Fundamentals of Artificial Intelligence
Chapter 14: **Probabilistic Reasoning**

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M.S. Course "Artificial Intelligence Systems", academic year 2021-2022

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Outline

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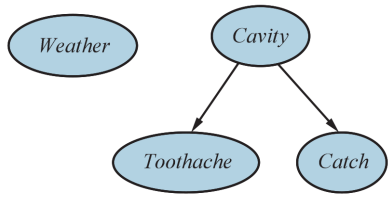
Bayesian Networks

Bayesian Networks (aka Belief Networks):

- Syntax: a directed acyclic graph (DAG):
  - each node represents a random variable (discrete or continuous)
  - directed arcs connect pairs of nodes: $X \rightarrow Y$ (X is a parent of Y)
  - a conditional distribution $\mathbf{P}(X_i|Parents(X_i))$ for each node X_i
- Conditional distribution represented as a conditional probability table (CPT)
  - distribution over X_i for each combination of parent values
- Allow for compact specification of full joint distributions
- Represent explicit conditional dependencies among variables:
  an arc from X to Y means that X has a direct influence on Y
- Topology encodes conditional independence assertions:
  - Toothache, Catch conditionally independent given Cavity
  - Tootchache, Catch depend on Cavity
  - Weather independent from others
- No arc $\Longleftrightarrow$ independence



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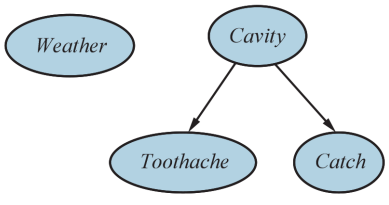

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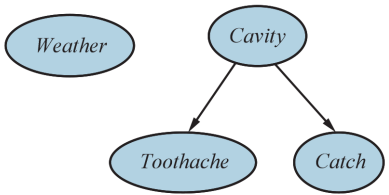

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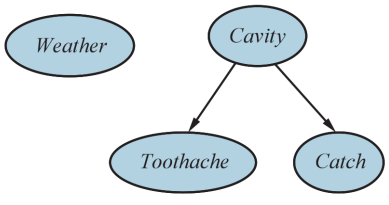

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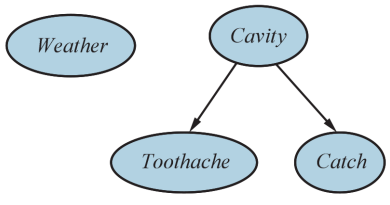

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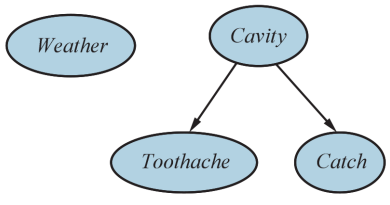

Example (from Judea Pearl, UCLA)

"*The burglary alarm goes off very likely on burglary and occasionally on earthquakes.*
John and Mary are neighbors who agreed to call when the alarm goes off.
Their reliability is different ..."

- Variables: Burglary, Earthquake, Alarm, JohnCalls, MaryCalls
- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call
- CPTs:
  - alarm setoff if bunglar
    in 94% of cases
  - alarm setoff if hearthquake
    in 29% of cases
  - false alarm setoff
    in 0.1% of cases
- Notice: in CPTs like $\mathbf{P}(A|B)$, only $P(a|B)$ are reported, because $P(\neg a|B) = 1 - P(a|B)$

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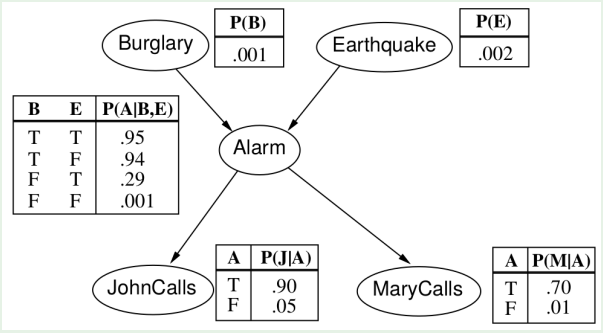

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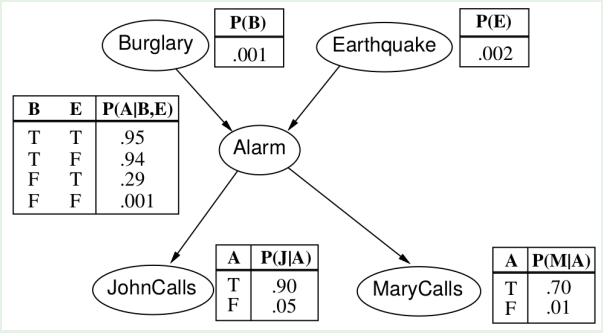

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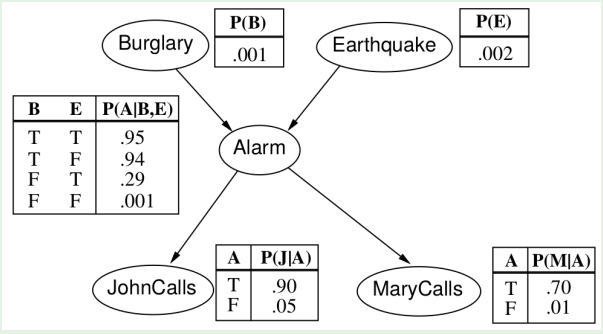

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Compactness of Bayesian Networks

- In most domains, it is reasonable to suppose that each random variable X_i is directly influenced by only a small number k_i of other variables, called parents of X_i (parents(X_i))
- A CPT for Boolean X_i with k_i Boolean parents has
  - 2^{k_i} rows for the combinations of parent values
  - each row requires one number p for $P(X_i = true)$ ($P(X_i = false) = 1 - P(X_i = true)$)

$\Longrightarrow$ If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

- a full joint distribution requires $2^n - 1$ numbers
- linear vs. exponential!

