Fundamentals of Artificial Intelligence
Chapter 05: **Adversarial Search and Games**

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Outline

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Games and AI

- Games are a form of multi-agent environment
  - Q.: What do other agents do and how do they affect our success?
  - recall: cooperative vs. competitive multi-agent environments
  - competitive multi-agent environments give rise to adversarial problems (aka games)
- Q.: Why study games in AI?
  - lots of fun, historically entertaining
  - easy to represent: agents restricted to small number of actions with precise rules
  - interesting also because computationally very hard
    (ex: chess has $b \approx 35$, $\#nodes \approx 10^{40}$)
  - metaphor for important application domains
    (e.g. competitive markets, life sciences, sport, politics, warfare, ...)

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Search and Games

- Search (with no adversary)
  - solution is a (heuristic) method for finding a goal
  - heuristics techniques can find optimal solutions
  - evaluation function: estimate of cost from start to goal through given node
  - examples: path planning, scheduling activities, ...
- Games (with adversary), aka adversarial search
  - solution is a strategy: specifies a move for every possible opponent reply
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  - examples: tic-tac-toe, chess, checkers, Othello, backgammon, ...
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Types of Games

- Many different kinds of games
- Relevant features:
  - deterministic vs. stochastic (with chance)
  - one, two, or more players
  - zero-sum vs. general games
  - perfect information (can you see the state?) vs. imperfect
- Most common: deterministic, turn-taking, two-player, zero-sum games, perfect information
- Want algorithms for calculating a strategy (aka policy):
  - recommends a move from each state: $policy : S \mapsto A$

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	deterministic	chance
perfect information	chess, checkers, go, othello	backgammon monopoly
imperfect information	battleships, blind tictactoe	bridge, poker, scrabble nuclear war

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Games: Main Concepts

- We first consider games with two players: "MAX" and "MIN"
  - MAX moves first;
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  - at the end of the game, points are awarded to the winner and penalties are given to the loser
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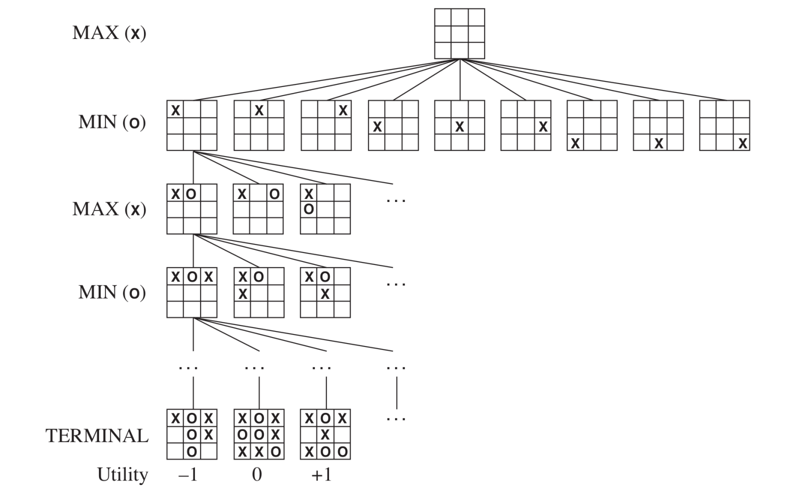
Game Tree: Example

Partial game tree for tic-tac-toe (2-player, deterministic, turn-taking)



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Zero-Sum Games vs. General Games

- General Games
  - agents have independent utilities
  - cooperation, indifference, competition, and more are all possible
- Zero-Sum Games: the total payoff to all players is the same for each game instance
  - adversarial, pure competition
  - agents have opposite utilities (values on outcomes)

$\Longrightarrow$ Idea: With two-player zero-sum games, we can use one single utility value

- one agent maximizes it, the other minimizes it

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Adversarial Search as Min-Max Search

- Assume MAX and MIN are very smart and always play optimally
- MAX must find a contingent strategy specifying:
  - MAX's move in the initial state
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  (a single-agent move is called half-move or ply)
- Analogous to the AND-OR search algorithm
  - MAX playing the role of OR
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- Optimal strategy: for which $Minimax(s)$ returns the highest value

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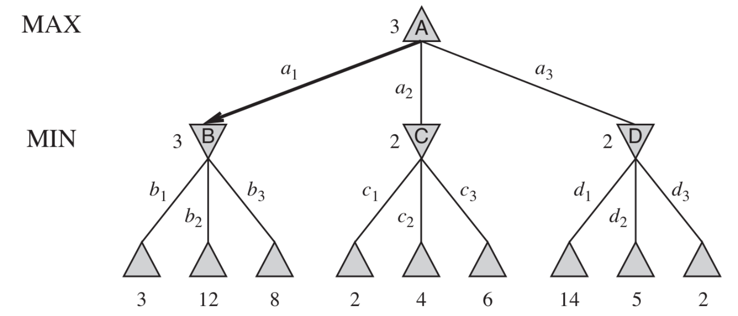
Min-Max Search: Example

A two-ply game tree

- Δ nodes are “MAX nodes”, ∇ nodes are “MIN nodes”,
  - terminal nodes show the utility values for MAX
  - the other nodes are labeled with their minimax value
- Minimax maximizes the worst-case outcome for MAX

⟹ MAX’s root best move is a_1



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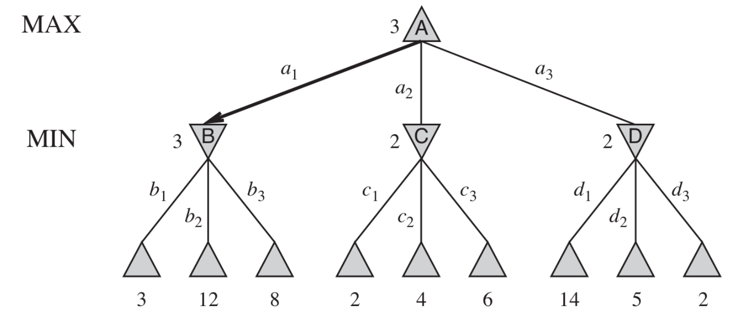
Min-Max Search: Example

A two-ply game tree

- Δ nodes are "MAX nodes", ∇ nodes are "MIN nodes",
  - terminal nodes show the utility values for MAX
  - the other nodes are labeled with their minimax value
- Minimax maximizes the worst-case outcome for MAX

