can be viewed as a normalization constant
- Idea: compute **whole distribution** on **query variable** by:
  - fixing **evidence variables** and summing over **hidden variables**
  - **normalize** the final distribution, so that  $\sum \dots = 1$
- Ex:<sup>a</sup>  $\mathbf{P}(\textit{Cavity}|\textit{toothache}) = \alpha \mathbf{P}(\textit{Cavity} \wedge \textit{toothache})$   
 $= \alpha[\mathbf{P}(\textit{Cavity}, \textit{toothache}, \textit{catch}) + \mathbf{P}(\textit{Cavity}, \textit{toothache}, \neg \textit{catch})]$   
 $= \alpha[\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle]$   
 $= \alpha \langle 0.12, 0.08 \rangle = (\textit{normalization}) = \langle 0.6, 0.4 \rangle [\alpha = 5]$   
 $\mathbf{P}(\textit{Cavity}|\neg \textit{toothache}) = \dots = \alpha \langle 0.08, 0.72 \rangle = \langle 0.1, 0.9 \rangle [\alpha = 1.25]$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

<sup>a</sup>n.b.: here “Cavity” is a variable, “toothache” is a proposition (i.e. Toothache=true)

# Normalization: Example

- $\alpha \stackrel{\text{def}}{=} 1/P(\textit{toothache})$  (previous example) can be viewed as a normalization constant
- Idea: compute **whole distribution** on **query variable** by:
  - fixing **evidence variables** and summing over **hidden variables**
  - **normalize** the final distribution, so that  $\sum \dots = 1$
- Ex:<sup>a</sup>  $\mathbf{P}(\textit{Cavity}|\textit{toothache}) = \alpha \mathbf{P}(\textit{Cavity} \wedge \textit{toothache})$   
 $= \alpha[\mathbf{P}(\textit{Cavity}, \textit{toothache}, \textit{catch}) + \mathbf{P}(\textit{Cavity}, \textit{toothache}, \neg \textit{catch})]$   
 $= \alpha[\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle]$   
 $= \alpha \langle 0.12, 0.08 \rangle = (\textit{normalization}) = \langle 0.6, 0.4 \rangle [\alpha = 5]$   
 $\mathbf{P}(\textit{Cavity}|\neg \textit{toothache}) = \dots = \alpha \langle 0.08, 0.72 \rangle = \langle 0.1, 0.9 \rangle [\alpha = 1.25]$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

<sup>a</sup>n.b.: here “Cavity” is a variable, “toothache” is a proposition (i.e. Toothache=true)

# Normalization: Example

- $\alpha \stackrel{\text{def}}{=} 1/P(\textit{toothache})$  (previous example) can be viewed as a normalization constant
- Idea: compute **whole distribution** on **query variable** by:
  - fixing **evidence variables** and summing over **hidden variables**
  - **normalize** the final distribution, so that  $\sum \dots = 1$
- Ex:<sup>a</sup>  $\mathbf{P}(\textit{Cavity}|\textit{toothache}) = \alpha \mathbf{P}(\textit{Cavity} \wedge \textit{toothache})$   
 $= \alpha[\mathbf{P}(\textit{Cavity}, \textit{toothache}, \textit{catch}) + \mathbf{P}(\textit{Cavity}, \textit{toothache}, \neg \textit{catch})]$   
 $= \alpha[\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle]$   
 $= \alpha \langle 0.12, 0.08 \rangle = (\textit{normalization}) = \langle 0.6, 0.4 \rangle [\alpha = 5]$   
 $\mathbf{P}(\textit{Cavity}|\neg \textit{toothache}) = \dots = \alpha \langle 0.08, 0.72 \rangle = \langle 0.1, 0.9 \rangle [\alpha = 1.25]$

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

(© S. Russell & P. Norwig, AIMA)

<sup>a</sup>n.b.: here “Cavity” is a variable, “toothache” is a proposition (i.e. Toothache=true)

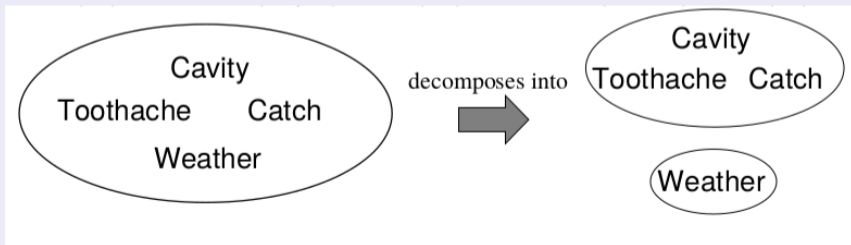
# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence**
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited



# Independence

- Variables  $X$  and  $Y$  are **independent** iff  $\mathbf{P}(X, Y) = \mathbf{P}(X)\mathbf{P}(Y)$   
(equivalently, iff  $\mathbf{P}(X|Y) = \mathbf{P}(X)$  and iff  $\mathbf{P}(Y|X) = \mathbf{P}(Y)$ )
  - ex:  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}, \textit{Weather}) = \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Weather})$   
 $\Rightarrow$  e.g.  $P(\textit{toothache}, \textit{catch}, \textit{cavity}, \textit{cloudy}) = P(\textit{toothache}, \textit{catch}, \textit{cavity})P(\textit{cloudy})$
  - typically **based on domain knowledge**
- May drastically reduce the number of entries and computation  
 $\Rightarrow$  ex: 32-element table decomposed into one 8-element and one 4-element table
- Unfortunately, absolute independence is quite rare



# Conditional Independence

- Variables  $X$  and  $Y$  are **conditionally independent given  $Z$**  iff  $\mathbf{P}(X, Y|Z) = \mathbf{P}(X|Z)\mathbf{P}(Y|Z)$  (equivalently, iff  $\mathbf{P}(X|Y, Z) = \mathbf{P}(X|Z)$  and iff  $\mathbf{P}(Y|X, Z) = \mathbf{P}(Y|Z)$ )
  - Consider  $\mathbf{P}(\textit{Toothache}, \textit{Cavity}, \textit{Catch})$ 
    - if I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:  $P(\textit{catch}|\textit{toothache}, \textit{cavity}) = P(\textit{catch}|\textit{cavity})$
    - the same independence holds if I haven't got a cavity:  
 $P(\textit{catch}|\textit{toothache}, \neg\textit{cavity}) = P(\textit{catch}|\neg\textit{cavity})$
- ⇒ **Catch is conditionally independent of Toothache given Cavity:**  
 $\mathbf{P}(\textit{Catch}|\textit{Toothache}, \textit{Cavity}) = \mathbf{P}(\textit{Catch}|\textit{Cavity})$   
or, equivalently:  $\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})$ , or  
 $\mathbf{P}(\textit{Toothache}, \textit{Catch}|\textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})$
- Hint: *Toothache* and *Catch* are two (mutually-independent) **effects** of the same **cause** *Cavity*

## Conditional Independence [cont.]

- In many cases, the use of conditional independence reduces the size of the representation of the joint distribution dramatically

- even from exponential to linear!

$$\mathbf{P}(Toothache, Catch, Cavity)$$

- Ex:
  - =  $\mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity)$
  - =  $\mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$
  - =  $\mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$

⇒ Passes from 7 to  $2+2+1=5$  independent numbers

- $\mathbf{P}(Toothache, Catch, Cavity)$  contains 7 independent entries (the 8th can be obtained as  $1 - \sum \dots$ )
- $\mathbf{P}(Toothache|Cavity), \mathbf{P}(Catch|Cavity)$  contain 2 independent entries ( $2 \times 2$  matrix, each row sums to 1)
- $\mathbf{P}(Cavity)$  contains 1 independent entry
- General Case: if one cause has  $n$  independent effects:  
 $\mathbf{P}(Cause, Effect_1, \dots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i|Cause)$

⇒ reduces from  $2^{n+1} - 1$  to  $2n + 1$  independent entries

## Exercise

Consider the joint probability distribution described in the table in previous section:

$\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$

- Consider the example in previous slide:

$\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$

=  $\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Catch}, \textit{Cavity})$

=  $\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})\mathbf{P}(\textit{Cavity})$

=  $\mathbf{P}(\textit{Toothache}|\textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})\mathbf{P}(\textit{Cavity})$

- Compute separately the distributions  $\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity})$ ,  $\mathbf{P}(\textit{Catch}|\textit{Cavity})$ ,  $\mathbf{P}(\textit{Cavity})$ ,  $\mathbf{P}(\textit{Toothache}|\textit{Cavity})$ .