- Ex: for burglary example:
  - $1 + 1 + 4 + 2 + 2 = 10$ numbers vs. $2^5 - 1 = 31$

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Global Semantics of Bayesian Networks

- Global semantics defines the full joint distribution as the product of the local conditional distributions:
  $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
  - if X_i has no parent, then conditional distributions reduce to prior probability $\mathbf{P}(X_i)$
- Intuition: order $X_1, ..., X_n$ s.t. $parents(X_i) \prec X_i$ for each i:
  $\mathbf{P}(X_1, ..., X_n)$
  $= \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, ..., X_{i-1}))$ // chain rule
  $= \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$ // conditional independence

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Global Semantics: Example

- $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
- Ex: "Prob. that both John and Mary call, the alarm sets off but no burglary nor earthquake"

$P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) =$
$P(j|m \wedge a \wedge \neg b \wedge \neg e) P(m|a \wedge \neg b \wedge \neg e) P(a|\neg b \wedge \neg e) P(\neg b|\neg e) P(\neg e) =$
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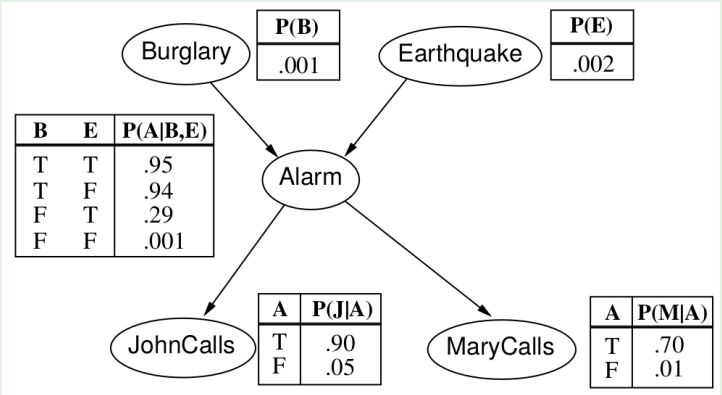

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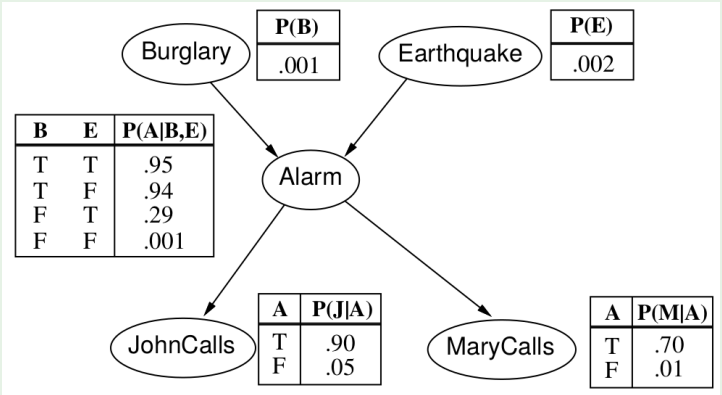

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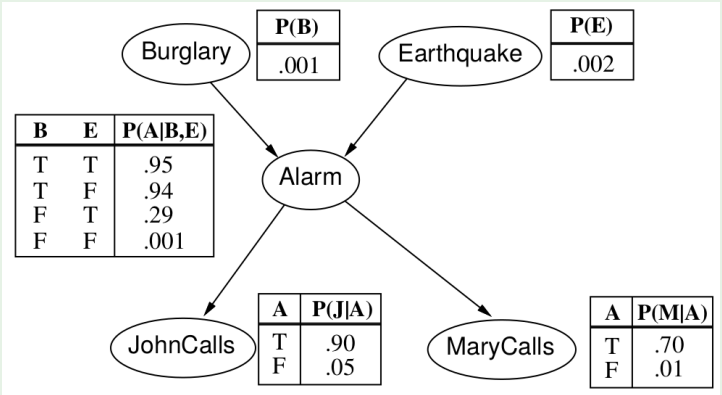

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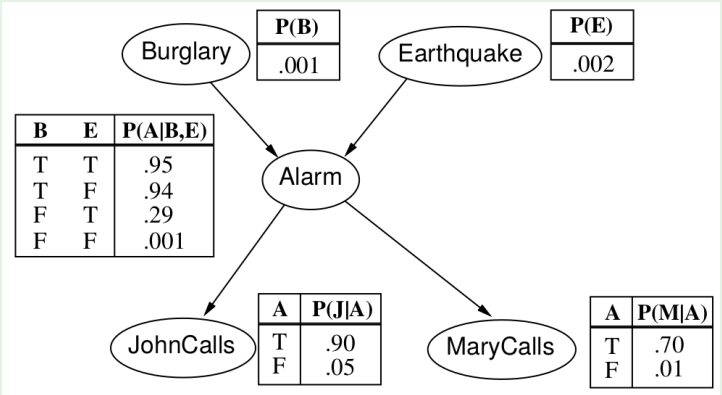

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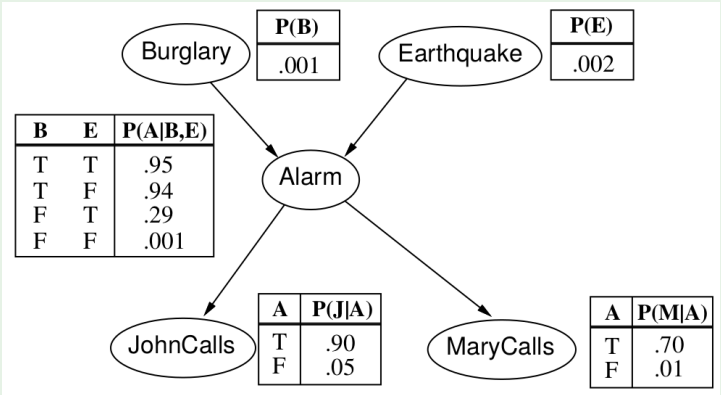

