Fundamentals of Artificial Intelligence
Chapter 08: **First-Order Logic**

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Outline

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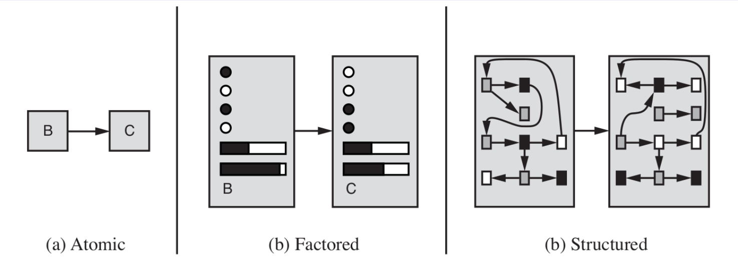
Recall: State Representations [Ch. 02]

Representations of states and transitions

- Three ways to represent states and transitions between them:
  - atomic: a state is a black box with no internal structure
  - factored: a state consists of a vector of attribute values
  - structured: a state includes objects, each of which may have attributes of its own as well as relationships to other objects
- increasing expressive power and computational complexity
- reality represented at different levels of abstraction



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Pros of Propositional Logic

- PL language is formal
  - non-ambiguous semantics
  - unlike natural language, which is intrinsically ambiguous (ex “key”)
- PL is declarative
  - knowledge and inference are separate
  - inference is entirely domain independent
- PL allows for partial/disjunctive/negated information
  - unlike, e.g., data bases
- PL is compositional
  - the meaning of $(A \wedge B) \rightarrow C$ derives from the meaning of A,B,C
- The meaning of PL sentence is context independent
  - unlike with natural language, where meaning depends on context

Cons of Propositional Logic

- Is "Atomic": based on atomic events which cannot be decomposed
- Assumes the world contains facts in the world that are either true or false, nothing else
  - ex: Man_Socrates, Man_Plato, Man_Aristotle, ... distinct atoms

$\Longrightarrow$ PL has has very limited expressive power

  - unlike natural language
  - cannot concisely describe an environment with many objects
  - e.g., cannot say "pits cause breezes in adjacent squares" (need writing one sentence for each square)

Logics

- A logic is a triple $\langle \mathcal{L}, \mathcal{S}, \mathcal{R} \rangle$ where
  - $\mathcal{L}$, the logic's language: a class of sentences described by a formal grammar
  - $\mathcal{S}$, the logic's semantics: a formal specification of how to assign meaning in the "real world" to the elements of $\mathcal{L}$
  - $\mathcal{R}$, the logic's inference system: is a set of formal derivation rules over $\mathcal{L}$
- There are several logics:
  - propositional logic (PL)
  - first-order logic (FOL)
  - modal logics (MLs)
  - description logics (DLs)
  - temporal logics (TLs)
  - (fuzzy logics, probabilistic logics, ...)
  - ...

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First-Order Logic (FOL)

- Is structured: a world/state includes objects, each of which may have attributes of its own as well as relationships to other objects
- Assumes the world contains:
  - Objects:
    e.g., people, houses, numbers, theories, Jim Morrison, colors, basketball games, wars, centuries, ...
  - Relations:
    e.g., red, round, bogus, prime, tall ...,
    brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
  - Functions:
    e.g., father of, best friend, one more than, end of, ...
- Allows to quantify on objects
  - ex: “All man are equal”, “some persons are left-handed”, ...

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Syntax of FOL: Basic Elements

- Constant symbols: KingJohn, 2, UniversityofTrento,...
- Predicate symbols: Man(.), Brother(.,.), $(. > .)$, AllDifferent(...),...
  - may have different arities (1,2,3,...)
  - may be prefix (e.g. Brother(.,.)) or infix (e.g. $(. > .)$)
- Function symbols: Sqrt, LeftLeg, MotherOf
  - may have different arities (1,2,3,...)
  - may be prefix (e.g. Sqrt(.)) or infix (e.g. $(. + .)$)
- Variable symbols: x, y, a, b, ...
- Propositional Connectives: $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$
- Equality: "$=$" (also "$\neq$" s.t. "$a \neq b$" shortcut for "$\neg(a = b)$")
- Quantifiers: "$\forall$" ("forall"), "$\exists$" ("exists", aka "for some")
- Punctuation Symbols: ",", "(", ")"

- Constants symbols are 0-ary function symbols
- Propositions are 0-ary predicates $\Longrightarrow$ PL subcase of FOL
- Signature: the set of predicate, function & constant symbols

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FOL: Syntax

- Terms:
  - constant or variable or *function*(*term*$_1$, ..., *term*$_n$)
  - ex: KingJohn, x, LeftLeg(Richard), (z*log(2))
  - denote objects in the real world (aka domain)
- Atomic sentences (aka atomic formulas):
  - $\top$, $\bot$
  - *proposition* or *predicate*(*term*$_1$, ..., *term*$_n$) or *term*$_1$ $=$ *term*$_2$
  - $(Length(LeftLeg(Richard)) > Length(LeftLeg(KingJohn)))$
  - denote facts
- Non-atomic sentences/formulas:
  - $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$, $\alpha \oplus \beta$,
    $\forall x.\alpha$, $\exists x.\alpha$ s.t. x (typically) occurs in α
  - Ex: $\forall y.(Italian(y) \rightarrow President(Mattarella, y))$
    $\exists x \forall y.President(x, y) \rightarrow \forall y \exists x.President(x, y)$
    $\forall x.(P(x) \wedge Q(x)) \leftrightarrow ((\forall x.P(x)) \wedge (\forall x.Q(x)))$
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  - denote (complex) facts

FOL: Ground and Closed Formulas

- A term/formula is ground iff no variable occurs in it (ex: $2 \geq 1$)
- A formula is closed iff all variables occurring in it (if any) are quantified (ex: $\forall x \exists y.(x > y)$)

$\Longrightarrow$ Ground formulas are closed, but not vice versa.

