Fundamentals of Artificial Intelligence
Chapter 08: **First-Order Logic**

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Outline

Outline

Recall: State Representations [Ch. 02]

Representations of states and transitions

- Three ways to represent states and transitions between them:
  - atomic: a state is a black box with no internal structure
  - factored: a state consists of a vector of attribute values
  - structured: a state includes objects, each of which may have attributes of its own as well as relationships to other objects
- increasing expressive power and computational complexity
- reality represented at different levels of abstraction

(a) Atomic   (b) Factored   (b) Structured
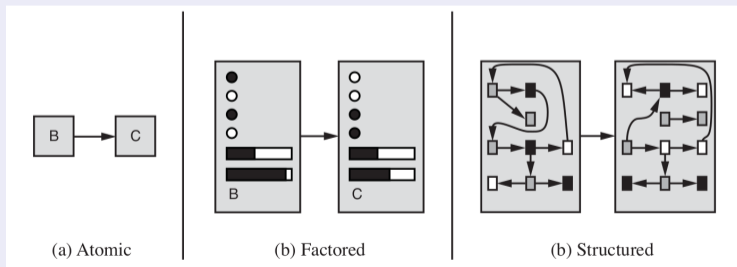
Pros of Propositional Logic

- PL language is formal
    - non-ambiguous semantics
    - unlike natural language, which is intrinsically ambiguous (ex "key")
- PL is declarative
    - knowledge and inference are separate
    - inference is entirely domain independent
- PL allows for partial/disjunctive/negated information
    - unlike, e.g., data bases
- PL is compositional
    - the meaning of $(A \land B) \rightarrow C$ derives from the meaning of A,B,C
- The meaning of PL sentence is context independent
    - unlike with natural language, where meaning depends on context

Cons of Propositional Logic

- Is "Atomic": based on atomic events which cannot be decomposed
- Assumes the world contains facts in the world that are either true or false, nothing else
    - ex: Man_Socrates, Man_Plato, Man_Aristotle, ... distinct atoms
$\implies$ PL has has very limited expressive power
    - unlike natural language
    - cannot concisely describe an environment with many objects
    - e.g., cannot say "pits cause breezes in adjacent squares"
      (need writing one sentence for each square)

Logics

- A logic is a triple $\langle \mathcal{L}, \mathcal{S}, \mathcal{R} \rangle$ where
    - $\mathcal{L}$, the logic's language: a class of sentences described by a formal grammar
    - $\mathcal{S}$, the logic's semantics: a formal specification of how to assign meaning in the "real world" to the elements of $\mathcal{L}$
    - $\mathcal{R}$, the logic's inference system: is a set of formal derivation rules over $\mathcal{L}$
- There are several logics:
    - propositional logic (PL)
    - first-order logic (FOL)
    - modal logics (MLs)
    - description logics (DLs)
    - temporal logics (TLs)
    - (fuzzy logics, probabilistic logics, ...)
    - ...

Logics in General

- Ontological Commitment: What exists in the world (facts)
- Epistemological Commitment: What an agent believes about facts

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

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First-Order Logic (FOL)

- Is structured: a world/state includes objects, each of which may have attributes of its own as well as relationships to other objects
- Assumes the world contains:
  - Objects:
    e.g., people, houses, numbers, theories, Jim Morrison, colors, basketball games, wars, centuries, ...
  - Relations:
    e.g., red, round, bogus, prime, tall ...,
    brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
  - Functions:
    e.g., father of, best friend, one more than, end of, ...
- Allows to quantify on objects
  - ex: "All man are equal", "some persons are left-handed", ...

Outline

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Syntax of FOL: Basic Elements

- Constant symbols: KingJohn, 2, UniversityofTrento,...
- Predicate symbols: Man(.), Brother(.,.), (. > .), AllDifferent(...),...
  - may have different arities (1,2,3,...)
  - may be prefix (e.g. Brother(.,.)) or infix (e.g. (. > .))
- Function symbols: Sqrt, LeftLeg, MotherOf
  - may have different arities (1,2,3,...)
  - may be prefix (e.g. Sqrt(.)) or infix (e.g. (. + .))
- Variable symbols: x, y, a, b, ...
- Propositional Connectives: $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$
- Equality: "$=$" (also "$\neq$" s.t. "$a \neq b$" shortcut for "$\neg(a = b)$")
- Quantifiers: "$\forall$" ("forall"), "$\exists$" ("exists", aka "for some")
- Punctuation Symbols: ",", "(", ")"

- Constants symbols are 0-ary function symbols
- Propositions are 0-ary predicates $\implies$ PL subcase of FOL
- Signature: the set of predicate, function & constant symbols

FOL: Syntax

- Terms:
  - constant or variable or *function*($term_1, ..., term_n$)
  - ex: KingJohn, x, LeftLeg(Richard), (z*log(2))
  - denote objects in the real world (aka domain)
- Atomic sentences (aka atomic formulas):
  - $\top$, $\bot$
  - *proposition* or *predicate*($term_1, ..., term_n$) or $term_1 = term_2$
  - (*Length*(*LeftLeg*(*Richard*)) > *Length*(*LeftLeg*(*KingJohn*)))
  - denote facts
- Non-atomic sentences/formulas:
  - $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$, $\alpha \oplus \beta$,
    $\forall x.\alpha$, $\exists x.\alpha$ s.t. *x* (typically) occurs in α
  - Ex: $\forall y.(Italian(y) \rightarrow President(Mattarella, y))$
    $\exists x \forall y. President(x, y) \rightarrow \forall y \exists x. President(x, y)$
    $\forall x.(P(x) \wedge Q(x)) \leftrightarrow ((\forall x. P(x)) \wedge (\forall x. Q(x)))$
    $\forall x.(((x \geq 0) \wedge (x \leq \pi)) \rightarrow (sin(x) \geq 0))$
  - denote (complex) facts

FOL: Ground and Closed Formulas

- A term/formula is ground iff no variable occurs in it (ex: $2 \geq 1$)
- A formula is closed iff all variables occurring in it (if any) are quantified
  (ex: $\forall x \exists y.(x > y)$)
$\implies$ Ground formulas are closed, but not vice versa.