$\Longrightarrow$ MAX's root best move is a_1



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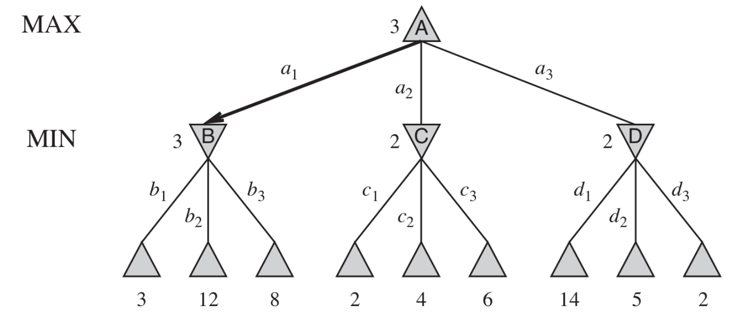

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The Minimax Algorithm

Depth-First Search Minimax Algorithm

function MINIMAX-DECISION(*state*) **returns** *an action*
  return $\arg\max_{a \in \text{ACTIONS}(s)}$ MIN-VALUE(RESULT(*state*, *a*))

---

function MAX-VALUE(*state*) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  for each *a* **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s, a)))
  return v

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Multi-Player Games: Optimal Decisions

- Replace the single value for each node with a vector of values
  - terminal states: utility for each agent
  - agents, in turn, choose the action with best value for themselves
- Alliances are possible!
  - e.g., if one agent is in dominant position, the other can ally

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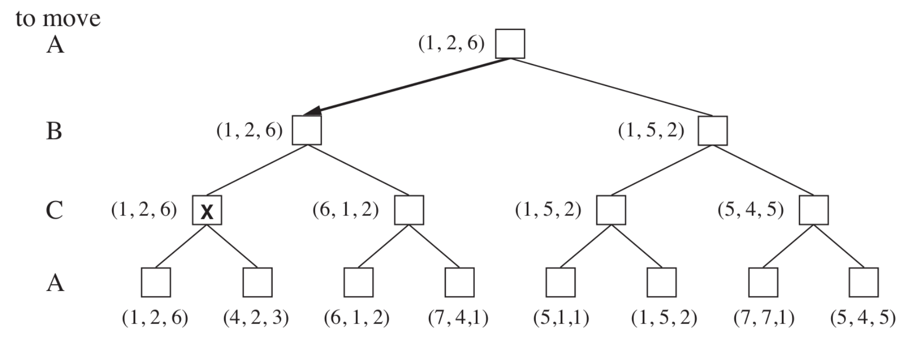
Multiplayer Min-Max Search: Example

The first three plies of a game tree with three players (A, B, C)

- Each node labeled with values from each player's viewpoint
- Agents choose the action with best value for themselves
- If A and B are allied, then they may agree that B and then A choose (5,4,5) instead of (1,5,2) $\Longrightarrow$ benefit for both



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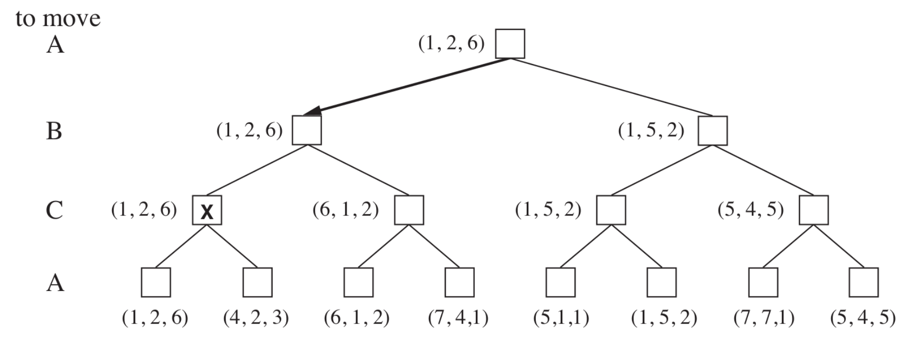

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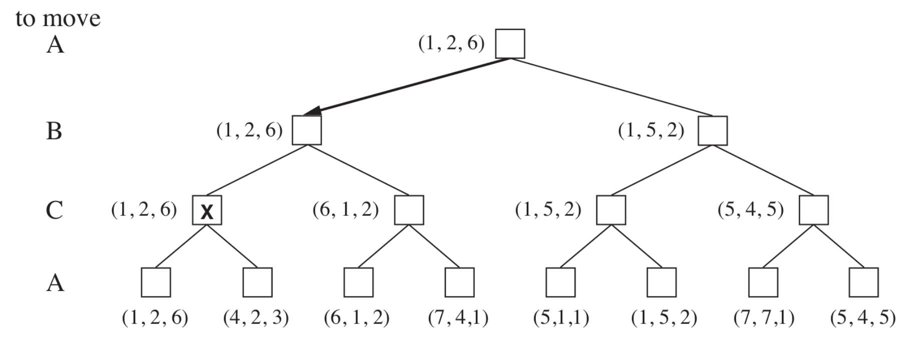

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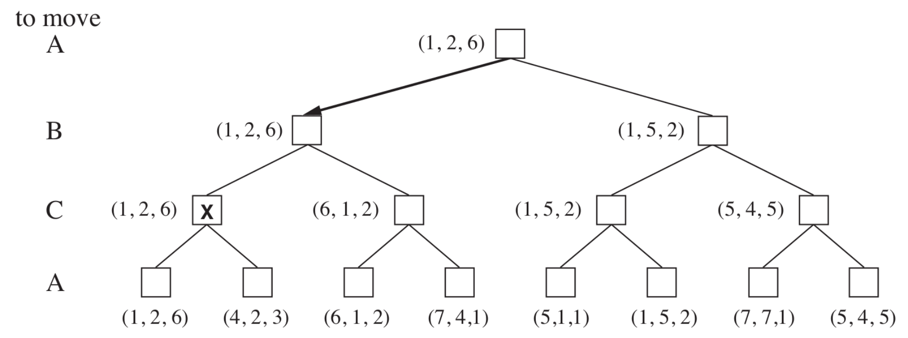

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Exercise

- Consider the Multiplayer Min-Max Search example of previous slide
  - Redo it with choice order A-C-B
  - Redo it with choice order C-A-B
  - Redo it with choice order C-B-A
  - Redo it with choice order B-A-C
  - Redo it with choice order B-C-A
- Do they have all the same outcome?
- For each case, try to define the best moves in case of alliance between the top two players

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The Minimax Algorithm: Properties

- Complete? Yes, if tree is finite
- Optimal? Yes, against an optimal opponent
  - What about non-optimal opponent?

  $\Longrightarrow$ even better, but non optimal in this case
- Time complexity? $O(b^m)$
- Space complexity? $O(bm)$ (DFS)

For chess, $b \approx 35$, $m \approx 100 \Longrightarrow 35^{100} = 10^{154}$ (!)

We need to prune the tree!