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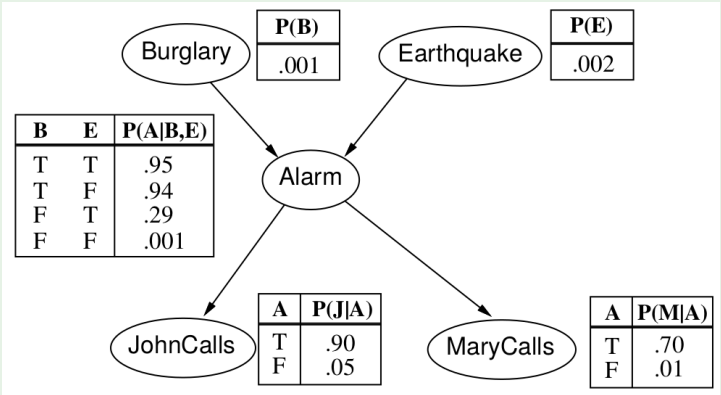

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- $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
- Ex: "Prob. that both John and Mary call, the alarm sets off but no burglary nor earthquake"
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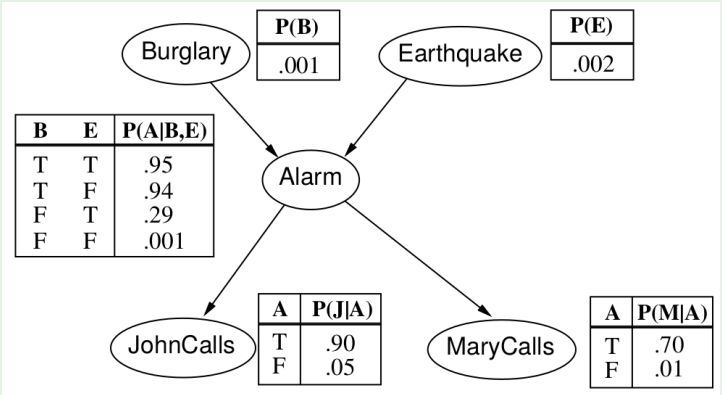

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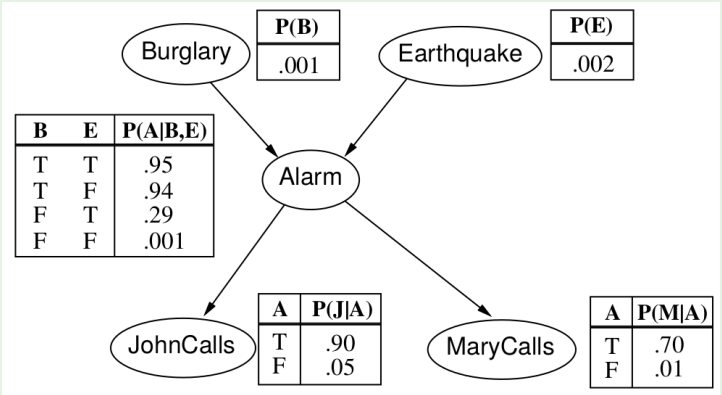

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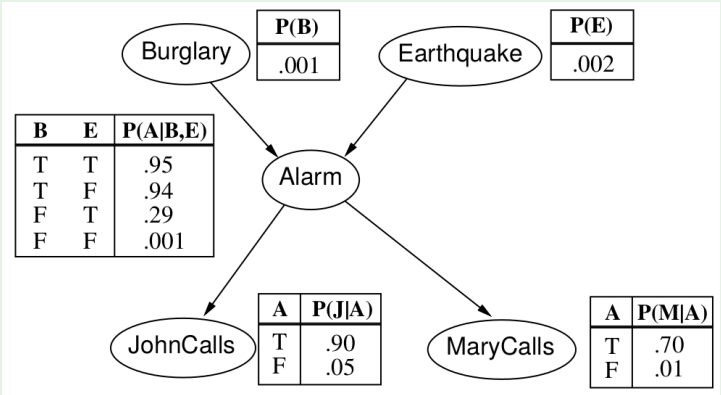


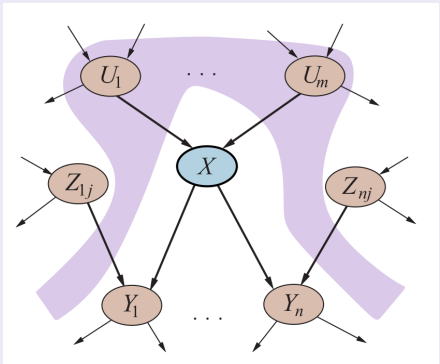
Exercises

- Compute:
  - The probability that John calls and Mary does not, the alarm is not set off with a burglar entering during an earthquake
  - The probability that John calls and Mary does not, given a burglar entering the house
  - The probability of an earthquake given the fact that John has called
  - ...

Outline

Local Semantics

- Local Semantics: each node is conditionally independent of its nondescendants given its parents: $\mathbf{P}(X|U_1,..,U_m,Z_{1j},...,Z_{nj}) = \mathbf{P}(X|U_1,..,U_m)$, for each X
  - "nondecendants" include ancestors
- Theorem: Local semantics holds iff global semantics holds: $\mathbf{P}(X_1,...,X_N) = \prod_{i=1} \mathbf{P}(X_i|parents(X_i))$



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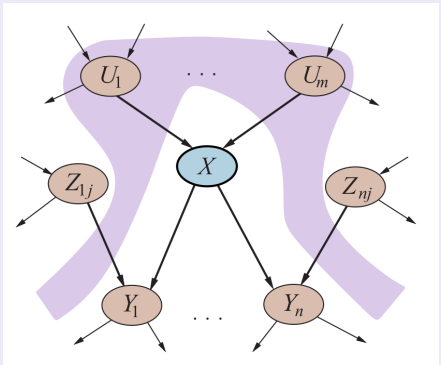

