# Fundamentals of Artificial Intelligence
# Chapter 05: **Adversarial Search and Games**


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# Outline

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# Games and AI

- Games are a form of multi-agent environment
  - Q.: What do other agents do and how do they affect our success?
  - recall: cooperative vs. competitive multi-agent environments
  - competitive multi-agent environments give rise to adversarial problems (aka games)
- Q.: Why study games in AI?
  - lots of fun, historically entertaining
  - easy to represent: agents restricted to small number of actions with precise rules
  - interesting also because computationally very hard
    (ex: chess has $b \approx 35$, $\#nodes \approx 10^{40}$)
  - metaphor for important application domains
    (e.g. competitive markets, life sciences, sport, politics, warfare, ...)

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- Search (with no adversary)
  - solution is a (heuristic) method for finding a goal
  - heuristics techniques can find optimal solutions
  - evaluation function: estimate of cost from start to goal through given node
  - examples: path planning, scheduling activities, ...
- Games (with adversary), aka adversarial search
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# Types of Games

- Many different kinds of games
- Relevant features:
  - deterministic vs. stochastic (with chance)
  - one, two, or more players
  - zero-sum vs. general games
  - perfect information (can you see the state?) vs. imperfect
- Most common: deterministic, turn-taking, two-player, zero-sum games, perfect information
- Want algorithms for calculating a strategy (aka policy):
  - recommends a move from each state: $policy : S \mapsto A$

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| | deterministic | chance |
|---|---|---|
| perfect information | chess, checkers, go, othello | backgammon monopoly |
| imperfect information | battleships, blind tictactoe | bridge, poker, scrabble nuclear war |

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# Games: Main Concepts

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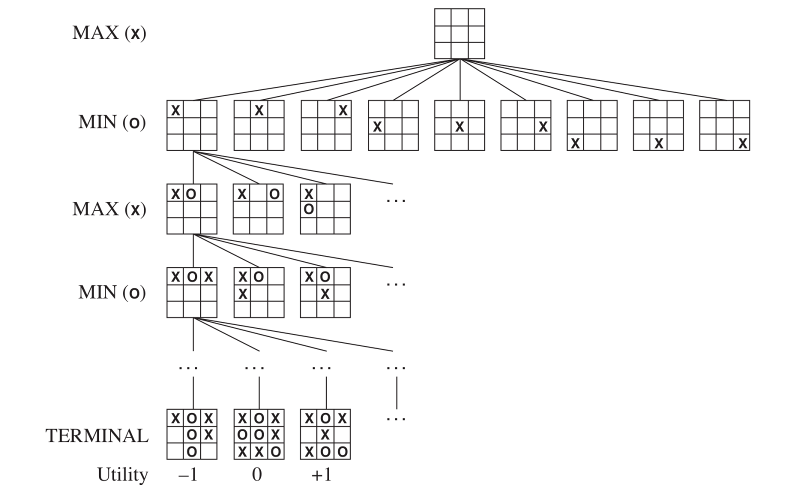
# Game Tree: Example

## Partial game tree for tic-tac-toe (2-player, deterministic, turn-taking)

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# Zero-Sum Games vs. General Games

- General Games
  - agents have independent utilities
  - cooperation, indifference, competition, and more are all possible
- Zero-Sum Games: the total payoff to all players is the same for each game instance
  - adversarial, pure competition
  - agents have opposite utilities (values on outcomes)

$\Longrightarrow$ Idea: With two-player zero-sum games, we can use one single utility value

- one agent maximizes it, the other minimizes it

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- MAX must find a contingent strategy specifying:
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  (a single-agent move is called half-move or ply)
- Analogous to the AND-OR search algorithm
  - MAX playing the role of OR
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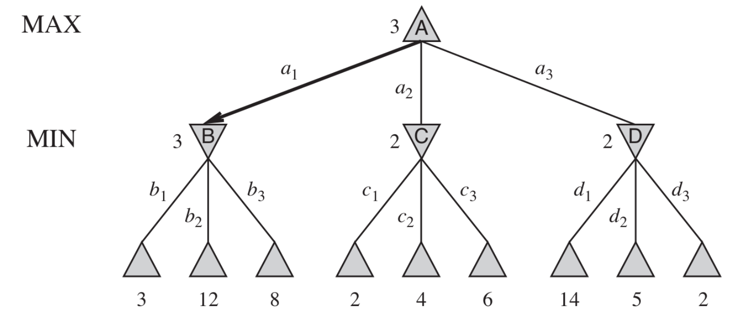
# Min-Max Search: Example

## A two-ply game tree

- $\Delta$ nodes are "MAX nodes", $\nabla$ nodes are "MIN nodes",
  - terminal nodes show the utility values for MAX
  - the other nodes are labeled with their minimax value
- Minimax maximizes the worst-case outcome for MAX

$\Longrightarrow$ MAX's root best move is $a_1$

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# Min-Max Search: Example

## A two-ply game tree

- Δ nodes are "MAX nodes", ∇ nodes are "MIN nodes",
  - terminal nodes show the utility values for MAX
  - the other nodes are labeled with their minimax value
- Minimax maximizes the worst-case outcome for MAX

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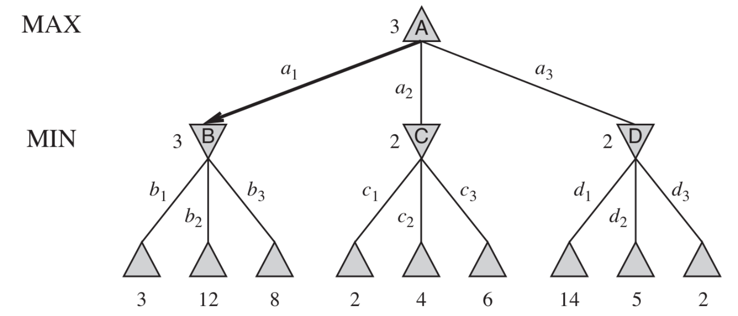

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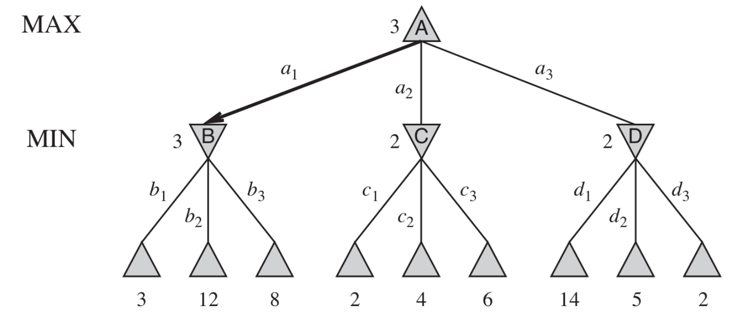


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# The Minimax Algorithm

## Depth-First Search Minimax Algorithm

**function** MINIMAX-DECISION(*state*) **returns** *an action*
  **return** $\arg\max_{a \,\in\, \text{ACTIONS}(s)}$ MIN-VALUE(RESULT(*state*, *a*))

---

**function** MAX-VALUE(*state*) **returns** *a utility value*
  **if** TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  **for each** $a$ **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX($v$, MIN-VALUE(RESULT($s$, $a$)))
  **return** $v$

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# Multi-Player Games: Optimal Decisions