- Recompute  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$  in two ways:

- $\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})\mathbf{P}(\textit{Cavity})$

- $\mathbf{P}(\textit{Toothache}|\textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})\mathbf{P}(\textit{Cavity})$

and compare the result with  $\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$

# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule**
- 6 An Example: The Wumpus World Revisited

# Bayes' Rule

## Bayes' Rule/Theorem/Law

- Bayes' rule:  $P(a|b) = \frac{P(a \wedge b)}{P(b)} = \frac{P(b|a)P(a)}{P(b)}$
- In distribution form  $\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha\mathbf{P}(X|Y)\mathbf{P}(Y)$ 
  - $\alpha \stackrel{\text{def}}{=} 1/\mathbf{P}(X)$ : normalization constant to make  $\mathbf{P}(Y|X)$  entries sum to 1 (different  $\alpha$ 's for different values of  $X$ )
- A version conditionalized on some background evidence  $\mathbf{e}$ :

$$\mathbf{P}(Y|X, \mathbf{e}) = \frac{\mathbf{P}(X|Y, \mathbf{e})\mathbf{P}(Y|\mathbf{e})}{\mathbf{P}(X|\mathbf{e})}$$

# Using Bayes' Rule: The Simple Case

- Used to assess **diagnostic probability** from **causal probability**:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- $P(\text{cause}|\text{effect})$  goes from effect to cause (**diagnostic** direction)
- $P(\text{effect}|\text{cause})$  goes from cause to effect (**causal** direction)

## Example

- An expert doctor is likely to have **causal knowledge** ...  $P(\text{symptoms}|\text{disease})$  (i.e.,  $P(\text{effect}|\text{cause})$ )  
... and needs producing **diagnostic knowledge**  $P(\text{disease}|\text{symptoms})$  (i.e.,  $P(\text{cause}|\text{effect})$ )
- Ex: let  $m$  be meningitis,  $s$  be stiff neck
  - $P(m) = 1/50000$ ,  $P(s) = 0.01$  (prior knowledge, from statistics)
  - “meningitis causes to the patient a stiff neck in 70% of cases”:  $P(s|m) = 0.7$  (doctor's experience)

$$\Rightarrow P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \cdot 1/50000}{0.01} = 0.0014$$

# Using Bayes' Rule: Combining Evidence

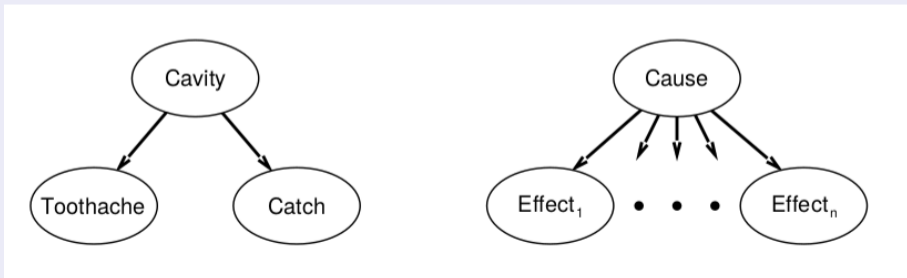
- A naive Bayes model is a probability model that assumes the effects are conditionally independent, given the cause

$$\Rightarrow \mathbf{P}(\text{Cause}, \text{Effect}_1, \dots, \text{Effect}_n) = \mathbf{P}(\text{Cause}) \prod_i \mathbf{P}(\text{Effect}_i | \text{Cause})$$

- total number of parameters is linear in  $n$
- ex:  $\mathbf{P}(\text{Cavity}, \text{Toothache}, \text{Catch}) = \mathbf{P}(\text{Cavity})\mathbf{P}(\text{Toothache} | \text{Cavity})\mathbf{P}(\text{Catch} | \text{Cavity})$

Q: How can we compute  $\mathbf{P}(\text{Cause} | \text{Effect}_1, \dots, \text{Effect}_k)$ ?

- ex  $\mathbf{P}(\text{Cavity} | \text{toothache} \wedge \text{catch})$ ?





## Using Bayes' Rule: Combining Evidence [cont.]

Q: How can we compute  $\mathbf{P}(\text{Cause}|\text{Effect}_1, \dots, \text{Effect}_k)$ ?

- ex:  $\mathbf{P}(\text{Cavity}|\text{toothache} \wedge \text{catch})$ ?