FOL: Syntax (BNF)

⟨Sentence⟩	::=	⟨AtomicSentence⟩ | ⟨ComplexSentence⟩
⟨AtomicSentence⟩	::=	⊤ | ⊥ |
		⟨PredicateSymbol⟩(⟨Term⟩, . . .) |
		⟨Term⟩ = ⟨Term⟩
⟨ComplexSentence⟩	::=	¬⟨Sentence⟩ |
		⟨Sentence⟩ ⟨Connective⟩ ⟨Sentence⟩ |
		⟨Quantifier⟩ ⟨Sentence⟩
⟨Term⟩	::=	⟨ConstantSymbol⟩ | ⟨Variable⟩ |
		⟨FunctionSymbol⟩(⟨Term⟩, . . .)
⟨Connective⟩	::=	∧ | ∨ | → | ← | ↔ | ⊕
⟨Quantifier⟩	::=	∀ ⟨Variable⟩. | ∃ ⟨Variable⟩.
⟨Variable⟩	::=	*a* | *b* | · · · | *x* | *y* | · · ·
⟨ConstantSymbol⟩	::=	*A* | *B* | · · · | *John* | 0 | 1 | · · · | π | . . .
⟨FunctionSymbol⟩	::=	*F* | *G* | · · · | *Cos* | *FatherOf* | + | . . .
⟨PredicateSymbol⟩	::=	*P* | *Q* | · · · | *Red* | *Brother* | > | · · ·

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
    - φ occurs positively in φ;
    - if $\neg\varphi_1$ occurs positively [negatively] in φ,
      then φ_1 occurs negatively [positively] in φ
    - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ,
      then φ_1 and φ_2 occur positively [negatively] in φ;
    - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ,
      then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
    - if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ,
      then φ_1 and φ_2 occur positively and negatively in φ;
    - if $\forall x.\varphi_1$ or $\exists x.\varphi_1$ occurs positively [negatively] in φ,
      then φ_1 occurs positively [negatively] in φ

Outline

Truth in FOL: Intuitions

- Sentences are true with respect to a model
  - containing a domain and an interpretation
- The domain contains ≥ 1 objects (domain elements) and relations and functions over them
- An interpretation specifies referents for
  - variables → objects
  - constant symbols → objects
  - predicate symbols → relations
  - function symbols → functional relations
- An atomic sentence $P(t_1, ..., t_n)$ is true in an interpretation iff the objects referred to by $t_1, ..., t_n$ are in the relation referred to by P

FOL: Semantics

FOL Models (aka possible worlds)

- A model $\mathcal{M}$ is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ ($\langle$ *domain*, *interpretation* $\rangle$)
- Domain $\mathcal{D}$: a non-empty set of objects (aka domain elements)
- Interpretation $\mathcal{I}$: a (non-injective) map on elements of the signature
    - constant symbols $\longmapsto$ domain elements:
      a constant symbol C is mapped into a particular object $[C]^{\mathcal{I}}$ in $\mathcal{D}$
    - predicate symbols $\longmapsto$ domain relations:
      a k-ary predicate $P(...)$ is mapped into a subset $[P]^{\mathcal{I}}$ of $\mathcal{D}^k$
      (i.e., the set of object tuples satisfying the predicate in this world)
    - functions symbols $\longmapsto$ domain functions:
      a k-ary function f is mapped into a domain function $[f]^{\mathcal{I}} : \mathcal{D}^k \longmapsto \mathcal{D}$ ($[f]^{\mathcal{I}}$ must be total)

    (we denote by $[.]^{\mathcal{I}}$ the result of the interpretation $\mathcal{I}$)

An Interpretation $\mathcal{I}$ is extended to assign domain values to variables, domain values to terms and truth values to formulas.

FOL: Semantics [cont.]

Interpretation of terms

$\mathcal{I}$ maps terms into domain elements

- Variables are assigned domain values
  - variables $\longmapsto$ domain elements:
    a variable x is mapped into a particular object $[x]^{\mathcal{I}}$ in $\mathcal{D}$
- A term $f(t_1, ..., t_k)$ is mapped by $\mathcal{I}$ into the value $[f(t_1, ..., t_k)]^{\mathcal{I}}$ returned by applying the domain function $[f]^{\mathcal{I}}$, into which f is mapped, to the values $[t_1]^{\mathcal{I}}, ..., [t_k]^{\mathcal{I}}$ obtained by applying recursively $\mathcal{I}$ to the terms $t_1, ..., t_k$:
  - $[f(t_1, ..., t_k)]^{\mathcal{I}} = [f]^{\mathcal{I}}([t_1]^{\mathcal{I}}, ..., [t_k]^{\mathcal{I}})$
  - Ex: if "Me, Mother, Father" are interpreted as usual, then "Mother(Father(Me))" is interpreted as my (paternal) grandmother
  - Ex: if "$+, -, \cdot, 0, 1, 2, 3, 4$" are interpreted as usual, then "$(3 - 1) \cdot (0 + 2)$" is interpreted as 4

FOL: Semantics [cont.]

Interpretation of formulas

$\mathcal{I}$ maps formulas into truth values

- An atomic formula $P(t_1, ..., t_k)$ is true in $\mathcal{I}$ iff the objects into which the terms $t_1,...t_k$ are mapped by $\mathcal{I}$ comply to the relation into which P is mapped
  - $[P(t_1, ..., t_k)]^{\mathcal{I}}$ is true iff $\langle [t_1]^{\mathcal{I}}, ..., [t_k]^{\mathcal{I}} \rangle \in [P]^{\mathcal{I}}$
  - Ex: if "Me, Mother, Father, Married" are interpreted as traditon, then "Married(Mother(Me),Father(Me))" is interpreted as true
  - Ex: if "$+, -, >, 0, 1, 2, 3, 4$" are interpreted as usual, then "$(4 - 0) > (1 + 2)$" is interpreted as true
- An atomic formula $t_1 = t_2$ is true in $\mathcal{I}$ iff the terms t_1, t_2 are mapped by $\mathcal{I}$ into the same domain element
  - $[t_1 = t_2]^{\mathcal{I}}$ is true iff $[t_1]^{\mathcal{I}}$ same as $[t_2]^{\mathcal{I}}$
  - Ex: if "Mother" is interpreted as usual, Richard, John are brothers, then "Mother(Richard)=Mother(John))" is interpreted as true
  - Ex: if "$+, -, 0, 1, 2, 3, 4$" are interpreted as usual, then "$(4 - 1) = (1 + 2)$" is interpreted as true
- $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$ interpreted by $\mathcal{I}$ as in PL