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FOL: Syntax (BNF)

$$
\begin{array}{lcl}
\langle \text{Sentence} \rangle & ::= & \langle \text{AtomicSentence} \rangle \mid \langle \text{ComplexSentence} \rangle \\
\langle \text{AtomicSentence} \rangle & ::= & \top \mid \bot \mid \\
 & & \langle \text{PredicateSymbol} \rangle(\langle \text{Term} \rangle, \ldots) \mid \\
 & & \langle \text{Term} \rangle = \langle \text{Term} \rangle \\
\langle \text{ComplexSentence} \rangle & ::= & \neg \langle \text{Sentence} \rangle \mid \\
 & & \langle \text{Sentence} \rangle \; \langle \text{Connective} \rangle \; \langle \text{Sentence} \rangle \mid \\
 & & \langle \text{Quantifier} \rangle \; \langle \text{Sentence} \rangle \\
\langle \text{Term} \rangle & ::= & \langle \text{ConstantSymbol} \rangle \mid \langle \text{Variable} \rangle \mid \\
 & & \langle \text{FunctionSymbol} \rangle(\langle \text{Term} \rangle, \ldots) \\
\langle \text{Connective} \rangle & ::= & \wedge \mid \vee \mid \rightarrow \mid \leftarrow \mid \leftrightarrow \mid \oplus \\
\langle \text{Quantifier} \rangle & ::= & \forall \; \langle \text{Variable} \rangle . \mid \exists \; \langle \text{Variable} \rangle . \\
\langle \text{Variable} \rangle & ::= & a \mid b \mid \cdots \mid x \mid y \mid \cdots \\
\langle \text{ConstantSymbol} \rangle & ::= & A \mid B \mid \cdots \mid John \mid 0 \mid 1 \mid \cdots \mid \pi \mid \ldots \\
\langle \text{FunctionSymbol} \rangle & ::= & F \mid G \mid \cdots \mid Cos \mid FatherOf \mid + \mid \ldots \\
\langle \text{PredicateSymbol} \rangle & ::= & P \mid Q \mid \cdots \mid Red \mid Brother \mid > \mid \cdots
\end{array}
$$

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
  - φ occurs positively in φ;
  - if $\neg\varphi_1$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ
  - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ, then φ_1 and φ_2 occur positively [negatively] in φ;
  - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
  - if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ, then φ_1 and φ_2 occur positively and negatively in φ;
  - if $\forall x.\varphi_1$ or $\exists x.\varphi_1$ occurs positively [negatively] in φ, then φ_1 occurs positively [negatively] in φ

Outline

Truth in FOL: Intuitions

- Sentences are true with respect to a model
  - containing a domain and an interpretation
- The domain contains ≥ 1 objects (domain elements) and relations and functions over them
- An interpretation specifies referents for
  - variables $\rightarrow$ objects
  - constant symbols $\rightarrow$ objects
  - predicate symbols $\rightarrow$ relations
  - function symbols $\rightarrow$ functional relations
- An atomic sentence $P(t_1, ..., t_n)$ is true in an interpretation iff the objects referred to by $t_1, ..., t_n$ are in the relation referred to by P

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FOL: Semantics

FOL Models (aka possible worlds)

- A model $\mathcal{M}$ is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ ($\langle$*domain*, *interpretation*$\rangle$)
- Domain $\mathcal{D}$: a non-empty set of objects (aka domain elements)
- Interpretation $\mathcal{I}$: a (non-injective) map on elements of the signature
  - constant symbols $\longmapsto$ domain elements:
    a constant symbol C is mapped into a particular object $[C]^{\mathcal{I}}$ in $\mathcal{D}$
  - predicate symbols $\longmapsto$ domain relations:
    a k-ary predicate $P(...)$ is mapped into a subset $[P]^{\mathcal{I}}$ of $\mathcal{D}^k$
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FOL: Semantics [cont.]

Interpretation of terms

$\mathcal{I}$ maps terms into domain elements

- Variables are assigned domain values
  - variables $\longmapsto$ domain elements:
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- A term $f(t_1, ..., t_k)$ is mapped by $\mathcal{I}$ into the value $[f(t_1, ..., t_k)]^{\mathcal{I}}$ returned by applying the domain function $[f]^{\mathcal{I}}$, into which f is mapped, to the values $[t_1]^{\mathcal{I}}, ..., [t_k]^{\mathcal{I}}$ obtained by applying recursively $\mathcal{I}$ to the terms $t_1, ..., t_k$:
  - $[f(t_1, ..., t_k)]^{\mathcal{I}} = [f]^{\mathcal{I}}([t_1]^{\mathcal{I}}, ..., [t_k]^{\mathcal{I}})$
  - Ex: if "Me, Mother, Father" are interpreted as usual, then "Mother(Father(Me))" is interpreted as my (paternal) grandmother
  - Ex: if "$+, -, \cdot, 0, 1, 2, 3, 4$" are interpreted as usual, then "$(3 - 1) \cdot (0 + 2)$" is interpreted as 4

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- An atomic formula $P(t_1, ..., t_k)$ is true in $\mathcal{I}$ iff the objects into which the terms t_1,...t_k are mapped by $\mathcal{I}$ comply to the relation into which P is mapped
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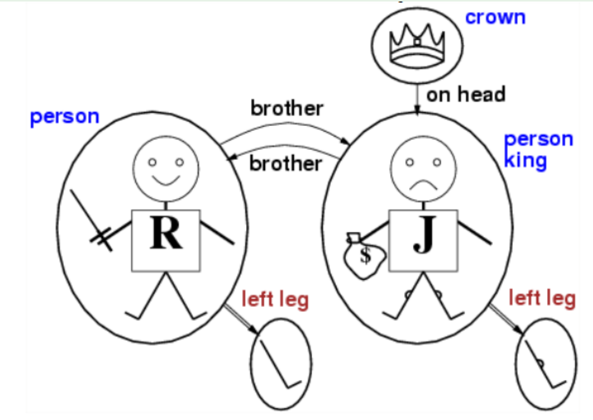
Models for FOL: Example

Richard Lionhearth and John Lackland

- $\mathcal{D}$: domain at right
- $\mathcal{I}$: s.t.
  - $[Richard]^{\mathcal{I}}$: Richard the Lionhearth
  - $[John]^{\mathcal{I}}$: evil King John
  - $[Brother]^{\mathcal{I}}$: brotherhood
- $[Brother(Richard, John)]^{\mathcal{I}}$ is true
- $[LeftLeg]^{\mathcal{I}}$ maps any individual to his left leg
- ...