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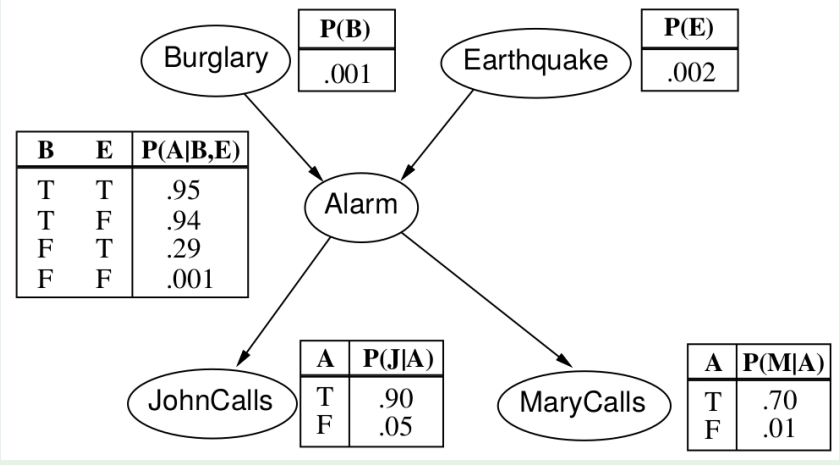
Local Semantics: Example

Ex: JohnCalls is independent of Burglary, Earthquake, and MaryCalls given the value of Alarm

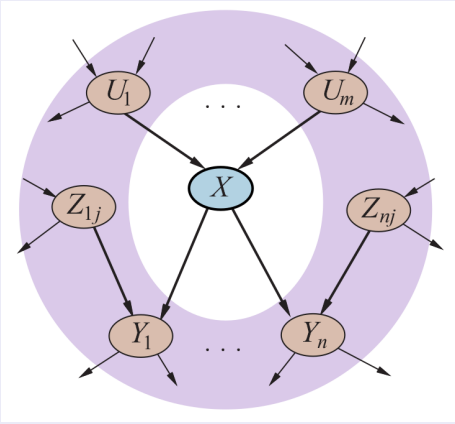
$$\mathbf{P}(JohnCalls|Alarm, Burglary, Earthquake, MaryCalls) = \mathbf{P}(JohnCalls|Alarm)$$



Outline

Independence Property: Markov Blanket

In an Baynesian Network, each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents:
$\mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj},W_1,...,W_k) = \mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj})$, for each X



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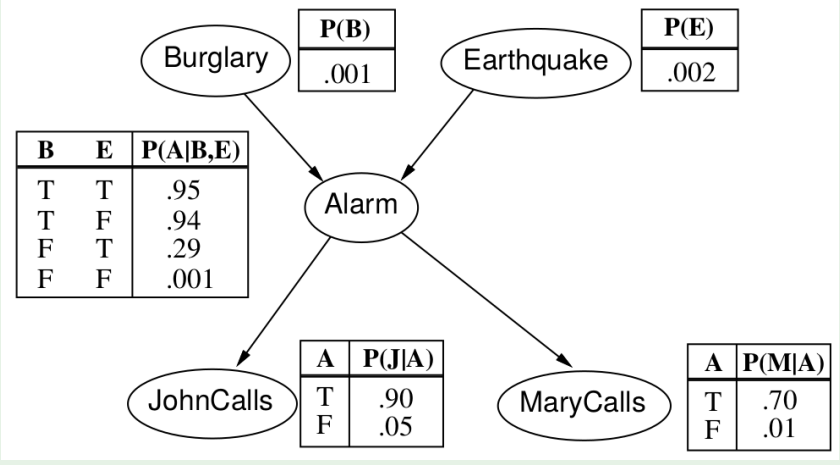
Markov Blanket: Example

Ex: Burglary is independent of JohnCalls and MaryCalls, given Alarm and Earthquake

$$\mathbf{P}(Burglary|Alarm, Earthquake, JohnCalls, MaryCalls) = \mathbf{P}(Burglary|Alarm, Earthquake)$$



Exercise

Verify numerically the two previous examples:

- Local Semantics
- Markov Blanket

Outline

Constructing Bayesian Networks

Building the graph

Given a set of random variables

1. Choose an ordering $\{X_1, ..., X_n\}$
   - in principle, any ordering will work (but some may cause blowups)
   - general rule: follow causality, $X \prec Y$ if $X \in causes(Y)$
2. For i=1 to n do
   1. add X_i to the network
   2. as $Parents(X_i)$, choose a subset of $\{X_1, ..., X_{i-1}\}$ s.t. $P(X_i|Parents(X_i)) = P(X_i|X_1, ..., X_{i-1})$

$\Longrightarrow$ Guarantees the global semantics by construction

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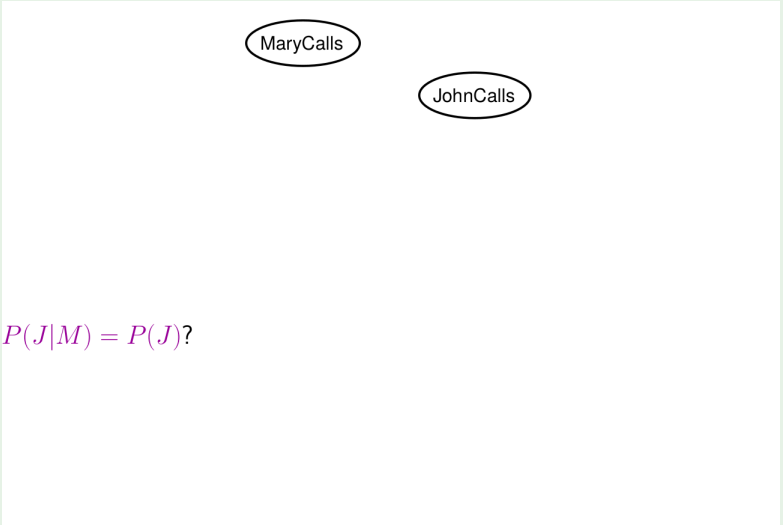

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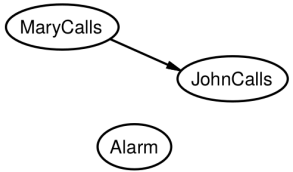

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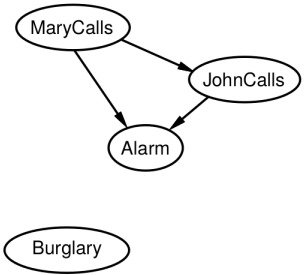

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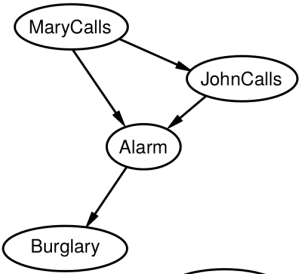

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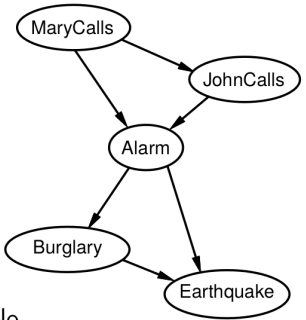

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Constructing Bayesian Networks: Example [cont.]

