

# Fundamentals of Artificial Intelligence

## Chapter 09: Inference in First-Order Logic

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# Outline

- 1 Propositional vs. First-Order Inference
- 2 Unification and Lifting
- 3 Forward & Backward Chaining for Definite FOL KBs
- 4 Resolution for General FOL KBs

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# A Brief History of Logical Reasoning

When	Who	What
322 B.C.	Aristotle	“Syllogisms” (inference rules), quantifiers
1867	Boole	PL
1879	Frege	FOL
1922	Wittgenstein	proof by truth tables
1930	Gödel	$\exists$ complete algorithm for FOL
1930	Herbrand	complete algorithm for FOL
1931	Gödel	$\neg\exists$ complete algorithm for arithmetic
1960	Davis/Putnam	“practical” algorithm for PL (DP/DPLL)
1965	Robinson	“practical” algorithm for FOL (resolution)

# Term/Subformula Substitutions

## Notation

- **Substitution:** “ $\text{Subst}(\{e_1/e_2\}, e)$ ” or “ $e\{e_1/e_2\}$ ”:  
the expression (term or formula) obtained by substituting every occurrence of  $e_1$  with  $e_2$  in  $e$ 
  - $e_1, e_2$  either both terms (**term substitution**)  
or both subformulas (**subformula substitution**)
  - $e$  is either a term or a formula (only term for term substitution)
- **Examples:**
  - (t. sub.):  $(y + 1 = 1 + y)\{y/S(x)\} \implies (S(x) + 1 = 1 + S(x))$
  - (s. sub.):  $(\text{Even}(x) \vee \text{Odd}(x))\{\text{Even}(x)/\text{Odd}(S(x))\} \implies ((\text{Odd}(S(x)) \vee \text{Odd}(x)))$
- **Multiple substitution:**  $e\{e_1/e_2, e_3/e_4\} \stackrel{\text{def}}{=} (e\{e_1/e_1\})\{e_3/e_4\}$ 
  - ex:  $(P(x, y) \rightarrow Q(x, y))\{x/1, y/2\} \implies (P(1, 2) \rightarrow Q(1, 2))$
- If  $\theta$  is a substitution list and  $e$  an expression (formula/term), then we denote the result of a substitution as  $e\theta$ 
  - $e\emptyset = e$
  - $e(\theta_1\theta_2) = (e\theta_1)\theta_2$ , denoted as  $e\theta_1\theta_2$

# Substitution with equivalent formulas/terms

- Equal-term substitution rule:

$$\frac{KB \wedge (t_1 = t_2) \wedge \alpha}{KB \wedge (t_1 = t_2) \wedge \alpha \wedge \alpha\{t_1/t_2\}}$$

- ex:  $(S(x) = x + 1) \wedge (0 \neq S(x))$  thus  
 $(S(x) = x + 1) \wedge (0 \neq S(x)) \wedge (0 \neq x + 1)$
- preserves validity:  
 $M(KB \wedge (t_1 = t_2) \wedge \alpha \wedge \alpha\{t_1/t_2\}) = M(KB \wedge (t_1 = t_2) \wedge \alpha)$
- $\alpha$  can be safely dropped from the result

- Equivalent-subformula substitution rule:

$$\frac{KB \wedge (\beta_1 \leftrightarrow \beta_2) \wedge \alpha}{KB \wedge (\beta_1 \leftrightarrow \beta_2) \wedge \alpha \wedge \alpha\{\beta_1/\beta_2\}}$$

- ex:  $(Even(x) \leftrightarrow Odd(S(x))) \wedge (Even(x) \vee Odd(x)) \implies$   
 $(Even(x) \leftrightarrow Odd(S(x))) \wedge (Even(x) \vee Odd(x)) \wedge (Odd(S(x)) \vee Odd(x))$
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# Universal Instantiation (UI)