- Replace the single value for each node with a vector of values
  - each value represent score from each player's viewpoint
  - terminal states: utility for each agent
  - agents, in turn, choose the action with best value for themselves
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## The first three plies of a game tree with three players (A, B, C)

- Each node labeled with values from each player's viewpoint
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  $\Longrightarrow$ A chooses the left move (1, 2, 6) (bad for A and B, good for C), or
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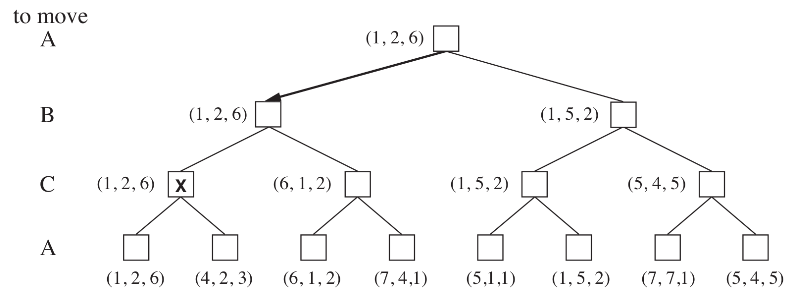
- If A and B are allied, then they may agree that B and then A choose (5,4,5) instead of (1,5,2)
  $\Longrightarrow$ benefit for both

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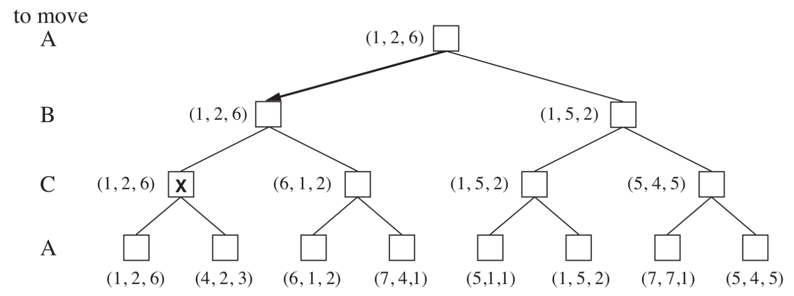


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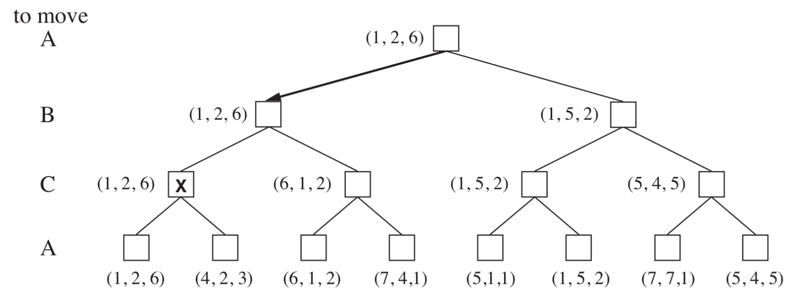


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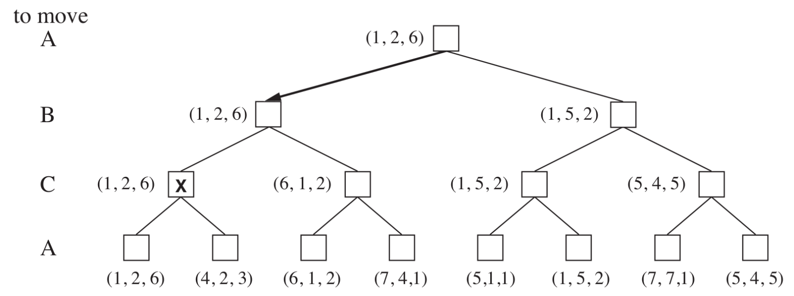

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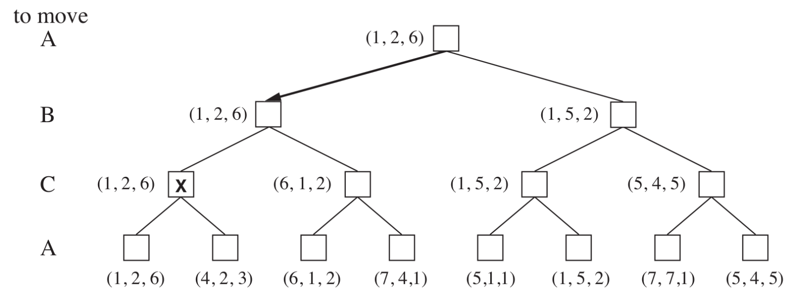


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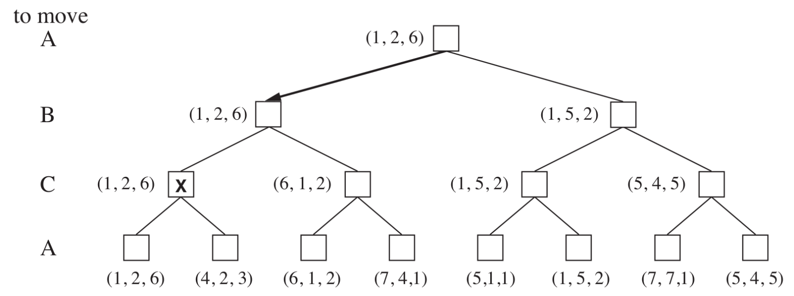


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# Exercise

- Consider the Multiplayer Min-Max Search example of previous slide
  - Redo it with choice order A-C-B
  - Redo it with choice order C-A-B
  - Redo it with choice order C-B-A
  - Redo it with choice order B-A-C
  - Redo it with choice order B-C-A
- Do they have all the same outcome?
- For each case, try to define the best moves in case of alliance between the top two players

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# The Minimax Algorithm: Properties

- Complete? Yes, if tree is finite
- Optimal? Yes, against an optimal opponent
  - What about non-optimal opponent?

  $\Longrightarrow$ even better, but non optimal in this case
- Time complexity? $O(b^m)$
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For chess, $b \approx 35$, $m \approx 100$ $\Longrightarrow$ $35^{100} = 10^{154}$ (!)

We need to prune the tree!

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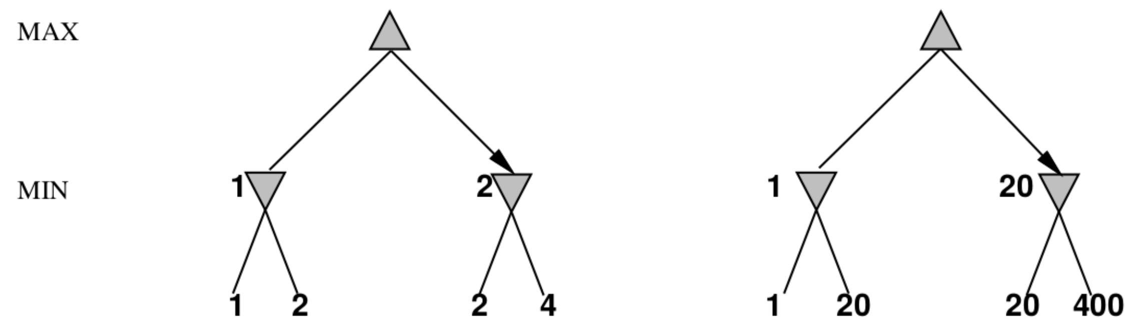
# Remark

## Exact values don't matter!

Behaviour preserved under any monotonic transformation of *Eval*()

- Only the order matters!