A: Apply Bayes' Rule

$$\begin{aligned} & \mathbf{P}(\text{Cavity}|\text{toothache} \wedge \text{catch}) \\ &= \mathbf{P}(\text{toothache} \wedge \text{catch}|\text{Cavity})\mathbf{P}(\text{Cavity})/P(\text{toothache} \wedge \text{catch}) \\ &= \alpha\mathbf{P}(\text{toothache} \wedge \text{catch}|\text{Cavity})\mathbf{P}(\text{Cavity}) \\ &= \alpha\mathbf{P}(\text{toothache}|\text{Cavity})\mathbf{P}(\text{catch}|\text{Cavity})\mathbf{P}(\text{Cavity}) \end{aligned}$$

- $\alpha \stackrel{\text{def}}{=} 1/P(\text{toothache} \wedge \text{catch})$  not computed explicitly
  - General case:  $\mathbf{P}(\text{Cause}|\text{Effect}_1, \dots, \text{Effect}_n) = \alpha\mathbf{P}(\text{Cause}) \prod_i \mathbf{P}(\text{Effect}_i|\text{Cause})$ 
    - $\alpha \stackrel{\text{def}}{=} 1/\mathbf{P}(\text{Effect}_1, \dots, \text{Effect}_n)$  not computed explicitly  
(one  $\alpha$  value for every value of  $\text{Effect}_1, \dots, \text{Effect}_n$ )
- $\Rightarrow$  reduces from  $2^{n+1} - 1$  to  $2n + 1$  independent entries

# Outline

- 1 Acting Under Uncertainty
- 2 Basics on Probability
- 3 Probabilistic Inference via Enumeration
- 4 Independence and Conditional Independence
- 5 Applying Bayes' Rule
- 6 An Example: The Wumpus World Revisited**

# An Example: The Wumpus World

## A probability model of the Wumpus World

- Consider again the Wumpus World (restricted to pit detection)
- Evidence: no pit in (1,1), (1,2), (2,1), breezy in (1,2), (2,1)

Q. Given the evidence, what is the probability of having a pit in (1,3), (2,2) or (3,1)?

- Two groups of variables:
  - $P_{ij} = \text{true}$  iff  $[i, j]$  contains a pit (“causes”)
  - $B_{ij} = \text{true}$  iff  $[i, j]$  is breezy (“effects”, consider only  $B_{1,1}, B_{1,2}, B_{2,1}$ )

- Joint Distribution:

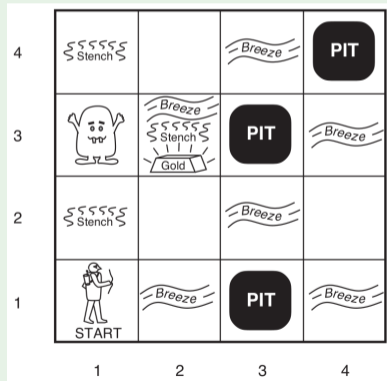
$$\mathbf{P}(P_{1,1}, \dots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$$

- Known facts (evidence):

- $b^* \stackrel{\text{def}}{=} \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

- $p^* \stackrel{\text{def}}{=} \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

- Queries:  $\mathbf{P}(P_{1,3} | b^*, p^*)$ ?  $\mathbf{P}(P_{2,2} | b^*, p^*)$ ?  
 $(\mathbf{P}(P_{3,1} | b^*, p^*) \text{ symmetric})$



# An Example: The Wumpus World

## A probability model of the Wumpus World

- Consider again the Wumpus World (restricted to pit detection)
- Evidence: no pit in (1,1), (1,2), (2,1), breezy in (1,2), (2,1)

Q. Given the evidence, what is the probability of having a pit in (1,3), (2,2) or (3,1)?

- Two groups of variables:
  - $P_{ij} = \text{true}$  iff  $[i, j]$  contains a pit (“causes”)
  - $B_{ij} = \text{true}$  iff  $[i, j]$  is breezy (“effects”, consider only  $B_{1,1}, B_{1,2}, B_{2,1}$ )

- Joint Distribution:

$$\mathbf{P}(P_{1,1}, \dots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$$

- Known facts (evidence):

- $b^* \stackrel{\text{def}}{=} \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

- $p^* \stackrel{\text{def}}{=} \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

- Queries:  $\mathbf{P}(P_{1,3}|b^*, p^*)$ ?  $\mathbf{P}(P_{2,2}|b^*, p^*)$ ?  
( $\mathbf{P}(P_{3,1}|b^*, p^*)$  symmetric)