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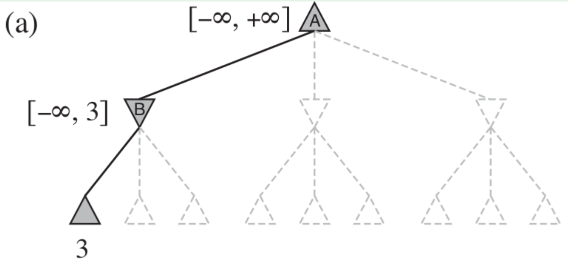
Outline

Pruning Min-Max Search: Example

- Consider the previous execution of the Minimax algorithm
- Let [*min*, *max*] track the currently-known bounds for the search
  - (a): B labeled with $[-\infty, 3]$ (MIN will not choose values ≥ 3 for B)
  - (c): B labeled with $[3, 3]$ (MIN cannot find values ≤ 3 for B)
  - (d): Is it necessary to evaluate the remaining leaves of *C*?
    NO! They cannot produce an upper bound ≥ 2
    $\Longrightarrow$ MAX cannot update the $min = 3$ bound due to C
  - (e): MAX updates the upper bound to 14 (D is last subtree)
  - (f): D labeled $[2, 2]$ $\Longrightarrow$ MAX updates the upper bound to 3

$\Longrightarrow$ 3 final value



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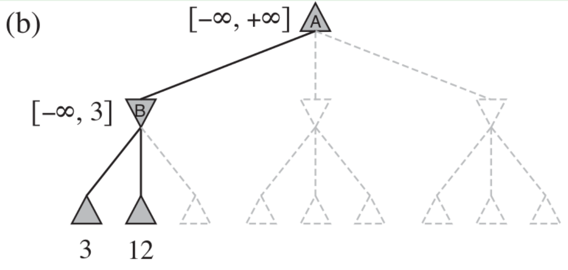

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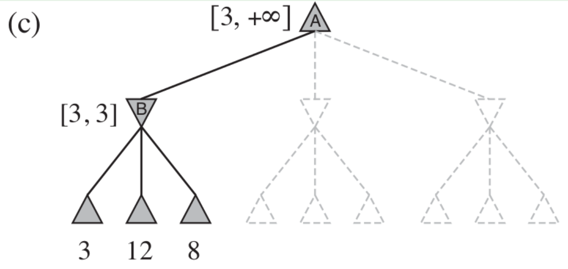

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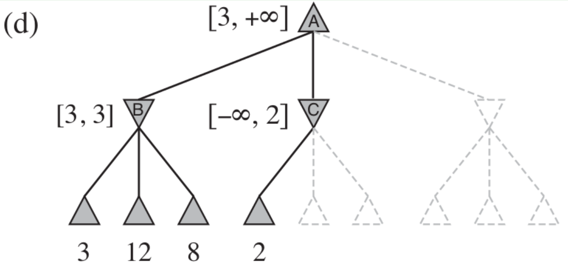

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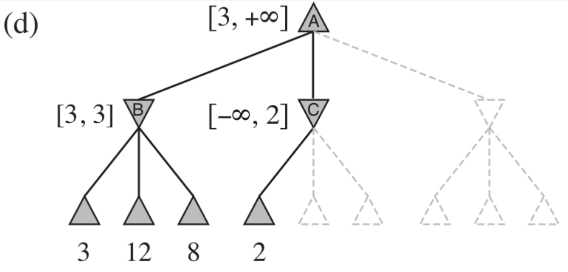

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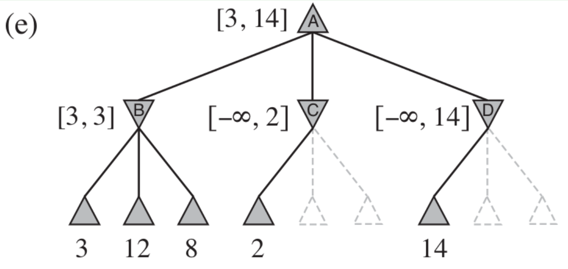

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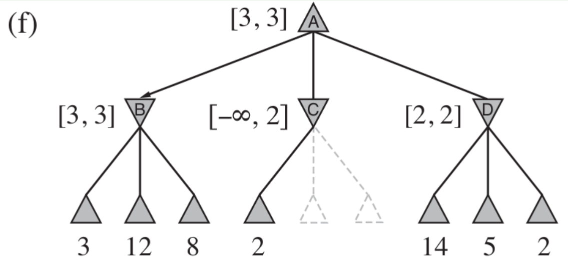

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Alpha-Beta Pruning Technique for Min-Max Search

- Idea: consider a node n (terminal or intermediate)
  - If player has a better choice m at the parent node of n or at any choice point further up, n will never be reached in actual play

  $\Longrightarrow$ if we know enough of n to draw this conclusion, we can prune n
- Alpha-Beta Pruning: nodes labeled with $[\alpha, \beta]$ s.t.:

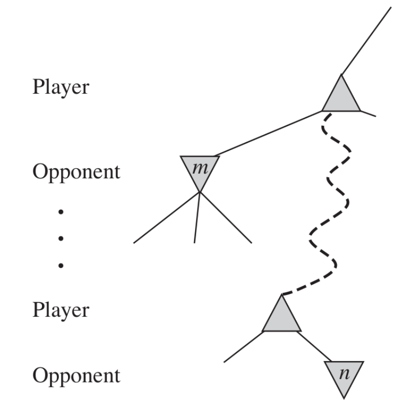
  α : best value for MAX (highest) so far off the current path

  $\Longrightarrow$ lower bound for future values

  β : best value for MIN (lowest) so far off the current path

  $\Longrightarrow$ upper bound for future values

$\Longrightarrow$ Prune *n* if its value is worse (lower) than the current α value for MAX (dual for β, MIN)



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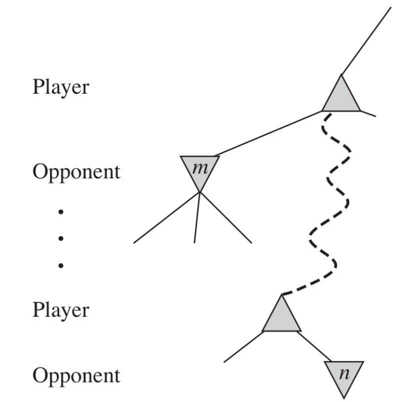

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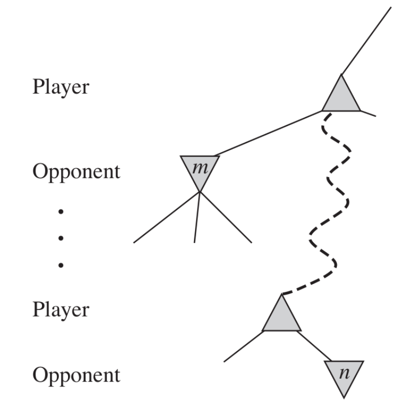