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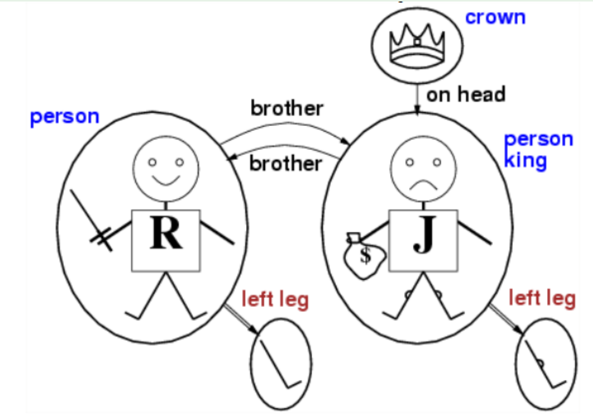

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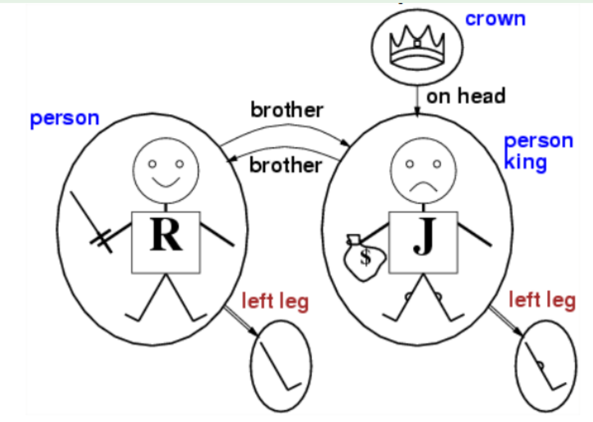

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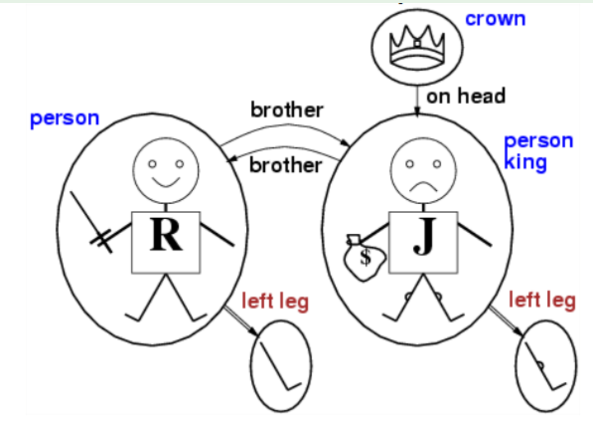

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Models for FOL: Remark

- $[f]^{\mathcal{I}}$ total: must provide an output for every input
- e.g.: $[\mathit{LeftLeg}(\mathit{crown})]^{\mathcal{I}}$?
- possible solution: assume "null" object ($[\mathit{LeftLeg}(\mathit{crown}) = \mathit{null}]^{\mathcal{I}}$
  (other solution, sorts, not considered here)

Universal Quantification

- $\forall x.\alpha(x, ...)$ (x variable, typically occurs in x)
  - ex: $\forall x.(King(x) \rightarrow Person(x))$ ("all kings are persons")
- $\forall x.\alpha(x, ...)$ true in $\mathcal{M}$ iff
  α is true in $\mathcal{M}$ for every possible domain value x is mapped to
- Roughly speaking, can be seen as a conjunction over all (typically infinite) possible instantiations of x in α

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\begin{array}{lll}
(King(John) & \rightarrow Person(John) &)\wedge \\
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(King(crown) & \rightarrow Person(crown) &)\wedge \\
(King(LeftLeg(John)) & \rightarrow Person(LeftLeg(John)) &)\wedge \\
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Universal Quantification [cont.]

- One may want to restrict the domain of universal quantification to elements of some kind P
  - ex "forall kings ...", "forall integer numbers..."
- Idea: use an implication, with restrictive predicate as implicant: $\forall x.(P(x) \rightarrow \alpha(x, ...))$
  - ex "$\forall x.(King(x) \rightarrow ...)$", "$\forall x.(Integer(x) \rightarrow ...)$",
- Beware of typical mistake: do not use "$\wedge$" instead of "$\rightarrow$"
  - ex: "$\forall x.(King(x) \wedge Person(x))$" means "everything/one is a King and is a Person"
  - ex: "$\forall x.(King(x) \rightarrow Person(x))$" means "everything/one who is a King is a Person (i.e. "every king is a person")"
- "$\forall$" distributes with "$\wedge$", but not with "$\vee$"
  - $\forall x.(P(x) \wedge Q(x))$ equivalent to $(\forall x.P(x)) \wedge (\forall x.Q(x))$
  - "Everybody is a king and is a person" same as "Everybody is a king and everybody is a person"
  - $\forall x.(P(x) \vee Q(x))$ not equivalent to $(\forall x.P(x)) \vee (\forall x.Q(x))$:
  - "Everybody is a king or is a peasant" much weaker than "Everybody is a king or everybody is a peasant" $(\forall x.P(x)) \vee (\forall x.Q(x)) \models \forall x.(P(x) \vee Q(x))$, $\forall x.(P(x) \vee Q(x)) \not\models (\forall x.P(x)) \vee (\forall x.Q(x))$