Models for FOL: Example

Richard Lionhearth and John Lackland

- $\mathcal{D}$: domain at right
- $\mathcal{I}$: s.t.
  - $[Richard]^{\mathcal{I}}$: Richard the Lionhearth
  - $[John]^{\mathcal{I}}$: evil King John
  - $[Brother]^{\mathcal{I}}$: brotherhood
- $[Brother(Richard, John)]^{\mathcal{I}}$ is true
- $[LeftLeg]^{\mathcal{I}}$ maps any individual to his left leg
- ...
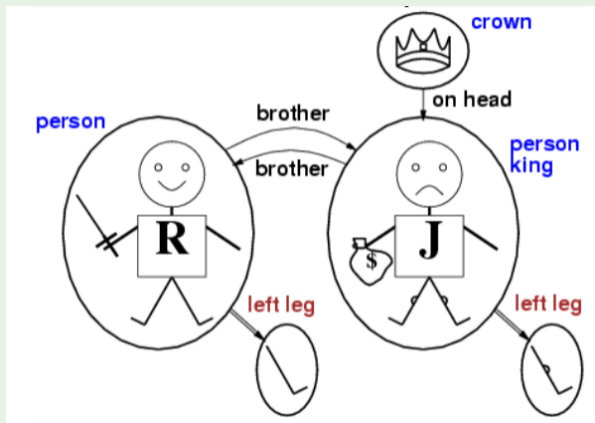
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Models for FOL: Remark

- $[f]^{\mathcal{I}}$ total: must provide an output for every input
- e.g.: $[LeftLeg(crown)]^{\mathcal{I}}$?
- possible solution: assume "null" object ($[LeftLeg(crown) = null]^{\mathcal{I}}$

Universal Quantification

- $\forall x.\alpha(x, ...)$ (*x* variable, typically occurs in x)
    - ex: $\forall x.(King(x) \rightarrow Person(x))$ ("all kings are persons")
- $\forall x.\alpha(x, ...)$ true in $\mathcal{M}$ iff
  α is true in $\mathcal{M}$ for every possible domain value *x* is mapped to
- Roughly speaking, can be seen as a conjunction over all (typically infinite) possible instantiations of *x* in α

(*King*(*John*)	$\rightarrow$ *Person*(*John*)	)$\wedge$
(*King*(*Richard*)	$\rightarrow$ *Person*(*Richard*)	)$\wedge$
(*King*(*crown*)	$\rightarrow$ *Person*(*crown*)	)$\wedge$
(*King*(*LeftLeg*(*John*))	$\rightarrow$ *Person*(*LeftLeg*(*John*))	)$\wedge$
(*King*(*LeftLeg*(*LeftLeg*(*John*)))	$\rightarrow$ *Person*(*LeftLeg*(*LeftLeg*(*John*)))	)$\wedge$
...	...	

Universal Quantification [cont.]

- One may want to restrict the domain of universal quantification to elements of some kind P
  - ex "forall kings ...", "forall integer numbers..."
- Idea: use an implication, with restrictive predicate as implicant:
  $\forall x.(P(x) \rightarrow \alpha(x, ...))$
  - ex "$\forall x.(King(x) \rightarrow ...)$", "$\forall x.(Integer(x) \rightarrow ...)$",
- Beware of typical mistake: do not use "$\wedge$" instead of "$\rightarrow$"
  - ex: "$\forall x.(King(x) \wedge Person(x))$" means "everything/one is a King and is a Person"
  - ex: "$\forall x.(King(x) \rightarrow Person(x))$" means "everything/one who is a King is a Person
    (i.e. "every king is a person")"
- "$\forall$" distributes with "$\wedge$", but not with "$\vee$"
  - $\forall x.(P(x) \wedge Q(x))$ equivalent to $(\forall x.P(x)) \wedge (\forall x.Q(x))$
  - "Everybody is a king and is a person" same as
    "Everybody is a king and everybody is a person"
  - $\forall x.(P(x) \vee Q(x))$ <u>not</u> equivalent to $(\forall x.P(x)) \vee (\forall x.Q(x))$
  - "Everybody is a king or is a peasant" much weaker than
    "Everybody is a king or everybody is a peasant"

Existential Quantification

- $\exists x.\alpha(x, ...)$ (*x* variable, typically occurs in x)
  - ex: $\exists x.(King(x) \land Evil(x))$ ("there is an evil king")
  - pronounced "exists x s.t. ..." or "for some x ..."
- $\exists x.\alpha(x, ...)$ true in $\mathcal{M}$ iff
  α is true in $\mathcal{M}$ for some possible domain value *x* is mapped to
- Roughly speaking, can be seen as a disjunction over all (typically infinite) possible instantiations of *x* in α

(*King*(*Richard*)	$\land Evil$(*Richard*)	)$\lor$
(*King*(*John*)	$\land Evil$(*John*)	)$\lor$
(*King*(*crown*)	$\land Evil$(*crown*)	)$\lor$
(*King*(*LeftLeg*(*John*))	$\land Evil$(*LeftLeg*(*John*))	)$\lor$
(*King*(*LeftLeg*(*LeftLeg*(*John*)))	$\land Evil$(*LeftLeg*(*LeftLeg*(*John*)))	)$\lor$
...	...	

Existential Quantification [cont.]