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The Alpha-Beta Search Algorithm

function ALPHA-BETA-SEARCH(*state*) **returns** an action
  $v \leftarrow$ MAX-VALUE($state, -\infty, +\infty$)
  return the *action* in ACTIONS(*state*) with value v

---

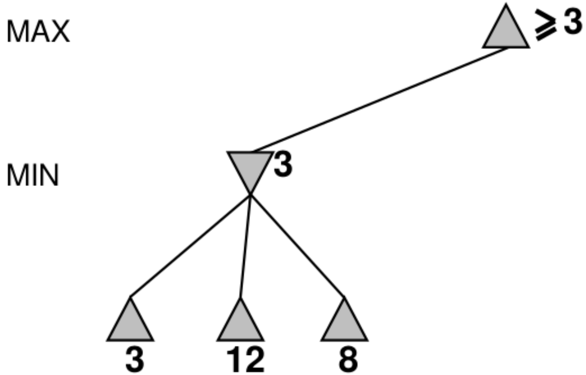
function MAX-VALUE($state, \alpha, \beta$) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  for each a **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s,a), α, β))
    if $v \geq \beta$ **then return** v
    $\alpha \leftarrow$ MAX(α, v)
  return v

---

function MIN-VALUE($state, \alpha, \beta$) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow +\infty$
  for each a **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
    if $v \leq \alpha$ **then return** v
    $\beta \leftarrow$ MIN(β, v)
  return v

Example revisited: Alpha-Beta Cuts

- Notation: $\geq \alpha$; $\leq \beta$;



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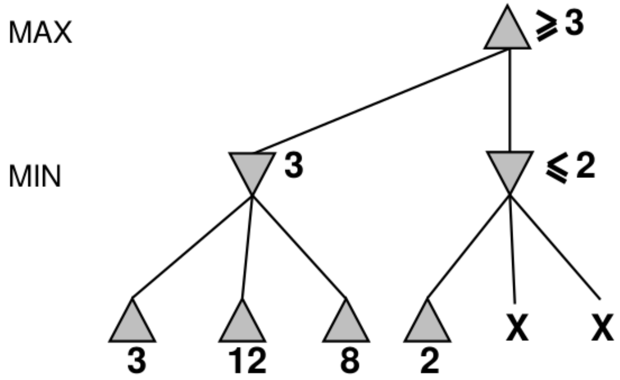

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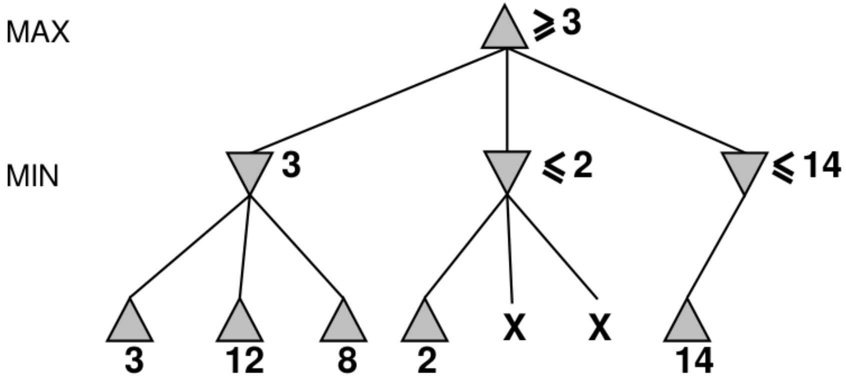

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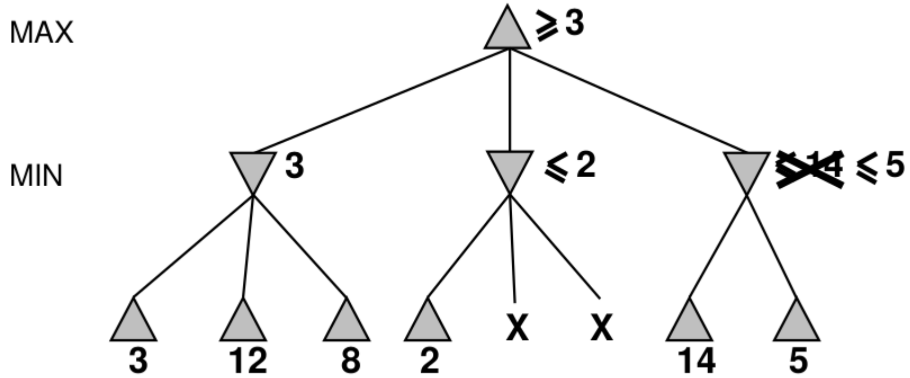

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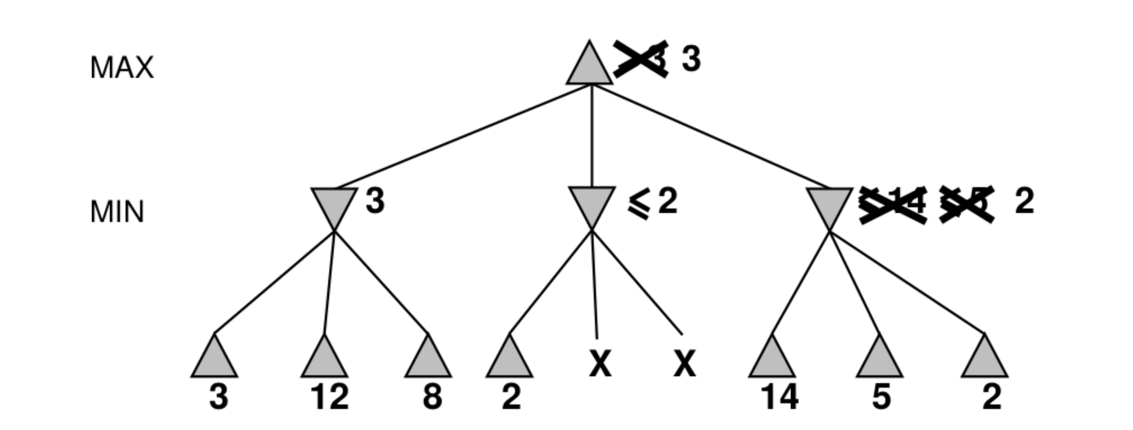

(© S. Russell & P. Norwig, AIMA)

Properties of Alpha-Beta Search

- Pruning does not affect the final result $\Longrightarrow$ correctness preserved
- Good move ordering improves effectiveness of pruning
  - Ex: if MIN expands 3^{rd} child of D first, the others are pruned
  - try to examine first the successors that are likely to be best
- With "perfect" ordering, time complexity reduces to $O(b^{m/2})$
  - aka "killer-move heuristic"