1,4	2,4	3,4	4,4
1,3	2,3	3,3	4,3
1,2 B OK	2,2	3,2	4,2
1,1 OK	2,1 B OK	3,1	4,1

## An Example: The Wumpus World [cont.]

### Specifying the probability model

- Apply the product rule to the joint distribution  $\mathbf{P}(P_{1,1}, \dots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1}) = \mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} | P_{1,1}, \dots, P_{4,4}) \mathbf{P}(P_{1,1}, \dots, P_{4,4})$
- $\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} | P_{1,1}, \dots, P_{4,4})$  deterministic:
  - 1 if one pit is adjacent to breeze,
  - 0 otherwise
- $\mathbf{P}(P_{1,1}, \dots, P_{4,4})$ : pits are placed randomly except in (1,1):  
$$\mathbf{P}(P_{1,1}, \dots, P_{4,4}) = \prod_{i=1}^4 \prod_{j=1}^4 P(P_{i,j})$$
$$P(P_{i,j}) = \begin{cases} 0.2 & \text{if } (i,j) \neq (1,1) \\ 0 & \text{otherwise} \end{cases}$$
  - ex:  $\mathbf{P}(P_{1,1}, \dots, P_{4,4}) = 0.2^3 \cdot (1 - 0.2)^{15-3} \approx 0.00055$  if 3 pits

# An Example: The Wumpus World [cont.]

## Inference by enumeration

Case  $P_{1,3}$ :

- General form of query:  $\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$ 
  - $\mathbf{Y}$ : query vars;  $\mathbf{E}, \mathbf{e}$ : evidence vars/values;  $\mathbf{H}, \mathbf{h}$ : hidden vars/values

- Our case:  $\mathbf{P}(P_{1,3}|p^*, b^*)$ , s.t. the evidence is

- $b^* \stackrel{\text{def}}{=} \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

- $p^* \stackrel{\text{def}}{=} \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

- Sum over hidden variables:

$$\mathbf{P}(P_{1,3}|p^*, b^*) =$$

$$\alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}|p^*, b^*, \text{unknown})$$

- *unknown* are all  $P_{ij}$ 's s.t.

$$(i, j) \notin \{(1, 1), (1, 2), (2, 1), (1, 3)\}$$

$$\Rightarrow 2^{16-4} = 4096 \text{ terms of the sum!}$$

- Grows exponentially in the number of hidden variables  $\mathbf{H}$ !

$\Rightarrow$  Inefficient

- Can we do better?

1,4	2,4	3,4	4,4
1,3 $P_{13}$	2,3	3,3	4,3
1,2 B OK	2,2	3,2	4,2
1,1 OK	2,1 B OK	3,1	4,1



## An Example: The Wumpus World [cont.]

$\mathbf{P}(p^*, b^*) = P(p^*, b^*)$  is scalar; use as a normalization constant

$$\mathbf{P}(P_{1,3}|p^*, b^*) = \mathbf{P}(P_{1,3}, p^*, b^*) / \underline{\mathbf{P}(p^*, b^*)} = \underline{\alpha} \mathbf{P}(P_{1,3}, p^*, b^*)$$



## An Example: The Wumpus World [cont.]

Sum over the unknowns

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}, \text{unknown}, p^*, b^*)\end{aligned}$$

# An Example: The Wumpus World [cont.]

Use the product rule

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, p^*, \underline{b^*}) \\ &= \alpha \sum_{unknown} \mathbf{P}(\underline{b^*} | P_{1,3}, p^*, unknown) \mathbf{P}(P_{1,3}, p^*, unknown)\end{aligned}$$

# An Example: The Wumpus World [cont.]

Separate unknown into *fringe* and *other*

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}, \text{unknown}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(b^* | P_{1,3}, p^*, \text{unknown}) \mathbf{P}(P_{1,3}, p^*, \text{unknown}) \\ &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}, \text{other}) \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other})\end{aligned}$$

## An Example: The Wumpus World [cont.]

$b^*$  is conditionally independent of *other* given *fringe*

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(b^* | P_{1,3}, p^*, unknown) \mathbf{P}(P_{1,3}, p^*, unknown) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, \underline{fringe}, other) \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, \underline{fringe}) \mathbf{P}(P_{1,3}, p^*, fringe, other)\end{aligned}$$