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  - ex: try $\{M, J, E, B, A\}$ (see AIMA)
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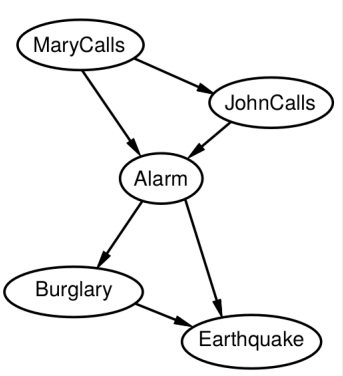

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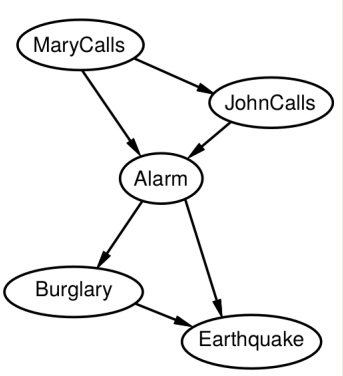

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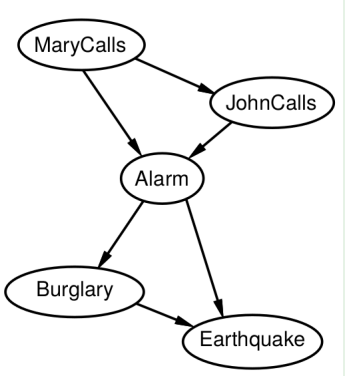

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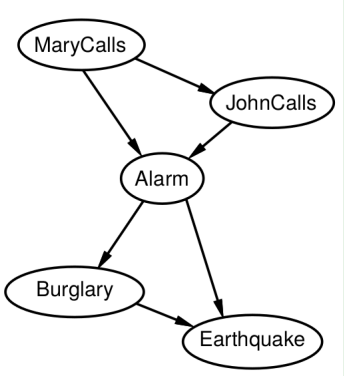

Building Conditional Probability Tables, CPTs

- Problem: CPT grow exponentially with number of parents
- If the causes don't interact: use a Noisy-OR distribution (generalization of logical or)
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T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network

Outline

Exact inference in Bayesian Networks

- Given:
  - X: the query variable (we assume one for simplicity)
  - **E**/**e**: the set of evidence variables $\{E_1, ..., E_m\}$ and of evidence values $\{e_1, ..., e_m\}$
  - **Y**/**y**: the set of unknown variables (aka hidden variables) $\{Y_1, ..., Y_l\}$ and unknown values $\{y_1, ..., y_l\}$

  $\Longrightarrow$ $\mathbf{X} = X \cup \mathbf{E} \cup \mathbf{Y}$

  A typical query asks for the posterior probability distribution:
  P(X | **E**=**e**) (also written **P**(X | **e**))
- Ex: $P(Burglar|JohnCalls = true, MaryCalls = true)$
  - query: $Burglar$
  - evidence variables: $\mathbf{E} = \{JohnCalls, MaryCalls\}$
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Inference by Enumeration

- We defined a procedure for the task as: $\mathbf{P}(X|\mathbf{e}) = \alpha\mathbf{P}(X,\mathbf{e}) = \alpha\sum_{\mathbf{y}}\mathbf{P}(X,\mathbf{e},\mathbf{y})$

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  - then apply factorization and simplify algebraically when possible

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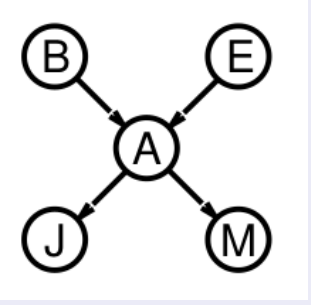

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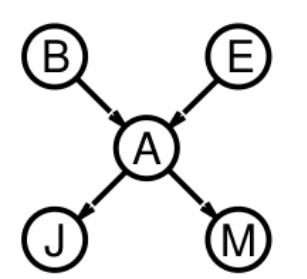

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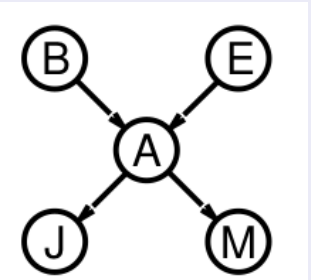

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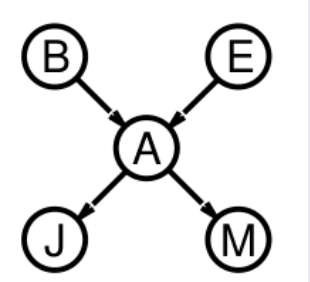


Enumeration Algorithm

function ENUMERATION-ASK(X, **e**, bn) **returns** a distribution over X computes **P**(X | **e**)
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* **Y** = *hidden variables* */