  $\Longrightarrow$ doubles solvable depth!
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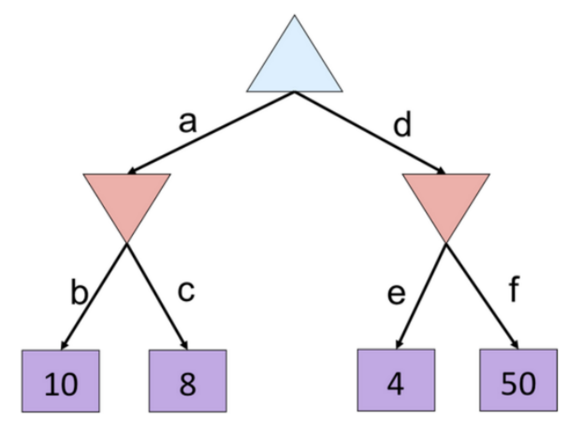
Exercise I

Apply alpha-beta search to the following tree



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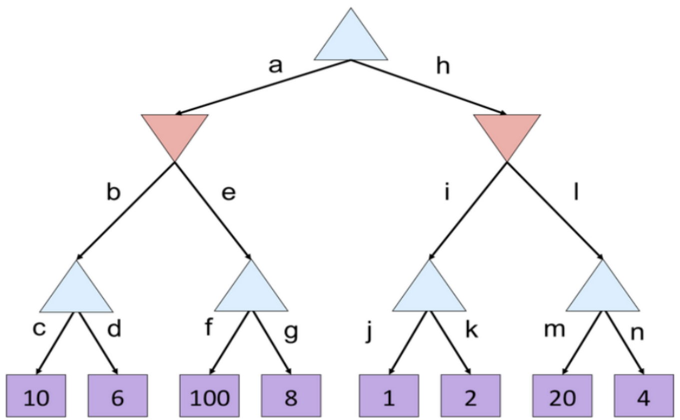
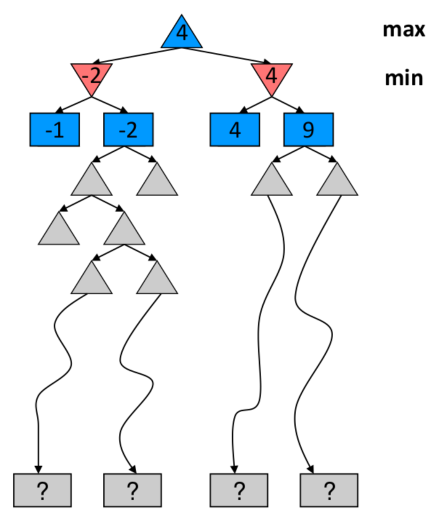


Outline

Adversarial Search with Resource Limits

Problem: In realistic games, full search is impractical!

- Complexity: b^d (ex. chess: $\approx 35^{100}$)
- Idea [Shannon, 1949]: Depth-limited search
  - cut off minimax search earlier, after limited depth
  - replace terminal utility function with evaluation for non-terminal nodes
- Ex (chess): depth $d = 8$ (decent) $\Longrightarrow$ α-β: $35^{8/2} = 10^5$ (feasible)



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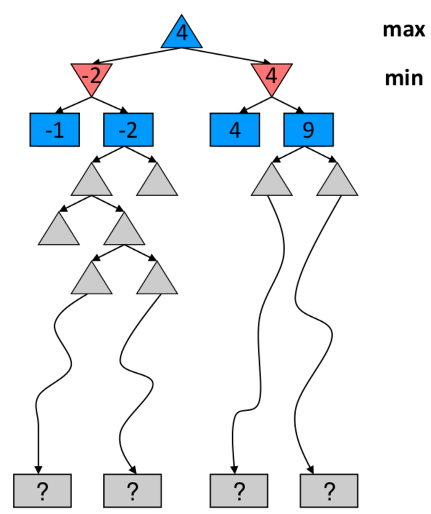

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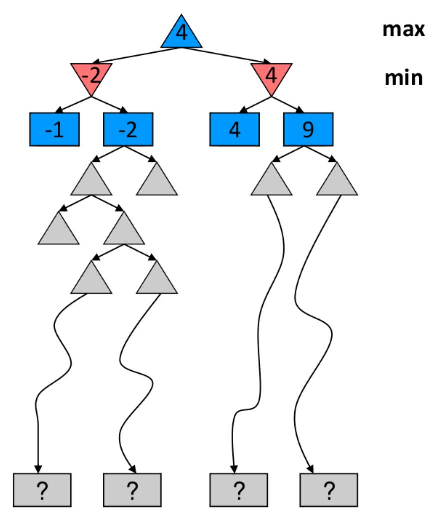

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$$H\text{-}Minimax(s,d) \stackrel{\text{def}}{=} \begin{cases} Eval(s) & \text{if } CutOffTest(s,d) \\ max_{a \in Actions(s)} H\text{-}Minimax(Result(s,a), d+1) & \text{if } Player(s) = MAX \\ min_{a \in Actions(s)} H\text{-}Minimax(Result(s,a), d+1) & \text{if } Player(s) = MIN \end{cases}$$

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Eval(*s*)

- Should be relatively cheap to compute
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  - Ideal function: returns the actual minimax value of the position
- Should order terminal states the same way as the utility function
  - e.g., wins $>$ draws $>$ losses
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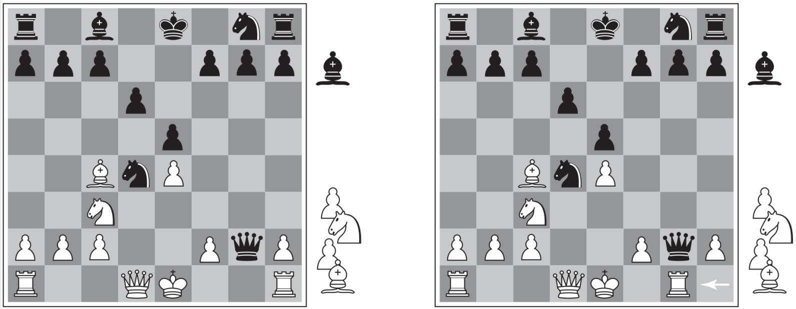
Example

- Two same-score positions (White: -8, Black: -3)
  - (a) Black has an advantage of a knight and two pawns,
    $\Longrightarrow$ should be enough to win the game
  - (b) White will capture the queen,
    $\Longrightarrow$ give it an advantage that should be strong enough to win

(Personal note: only very-stupid black player would get into (b))