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Existential Quantification

- $\exists x.\alpha(x, ...)$ (x variable, typically occurs in x)
  - ex: $\exists x.(King(x) \land Evil(x))$ ("there is an evil king")
  - pronounced "exists x s.t. ..." or "for some x ..."
- $\exists x.\alpha(x, ...)$ true in $\mathcal{M}$ iff
  α is true in $\mathcal{M}$ for some possible domain value x is mapped to
- Roughly speaking, can be seen as a disjunction over all (typically infinite) possible instantiations of x in α

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(King(Richard) & \land Evil(Richard) &)\lor \\
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(King(crown) & \land Evil(crown) &)\lor \\
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Examples

- Brothers are siblings
  - $\forall x, y.\ (Brothers(x, y) \rightarrow Siblings(x, y))$
- "Siblings" is symmetric
  - $\forall x, y.\ (Siblings(x, y) \leftrightarrow Siblings(y, x))$
- One's mother is one's female parent
  - $\forall x, y.\ (Mother(x, y) \leftrightarrow (Female(x) \wedge Parent(x, y)))$
- A first cousin is a child of a parent's sibling
  - $\forall x_1, x_2.\ (FirstCousin(x_1, x_2) \leftrightarrow$
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Equality

- Equality is a special predicate: $t_1 = t_2$ is true under a given interpretation if and only if t_1 and t_2 refer to the same object
  - Ex: $1 = 2$ and $x * x = x$ are satisfiable (!)
  - Ex: $2 = 2$ is valid
- Ex: definition of *Sibling* in terms of *Parent*
  $\forall x, y.\ (Siblings(x, y) \leftrightarrow [\neg(x = y) \wedge \exists m, f.\ (\neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]))$

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Example

- No one is his/her own sibling
  - $\forall x.\ \neg Siblings(x, x)$
- Sisters are female, brothers are male
  - $\forall x, y.\ ((Sisters(x, y) \to (Female(x) \land Female(y))) \land (Brothers(x, y) \to (Male(x) \land Male(y))))$
- Every married person has a spouse
  - $\forall x.\ ((Person(x) \land Married(x)) \to \exists y.\ Spouse(x, y))$
- Married people have spouses
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- People cannot be married to their siblings
  - $\forall x, y.\ (Spouse(x, y) \rightarrow \neg Siblings(x, y))$

Example

- No one is his/her own sibling
  - $\forall x.\ \neg Siblings(x, x)$
- Sisters are female, brothers are male
  - $\forall x, y.\ ((Sisters(x, y) \rightarrow (Female(x) \wedge Female(y))) \wedge (Brothers(x, y) \rightarrow (Male(x) \wedge Male(y))))$
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  - $\exists x.\ (Person(x) \wedge \neg\exists y.\ Spouse(x, y))$
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  - $\forall x.\ (Person(x) \rightarrow \exists y.\ Mother(y, x))$
- Everybody has a mother and only one
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Properties of Quantifiers

Notation variants: $\forall x(\forall y.\alpha) \Longleftrightarrow \forall x \forall y.\alpha \Longleftrightarrow \forall x, y.\alpha \Longleftrightarrow \forall xy.\alpha$
(same with $\exists$)

- if x does not occur in φ, $\forall x.\varphi$ equivalent to $\exists x.\varphi$ equivalent to φ
- $\forall xy.P(x, y)$ equivalent to $\forall yx.P(x, y)$
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Duality of Universal and Existential Quantification

- $\forall$ and $\exists$ are dual
  - $\forall x.\alpha \iff \neg\exists x.\neg\alpha$
  - $\neg\forall x.\alpha \iff \exists x.\neg\alpha$
  - $\exists x.\alpha \iff \neg\forall x.\neg\alpha$
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- Examples
  - $\forall x.Likes(x, Icecream)$ equivalent to $\neg\exists x.\neg Likes(x, Icecream)$
  - $\exists x.Likes(x, Broccoli)$ equivalent to $\neg\forall x.\neg Likes(x, Broccoli)$
- Negated restricted quantifiers switch "$\rightarrow$" with "$\wedge$"
  - $\forall x.(P(x) \rightarrow \alpha) \iff \neg\exists x.(P(x) \wedge \neg\alpha)$
  - $\neg\forall x.(P(x) \rightarrow \alpha) \iff \exists x.(P(x) \wedge \neg\alpha)$
  - ...
- Ex: "not all kings are evil" same as "some king is not evil"
  - $\neg\forall x.(King(x) \rightarrow Evil(x)) \iff \exists x.(King(x) \wedge \neg Evil(x))$
- Unsurprising, since $\langle\forall, \exists\rangle$ are $\langle\wedge, \vee\rangle$ over infinite instantiations

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Outline

Satisfiability, Validity, Entailment

- A model $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{D}, \mathcal{I} \rangle$ satisfies φ ($\mathcal{M} \models \varphi$) iff $[\varphi]^{\mathcal{I}}$ is true
- $M(\varphi) \stackrel{\text{def}}{=} \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$ (the set of models of φ)
- φ is satisfiable iff $\mathcal{M} \models \varphi$ for some $\mathcal{M}$ (i.e. $M(\varphi) \neq \emptyset$)
- α entails β ($\alpha \models \beta$) iff, for all $\mathcal{M}$, $\mathcal{M} \models \alpha \Longrightarrow \mathcal{M} \models \beta$ (i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is valid ($\models \varphi$) iff $\mathcal{M} \models \varphi$ forall $\mathcal{M}$s (i.e., $\mathcal{M} \in M(\varphi)$ forall $\mathcal{M}$s)
- α, β are equivalent iff $\alpha \models \beta$ and $\beta \models \alpha$ (i.e. $M(\alpha) = M(\beta)$)