- Every instantiation of a universally quantified-sentence is entailed by it:

$$\frac{KB \wedge \forall x.\alpha}{KB \wedge \forall x.\alpha \wedge \alpha\{x/t\}}$$

for every variable  $x$  and term  $t$

- Ex:  $\forall(x.(King(x) \wedge Greedy(x)) \rightarrow Evil(x))$ 
  - $(King(John) \wedge Greedy(John)) \rightarrow Evil(John)$
  - $(King(Richard) \wedge Greedy(Richard)) \rightarrow Evil(Richard)$
  - $(King(Father(John)) \wedge Greedy(Father(John))) \rightarrow Evil(Father(John))$
  - $(King(Father(Father(John))) \wedge Greedy(Father(Father(John)))) \rightarrow Evil(Father(Father(John)))$
  - ...
- Preserves validity:  
 $M(KB \wedge \forall x.\alpha \wedge \alpha\{x/t\}) = M(KB \wedge \forall x.\alpha)$

# Existential Instantiation (EI)

- An existentially quantified-sentence can be substituted by one of its instantiation with a fresh constant:

$$\frac{KB \wedge \exists x.\alpha}{KB \wedge \alpha\{x/c\}}$$

for every variable  $x$  and for a “fresh” constant  $c$ , i.e. a constant which does not appear in  $KB \wedge \exists x.\alpha$

- $c$  is called **Skolem constant**, EI subcase of **Skolemization**
- Intuition: if there is an object satisfying some condition, then we give a (new) name to such object
- Ex:  $\exists x.(Crown(x) \wedge OnHead(x, John))$ 
  - $(Crown(c) \wedge OnHead(c, John))$
  - given “There is a crown on John’s head”, I call “ $c$ ” such crown
- **Preserves satisfiability** (aka preserves **inferential equivalence**)  
 $M(KB \wedge \alpha\{x/t\}) \neq \emptyset$  iff  $M(KB \wedge \exists x.\alpha) \neq \emptyset$   
(i.e..  $(KB \wedge \alpha\{x/t\}) \models \beta$  iff  $(KB \wedge \exists x.\alpha) \models \beta$ , for every  $\beta$ )
- Ex from math:  $\exists x.(\frac{d(x^y)}{dy} = x^y)$ , we call it “ $e$ ”  $\implies (\frac{d(e^y)}{dy} = e^y)$

# Remarks

- **About Universal Instantiation:**

- UI can be applied several times to add new sentences;
- the new KB is **logically equivalent** to the old KB

- **About Existential Instantiation:**

- EI can be applied once to replace the existential sentence;
- the new KB is **not equivalent** to the old,
- but is **(un)satisfiable iff the old KB is (un)satisfiable**

⇒ the new KB can infer  $\beta$  iff the old KB can infer  $\beta$

Before applying UI or EI, sentences must be rewritten s.t. negations must be pushed inside the quantifications:

- $\neg\forall x.\alpha \implies \exists x.\neg\alpha$

- $\neg\exists x.\alpha \implies \forall x.\neg\alpha$

# Reduction to Propositional Inference

- Idea: **Convert  $(KB \wedge \neg\alpha)$  to PL** (aka **propositionalization**)  
 $\implies$  use a PL SAT solver to check PL (un)satisfiability
- Trick:
  - replace variables with ground terms, creating all possible instantiations of quantified sentences
  - convert atomic sentences into propositional symbols  
e.g. “**King(John)**”  $\implies$  “**King\_John**”,  
e.g. “**Brother(John,Richard)**”  $\implies$  “**Brother\_John-Richard**”,
- Theorem: (Herbrand, 1930)  
**If a ground sentence  $\alpha$  is entailed by an FOL KB, then it is entailed by a finite subset of the propositional KB**
  - $\implies$  A ground sentence is entailed by the propositionalized KB if it is entailed by original KB
  - $\implies$  Every FOL KB can be propositionalized s.t. to preserve entailment
- The vice-versa does not hold  
 $\implies$  works if  $\alpha$  is entailed, loops if  $\alpha$  is not entailed

## Reduction to Propositional Inference: Example

- Suppose the KB contains only:

$\forall x.((King(x) \wedge Greedy(x)) \rightarrow Evil(x))$

$King(John)$

$Greedy(John)$

$Brother(Richard, John)$

- Instantiating the universal sentence in all possible ways:

$(King(John) \wedge Greedy(John)) \rightarrow Evil(John)$

$(King(Richard) \wedge Greedy(Richard)) \rightarrow Evil(Richard)$

$King(John)$

$Greedy(John)$

$Brother(Richard, John)$

- The new KB is propositionalized:

$(King\_John \wedge Greedy\_John) \rightarrow Evil\_John$

$(King\_Richard \wedge Greedy\_Richard) \rightarrow Evil\_Richard$

$King\_John$

$Greedy\_John$

$Brother\_Richard-John$

- $Evil\_John$  entailed by new KB ( $Evil(John)$  entailed by old KB)

# Problems with Propositionalization

- Propositionalization generates lots of irrelevant sentences  
produces irrelevant atoms like Greedy(Richard)

$\forall x. ((King(x) \wedge Greedy(x)) \rightarrow Evil(x))$

$King(John)$

$\forall y. Greedy(y)$

$Brother(Richard, John)$

$\Rightarrow$  produces irrelevant atoms like Greedy(Richard)

- With  $p$   $k$ -ary predicates and  $n$  constants,  $p \cdot n^k$  instantiations
- What happens with function symbols?

## Problems with Propositionalization [cont.]

- Problem: nested function applications
  - e.g.  $\text{Father}(\text{John})$ ,  $\text{Father}(\text{Father}(\text{John}))$ ,  
 $\text{Father}(\text{Father}(\text{Father}(\text{John})))$ , ...
  - ⇒ infinite instantiations
- Actual Trick: for  $k = 0$  to  $\infty$ , use terms of function nesting depth  $k$ 
  - create propositionalized KB by instantiating depth- $k$  terms
  - if  $KB \models \alpha$ , then will find a contradiction for some finite  $k$
  - if  $KB \not\models \alpha$ , may find a loop forever
- Theorem: (Turing, 1936), (Church, 1936):  
Entailment in FOL is semidecidable



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# Generalized Modus Ponens (GMP)

- “Lifted inference”: Combine PL inference with UI/EI
- Aristotle’s “Modus Ponens” syllogism:  
“All men are mortal; Socrates is a man; thus Socrates is mortal.”

$$\frac{Man(Socrates) \quad \forall x.(Man(x) \rightarrow Mortal(x))}{Mortal(Socrates)}$$

- Generalized Modus Ponens:  
if exists a substitution  $\theta$  s.t., for all  $i \in 1..k$ ,  $\alpha'_i\theta = \alpha_i\theta$ , then

$$\frac{\alpha'_1, \alpha'_2, \dots, \alpha'_k, (\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k) \rightarrow \beta}{\beta\theta}$$

- all variables (implicitly) assumed as universally quantified
- $\theta$  substitutes (universally quantified) variables with terms
- Ex: using  $\theta \stackrel{\text{def}}{=} \{x/John, y/John\}$  we can infer  $Evil(John)$  from:  
 $\forall x.((King(x) \wedge Greedy(x)) \rightarrow Evil(x)), King(John), \forall y.Greedy(y)$
- GMP used w. KB of definite clauses (exactly one positive literal)
  - Used in Prolog, Datalog, Production-rule systems,...

# Unification

- Unification: Given  $\langle \alpha'_1, \alpha'_2, \dots, \alpha'_k \rangle$  and  $\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ , find a substitution  $\theta$  s.t.  $\alpha'_i\theta = \alpha_i\theta$ , for all  $i \in 1..k$ 
  - $\theta$  is called a **unifier** for  $\langle \alpha'_1, \alpha'_2, \dots, \alpha'_k \rangle$  and  $\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$
  - $\text{Unify}(\alpha, \beta) = \theta$  iff  $\alpha\theta = \beta\theta$

- Ex:

$\text{Unify}(\text{Knows}(\text{John}, x), \text{Knows}(\text{John}, \text{Jane})) = \{x/\text{Jane}\}$

$\text{Unify}(\text{Knows}(\text{John}, x), \text{Knows}(y, \text{OJ})) = \{x/\text{OJ}, y/\text{John}\}$

$\text{Unify}(\text{Knows}(\text{John}, x), \text{Knows}(y, \text{Mother}(y))) =$   
 $\{y/\text{John}, x/\text{Mother}(\text{John})\}$

$\text{Unify}(\text{Knows}(\text{John}, x), \text{Knows}(x, \text{OJ})) = \text{FAIL} : x/?$

- different (implicitly-universally-quantified) formulas should use different variables

⇒ (standardizing apart): rename variables to avoid name clashes

$\text{Unify}(\text{Knows}(\text{John}, x_1), \text{Knows}(x_2, \text{OJ})) = \{x_1/\text{OBJ}, x_2/\text{John}\}$