- One may want to restrict the domain of existential quantification to elements of some kind P
  - ex "exists a king s.t. ...", "for some integer numbers..."
- Idea: use a conjunction with restrictive predicate:
  $\exists x.(P(x) \land \alpha(x, ...))$
  - ex "$\exists x.(King(x) \land ...)$", "$\exists x.(Integer(x) \land ...)$",
- Beware of typical mistake: do not use "$\to$" instead of "$\land$"
  - ex: "$\exists x.(King(x) \to Evil(x))$" means "Someone is not a king or is evil"
  - ex: "$\exists x.(King(x) \land Evil(x))$" means "Someone is king and is evil"
    (i.e., "Some king is evil")
- "$\exists$" distributes with "$\lor$", but not with "$\land$"
  - $\exists x.(P(x) \lor Q(x))$ equivalent to $(\exists x.P(x)) \lor (\exists x.Q(x))$
  - "Somebody is a king or is a knight" same as
    "Somebody is a king or somebody is a knight"
  - $\exists x.(P(x) \land Q(x))$ <u>not</u> equivalent to $(\exists x.P(x)) \land (\exists x.Q(x))$
  - "Somebody is a king and is evil" much stronger than
    "Somebody is a king and somebody is evil"

Examples

- Brothers are siblings
  - $\forall x, y.$ (*Brothers*$(x, y) \rightarrow$ *Siblings*(x, y))
- "Siblings" is symmetric
  - $\forall x, y.$ (*Siblings*$(x, y) \leftrightarrow$ *Siblings*(y, x))
- One's mother is one's female parent
  - $\forall x, y.$ (*Mother*$(x, y) \leftrightarrow$ (*Female*$(x) \wedge$ *Parent*(x, y)))
- A first cousin is a child of a parent's sibling
  - $\forall x_1, x_2.$ (*FirstCousin*$(x_1, x_2) \leftrightarrow$
    $\exists p_1, p_2.$ (*Siblings*$(p_1, p_2) \wedge$ *Parent*$(p_1, x_1) \wedge$ *Parent*(p_2, x_2)))
- Dogs are mammals
  - $\forall x.$ (*Dog*$(x) \rightarrow$ *Mammal*(x))

Equality

- Equality is a special predicate: $t_1 = t_2$ is true under a given interpretation if and only if t_1 and t_2 refer to the same object
    - Ex: $1 = 2$ and $x * x = x$ are satisfiable (!)
    - Ex: $2 = 2$ is valid
- Ex: definition of *Sibling* in terms of *Parent*
  $\forall x, y. \ (Siblings(x, y) \leftrightarrow [\neg(x = y) \land \exists m, f. \ (\neg(m = f) \land Parent(m, x) \land Parent(f, x) \land Parent(m, y) \land Parent(f, y)]))$

Example

- No one is his/her own sibling
  - $\forall x.\ \neg Siblings(x, x)$
- Sisters are female, brothers are male
  - $\forall x, y.\ ((Sisters(x, y) \rightarrow (Female(x) \wedge Female(y))) \wedge$
    $(Brothers(x, y) \rightarrow (Male(x) \wedge Male(y))))$
- Every married person has a spouse
  - $\forall x.\ ((Person(x) \wedge Married(x)) \rightarrow \exists y.\ Spouse(x, y))$
- Married people have spouses
  - $\forall x.\ ((Person(x) \wedge Married(x)) \rightarrow \exists y.\ Spouse(x, y))$
- Only married people have spouses
  - $\forall x, y.\ ((Person(x) \wedge Person(y) \wedge Spouse(x, y)) \rightarrow (Married(x) \wedge Married(y)))$
- People cannot be married to their siblings
  - $\forall x, y.\ (Spouse(x, y) \rightarrow \neg Siblings(x, y))$

Example (cont.)

- Not everybody has a spouse
    - $\neg \forall x.\ (Person(x) \rightarrow \exists y.\ Spouse(x, y))$ or
    - $\exists x.\ (Person(x) \land \neg \exists y.\ Spouse(x, y))$
- Everybody has a mother
    - $\forall x.\ (Person(x) \rightarrow \exists y.\ Mother(y, x))$
- Everybody has a mother and only one
    - $\forall x.\ Person(x) \rightarrow (\exists y.\ Mother(y, x) \land$
      $\neg \exists z.\ (\neg(y = z) \land Mother(z, x)))$

Properties of Quantifiers

Notation variants: $\forall x(\forall y.\alpha) \iff \forall x \forall y.\alpha \iff \forall x, y.\alpha \iff \forall xy.\alpha$
(same with $\exists$)

- if x does not occur in φ, $\forall x.\varphi$ equivalent to $\exists x.\varphi$ equivalent to φ
- $\forall xy.P(x, y)$ equivalent to $\forall yx.P(x, y)$
    - ex: $\forall xy.(x < y)$ same as $\forall yx.(x < y)$
- $\exists xy.P(x, y)$ equivalent to $\exists yx.P(x, y)$
    - ex: $\exists xy.Twins(x, y)$ same as $\exists yx.Twins(x, y)$
- $\exists x \forall y.P(x, y)$ <u>not</u> equivalent to $\forall y \exists x.P(x, y)$
    - ex: $\forall y \exists x.Father(x, y)$ much weaker than $\exists x \forall y.Father(x, y)$
      "everybody has a father" vs. "exists a father of everybody"

Duality of Universal and Existential Quantification

- $\forall$ and $\exists$ are dual
  - $\forall x.\alpha \iff \neg\exists x.\neg\alpha$
  - $\neg\forall x.\alpha \iff \exists x.\neg\alpha$
  - $\exists x.\alpha \iff \neg\forall x.\neg\alpha$
  - $\neg\exists x.\alpha \iff \forall x.\neg\alpha$
- Examples
  - $\forall x.Likes(x, Icecream)$ equivalent to $\neg\exists x.\neg Likes(x, Icecream)$
  - $\exists x.Likes(x, Broccoli)$ equivalent to $\neg\forall x.\neg Likes(x, Broccoli)$
- Negated restricted quantifiers switch "$\rightarrow$" with "$\wedge$"
  - $\forall x.(P(x) \rightarrow \alpha) \iff \neg\exists x.(P(x) \wedge \neg\alpha)$
  - $\neg\forall x.(P(x) \rightarrow \alpha) \iff \exists x.(P(x) \wedge \neg\alpha)$
  - ...
- Ex: "not all kings are evil" same as "some king is not evil"
  - $\neg\forall x.(King(x) \rightarrow Evil(x)) \iff \exists x.(King(x) \wedge \neg Evil(x))$
- Unsurprising, since $\langle\forall, \exists\rangle$ are $\langle\wedge, \vee\rangle$ over infinite instantiations