Sets of formulas as conjunctions

Let $\Gamma \stackrel{\text{def}}{=} \{\varphi_1, ..., \varphi_n\}$. Then:

- Γ satisfiable iff $\bigwedge_{i=1}^{n} \varphi_i$ satisfiable
- $\Gamma \models \phi$ iff $\bigwedge_{i=1}^{n} \varphi_i \models \phi$
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Satisfiability, Validity, Entailment

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Examples

- $P(x), \forall x.(x \geq y), \{\forall x.(x \geq 0), \forall x.(x+1 > x)\}$ satisfiable
- $P(x) \wedge \neg P(x), \neg(x = x), (\forall x, y.Q(x, y)) \rightarrow \neg Q(a, b)$ unsatisfiable
- $\forall x.P(x) \rightarrow \exists x.P(x)$ valid
- $\forall x.P(x) \models \exists x.P(x)$
- $\neg(\forall x.P(x)) \rightarrow \exists x.P(x))$ unsatisfiable
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Exercises

- Is $\forall x.P(x)$ equivalent to $\forall y.P(y)$?
- Is $\forall xy.P(x, y)$ equivalent to $\forall yx.P(y, x)$?
- $\forall x.\exists x.P(x)$ is equivalent to:
  - $\exists x.P(x)$
  - $\forall x.P(x)$
  - neither
- $\exists x.\forall x.P(x)$ is equivalent to:
  - $\exists x.P(x)$
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Enumeration of Models?

- We *can enumerate* the models for a given FOL sentence:

  For each number of universe elements n from 1 to ∞
      For each k-ary predicate P_k in the sentence
          For each possible k-ary relation on n objects
              For each constant symbol C in the sentence
                  For each one of n objects C is mapped to
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Theorem

Entailment (validity, unsatisfiability) in FOL is only semi-decidable:

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Outline

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[Recall:] Knowledge-Based Agent: General Schema

- Given a percept, the agent
  - Tells the KB of the percept at time step t
  - ASKs the KB for the best action to do at time step t
  - Tells the KB that it has in fact taken that action
- Details hidden in three functions:
  MAKE-PERCEPT-SENTENCE, MAKE-ACTION-QUERY, MAKE-ACTION-SENTENCE
  - construct logic sentences
  - implement the interface between sensors/actuators and KRR core
- Tell and Ask may require complex logical inference

```
function KB-AGENT(percept) returns an action
   persistent: KB, a knowledge base
               t, a counter, initially 0, indicating time

   TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
   action ← ASK(KB, MAKE-ACTION-QUERY(t))
   TELL(KB, MAKE-ACTION-SENTENCE(action, t))
   t ← t + 1
   return action
```

FOL Knowledge-Based Agent

- We can assert FOL sentences (assertions) into the KB. Ex:
  - ex: $\text{Tell}(KB, King(John))$
  - ex: $\text{Tell}(KB, Person(Richard))$
  - ex: $\text{Tell}(KB, \forall x.(King(x) \rightarrow Person(x)))$
- We can ask queries (aka goals) to the KB. Ex:
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  - ex: $\text{Ask}(KB, \exists x.Person(x))$

$\Longrightarrow$ Ask(KB,α) returns true only if $KB \models \alpha$

- Other queries: AskVars, asking for variable values
  $\Longrightarrow$ returns one (or more) binding lists (aka substitutions) $\{var/term; var/term, ...\}$
  - ex: $\text{AskVars}(KB, \exists x.Person(x)) \Longrightarrow \{x/John\}; \{x/Richard\}$
  - typical for Horn clauses
    (e.g. with $King(John) \vee King(Richard)$,
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Example: The Kinship Domain

Domain of family relationships

- Binary predicate symbols (family relationships):
  - Parent , Sibling, Brother, Sister, Child , Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle
- function symbols:
  - Mother, Father
- Knowledge base KB:
  1. $\forall x, y.(x = Mother(y) \leftrightarrow (Female(x) \wedge Parent(x, y)))$
  2. $\forall x, y.(Brother(x, y) \leftrightarrow (Male(x) \wedge Sibling(x, y)))$
  3. $\forall x, y.(Grandparent(x, y) \leftrightarrow \exists z.(Parent(x, z) \wedge Parent(z, y)))$
  4. $\forall x, y.(Sibling(x, y) \leftrightarrow ((x \neq y) \wedge \exists m, f.((m \neq f) \wedge Parent(m, x) \wedge Parent(m, y) \wedge (Parent(f, x) \wedge Parent(f, y))))$
  5. ...
- Queries inferred from KB
  - ex: (4) $\models \forall x, y.(Sibling(x, y) \leftrightarrow Sibling(y, x))$

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Peano Arithmetic

- Basic symbols
  - Unary predicate symbol: NatNum (natural number)
  - Unary function symbol: S (Successor)
  - Constant symbol: 0
- Defined symbols:
  - Binary function symbols: +,* (infix)
  - Constant symbols: 1,2,3,4,5,6,...
- Knowledge base KB:
  1. $NatNum(0)$
  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
  3. $\forall x.(NatNum(x) \rightarrow (0 \neq S(x)))$
  4. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
  5. $\forall x.(NatNum(x) \rightarrow (x = (0 + x)))$
  6. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow (S(x) + y) = S(x + y))$
  7. $1 = S(0), 2 = S(1), 3 = S(2), \ldots$
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  - Binary function symbols: +,* (infix)
  - Constant symbols: 1,2,3,4,5,6,...
- Knowledge base KB:
  1. $NatNum(0)$
  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
  3. $\forall x.(NatNum(x) \rightarrow (0 \neq S(x)))$
  4. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
  5. $\forall x.(NatNum(x) \rightarrow (x = (0 + x)))$
  6. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow (S(x) + y) = S(x + y))$
  7. $1 = S(0)$, $2 = S(1)$, $3 = S(2)$, ...
- Queries inferred from KB
  - ex: (4) $\models \forall x, y.((NatNum(x) \wedge (NatNum(y))) \rightarrow ((x + y) = (y + x)))$