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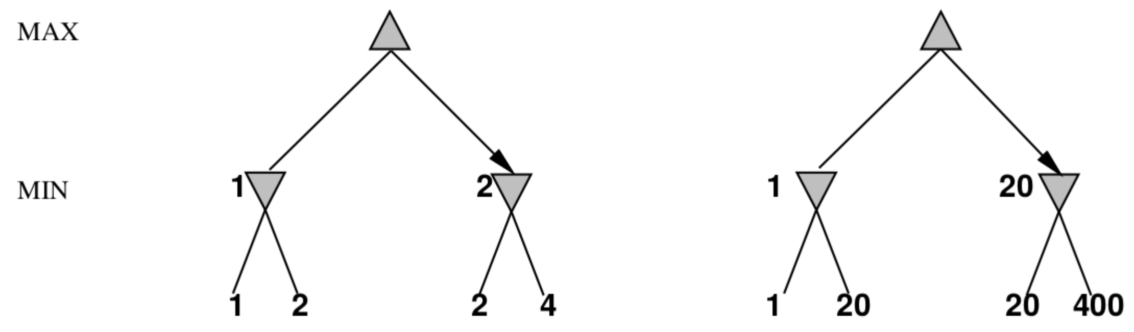


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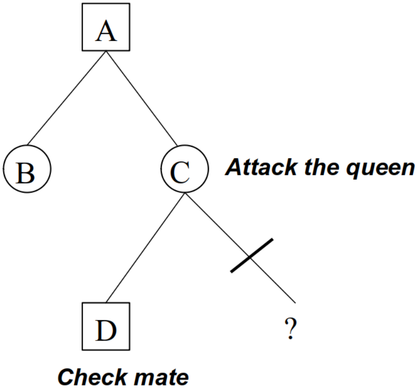
# Outline

1. Games
2. Optimal Decisions in Games
   - Min-Max Search
   - Alpha-Beta Pruning
3. Adversarial Search with Resource Limits
4. Stochastic Games

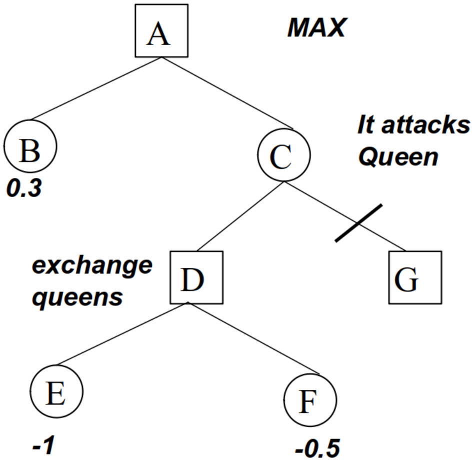
# Example: Chess (1)




- No matter which is the evaluation of the other children of C (I realize that I should never move to C).

(Courtesy of Michela Milano, UniBO)

# Example: Chess (2)




- Max in A avoids C because B is better. At most max gets from C a - 0.5 so 0.3 is better
- The subtree in G can be cut as soon as I receive the value of D.
  Indeed: C = min (-0.5, G);
  A = max (0.3, min (-0.5, G)) = 0.3

  Since A is independent of G, the tree under G can be cut.

(Courtesy of Michela Milano, UniBO)

# Pruning Min-Max Search: Example

- Consider the Min-Max example, let $[\alpha, \beta]$ track the currently-known bounds:
  ($\alpha$ (resp $\beta$): best value for MAX (resp MIN) so far at any choice point along the path)
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  - (d): Is it necessary to evaluate the remaining leaves of $C$?
    NO! They cannot produce an upper bound $\geq 2$
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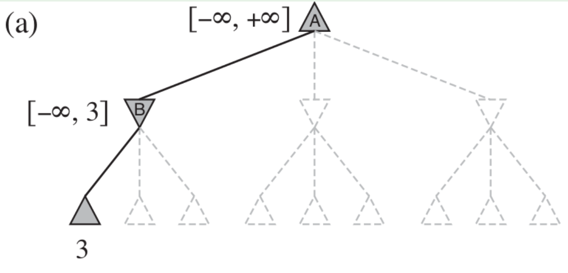
  - (e): MAX updates the upper bound to 14 (D is last subtree)
  - (f): D labeled $[2, 2]$ $\Longrightarrow$ MAX updates the upper bound to 3

$\Longrightarrow$ final value: 3

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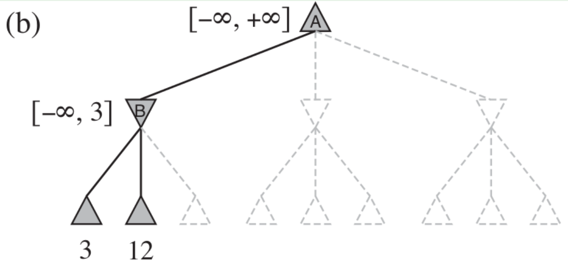


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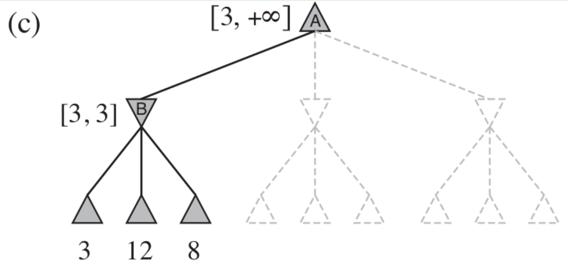


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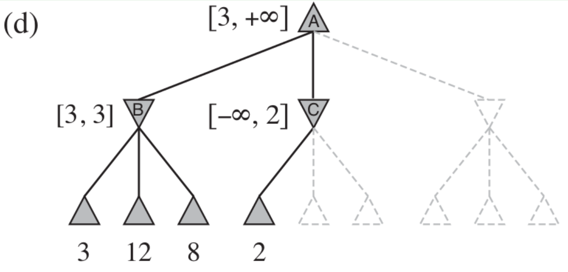


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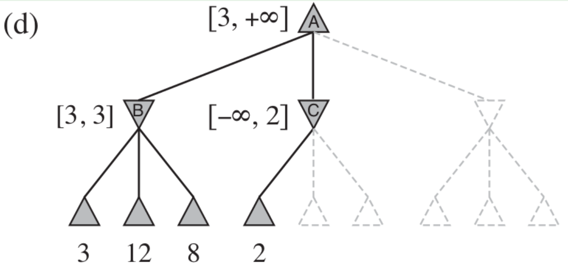


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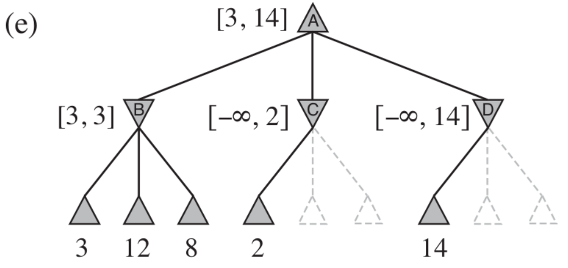