Outline

Satisfiability, Validity, Entailment

- A model $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{D}, \mathcal{I} \rangle$ satisfies φ ($\mathcal{M} \models \varphi$) iff $[\varphi]^{\mathcal{I}}$ is true
- $M(\varphi) \stackrel{\text{def}}{=} \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$ (the set of models of φ)
- φ is satisfiable iff $\mathcal{M} \models \varphi$ for some $\mathcal{M}$ (i.e. $M(\varphi) \neq \emptyset$)
- α entails β ($\alpha \models \beta$) iff, for all $\mathcal{M}$, $\mathcal{M} \models \alpha \Longrightarrow \mathcal{M} \models \beta$
  (i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is valid ($\models \varphi$) iff $\mathcal{M} \models \varphi$ forall $\mathcal{M}$s (i.e., $\mathcal{M} \in M(\varphi)$ forall $\mathcal{M}$s)
- α, β are equivalent iff $\alpha \models \beta$ and $\beta \models \alpha$ (i.e. $M(\alpha) = M(\beta)$)

Sets of formulas as conjunctions

Let $\Gamma \stackrel{\text{def}}{=} \{\varphi_1, ..., \varphi_n\}$. Then:
- Γ satisfiable iff $\bigwedge_{i=1}^{n} \varphi_i$ satisfiable
- $\Gamma \models \phi$ iff $\bigwedge_{i=1}^{n} \varphi_i \models \phi$
- Γ valid iff $\bigwedge_{i=1}^{n} \varphi_i$ valid

Properties & Results

Property

φ is valid iff $\neg\varphi$ is unsatisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is unsatisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

Examples

- $P(x)$, $\forall x.(x \geq y)$, $\{\forall x.(x \geq 0), \forall x.(x + 1 > x)\}$ satisfiable
- $P(x) \wedge \neg P(x)$, $\neg(x = x)$, $\forall x, y.(Q(x, y)) \rightarrow \neg Q(a, b))$ unsatisfiable
- $\forall x.P(x) \rightarrow \exists x.P(x)$ valid
- $\forall x.P(x) \models \exists x.P(x)$
- $\neg(\forall x.P(x)) \rightarrow \exists x.P(x))$ unsatisfiable
- $\forall x.P(x) \wedge \neg \exists x.P(x))$ unsatisfiable

$(1 > 2)$ is satisfiable. Why?

Exercises

- Is $\forall x.P(x)$ equivalent to $\forall y.P(y)$?
- Is $\forall xy.P(x,y)$ equivalent to $\forall yx.P(y,x)$?
- $\forall x.\exists x.P(x)$ is equivalent to:
    - $\exists x.P(x)$
    - $\forall x.P(x)$
    - neither
- $\exists x.\forall x.P(x)$ is equivalent to:
    - $\exists x.P(x)$
    - $\forall x.P(x)$
    - neither

Enumeration of Models?

- We *can enumerate* the models for a given FOL sentence:

  For each number of universe elements n from 1 to ∞

      For each k-ary predicate P_k in the sentence

          For each possible k-ary relation on n objects

              For each constant symbol C in the sentence

                  For each one of n objects C is mapped to

                       $\cdots$

- $\implies$ Enumerating models is not going to be easy!

Semi-decidability of FOL

Theorem

Entailment (validity, unsatisfiability) in FOL is only semi-decidable:

- if $\Gamma \models \alpha$, this can be checked in finite time
- if $\Gamma \not\models \alpha$, no algorithm is guaranteed to check it in finite time

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Outline

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[Recall:] Knowledge-Based Agent: General Schema

- Given a percept, the agent
  - Tells the KB of the percept at time step t
  - ASKs the KB for the best action to do at time step t
  - Tells the KB that it has in fact taken that action
- Details hidden in three functions:
  MAKE-PERCEPT-SENTENCE, MAKE-ACTION-QUERY, MAKE-ACTION-SENTENCE
  - construct logic sentences
  - implement the interface between sensors/actuators and KRR core
- Tell and Ask may require complex logical inference

---

function KB-AGENT($percept$) **returns** an $action$
  persistent: KB, a knowledge base
                t, a counter, initially 0, indicating time

  TELL(KB, MAKE-PERCEPT-SENTENCE($percept, t$))
  $action \leftarrow$ ASK(KB, MAKE-ACTION-QUERY(t))
  TELL(KB, MAKE-ACTION-SENTENCE($action, t$))
  $t \leftarrow t + 1$
  return $action$

FOL Knowledge-Based Agent

- We can assert FOL sentences (assertions) into the KB. Ex:
    - ex: Tell(KB, $King(John)$)
    - ex: Tell(KB, $Person(Richard)$)
    - ex: Tell(KB, $\forall x.(King(x) \rightarrow Person(x))$)
- We can ask queries (aka goals) to the KB. Ex:
    - ex: Ask(KB, $King(John)$)
    - ex: Ask(KB, $Person(John)$)
    - ex: Ask(KB, $\exists x.Person(x)$)
$\implies$ Ask(KB,α) returns true only if $KB \models \alpha$

- Other queries: AskVars, asking for variable values
    $\implies$ returns one (or more) binding lists (aka substitutions) {$var/term$; $var/term, ...$}
    - ex: AskVars(KB, $\exists x.Person(x)$) $\implies$ {$x/John$}; {$x/Richard$}
    - typical for Horn clauses
      (e.g. with $King(John) \lor King(Richard)$,
      the query AskVars(KB, $\exists x.King(x)$) would not cause a binding list)

Example: The Kinship Domain

Domain of family relationships

- Binary predicate symbols (family relationships):
  - Parent , Sibling, Brother, Sister, Child , Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle
- function symbols:
  - Mother, Father
- Knowledge base KB:
  1. $\forall x, y.(x = Mother(y) \leftrightarrow (Female(x) \land Parent(x, y)))$
  2. $\forall x, y.(Brother(x, y) \leftrightarrow (Male(x) \land Sibling(x, y)))$
  3. $\forall x, y.(Grandparent(x, y) \leftrightarrow \exists z.(Parent(x, z) \land Parent(z, y)))$
  4. $\forall x, y.(Sibling(x, y) \leftrightarrow ((x \neq y) \land \exists m, f.((m \neq f) \land Parent(m, x) \land Parent(m, y) \land (Parent(f, x) \land Parent(f, y)))))$
  5. ...
- Queries inferred from KB
  - ex: $(4) \models \forall x, y.(Sibling(x, y) \leftrightarrow Sibling(y, x))$