## An Example: The Wumpus World [cont.]

Move  $\mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe})$  outward

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{\textit{unknown}} \mathbf{P}(P_{1,3}, \textit{unknown}, p^*, b^*) \\ &= \alpha \sum_{\textit{unknown}} \mathbf{P}(b^*|P_{1,3}, p^*, \textit{unknown}) \mathbf{P}(P_{1,3}, p^*, \textit{unknown}) \\ &= \alpha \sum_{\textit{fringe}} \sum_{\textit{other}} \mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe}, \textit{other}) \mathbf{P}(P_{1,3}, p^*, \textit{fringe}, \textit{other}) \\ &= \alpha \sum_{\textit{fringe}} \sum_{\textit{other}} \frac{\mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe}) \mathbf{P}(P_{1,3}, p^*, \textit{fringe}, \textit{other})}{\mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe})} \\ &= \alpha \sum_{\textit{fringe}} \frac{\mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe})}{\mathbf{P}(b^*|p^*, P_{1,3}, \textit{fringe})} \sum_{\textit{other}} \mathbf{P}(P_{1,3}, p^*, \textit{fringe}, \textit{other})\end{aligned}$$

## An Example: The Wumpus World [cont.]

All of the pit locations are independent

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(b^* | P_{1,3}, p^*, unknown) \mathbf{P}(P_{1,3}, p^*, unknown) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, fringe, other) \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \sum_{other} \frac{\mathbf{P}(P_{1,3}) \mathbf{P}(p^*) \mathbf{P}(fringe) \mathbf{P}(other)}\end{aligned}$$

## An Example: The Wumpus World [cont.]

Move  $P(p^*)$ ,  $\mathbf{P}(P_{1,3})$ , and  $P(\text{fringe})$  outward

$$\begin{aligned}\mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}, \text{unknown}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(b^* | P_{1,3}, p^*, \text{unknown}) \mathbf{P}(P_{1,3}, p^*, \text{unknown}) \\ &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}, \text{other}) \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \sum_{\text{other}} \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \sum_{\text{other}} \frac{\mathbf{P}(P_{1,3}) P(p^*) P(\text{fringe})}{P(\text{other})} P(\text{other}) \\ &= \alpha \frac{P(p^*) \mathbf{P}(P_{1,3})}{\sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe})} \frac{P(\text{fringe})}{\sum_{\text{other}} P(\text{other})}\end{aligned}$$

# An Example: The Wumpus World [cont.]

Remove  $\sum_{other} P(other)$  because it equals 1

$$\begin{aligned} \mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, p^*, b^*) \\ &= \alpha \sum_{unknown} \mathbf{P}(b^* | P_{1,3}, p^*, unknown) \mathbf{P}(P_{1,3}, p^*, unknown) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, fringe, other) \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}, p^*, fringe, other) \\ &= \alpha \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}) P(p^*) P(fringe) P(other) \\ &= \alpha P(p^*) \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) P(fringe) \underbrace{\sum_{other} P(other)} \\ &= \alpha P(p^*) \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b^* | p^*, P_{1,3}, fringe) P(fringe) \end{aligned}$$



## An Example: The Wumpus World [cont.]

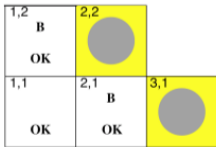
$P(p^*)$  is scalar, so make it part of the normalization constant

$$\begin{aligned} \mathbf{P}(P_{1,3}|p^*, b^*) &= \mathbf{P}(P_{1,3}, p^*, b^*) / \mathbf{P}(p^*, b^*) = \alpha \mathbf{P}(P_{1,3}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(P_{1,3}, \text{unknown}, p^*, b^*) \\ &= \alpha \sum_{\text{unknown}} \mathbf{P}(b^* | P_{1,3}, p^*, \text{unknown}) \mathbf{P}(P_{1,3}, p^*, \text{unknown}) \\ &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}, \text{other}) \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \sum_{\text{other}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \sum_{\text{other}} \mathbf{P}(P_{1,3}, p^*, \text{fringe}, \text{other}) \\ &= \alpha \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) \sum_{\text{other}} \mathbf{P}(P_{1,3}) P(p^*) P(\text{fringe}) P(\text{other}) \\ &= \alpha P(p^*) \mathbf{P}(P_{1,3}) \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) P(\text{fringe}) \sum_{\text{other}} P(\text{other}) \\ &= \underline{\alpha P(p^*)} \mathbf{P}(P_{1,3}) \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) P(\text{fringe}) \\ &= \underline{\alpha'} \mathbf{P}(P_{1,3}) \sum_{\text{fringe}} \mathbf{P}(b^* | p^*, P_{1,3}, \text{fringe}) P(\text{fringe}) \end{aligned}$$

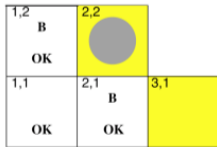
## An Example: The Wumpus World [cont.]