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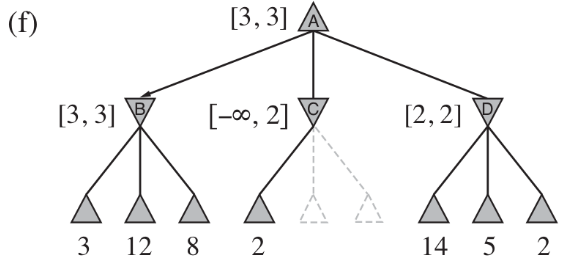
$\Longrightarrow$ final value: 3

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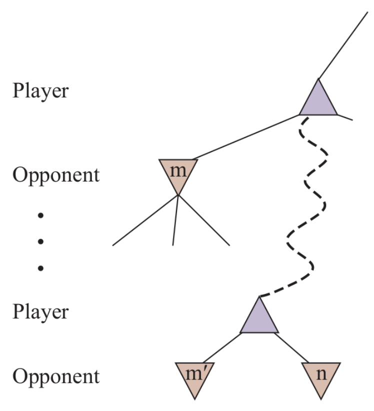
# Alpha-Beta Pruning Technique for Min-Max Search

- Idea: consider a node *n* (terminal or intermediate) and its current value
  - If player has a better choice at the same level of *n* ($m'$) or at any point higher up in the tree ($m$), then n will never be reached in actual play

$\Longrightarrow$ as soon as we know enough of n to draw this conclusion, we can prune n

- Alpha-Beta Pruning: nodes labeled with $[\alpha, \beta]$ s.t.:
  - $\alpha$ : best value for MAX (highest) so far at any choice point along the path
    $\Longrightarrow$ lower bound for future values
  - $\beta$ : best value for MIN (lowest) so far at any choice point along the path
    $\Longrightarrow$ upper bound for future values

$\Longrightarrow$ Prune *n* if its value is worse (lower) than the current $\alpha$ value for MAX (dual for $\beta$, MIN)

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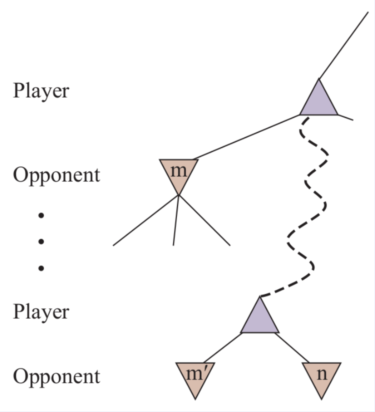


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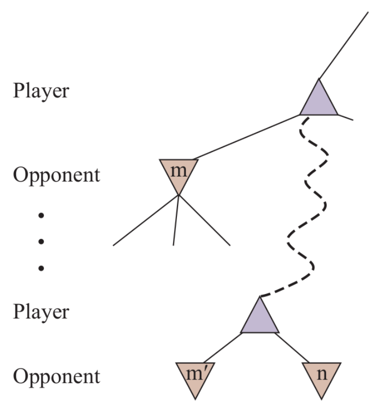


# The Alpha-Beta Search Algorithm

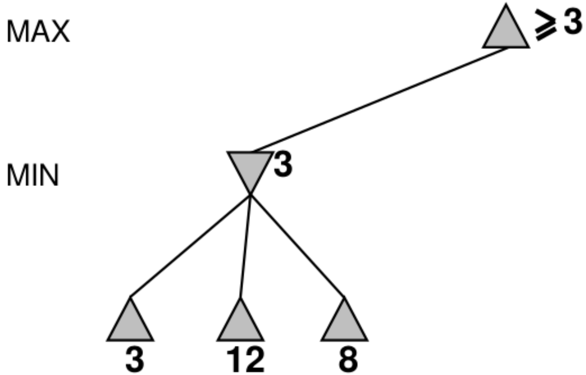
```
function ALPHA-BETA-SEARCH(state) returns an action
  v ← MAX-VALUE(state, −∞, +∞)
  return the action in ACTIONS(state) with value v

function MAX-VALUE(state, α, β) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for each a in ACTIONS(state) do
    v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
    if v ≥ β then return v   // MIN will never choose a bigger value
    α ← MAX(α, v)
  return v

function MIN-VALUE(state, α, β) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← +∞
  for each a in ACTIONS(state) do
    v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
    if v ≤ α then return v   // MAX will never choose a smaller value
    β ← MIN(β, v)
  return v
```

# Example revisited: Alpha-Beta Cuts

- Notation: $\geq \alpha$; $\leq \beta$;

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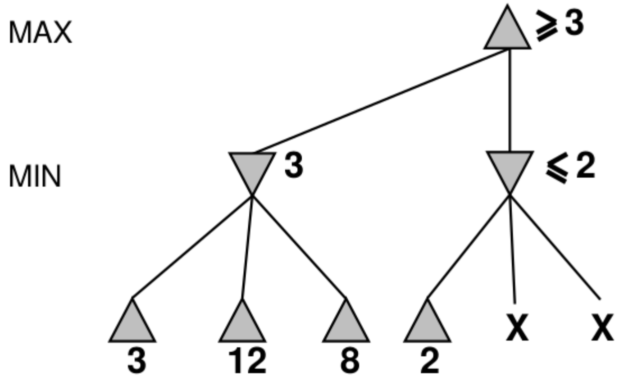

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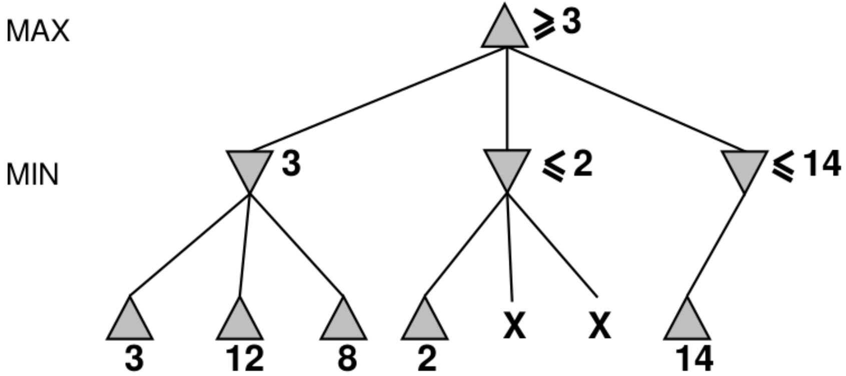


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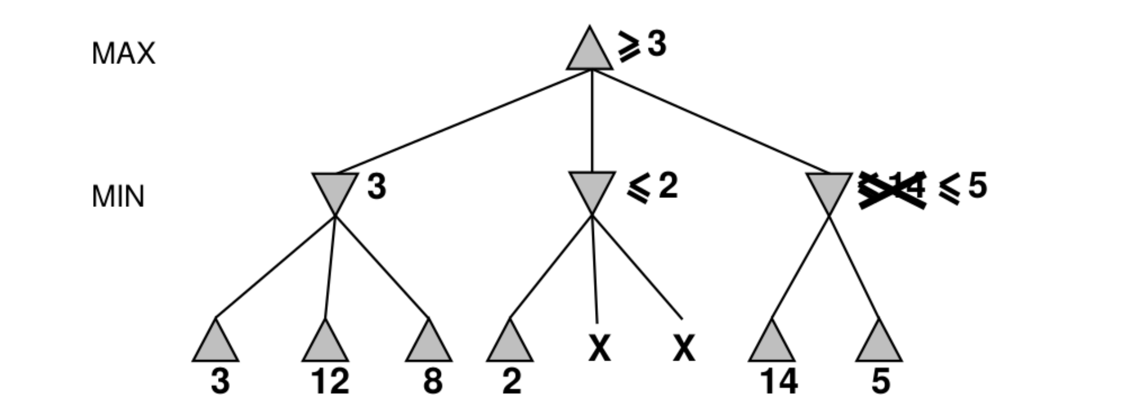