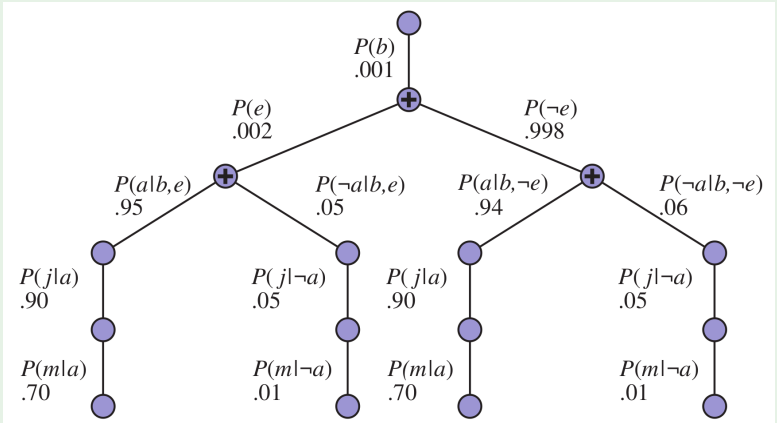
  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$) computes P(xi, **Y**, **e**) (single probability value)
      where $\mathbf{e}_{x_i}$ is **e** extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, **e**) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in **e**
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), **e**) X or evidence var
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$) hidden var
      where $\mathbf{e}_y$ is **e** extended with $Y = y$

Inference by Enumeration: Example

$P(\ b|j,m) = \alpha P(\ b) \sum_e P(e) \sum_a P(a|\ b,e) P(j|a) P(m|a) = \alpha \cdot 0.00059224$
$P(\neg b|j,m) = \alpha P(\neg b) \sum_e P(e) \sum_a P(a|\neg b,e) P(j|a) P(m|a) = \alpha \cdot 0.0014919$
$\Longrightarrow \mathbf{P}(B|j,m) = \alpha \cdot \langle 0.00059224, 0.0014919 \rangle = [normal.] \approx \langle 0.284, 0.716 \rangle$



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Repeated computation: $P(j|a)P(m|a)$ & $P(j|\neg a)P(m|\neg a)$ for each value of e

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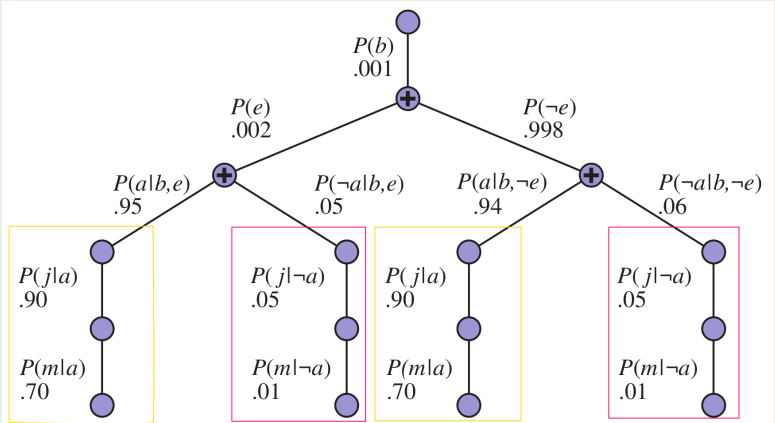

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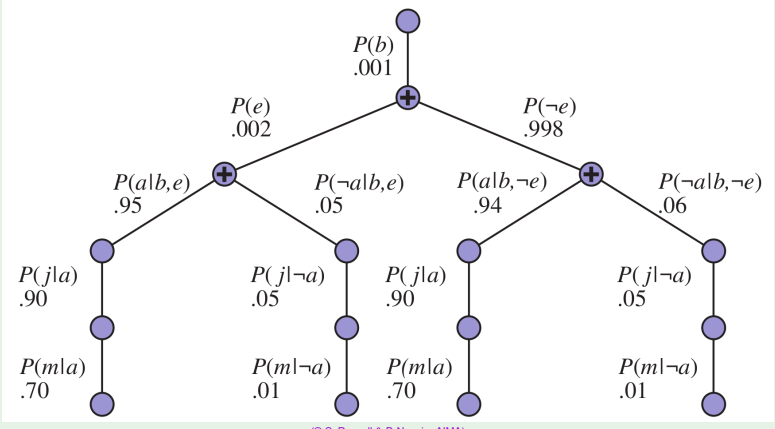

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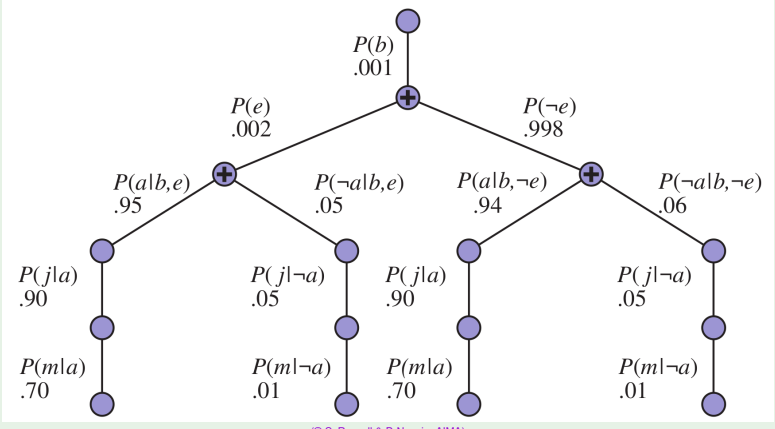

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Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network
   (b) Compute the query $P(P_{1,3}|b^*, p^*)$ via enumeration
   (c) Compare the result with that of the example in Ch. 13

Outline

Inference by Variable Elimination

- Variable elimination:
  - carry out summations right-to-left (i.e., bottom-up in the tree)
  - store intermediate results (factors) to avoid recomputation
- Ex: $P(B|j,m)$

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\begin{aligned}
&= \alpha \overbrace{P(B)}^{B} \sum_e \overbrace{P(e)}^{E} \sum_a \overbrace{P(a|B,e)}^{A} \overbrace{P(j|a)}^{J} \overbrace{P(m|a)}^{M} \\
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  - "$\times$" is the pointwise product (see later)

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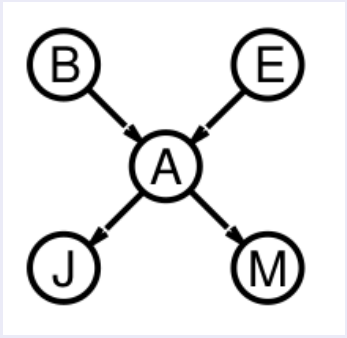