Notation: "$t \neq s$" shortcut for "$\neg(t = s)$"

Example: Integer Numbers

Peano Arithmetic

- Basic symbols
  - Unary predicate symbol: NatNum (natural number)
  - Unary function symbol: S (Successor)
  - Constant symbol: 0
- Defined symbols:
  - Binary function symbols: $+,^*$ (infix)
  - Constant symbols: 1,2,3,4,5,6,...
- Knowledge base KB:
  1. $NatNum(0)$
  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
  3. $\forall x.(NatNum(x) \rightarrow (0 \neq S(x)))$
  4. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
  5. $\forall x.(NatNum(x) \rightarrow (x = (0 + x)))$
  6. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow (S(x) + y) = S(x + y))$
  7. $1 = S(0), 2 = S(1), 3 = S(2), ...$
- Queries inferred from KB
  - ex: $(4) \models \forall x, y.((NatNum(x) \wedge (NatNum(y))) \rightarrow ((x + y) = (y + x)))$

Exercises

About the Kinship domain

- Try to add the axioms defining other predicates or functions
  (e.g. Brother, Sister, Child , Daughter, Son, Spouse, Wife, Husband, Grandparent,
  Grandchild, Cousin, Aunt, Uncle, ...)
- Add some ground atom or its negation to the KB
  (ex: Brother(Steve,Mary), Mary=Mother(Paul),...)
- Try to solve some query by entailment
  (e.g. Uncle(Steve,Paul), $\exists x.Uncle(x, Paul)$, ...)

About the Peano Arithmetic domain

- Try to add the axioms defining other predicate or functions
  (e.g. "$n \leq m$" or "$m * n$"", n^m)
- Add some ground atom or its negation to the KB
  (ex: $1 = S(0), 2 = S(1), ...$)
- Try to solve some query by entailment
  (e.g. $3 + 2 = 5, 2 * 3 = 6$, ...)

Outline

Example: The Wumpus World

The FOL KB

- Perception: binary predicate Percept([s, b, g, b, sc],t)
  - (recall: perception is [Stench,Breeze,Glitter,Bump,Scream])
  - Stench, Breeze, Glitter, Bump, Scream constant symbols
  - time step t represented as integer
- Percepts imply facts about the current state.
  - $\forall t, s, g, m, c.(Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$
  - $\forall t, s, g, m, c.(Percept([s, Null, g, m, c], t) \rightarrow \neg Breeze(t))$
  - ...
- Environment:
  - Square: term (pair of integers): $[1, 2]$
  - Adjacency: binary predicate Adjacent:
    $\forall x, y, a, b.(Adjacent([x, y], [a, b]) \leftrightarrow$
    $(x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$
  - Position: predicate $At(Agent, s, t)$, ex: $At(Agent, [1, 1], 1)$
  - Unique position: $\forall x, s_1, s_2, t.((At(x, s_1, t) \wedge At(x, s_2, t)) \rightarrow s_1 = s_2)$
  - Wumpus: predicate $Wumpus(s)$, ex: $Wumpus([3, 1])$
  - Pits: predicate $Pit(s)$, ex: $Pit([3, 1])$

Personal Remark

- For Wumpus, AIMA suggests;
  - Wumpus: constant, ex $\forall t.At(Wumpus, [2, 2], t)$
- Simplification: assume Wumpus status does not evolve with time
  - predicate $Wumpus(s)$, ex: $Wumpus([3, 1])$
  - $\implies$ makes inference much easier
  - if we consider the case the Wumpus is killed by arrow, then we need reintroducing the "At" formalization

Example: The Wumpus World [cont.]

The FOL KB [cont.]

- Infer properties from percepts:
  - $\forall s, t.((At(Agent, s, t) \land Breeze(t)) \rightarrow Breezy(s))$
  - $\forall s, t.((At(Agent, s, t) \land \neg Breeze(t)) \rightarrow \neg Breezy(s))$
- Infer information about pits & Wumpus
  - $\forall s. (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Pit(r)))$
  - $\forall s. (Stench(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Wumpus(r)))$
- Evolution on time: successor states:
  - $\forall t.(HaveArrow(t + 1) \leftrightarrow (HaveArrow(t) \land \neg Action(Shoot, t)))$
- Actions: terms Turn(Right), Turn(Left), Forward, Shoot, Grab, Climb
  - simple reflex action: $\forall t.(Glitter(t) \rightarrow BestAction(Grab, t))$
  - Query: $AskVars(\exists a.BestAction(a, 5)) \implies \{a/Grab\}$

Example: Exploring the Wumpus World

KB initially contains:

$\forall x, y, a, b.(Adjacent([x, y], [a, b]) \leftrightarrow$
$(x = a \land (y = b - 1 \lor y = b + 1)) \lor (y = b \land (x = a - 1 \lor x = a + 1)))$
$\forall t, s, g, m, c.(Percept([s, Null, g, m, c], t) \rightarrow \neg Breeze(t))$
$\forall t, b, g, m, c.(Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$
$\forall s, t.((At(Agent, s, t) \land \neg Breeze(t)) \rightarrow \neg Breezy(s))$ $\forall s, t.((At(Agent, s, t) \land \neg Stench(t)) \rightarrow \neg Stenchy(s))$
$\forall s. (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Pit(r)))$ $\forall s. (Stench(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Wumpus(r)))$
$\forall s.(Ok(s) \leftrightarrow (\neg Pit(s) \land \neg Wumpus(s)))$