(a) White to move (b) White to move

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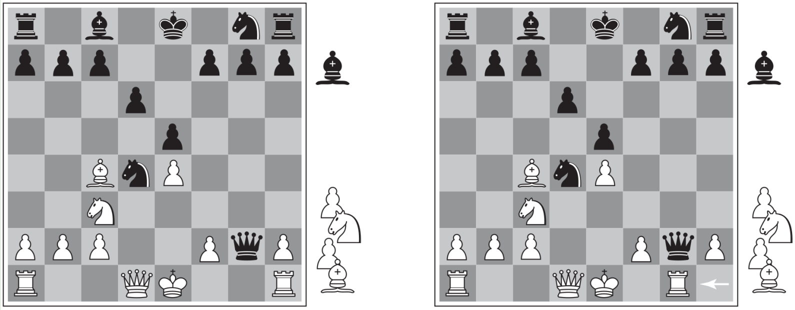

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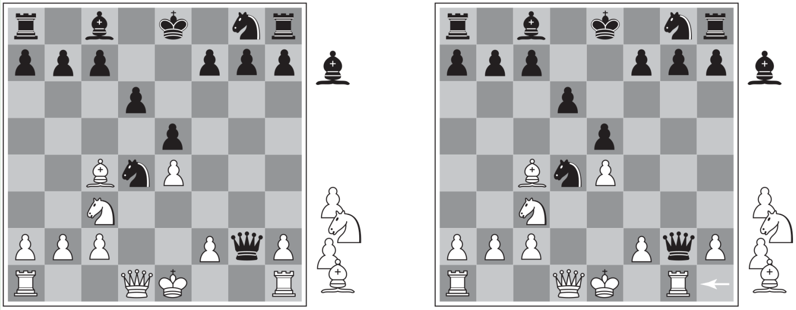

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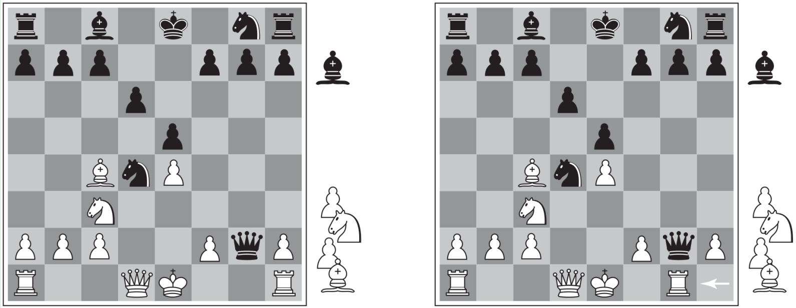

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Cutting-off the Search

CutOffTest(state, depth)

- Most straightforward approach: set a fixed depth limit
  - d chosen s.t. a move is selected within the allocated time
  - sometimes may produce very inaccurate outcomes (see previous example)
- More robust approach: apply Iterative Deepening
- More sophisticate: apply *Eval*() only to quiescent states
  - quiescent: unlikely to exhibit wild swings in value in the near future
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$\Longrightarrow$ further expand non-quiescent states until quiescence is reached

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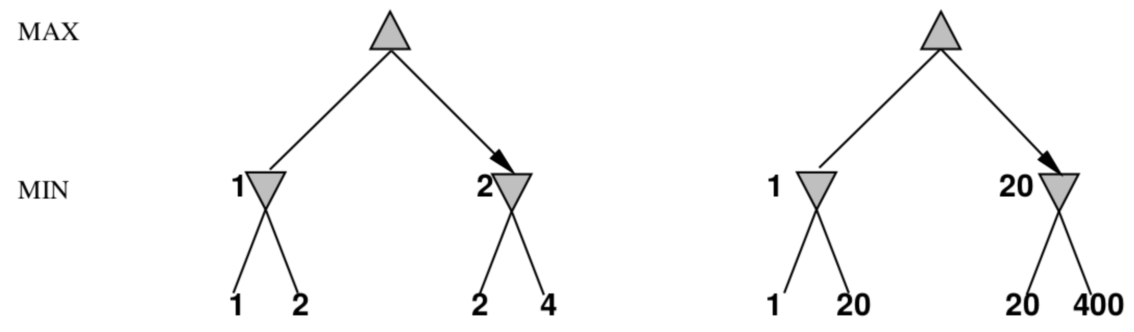
- Only the order matters!
- payoff in deterministic games acts as an ordinal utility function



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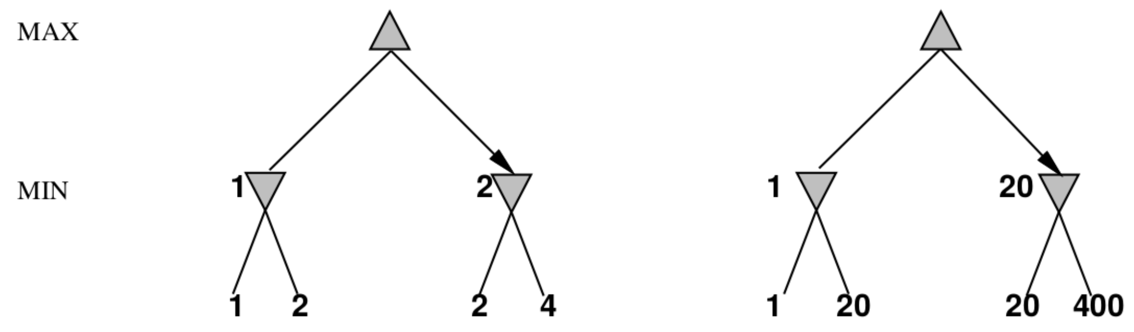

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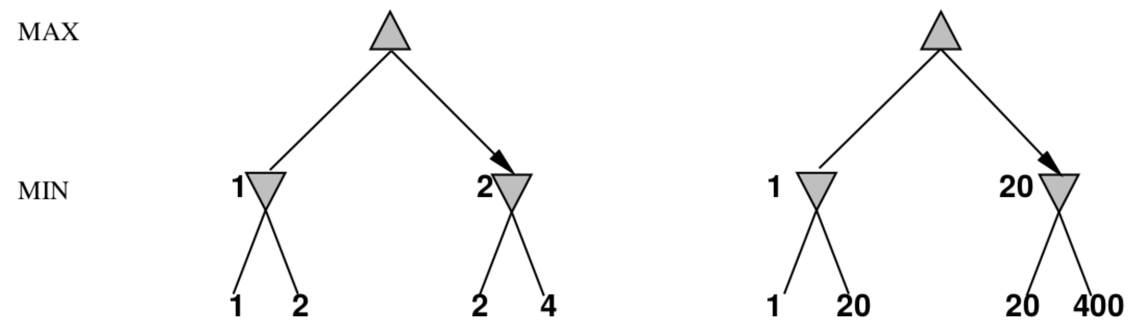