Example: Integer Numbers

Peano Arithmetic

- Basic symbols
  - Unary predicate symbol: NatNum (natural number)
  - Unary function symbol: S (Successor)
  - Constant symbol: 0
- Defined symbols:
  - Binary function symbols: +,* (infix)
  - Constant symbols: 1,2,3,4,5,6,...
- Knowledge base KB:
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  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
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  - ex: (4) $\models \forall x, y.((NatNum(x) \wedge (NatNum(y))) \rightarrow ((x + y) = (y + x)))$

Exercises

About the Kinship domain

- Try to add the axioms defining other predicates or functions (e.g. Brother, Sister, Child , Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle, ...)
- Add some ground atom or its negation to the KB (ex: Brother(Steve,Mary), Mary=Mother(Paul),...)
- Try to solve some query by entailment (e.g. Uncle(Steve,Paul), $\exists x.Uncle(x, Paul)$, ...)

About the Peano Arithmetic domain

- Try to add the axioms defining other predicate or functions (e.g. "$n \leq m$" or "$m * n$", "n^m")
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Outline

Example: The Wumpus World

The FOL KB

- Perception: binary predicate Percept([s, b, g, b, sc],t)
  - (recall: perception is [Stench,Breeze,Glitter,Bump,Scream])
  - Stench, Breeze, Glitter, Bump, Scream constant symbols
  - time step *t* represented as integer
- Percepts imply facts about the current state.
  - $\forall t, s, g, m, c.(Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$
  - $\forall t, s, g, m, c.(Percept([s, Null, g, m, c], t) \rightarrow \neg Breeze(t))$
  - ...
- Environment:
  - Square: term (pair of integers): $[1, 2]$
  - Adjacency: binary predicate Adjacent:
    $$\forall x, y, a, b.(Adjacent([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$$
  - Position: predicate $At(Agent, s, t)$, ex: $At(Agent, [1, 1], 1)$
  - Unique position: $\forall x, s_1, s_2, t.((At(x, s_1, t) \wedge At(x, s_2, t)) \rightarrow s_1 = s_2)$
  - Wumpus: predicate $Wumpus(s)$, ex: $Wumpus([3, 1])$
  - Pits: predicate $Pit(s)$, ex: $Pit([3, 1])$

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Personal Remark

- For Wumpus, AIMA suggests;
  - Wumpus: constant, ex $\forall t. At(Wumpus, [2, 2], t)$
- Simplification: assume Wumpus status does not evolve with time
  - predicate $Wumpus(s)$, ex: $Wumpus([3, 1])$
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Example: The Wumpus World [cont.]

The FOL KB [cont.]

- Infer properties from percepts:
  - $\forall s,t.((At(Agent,s,t) \land Breeze(t)) \rightarrow Breezy(s))$
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- Infer information about pits & Wumpus
  - $\forall s.\ (Breezy(s) \leftrightarrow \exists r.(Adjacent(r,s) \land Pit(r)))$
  - $\forall s.\ (Stench(s) \leftrightarrow \exists r.(Adjacent(r,s) \land Wumpus(r)))$
- Evolution on time: successor states:
  - $\forall t.(HaveArrow(t+1) \leftrightarrow (HaveArrow(t) \land \neg Action(Shoot,t)))$
- Actions: terms Turn(Right), Turn(Left), Forward, Shoot, Grab, Climb
  - simple reflex action: $\forall t.(Glitter(t) \rightarrow BestAction(Grab,t))$
  - Query: $AskVars(\exists a.BestAction(a,5)) \Longrightarrow \{a/Grab\}$

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Simplified action axiomatization: "Move(...)" instead of "*Turn*(...), *Forward*"

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The FOL KB [cont.]

- Infer properties from percepts:
  - $\forall s,t.((At(Agent,s,t) \wedge Breeze(t)) \rightarrow Breezy(s))$
  - $\forall s,t.((At(Agent,s,t) \wedge \neg Breeze(t)) \rightarrow \neg Breezy(s))$
- Infer information about pits & Wumpus
  - $\forall s.\ (Breezy(s) \leftrightarrow \exists r.(Adjacent(r,s) \wedge Pit(r)))$
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Example: The Wumpus World [cont.]

The FOL KB [cont.]

- Infer properties from percepts:
  - $\forall s, t.((At(Agent, s, t) \land Breeze(t)) \to Breezy(s))$
  - $\forall s, t.((At(Agent, s, t) \land \neg Breeze(t)) \to \neg Breezy(s))$
- Infer information about pits & Wumpus
  - $\forall s.\ (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Pit(r)))$
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Personal remark

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Example: Exploring the Wumpus World

KB initially contains:

$\forall x, y, a, b.(Adjacent([x, y], [a, b]) \leftrightarrow (x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$

$\forall t, s, g, m, c.(Percept([s, Null, g, m, c], t) \rightarrow \neg Breeze(t))$

$\forall t, b, g, m, c.(Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$

$\forall s, t.((At(Agent, s, t) \wedge \neg Breeze(t)) \rightarrow \neg Breezy(s))$

$\forall s, t.((At(Agent, s, t) \wedge \neg Stench(t)) \rightarrow \neg Stenchy(s))$

$\forall s. (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \wedge Pit(r)))$

$\forall s. (Stench(s) \leftrightarrow \exists r.(Adjacent(r, s) \wedge Wumpus(r)))$