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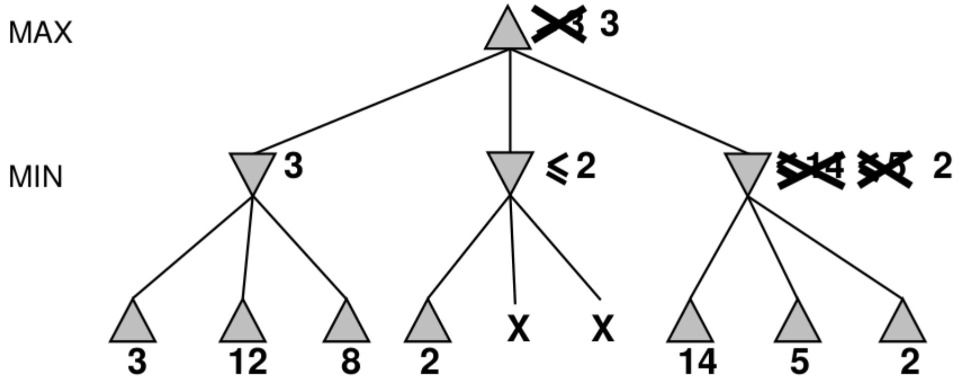


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# Properties of Alpha-Beta Search

- Pruning does not affect the final result $\Longrightarrow$ correctness preserved
- Good move ordering improves effectiveness of pruning
  - Ex: if MIN expands $3^{rd}$ child of D first (see $2^{nd}$ last example), then the others are pruned

  $\Longrightarrow$ try to examine first the successors that are likely to be best
- With "perfect" ordering, time complexity reduces to $O(b^{m/2})$
  - aka "killer-move heuristic"

  $\Longrightarrow$ doubles solvable depth!
- With "random" ordering, time complexity reduces to $O(b^{3m/4})$
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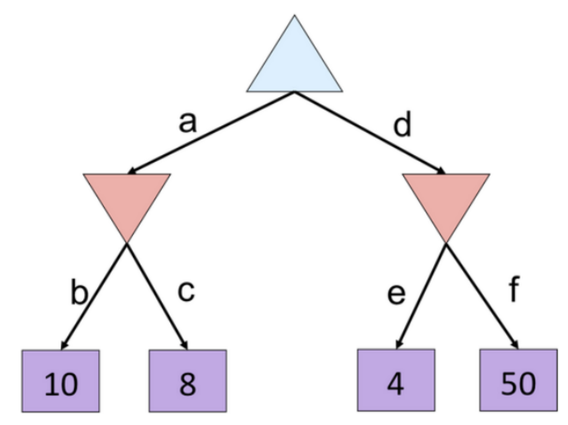
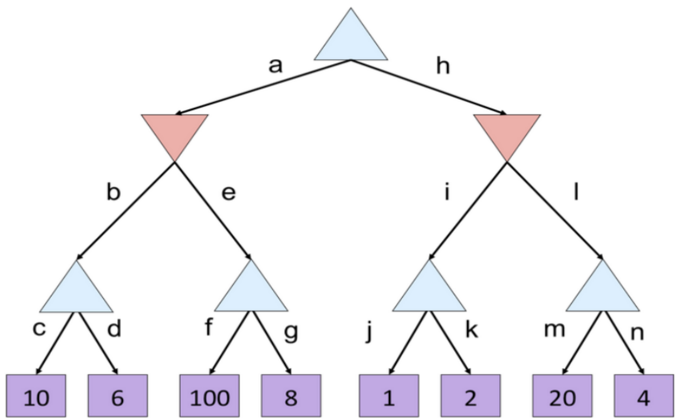
# Exercise I

Apply alpha-beta search to the following tree



# Exercise II

Apply alpha-beta search to the following tree

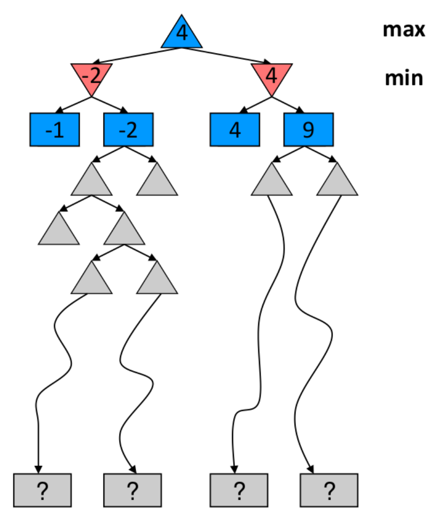


# Outline

# Adversarial Search with Resource Limits

Problem: In realistic games, full search is impractical!

- Complexity: $b^d$ (ex. chess: $\approx 35^{100}$)
- Idea [Shannon, 1949]: Depth-limited search
  - cut off minimax search earlier, after limited depth
  - replace terminal utility function with evaluation for non-terminal nodes
- Ex (chess): depth $d = 8$ (decent)
  $\Longrightarrow \alpha$-$\beta$: $35^{8/2} \approx 10^5$ (feasible)



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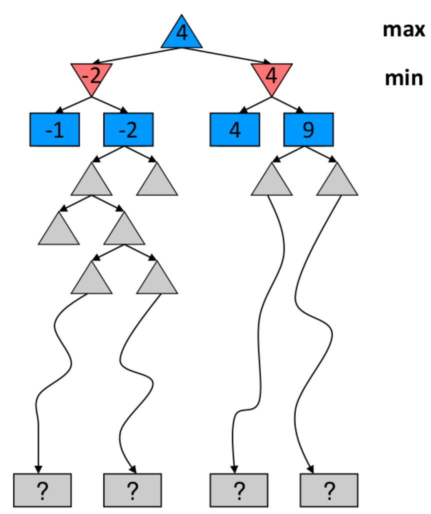

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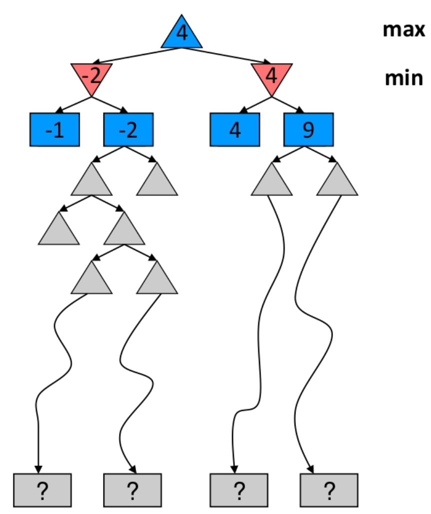

# Adversarial Search with Resource Limits [cont.]

- Idea:
  - cut off the search earlier, at limited depths
  - apply a heuristic evaluation function to states in the search