- A is initially in 1,1: $At(A, [1, 1], 0)$
- Perceives no stench, no breeze:
  $Tell(KB, Percept([Null, Null, Null, Null, Null], 0))$
  $\implies \neg Breeze(0), \neg Stench(0),$
  $\implies \neg Breezy([1, 1]), \neg Stenchy([1, 1]),$
  $\implies \neg Pit([1, 2]), \neg Pit([2, 1]) \quad \neg Wumpus([1, 2]),$
  $\neg Wumpus([2, 1]),$
  $\implies Ok([1, 2]), Ok([2, 1])$
  $AskVars(KB, \exists a.Action(a, 0))$
  $\implies \{a/Move([1, 2])\}, \{a/Move([2, 1])\}$

Example: Exploring the Wumpus World

KB initially contains:

$\neg Pit([1,1]), \neg Wumpus([1,1]), ...$

$\forall x, y, a, b.(Adjacent([x, y], [a, b]) \leftrightarrow$
 $(x = a \wedge (y = b - 1 \vee y = b + 1)) \vee (y = b \wedge (x = a - 1 \vee x = a + 1)))$

$\forall t, s, g, m, c.(Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$

$\forall t, b, g, m, c.(Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t))$ $\forall s, t.((At(Agent, s, t) \wedge Breeze(t)) \rightarrow Breezy(s))$

$\forall s, t.((At(Agent, s, t) \wedge \neg Stench(t)) \rightarrow \neg Stenchy(s))$ $\forall s. (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \wedge Pit(r)))$

$\forall s. (Stench(s) \leftrightarrow \exists r.(Adjacent(r, s) \wedge Wumpus(r)))$

- Agent moves to [2,1]: $At(A, [2, 1], 1)$
- Perceives a breeze and no stench:
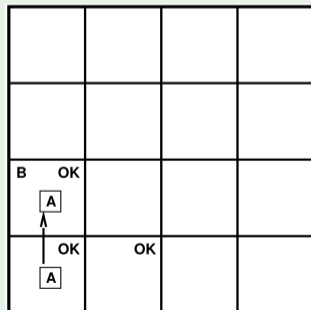 $Tell(KB, Percept([Null, Breeze, Null, Null, Null], 1))$
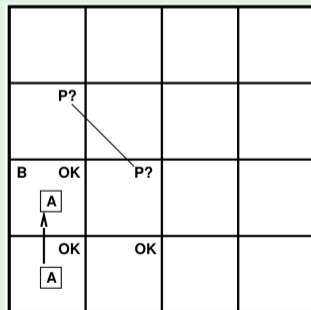 $\Longrightarrow Breeze(1), \neg Stench(1),$
  $\Longrightarrow Breezy([2, 1]), \neg Stenchy([2, 1]),$
  $\Longrightarrow \exists r.(Adjacent(r, [2, 1]) \wedge Pit(r)),$
   $\neg Wumpus([3, 1]), \neg Wumpus([2, 2]),$
  $\Longrightarrow (Pit([3, 1]) \vee Pit([2, 2]))$
  $AskVars(KB, \exists a.Action(a, 1)) \Longrightarrow \{a/Move([1, 1])\}$

Example: Exploring the Wumpus World

KB initially contains:

$\neg Pit([1,1]), \neg Wumpus([1,1]), ...$

$\forall x, y, a, b.(Adjacent([x,y],[a,b]) \leftrightarrow$
$(x = a \land (y = b - 1 \lor y = b + 1)) \lor (y = b \land (x = a - 1 \lor x = a + 1)))$

$\forall t, s, g, m, c.(Percept([s, Breeze, g, m, c], t) \rightarrow Breeze(t))$

$\forall t, b, g, m, c.(Percept([Null, b, g, m, c], t) \rightarrow \neg Stench(t)) \; \forall s, t.((At(Agent, s, t) \land Breeze(t)) \rightarrow Breezy(s))$

$\forall s, t.((At(Agent, s, t) \land \neg Stench(t)) \rightarrow \neg Stenchy(s)) \; \forall s. (Breezy(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Pit(r)))$

$\forall s. (Stench(s) \leftrightarrow \exists r.(Adjacent(r, s) \land Wumpus(r)))$

- Agent moves to [2,1]: $At(A, [2,1], 1)$
- Perceives a breeze and no stench:
  $Tell(KB, Percept([Null, Breeze, Null, Null, Null], 1))$
  $\implies Breeze(1), \neg Stench(1),$
  $\implies Breezy([2,1]), \neg Stenchy([2,1]),$
  $\implies \exists r.(Adjacent(r, [2,1]) \land Pit(r)),$
    $\neg Wumpus([3,1]), \neg Wumpus([2,2]),$
  $\implies (Pit([3,1]) \lor Pit([2,2]))$
  $AskVars(KB, \exists a.Action(a, 1)) \implies \{a/Move([1,1])\}$

Complete the example in the FOL case (see the PL case).

Outline

Knowledge Engineering in FOL

The knowledge-engineering process

1. **Identify the task** (analogous to PEAS process to design agents)
   - determine what knowledge must be represented in order to connect problem instances to answers
2. **Assemble the relevant knowledge** (aka knowledge acquisition)

   (either by own domain knowledge or by experts interviews)
   - understand the scope of the knowledge base
   - understand how the domain actually works
3. **Decide on a vocabulary of predicates, functions, and constants**
   - translate relevant domain-level concepts into logic-level names
   - what should be represented as predicate/function/constant?
   $\implies$ define the ontology of the domain
4. **Encode into FOL general knowledge about the domain**
   - write down the axioms for all the vocabulary terms
   $\implies$ should enable the domain expert to check the content
5. ...

Knowledge Engineering in FOL [cont.]

The knowledge-engineering process [cont.]

4. ...