$\forall s.(Ok(s) \leftrightarrow (\neg Pit(s) \wedge \neg Wumpus(s)))$

- A is initially in 1,1: $At(A, [1, 1], 0)$
- Perceives no stench, no breeze:

  $Tell(KB, Percept([Null, Null, Null, Null, Null], 0))$

  $\Longrightarrow \neg Breeze(0), \neg Stench(0),$

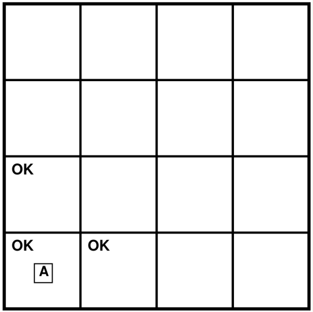
  $\Longrightarrow \neg Breezy([1, 1]), \neg Stenchy([1, 1]),$

  $\Longrightarrow \neg Pit([1, 2]), \neg Pit([2, 1] \neg Wumpus([1, 2]), \neg Wumpus([2, 1]),$

  $\Longrightarrow Ok([1, 2]), Ok([2, 1])$

  $AskVars(KB, \exists a.BestAction(a, 0))$

  $\Longrightarrow \{a/Move([1, 2])\}, \{a/Move([2, 1])\}$



Example: Exploring the Wumpus World

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$\forall x,y,a,b.(Adjacent([x,y],[a,b]) \leftrightarrow (x=a \wedge (y=b-1 \vee y=b+1)) \vee (y=b \wedge (x=a-1 \vee x=a+1)))$

$\forall t,s,g,m,c.(Percept([s,Breeze,g,m,c],t) \rightarrow Breeze(t))$

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$\forall s.\ (Stench(s) \leftrightarrow \exists r.(Adjacent(r,s) \wedge Wumpus(r)))$

- Agent moves to [2,1]: $At(A,[2,1],1)$
- Perceives a breeze and no stench:

  $Tell(KB, Percept([Null, Breeze, Null, Null, Null], 1))$

  $\Longrightarrow Breeze(1), \neg Stench(1),$

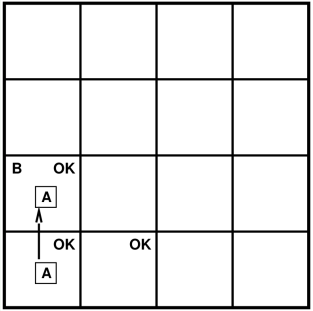
  $\Longrightarrow Breezy([2,1]), \neg Stenchy([2,1]),$

  $\Longrightarrow \exists r.(Adjacent(r,[2,1]) \wedge Pit(r)),$

  $\neg Wumpus([3,1]), \neg Wumpus([2,2]),$

  $\Longrightarrow (Pit([3,1]) \vee Pit([2,2]))$

  $AskVars(KB, \exists a.Action(a,1)) \Longrightarrow \{a/Move([1,1])\}$



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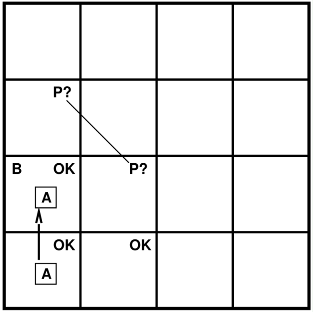

Exercise

Complete the example in the FOL case (see the PL case).

Outline

Knowledge Engineering in FOL

The knowledge-engineering process

1. Identify the task (analogous to PEAS process to design agents)
   - determine what knowledge must be represented in order to connect problem instances to answers
2. Assemble the relevant knowledge (aka knowledge acquisition)
   (either by own domain knowledge or by experts interviews)
   - understand the scope of the knowledge base
   - understand how the domain actually works
3. Decide on a vocabulary of predicates, functions, and constants
   - translate relevant domain-level concepts into logic-level names
   - what should be represented as predicate/function/constant?

   $\Longrightarrow$ define the ontology of the domain
4. Encode into FOL general knowledge about the domain
   - write down the axioms for all the vocabulary terms

   $\Longrightarrow$ should enable the domain expert to check the content
5. ...

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Knowledge Engineering in FOL [cont.]

The knowledge-engineering process [cont.]

4. ...
5. Encode into FOL a description of the specific problem instance

   (straightforward iff the ontology is well-conceived)
   - mostly assertions of (possibly negated) ground atomic formulas
   - for a logical agent, problem instances are supplied by the sensors
   - general knowledge base is supplied with additional sentences
6. Pose queries to the inference procedure and get answers
   - the final outcome
   - check the queries
7. Debug the knowledge base
   - detect un-answered/wrong queries
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No need for writing an application-specific solution algorithm!

Knowledge Engineering in FOL [cont.]

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The knowledge-engineering process [cont.]

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Knowledge Engineering in FOL [cont.]

The knowledge-engineering process [cont.]

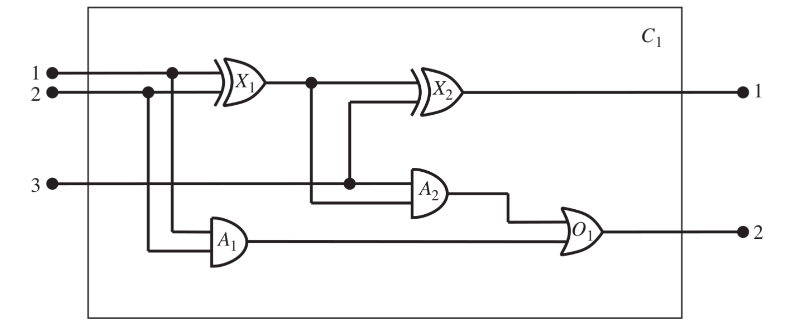
4. ...
5. Encode into FOL a description of the specific problem instance
   (straightforward iff the ontology is well-conceived)
   - mostly assertions of (possibly negated) ground atomic formulas
   - for a logical agent, problem instances are supplied by the sensors
   - general knowledge base is supplied with additional sentences
6. Pose queries to the inference procedure and get answers
   - the final outcome
   - check the queries
7. Debug the knowledge base
   - detect un-answered/wrong queries
   - identify too-weak or missing axioms by backward-analysis

No need for writing an application-specific solution algorithm!