- We have obtained:  $\mathbf{P}(P_{1,3}|p^*, b^*) = \alpha' \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b^*|p^*, P_{1,3}, fringe) P(fringe)$
- We know that  $\mathbf{P}(P_{1,3}) = \langle 0.2, 0.8 \rangle$  (see slide 38)
- We can compute the normalization coefficient  $\alpha'$  afterwards
- $\sum_{fringe} \mathbf{P}(b^*|p^*, P_{1,3}, fringe) P(fringe)$ : only 4 possible fringes
- Start by rewriting as two separate equations:
  - $\mathbf{P}(p_{1,3}|p^*, b^*) = \alpha' P(p_{1,3}) \sum_{fringe} \mathbf{P}(b^*|p^*, p_{1,3}, fringe) P(fringe)$
  - $\mathbf{P}(\neg p_{1,3}|p^*, b^*) = \alpha' P(\neg p_{1,3}) \sum_{fringe} \mathbf{P}(b^*|p^*, \neg p_{1,3}, fringe) P(fringe)$

Four possible fringes:



$$0.2 \times 0.2 = 0.04$$



$$0.2 \times 0.8 = 0.16$$



$$0.8 \times 0.2 = 0.16$$



$$0.8 \times 0.8 = 0.64$$

# An Example: The Wumpus World [cont.]

- Start by rewriting as two separate equations:

$$P(p_{1,3}|p^*, b^*) = \alpha' P(p_{1,3}) \sum_{fringe} P(b^*|p^*, p_{1,3}, fringe) P(fringe)$$

$$P(\neg p_{1,3}|p^*, b^*) = \alpha' P(\neg p_{1,3}) \sum_{fringe} P(b^*|p^*, \neg p_{1,3}, fringe) P(fringe)$$

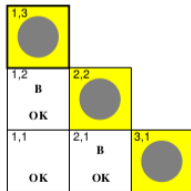
- For each of them,  $P(b^*|...)$  is 1 if the breezes occur, 0 otherwise:

$$\sum_{fringe} P(b^*|p^*, p_{1,3}, fringe) P(fringe) = 1 \cdot 0.04 + 1 \cdot 0.16 + 1 \cdot 0.16 + 0 \cdot 0.64 = 0.36$$

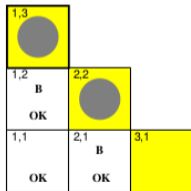
$$\sum_{fringe} P(b^*|p^*, \neg p_{1,3}, fringe) P(fringe) = 1 \cdot 0.04 + 1 \cdot 0.16 + 0 \cdot 0.16 + 0 \cdot 0.64 = 0.2$$

$$\Rightarrow P(P_{1,3}|p^*, b^*) = \alpha' P(P_{1,3}) \sum_{fringe} P(b^*|p^*, P_{1,3}, fringe) P(fringe)$$

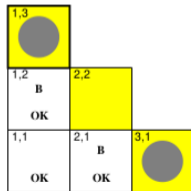
$$= \alpha' \langle 0.2, 0.8 \rangle \langle 0.36, 0.2 \rangle = \alpha' \langle 0.072, 0.16 \rangle = (\text{normalization, s.t. } \alpha' \approx 4.31) \approx \langle 0.31, 0.69 \rangle$$



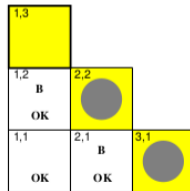
$$0.2 \times 0.2 = 0.04$$



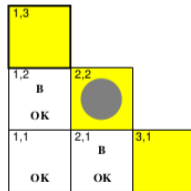
$$0.2 \times 0.8 = 0.16$$



$$0.8 \times 0.2 = 0.16$$



$$0.2 \times 0.2 = 0.04$$



$$0.2 \times 0.8 = 0.16$$

## Exercise

Compute  $\mathbf{P}(P_{2,2}|p^*, b^*)$  in the same way.