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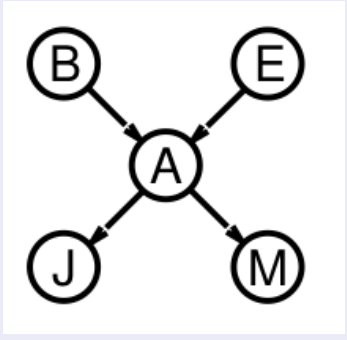

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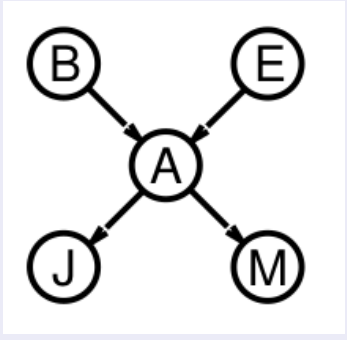

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  - $\mathbf{f}_M(A) \stackrel{\text{def}}{=} \begin{bmatrix} P(m|\ a) \\ P(m|\neg a) \end{bmatrix}$, $\mathbf{f}_J(A) \stackrel{\text{def}}{=} \begin{bmatrix} P(j|\ a) \\ P(j|\neg a) \end{bmatrix}$, ...
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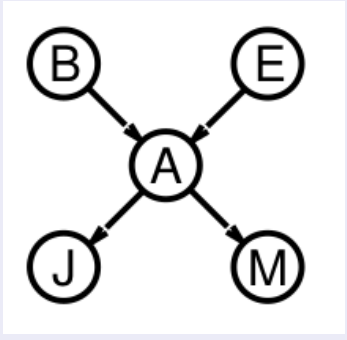

Inference by Variable Elimination

- Variable elimination:
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- Ex: $P(B|j,m)$

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\begin{aligned}
&= \alpha \overbrace{P(B)}^{B} \sum_e \overbrace{P(e)}^{E} \sum_a \overbrace{\mathbf{P}(a|B,e)}^{A} \overbrace{P(j|a)}^{J} \overbrace{P(m|a)}^{M} \\
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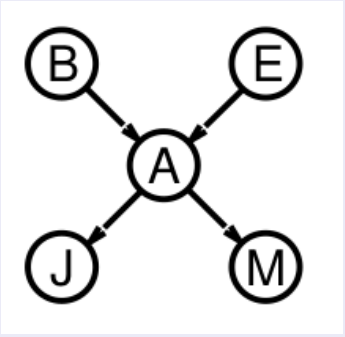

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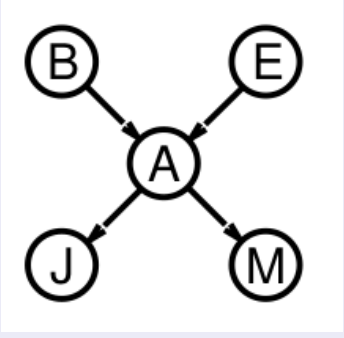

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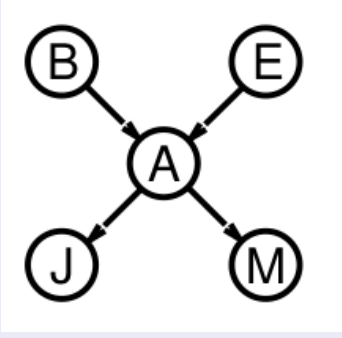

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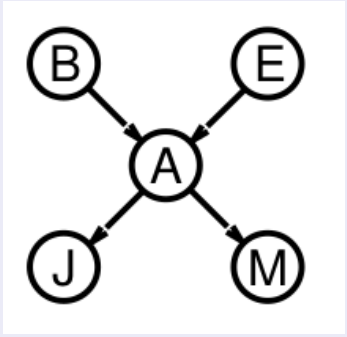

Inference by Variable Elimination

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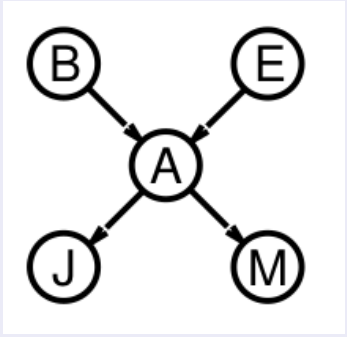

Variable Elimination: Basic Operations

- Factor summation: $\mathbf{f}_3(X_1, ..., X_j) = \mathbf{f}_1(X_1, ..., X_j) + \mathbf{f}_2(X_1, ..., X_j)$
  - standard matrix summation:

$$\begin{bmatrix} a_{11} & a_{21} & ... \\ ... & ... & ... \\ a_{n1} & a_{n1} & ... \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & ... \\ ... & ... & ... \\ b_{n1} & b_{n1} & ... \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{21} + b_{21} & ... \\ ... & ... & ... \\ a_{n1} + b_{n1} & a_{n1} + b_{n1} & ... \end{bmatrix}$$

  - must have the same argument variables
- Pointwise product: Multiply the array elements for the same variable values
- Ex: $\mathbf{f}_J(A) \times \mathbf{f}_M(A) = \begin{bmatrix} P(j|\ a) \\ P(j|\neg a) \end{bmatrix} \times \begin{bmatrix} P(m|\ a) \\ P(m|\neg a) \end{bmatrix} = \begin{bmatrix} P(j|\ a)P(m|\ a) \\ P(j|\neg a)P(m|\neg a) \end{bmatrix}$
- General case:
  $\mathbf{f}_3(X_1, ..., X_j, Y_1, ..., Y_k, Z_1, ..., Z_l) = \mathbf{f}_1(X_1, ..., X_j, Y_1, ..., Y_k) \times \mathbf{f}_2(Y_1, ..., Y_k, Z_1, ..., Z_l)$
  - union of arguments
  - values: $\mathbf{f}_3(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{f}_1(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}_2(\mathbf{y}, \mathbf{z})$
  - matrix size: $\mathbf{f}_1 : 2^{j+k}$, $\mathbf{f}_1 : 2^{k+l}$, $\mathbf{f}_3 : 2^{j+k+l}$