Example: The Electronic Circuits Domain

Task: Develop (an ontology and) a knowledge base allowing to reason about digital circuits (e.g., that shown in Figure)

- Ex: One-bit full adder:
  - first two inputs are to be added, the third input is a carry bit
  - first output is the sum, the second output is a carry bit



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Example: The Electronic Circuits Domain [cont.]

1. Identify the task
   - At the highest level, analyze the circuit's functionality
   - ex: does the circuit contain feedback loops?
   - ...
2. Assemble the relevant knowledge
   - signals flow along wires to the input terminals of gates
   - each gate produces a signal on the output
   - AND, OR, XOR gates have two inputs, NOT gates have one
   - ...
3. Decide on a vocabulary of predicates, functions, and constants
   - e.g. each gate instance represented as constant (ex "X_1")
   - each gate type represented as constant (ex "*AND*")
   - a function Type (ex: $Type(X_1) = XOR$)
   - gate terminals represented as integer constants,
   - two functions In, Out, and one predicate *Connected* (ex: $Connected(In(1, X_1), In(1, A_2))$),
   - two values 0,1, a predicate Signal(t) (ex: $Signal(In(1, X_1)) = 1$)
   - ...

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Example: The Electronic Circuits Domain [cont.]

4. Encode general knowledge about the domain

$\forall t_1, t_2.((Terminal(t_1) \land Terminal(t_2) \land Connected(t_1, t_2))) \rightarrow (Signal(t_1) = Signal(t_2))$
$\forall t.(Terminal(t) \rightarrow ((Signal(t) = 1) \lor (Signal(t) = 0)))$
$\forall t_1, t_2.(Connected(t_1, t_2) \leftrightarrow Connected(t_2, t_1))$
$\forall g.(Gate(g) \rightarrow ((Type(g) = AND) \lor (Type(g) = OR) \lor (Type(g) = XOR) \lor (Type(g) = NOT)))$
$\forall g.((Gate(g) \land Type(g) = AND) \rightarrow ((Signal(Out(1, g)) = 0) \leftrightarrow \exists n.(Signal(In(n, g)) = 0)))$
... analogous definitions for OR, XOR, NOT
$\forall g.((Gate(g) \land (Type(g) = NOT)) \rightarrow Arity(g, 1, 1))$
$\forall g.((Gate(g) \land ((Type(g) = AND) \lor (Type(g) = OR) \lor (Type(g) = XOR))) \rightarrow Arity(g, 2, 1))$
$\forall c, i, j.((Circuit(c) \land Arity(c, i, j)) \rightarrow$
$\quad \forall n.((n \leq i \rightarrow Terminal(In(c, n))) \land (n > i \rightarrow In(c, n) = Nothing)) \land$
$\quad \forall n.((n \leq j \rightarrow Terminal(Out(c, n))) \land (n > j \rightarrow Out(c, n) = Nothing)))$
$\forall g, t.((Gate(g) \land Terminal(t)) \rightarrow (g \neq t \neq 1 \neq 0 \neq OR \neq AND \neq XOR \neq NOT \neq Nothing))$
$forallg.(Gate(g) \rightarrow Circuit(g))$

Notation: $(t_1 \neq t_2 \neq t_3 \neq ... \neq t_n)$: shortcut for $\neg(t_1 = t_2) \land \neg(t_1 = t_3) \land ...\neg(t_{n-1} = t_n)$.

Example: The Electronic Circuits Domain [cont.]

5. Encode a description of the specific problem instance
   - $Circuit(C_1) \land Arity(C_1, 3, 2) \land$
     $Gate(X_1) \land Type(X_1) = XOR \land Gate(X_2) \land Type(X_2) = XOR \land ... \land$
     $Gate(O_1) \land Type(O_1) = OR$
   - $Connected(Out(1, X_1), In(1, X_2)) \land$
     $... \land$
     $Connected(In(3, C_1), In(1, A_2))$

6. Pose queries to the inference procedure and get answers
   - Ex: Which inputs would cause the first output of C_1 (the sum bit) to be 0 and the second output of C_1 (the carry bit) to be 1?
     $AskVars(KB, \exists i_1, i_2, i_3.(Signal(In(1, C_1)) = i_1 \land$
     $Signal(In(2, C_1)) = i_2 \land Signal(In(3, C_1)) = i_3 \land$
     $Signal(Out(1, C_1)) = 0 \land Signal(Out(2, C_1)) = 1))$
     $\Longrightarrow \{i_1/1, i_2/1, i_3/0\}$ or $\{i_1/1, i_2/0, i_3/1\}$ or $\{i_1/0, i_2/1, i_3/1\}$
   - What are the possible value sets of all terminals?
     $AskVars(KB, \exists i_1, i_2, i_3, o_1, o_2.(Signal(In(1, C_1)) = i_1 \land$
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     $\Longrightarrow \{i_1/1, i_2/1, i_3/1, o_1/1, o_2/1\}$ or $\{i_1/1, i_2/1, i_3/0, o_1/0, o_2/1\}$ or ...

Example: The Electronic Circuits Domain [cont.]

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Example: The Electronic Circuits Domain [cont.]

7. Debug the knowledge base
   - Suppose no output produced by previous query
   - We progressively try to restrict our analysis my more local queries, until we pinpoint the problem.
   - Ex: $\exists i_1, i_2, o.(Signal(In(1, C1)) = i_1 \wedge Signal(In(2, C1)) = i_2 \wedge$
     $Signal(Out(1, X1)) = o)$

     (see AIMA book for a detailed example)