  $\Longrightarrow$ effectively turning nonterminal nodes into terminal leaves
- Modify *Minimax*() or Alpha-Beta search in two ways:
  - replace the utility function *Utility*(*s*) by a heuristic evaluation function *Eval*(*s*), which estimates the position's utility
  - replace the terminal test *TerminalTest*(*s*) by a cutoff test *CutOffTest*(*s*, *d*), that decides when to apply *Eval*()
  - plus some bookkeeping to increase depth *d* at each recursive call

$\Longrightarrow$ Heuristic variant of *Minimax*():

$$H\text{-}Minimax(s,d) \stackrel{\text{def}}{=} \begin{cases} Eval(s) & \text{if } CutOffTest(s,d) \\ \max_{a \in Actions(s)} H\text{-}Minimax(Result(s,a), d+1) & \text{if } Player(s) = MAX \\ \min_{a \in Actions(s)} H\text{-}Minimax(Result(s,a), d+1) & \text{if } Player(s) = MIN \end{cases}$$

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# Evaluation Functions

## *Eval*(*s*)

- Should be relatively cheap to compute
- Returns an estimate of the expected utility from a given position
  - Ideal function: returns the actual minimax value of the position
- Should order terminal states the same way as the utility function
  - e.g., wins $>$ draws $>$ losses
- For nonterminal states, should be strongly correlated with the actual chances of winning
- Defines equivalence classes of positions (same $Eval(s)$ value)
  - e.g. returns a value reflecting the % of states with each outcome
- Typically weighted linear sum of features:
  $Eval(s) = w_1 \cdot f_1(s) + w_2 \cdot f_2(s) + ... + w_n \cdot f_n(s)$
  - ex (chess): $f_{pawns}(s) = \#white\ pawns - \#black\ pawns$,
    $w_{pawns} = 1$: $w_{bishops} = w_{knights} = 3$, $w_{rooks} = 5$, $w_{queens} = 9$
- May depend on depth
  - ex: knights more valuable with low depths, rooks more valuable with high depths
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    $w_{pawns} = 1$: $w_{bishops} = w_{knights} = 3$, $w_{rooks} = 5$, $w_{queens} = 9$
- May depend on depth
  - ex: knights more valuable with low depths, rooks more valuable with high depths
- May be very inaccurate for some positions

# Evaluation Functions

## $Eval(s)$

- Should be relatively cheap to compute
- Returns an estimate of the expected utility from a given position
  - Ideal function: returns the actual minimax value of the position
- Should order terminal states the same way as the utility function
  - e.g., wins $>$ draws $>$ losses
- For nonterminal states, should be strongly correlated with the actual chances of winning
- Defines equivalence classes of positions (same $Eval(s)$ value)
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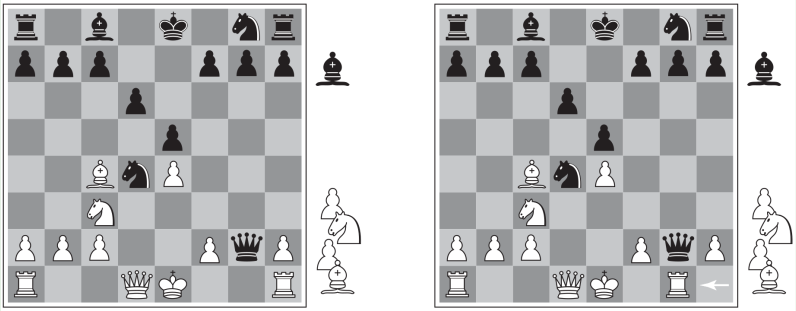
# Example

- Two same-score positions (White: -8, Black: -3)
  - (a) Black has an advantage of a knight and two pawns,
    ⟹ should be enough to win the game
  - (b) White will capture the queen,
    ⟹ give it an advantage that should be strong enough to win

(Personal note: only very-stupid black player would get into (b))



(a) White to move (b) White to move

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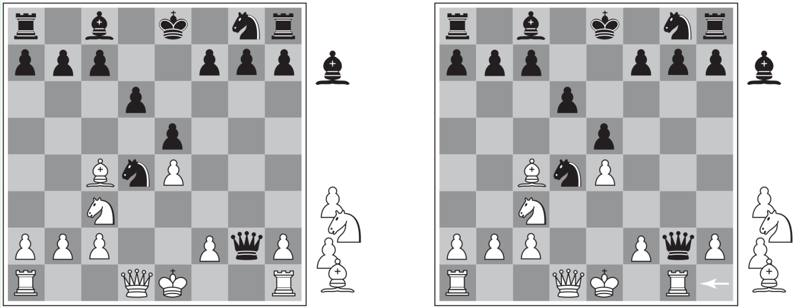

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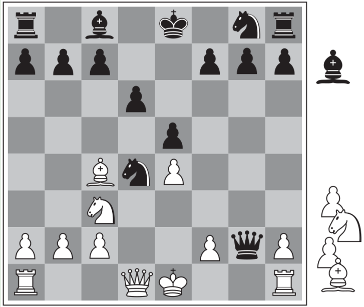

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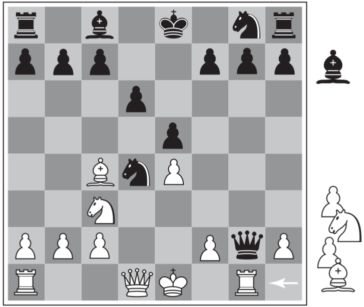

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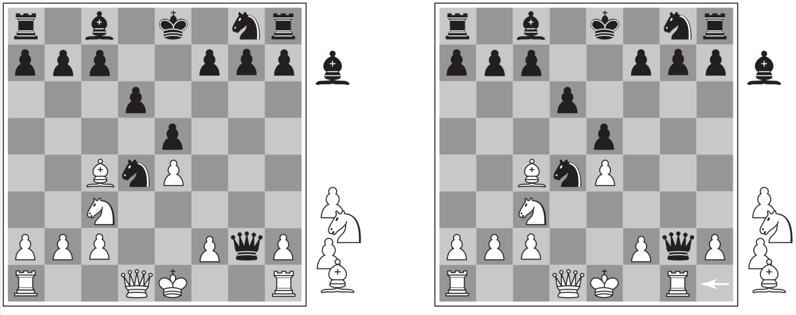

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# Cutting-off the Search

*CutOffTest(state, depth)*

- Most straightforward approach: set a fixed depth limit
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  - sometimes may produce very inaccurate outcomes (see previous example)
- More robust approach: apply Iterative Deepening
- More sophisticate: apply *Eval*() only to quiescent states
  - quiescent: unlikely to exhibit wild swings in value in the near future
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# Deterministic Games in Practice

- Checkers: (1994) Chinook ended 40-year-reign of world champion Marion Tinsley
  - used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board
  - a total of 443,748,401,247 positions
- Chess: (1997) Deep Blue defeated world champion Gary Kasparov in a six-game match
  - searches 200 million positions per second
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# AlphaGo beats GO world champion, Lee Sedol (2016)

# Outline

1. Games
2. Optimal Decisions in Games
   - Min-Max Search
   - Alpha-Beta Pruning
3. Adversarial Search with Resource Limits
4. **Stochastic Games**