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Variable Elimination: Basic Operations

- $\mathbf{f}_3(A, B, C) = \mathbf{f}_1(A, B) \times \mathbf{f}_2(B, C)$
- Summing out one variable: $f(B, C) = \sum_a f_3(A, B, C) = f_3(a, B, C) + f_3(\neg a, B, C) =$
$$\begin{bmatrix} 0.06 & 0.24 \\ 0.42 & 0.28 \end{bmatrix} + \begin{bmatrix} 0.18 & 0.72 \\ 0.06 & 0.04 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.96 \\ 0.48 & 0.32 \end{bmatrix}$$

A	B	$\mathbf{f}_1(A, B)$
T	T	.3
T	F	.7
F	T	.9
F	F	.1

B	C	$\mathbf{f}_2(B, C)$
T	T	.2
T	F	.8
F	T	.6
F	F	.4

A	B	C	$\mathbf{f}_3(A, B, C)$
T	T	T	$.3 \times .2 = .06$
T	T	F	$.3 \times .8 = .24$
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T	F	F	$.7 \times .4 = .28$
F	T	T	$.9 \times .2 = .18$
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F	F	T	$.1 \times .6 = .06$
F	F	F	$.1 \times .4 = .04$

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Variable Elimination Algorithm

function ELIMINATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$
  for each var **in** ORDER(bn.VARS) **do**
    $factors \leftarrow [\text{MAKE-FACTOR}(var, \mathbf{e}) | factors]$
    if var is a hidden variable **then** $factors \leftarrow \text{SUM-OUT}(var, factors)$
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

(© S. Russell & P. Norwig, AIMA)

- Efficiency depends on variable ordering ORDER(...)
- Efficiency improvements:
  - factor out of summations factors not depending on sum variable
  - remove irrelevant variables

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Factor Out Constant Factors

- If $f_1, ..., f_i$ do not depend on X, then move them out of a $\sum_x(...)$:
  $\sum_x f_1 \times \cdots \times f_k =$
  $f_1 \times \cdots \times f_i \sum_x (f_{i+1} \times \cdots \times f_k) =$
  $f_1 \times \cdots \times f_i \times f_X$
- Ex: $\sum_a \mathbf{f}_1(A, B) \times \mathbf{f}_2(B, C)$
  $= \mathbf{f}_2(B, C) \times \sum_a \mathbf{F}_1(A, B)$
- Ex: $\mathbf{P}(JohnCalls|Burglary = true)$:
  $\mathbf{P}(J|b) = \alpha \sum_e \sum_a \sum_m \mathbf{P}(J, m, b, e, a) =$
  $\alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b, e)\mathbf{P}(J|a)P(m|a) =$
  $\alpha P(b) \sum_e P(e) \sum_a P(a|b, e)\mathbf{P}(J|a) \sum_m P(m|a)$

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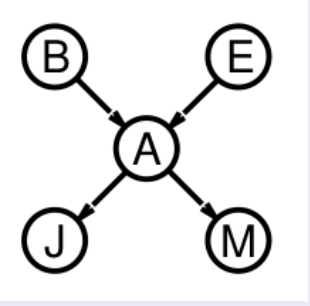

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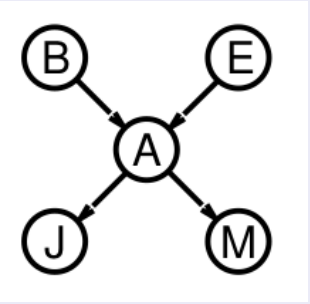

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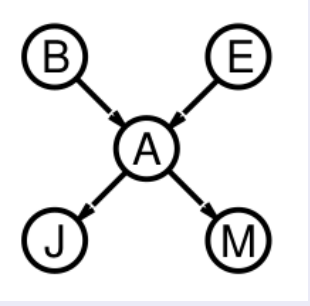

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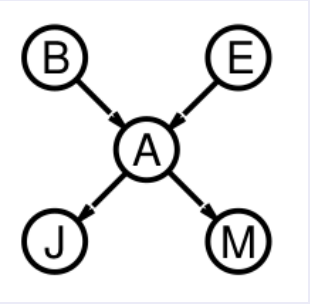

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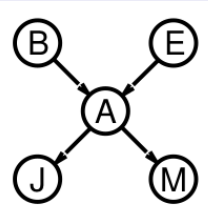

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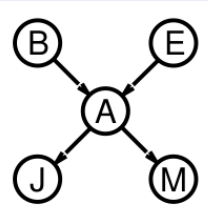

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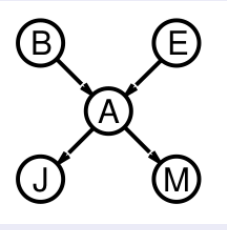

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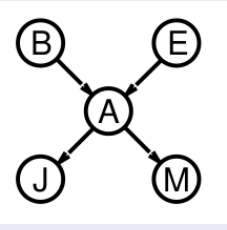

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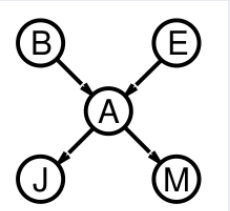

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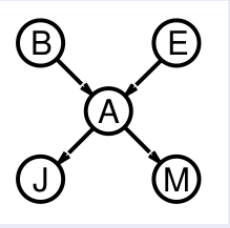

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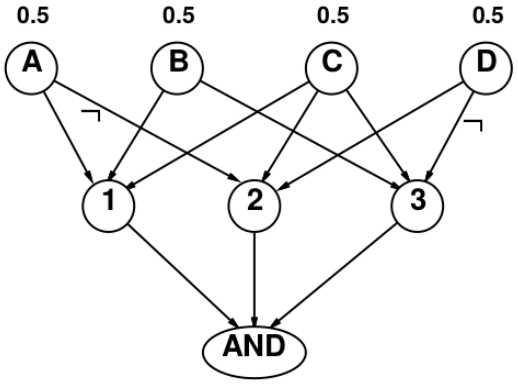

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