Deterministic Games in Practice

- Checkers: (1994) Chinook ended 40-year-reign of world champion Marion Tinsley
  - used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board
  - a total of 443,748,401,247 positions
- Chess: (1997) Deep Blue defeated world champion Gary Kasparov in a six-game match
  - searches 200 million positions per second
  - uses very sophisticated evaluation, and undisclosed methods
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AlphaGo beats GO world champion, Lee Sedol (2016)

Outline

Stochastic Games: Generalities

- In real life, unpredictable external events may occur
- Stochastic Games mirror unpredictability by random steps:
  - e.g. dice throwing, card-shuffling, coin flipping, tile extraction, ...
- Ex: Backgammon
- Cannot calculate definite minimax value, only expected values
- Uncertain outcomes controlled by chance, not an adversary!
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  - chance $\Longrightarrow$ average case
- Ex: if chance is 0.5 each (coin):
  - minimax: 10
  - average: (100+9)/2=54.5

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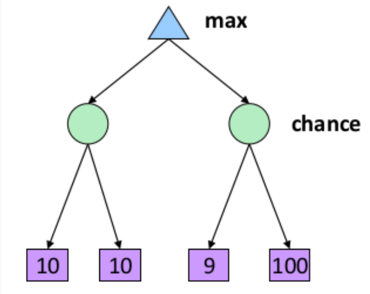

An Example: Backgammon

- Rules
  - 15 pieces each
  - white moves clockwise to 25, black moves counterclockwise to 0
  - a piece can move to a position unless ≥ 2 opponent pieces there
  - if there is one opponent, it is captured and must start over
  - termination: all whites in 25 or all blacks in 0
- Ex: Possible white moves (dice: 6,5):
  (5-10,5-11)
  (5-11,19-24)
  (5-10,10-16)
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- Combines strategy with luck
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  - double rolls (1-1),...,(6-6)
    have 1/36 probability each
  - other 15 distinct rolls
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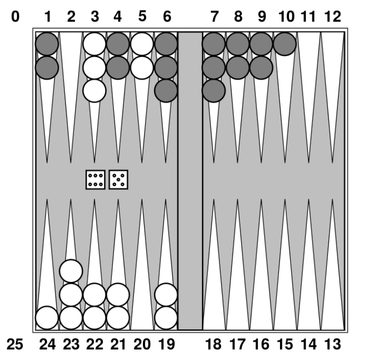

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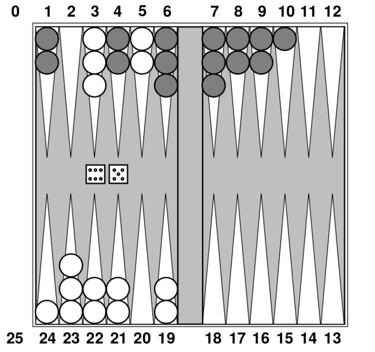

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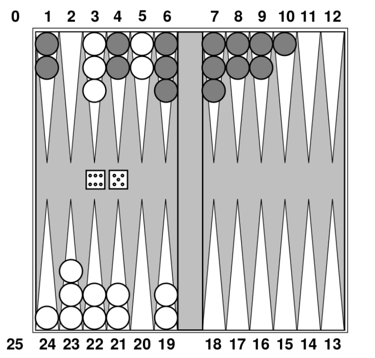
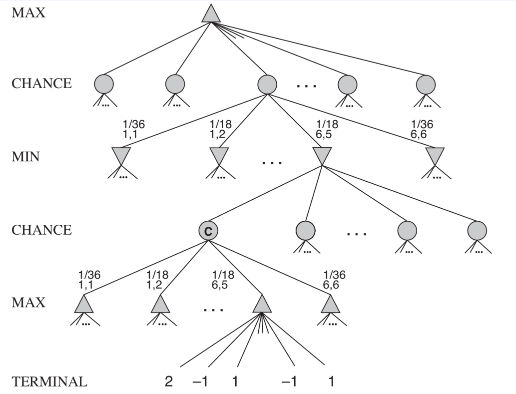


Stochastic Games Trees

- Idea: A game tree for a stochastic game includes chance nodes in addition to MAX and MIN nodes.
  - chance nodes above agent represent stochastic events for agent (e.g. dice roll)
  - outcoming arcs represent stochastic event outcomes
  - labeled with stochastic event and relative probability



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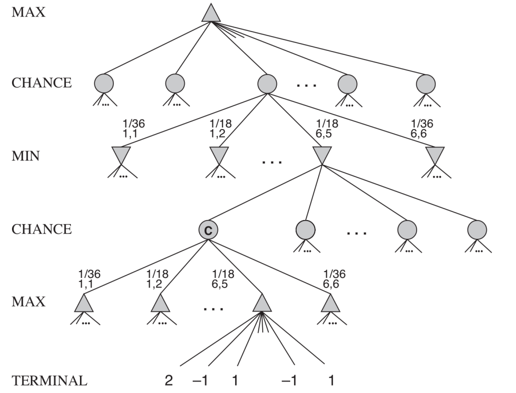

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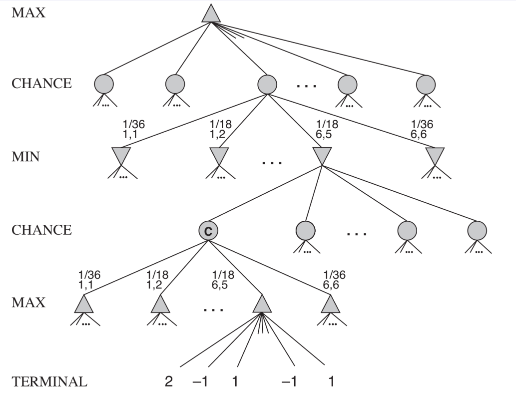

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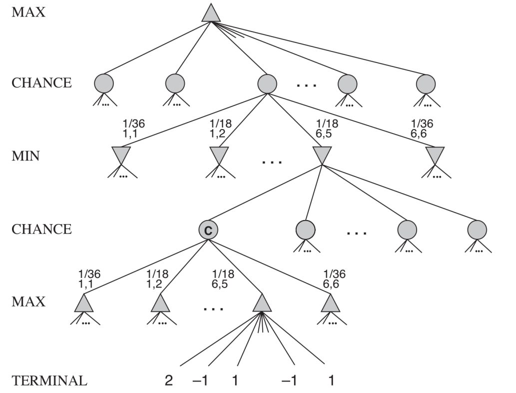

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Algorithm for Stochastic Games: *ExpectMinimax*()