# Stochastic Games: Generalities

- In real life, unpredictable external events may occur
- Stochastic Games mirror unpredictability by random steps:
  - e.g. dice throwing, card-shuffling, coin flipping, tile extraction, ...
- Ex: Backgammon
- Cannot calculate definite minimax value, only expected values
- Uncertain outcomes controlled by chance, not an adversary!
  - adversarial $\Longrightarrow$ worst case
  - chance $\Longrightarrow$ average case
- Ex: if chance is 0.5 each (coin):
  - minimax: 10
  - average: (100+9)/2=54.5

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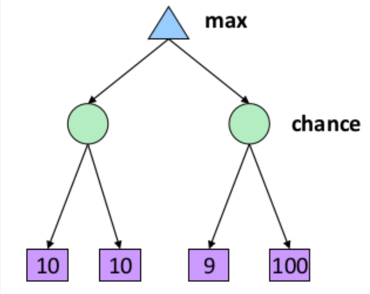

# An Example: Backgammon

- Rules
  - 15 pieces each
  - white moves clockwise to 25, black moves counterclockwise to 0
  - a piece can move to a position unless $\geq 2$ opponent pieces there
  - if there is one opponent, it is captured and must start over
  - termination: all whites in 25 or all blacks in 0
- Ex: Possible white moves (dice: 6,5):
  - (5-10,5-11)
  - (5-11,19-24)
  - (5-10,10-16)
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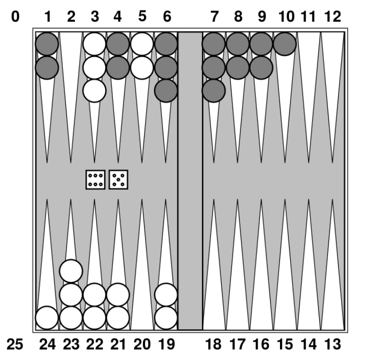
- Combines strategy with luck
  $\Longrightarrow$ stochastic component (dice)
  - double rolls (1-1),...,(6-6) have 1/36 probability each
  - other 15 distinct rolls have a 1/18 probability each



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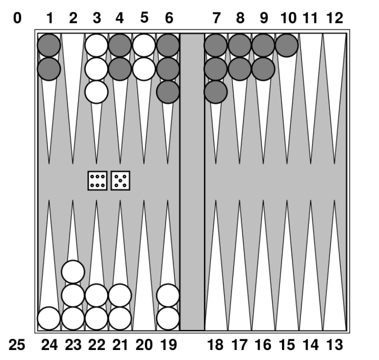

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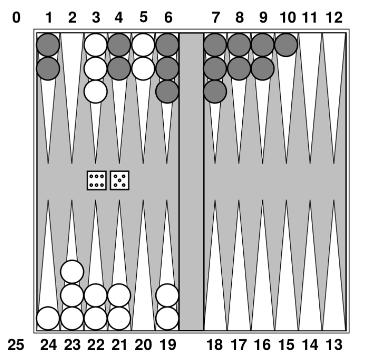

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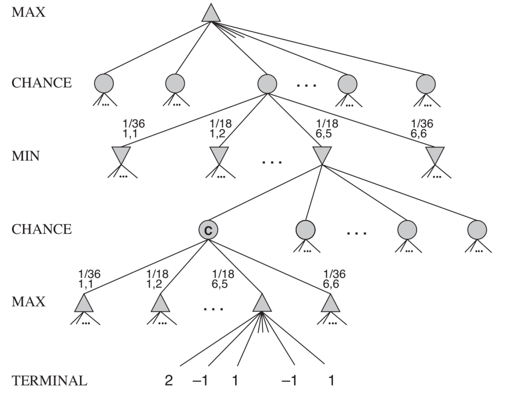
# Stochastic Games Trees

## Idea:

- A tree for a stochastic game includes chance nodes in addition to MAX and MIN nodes.
  - chance nodes above agent represent stochastic events for agent (e.g. dice roll)
  - outcoming arcs represent stochastic event outcomes
  - labeled with stochastic event and relative probability

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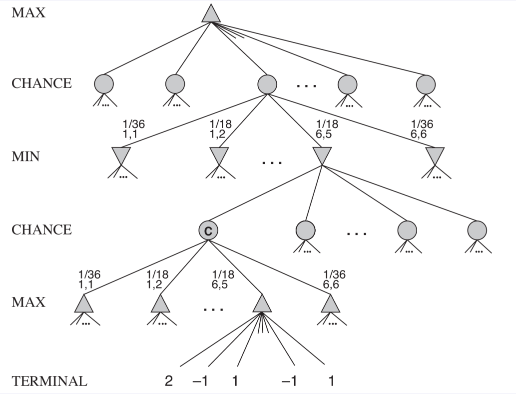


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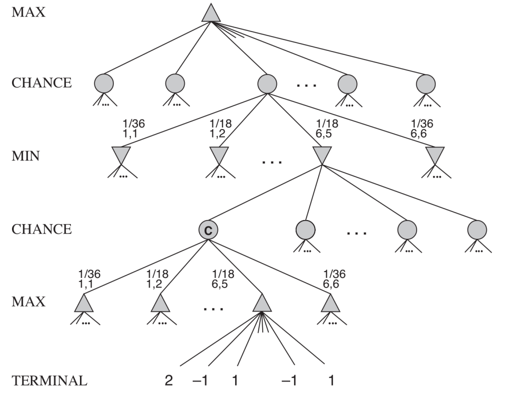


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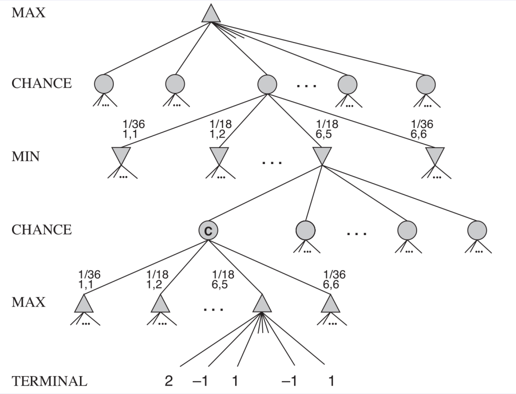


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# Algorithm for Stochastic Games: *ExpectMinimax*()

- Extension of *Minimax*(), handling also chance nodes:

$$ExpectMinimax(s) \stackrel{\text{def}}{=} \begin{cases} Utility(s) & \text{if } TerminalTest(s) \\ max_{a \in Actions(s)} ExpectMinimax(Result(s,a)) & \text{if } Player(s) = MAX \\ min_{a \in Actions(s)} ExpectMinimax(Result(s,a)) & \text{if } Player(s) = MIN \\ \sum_r P(r) \cdot ExpectMinimax(Result(s,r)) & \text{if } Player(s) = Chance \end{cases}$$

  - $P(r)$: probability of stochastic event outcome $r$
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$\Longrightarrow$ Returns the weighted average of the minimax outcomes (recall that $\sum_r P(r) = 1$))