5. Encode into FOL a description of the specific problem instance
   (straightforward iff the ontology is well-conceived)
   - mostly assertions of (possibly negated) ground atomic formulas
   - for a logical agent, problem instances are supplied by the sensors
   - general knowledge base is supplied with additional sentences

6. Pose queries to the inference procedure and get answers
   - the final outcome
   - check the queries

7. Debug the knowledge base
   - detect un-answered/wrong queries
   - identify too-weak or missing axioms by backward-analysis
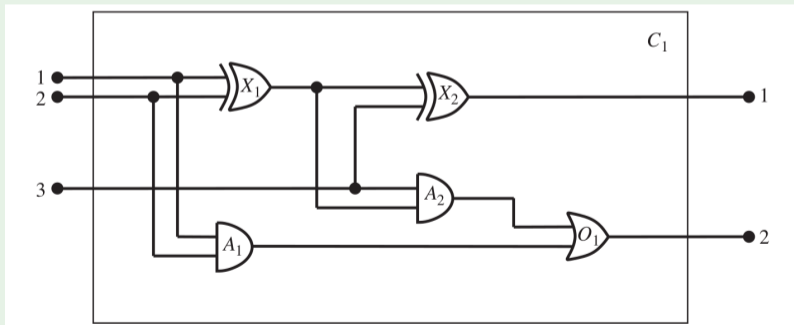
No need for writing an application-specific solution algorithm!

Example: The Electronic Circuits Domain

Task: Develop (an ontology and) a knowledge base allowing to reason about digital circuits (e.g., that shown in Figure)

- Ex: One-bit full adder:
  - first two inputs are to be added, the third input is a carry bit
  - first output is the sum, the second output is a carry bit

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Example: The Electronic Circuits Domain [cont.]

1. **Identify the task**
   - At the highest level, analyze the circuit's functionality
   - ex: does the circuit contain feedback loops?
   - ...

2. **Assemble the relevant knowledge**
   - signals flow along wires to the input terminals of gates
   - each gate produces a signal on the output
   - AND, OR, XOR gates have two inputs, NOT gates have one
   - ...

3. **Decide on a vocabulary of predicates, functions, and constants**
   - e.g. each gate instance represented as constant (ex "X_1")
   - each gate type represented as constant (ex "*AND*")
   - a function Type (ex: $Type(X_1) = XOR$)
   - gate terminals represented as integer constants,
   - two functions In, Out, and one predicate *Connected*
     (ex: $Connected(In(1, X_1), In(1, A_2))$),
   - two values 0,1, a predicate Signal(t) (ex: $Signal(In(1, X_1)) = 1$)
   - ...

Example: The Electronic Circuits Domain [cont.]

④ Encode general knowledge about the domain

$\forall t_1, t_2.((Terminal(t_1) \land Terminal(t_2) \land Connected(t_1, t_2))) \rightarrow$
$\qquad (Signal(t_1) = Signal(t_2))$

$\forall t.(Terminal(t) \rightarrow ((Signal(t) = 1) \lor (Signal(t) = 0)))$

$\forall t_1, t_2.(Connected(t_1, t_2) \leftrightarrow Connected(t_2, t_1))$

$\forall g.(Gate(g) \rightarrow ((Type(g) = AND) \lor (Type(g) = OR) \lor$
$\qquad\qquad (Type(g) = XOR) \lor (Type(g) = NOT)))$

$\forall g.((Gate(g) \land Type(g) = AND) \rightarrow$
$\qquad ((Signal(Out(1, g)) = 0) \leftrightarrow \exists n.(Signal(In(n, g)) = 0)))$

... analogous definitions for OR, XOR, NOT

$\forall g.((Gate(g) \land (Type(g) = NOT)) \rightarrow Arity(g, 1, 1))$

$\forall g.((Gate(g) \land ((Type(g) = AND) \lor (Type(g) = OR) \lor$
$\qquad\qquad (Type(g) = XOR))) \rightarrow Arity(g, 2, 1))$

$\forall c, i, j.((Circuit(c) \land Arity(c, i, j)) \rightarrow$
$\quad \forall n.((n \le i \rightarrow Terminal(In(c, n))) \land (n > i \rightarrow In(c, n) = Nothing)) \land$
$\quad \forall n.((n \le j \rightarrow Terminal(Out(c, n))) \land (n > j \rightarrow Out(c, n) = Nothing)))$

$\forall g, t.((Gate(g) \land Terminal(t)) \rightarrow$
$\qquad (g \ne t \ne 1 \ne 0 \ne OR \ne AND \ne XOR \ne NOT \ne Nothing))$

Example: The Electronic Circuits Domain [cont.]

⑤ Encode a description of the specific problem instance

- $Circuit(C_1) \land Arity(C_1, 3, 2) \land$
  $Gate(X_1) \land Type(X_1) = XOR \land Gate(X_2) \land Type(X_2) = XOR \land ... \land$
  $Gate(O_1) \land Type(O_1) = OR$
- $Connected(Out(1, X_1), In(1, X_2)) \land$
  $... \land$
  $Connected(In(3, C_1), In(1, A_2))$

⑥ Pose queries to the inference procedure and get answers

- Ex: Which inputs would cause the first output of C_1 (the sum bit) to be 0 and the second output of C_1 (the carry bit) to be 1?
  $AskVars(KB, \exists i_1, i_2, i_3.(Signal(In(1, C_1)) = i_1 \land$
  $Signal(In(2, C_1)) = i_2 \land Signal(In(3, C_1)) = i_3 \land$
  $Signal(Out(1, C_1)) = 0 \land Signal(Out(2, C_1)) = 1))$
  $\implies \{i_1/1, i_2/1, i_3/0\}$ or $\{i_1/1, i_2/0, i_3/1\}$ or $\{i_1/0, i_2/1, i_3/1\}$
- What are the possible value sets of all terminals?
  $AskVars(KB, \exists i_1, i_2, i_3, o_1, o_2.(Signal(In(1, C_1)) = i_1 \land$
  $Signal(In(2, C_1)) = i_2 \land Signal(In(3, C_1)) = i_3 \land$
  $Signal(Out(1, C_1)) = o_1 \land Signal(Out(2, C_1)) = o_2))$
  $\implies \{i_1/1, i_2/1, i_3/1, o_1/1, o_2/1\}$ or $\{i_1/1, i_2/1, i_3/0, o_1/0, o_2/1\}$ or ...

Example: The Electronic Circuits Domain [cont.]

7 Debug the knowledge base
   - Suppose no output produced by previous query
   - We progressively try to restrict our analysis my more local queries, until we pinpoint the problem.
   - Ex: $\exists i_1, i_2, o.(Signal(In(1, C1)) = i_1 \land Signal(In(2, C1)) = i_2 \land$
                    $Signal(Out(1, X1)) = o)$

     (see AIMA book for a detailed example)