- Extension of *Minimax*(), handling also chance nodes:

$$ExpectMinimax(s) \stackrel{\text{def}}{=} \begin{cases} Utility(s) & \text{if } TerminalTest(s) \\ max_{a \in Actions(s)} ExpectMinimax(Result(s,a)) & \text{if } Player(s) = MAX \\ min_{a \in Actions(s)} ExpectMinimax(Result(s,a)) & \text{if } Player(s) = MIN \\ \sum_r P(r) \cdot ExpectMinimax(Result(s,r)) & \text{if } Player(s) = Chance \end{cases}$$

  - $P(r)$: probability of stochastic event outcome r
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$\Longrightarrow$ Returns the weighted average of the minimax outcomes (recall that $\sum_r P(r) = 1$))

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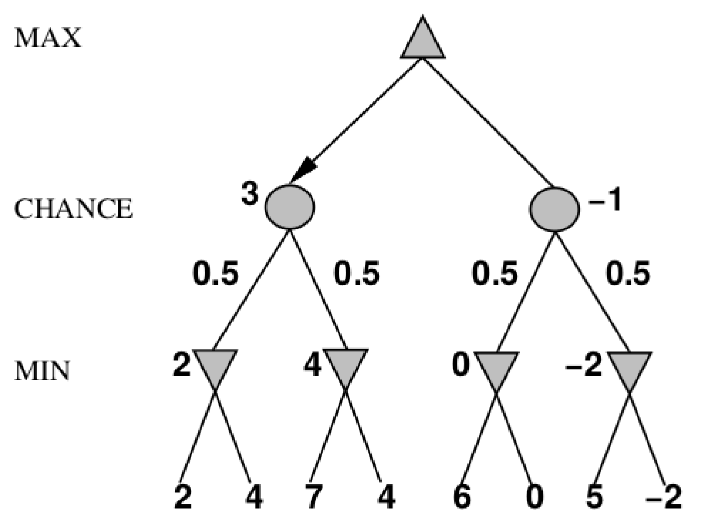
Simple Example with Coin-Flipping



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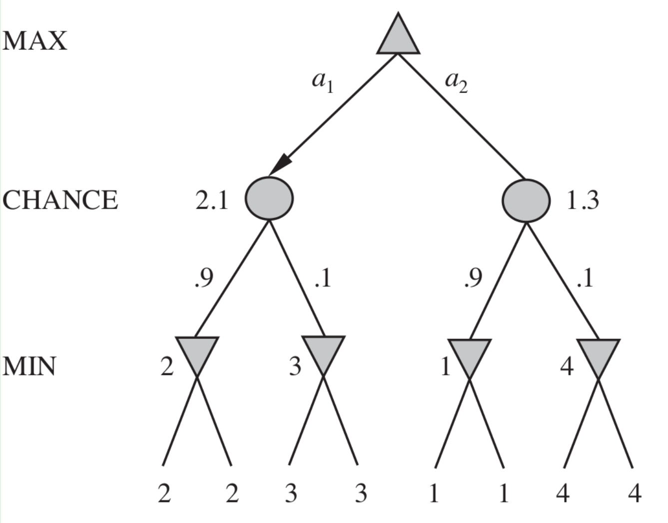
Example (Non-uniform Probabilities)



Remark (compare with deterministic case)

Exact values do matter!

Behaviour not preserved under monotonic transformations of *Utility*()

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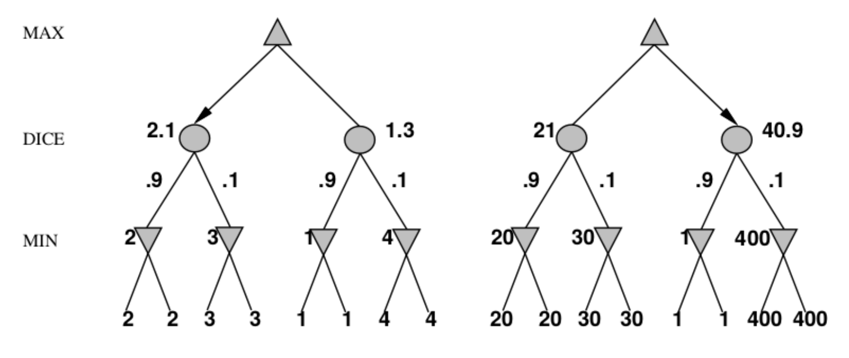
$\Longrightarrow$ *Utility*() should be proportional to the expected payoff



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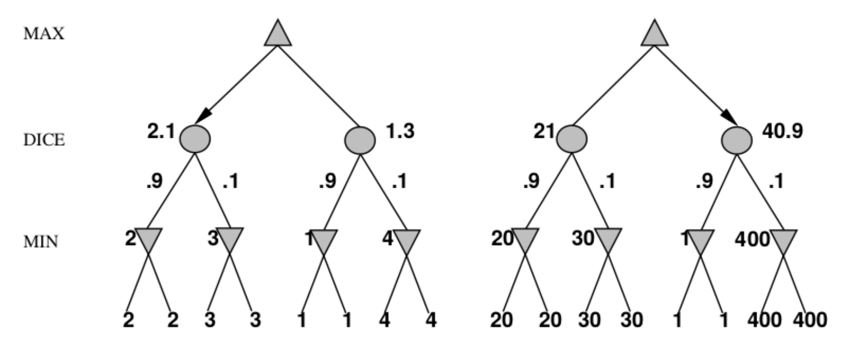

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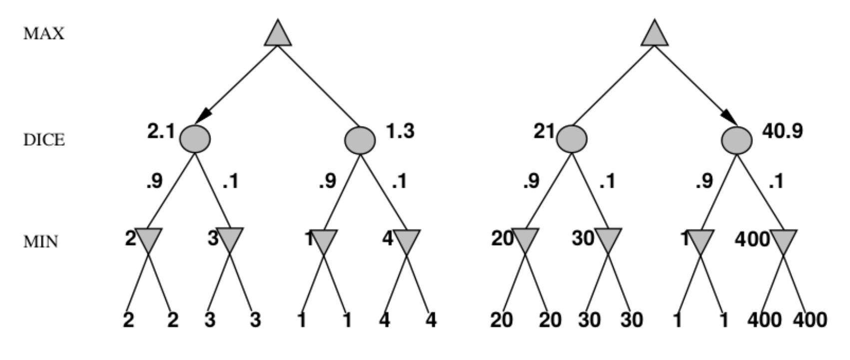

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Stochastic Games in Practice

- Dice rolls increase b: 21 possible rolls with 2 dice
  $\Longrightarrow$ $O(b^m \cdot n^m)$, n being the number of distinct roll
- Ex: Backgammon has ≈ 20 moves
  $\Longrightarrow$ depth 4: $20 \cdot (21 \times 20)^3 \approx 10^9$ (!)
- Alpha-beta pruning much less effective than with deterministic games

$\Longrightarrow$ Unrealistic to consider high depths in most stochastic games

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- Ex: Backgammon has ≈ 20 moves
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