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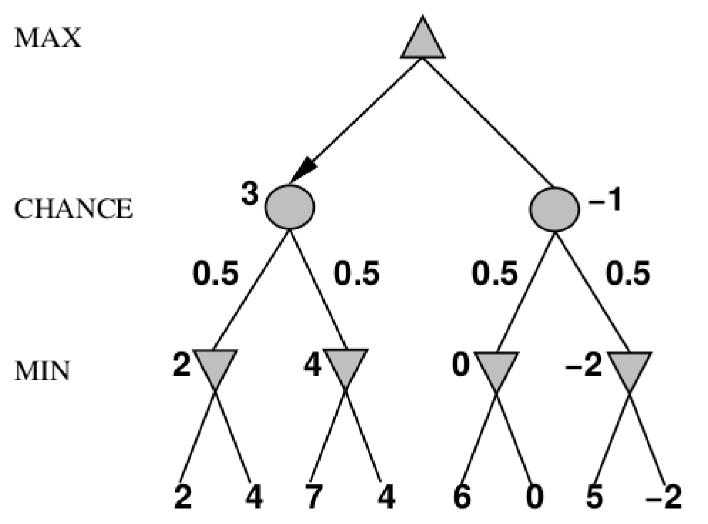
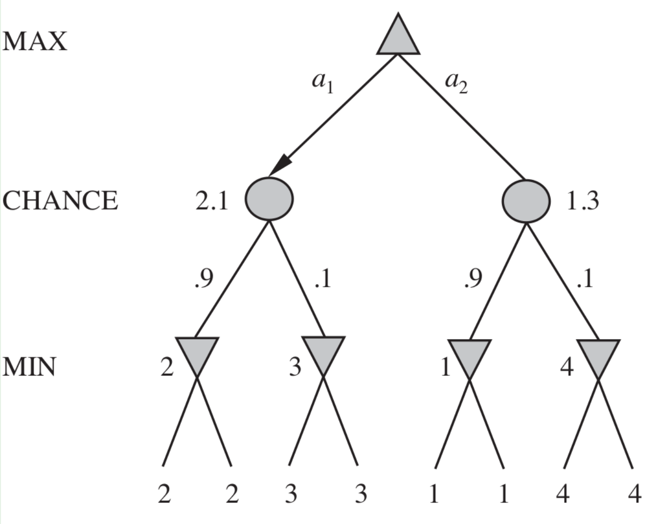
# Simple Example with Coin-Flipping



# Example (Non-uniform Probabilities)

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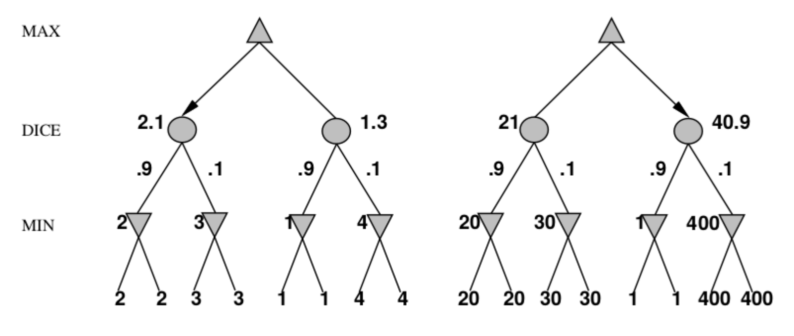
# Remark (compare with deterministic case)

**Exact values do matter!**

Behaviour not preserved under monotonic transformations of *Utility*()

- preserved only by positive linear transformation of *Utility*()
  - hint: $p_1 v_1 \geq p_2 v_2 \Longrightarrow p_1(av_1 + b) \geq p_2(av_2 + b)$ if $a \geq 0$

$\Longrightarrow$ *Utility*() should be proportional to the expected payoff

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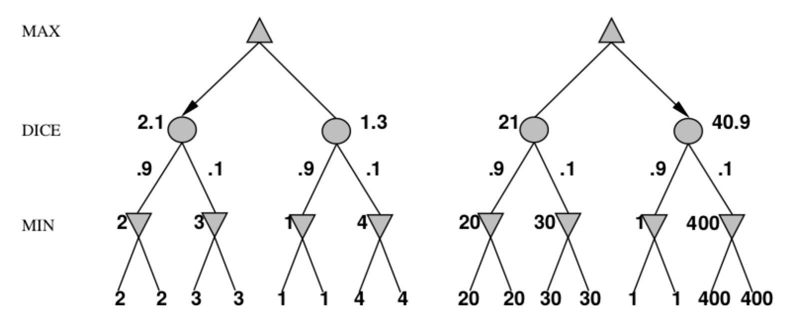


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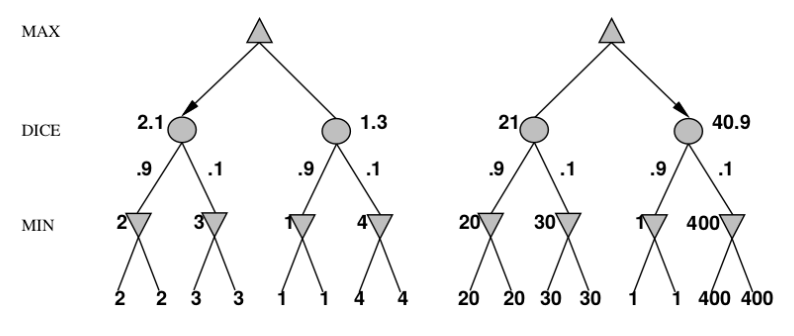


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# Example

## Beware of money as utility function!

- Ex: choose between two alternatives in a coin-toss tree:
  (a) gain 0.8M€ (heads) vs. gain 1.2M€ (tails)
  (b) gain 0.0M€ (heads) vs. gain 3.0M€ (tails).
- Which one will you choose? Why?
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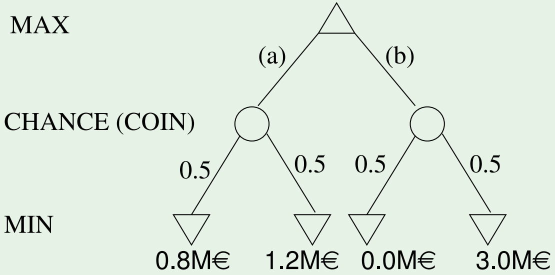

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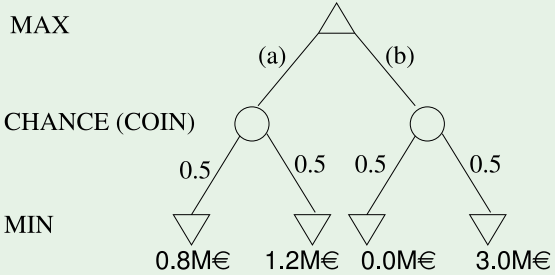

# Stochastic Games in Practice

- Dice rolls increase b: 21 possible rolls with 2 dice

  $\Longrightarrow$ $O(b^m \cdot n^m)$, $n$ being the number of distinct roll
- Ex: Backgammon has $\approx$ 20 moves

  $\Longrightarrow$ depth 4: $20 \cdot (21 \cdot 20)^3 \approx 10^9$ (!)
- Alpha-beta pruning much less effective than with deterministic games

$\Longrightarrow$ Unrealistic to consider high depths in most stochastic games

- Heuristic variants of *ExpectMinimax*() effective, low cutoff depths
- Ex: TD-GAMMON uses depth-2 search + very-good *Eval*()
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