- $\{\forall x.\alpha, \forall x.\beta\} \iff \{\forall x_1.\alpha\{x/x_1\}, \forall x_2.\beta\{x/x_2\}\}$ , s.t.  $x_1, x_2$  new

# Most-General Unifier (MGU)

- Unifiers are not unique
  - ex:  $\text{Unify}(\text{Knows}(\text{John}, x), \text{Knows}(y, z))$   
could return  $\{y/\text{John}, x/z\}$  or  $\{y/\text{John}, x/\text{John}, z/\text{John}\}$
- Given  $\alpha, \beta$ , the unifier  $\theta_1$  is **more general** than the unifier  $\theta_2$  for  $\alpha, \beta$  if exists  $\theta_3$  s.t.  $\theta_2 = \theta_1\theta_3$ 
  - ex:  $\{y/\text{John}, x/z\}$  more general than  $\{y/\text{John}, x/\text{John}, z/\text{John}\}$ :  
 $\{y/\text{John}, x/\text{John}, z/\text{John}\} = \{y/\text{John}, x/z\}\{z/\text{John}\}$
- Theorem: **If exists an unifier for  $\alpha, \beta$ , then exists a most general unifier (MGU)  $\theta$  for  $\alpha, \beta$** 
  - Ex:  $\{y/\text{John}, x/z\}$  MGU for  $\text{Knows}(\text{John}, x), \text{Knows}(y, z)$
  - Ex: an MGU is unique modulo variable renaming
- **UNIFY()** returns the MGU between two (lists of) formulas
  - efficiency optimizations based on predicate/term indexing techniques (see AIMA if interested)

# The Procedure Unify

**function** UNIFY( $x, y, \theta$ ) **returns** a substitution to make  $x$  and  $y$  identical

**inputs:**  $x$ , a variable, constant, list, or compound expression

$y$ , a variable, constant, list, or compound expression

$\theta$ , the substitution built up so far (optional, defaults to empty)

**if**  $\theta = \text{failure}$  **then return** failure

**else if**  $x = y$  **then return**  $\theta$

**else if** VARIABLE?( $x$ ) **then return** UNIFY-VAR( $x, y, \theta$ )

**else if** VARIABLE?( $y$ ) **then return** UNIFY-VAR( $y, x, \theta$ )

**else if** COMPOUND?( $x$ ) **and** COMPOUND?( $y$ ) **then**

**return** UNIFY( $x$ .ARGS,  $y$ .ARGS, UNIFY( $x$ .OP,  $y$ .OP,  $\theta$ ))

**else if** LIST?( $x$ ) **and** LIST?( $y$ ) **then**

**return** UNIFY( $x$ .REST,  $y$ .REST, UNIFY( $x$ .FIRST,  $y$ .FIRST,  $\theta$ ))

**else return** failure

---

**function** UNIFY-VAR( $var, x, \theta$ ) **returns** a substitution

**if**  $\{var/val\} \in \theta$  **then return** UNIFY( $val, x, \theta$ )

**else if**  $\{x/val\} \in \theta$  **then return** UNIFY( $var, val, \theta$ )

**else if** OCCUR-CHECK?( $var, x$ ) **then return** failure

**else return** add  $\{var/x\}$  to  $\theta$

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# First-Order Definite Clauses

- FOL **Definite Clauses**: clauses with exactly one positive literal
  - we omit universal quantifiers
    - ⇒ variables are (implicitly) universally quantified
  - we remove existential quantifiers by EI
  - ⇒ existentially-quantified variables are substituted by fresh constants (we assume no  $\exists$  under the scope of  $\forall$ , see later for general case)
- Represent **implications of atomic formulas**
  - Ex:  $\forall x.((King(x) \wedge Greedy(x)) \rightarrow Evil(x))$
  - ⇒  $(\neg King(x) \vee \neg Greedy(x) \vee Evil(x))$
- Important subcase: **Datalog KBs**:  
**sets of FOL definite clauses without function symbols**
  - can represent statements typically made in relational databases
  - makes inference much easier

## Example (Datalog)

KB: The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Goal: Prove that Col. West is a criminal.



## Example (Datalog) [cont.]

- it is a crime for an American to sell weapons to hostile nations:

$$\forall x, y, z. ((\text{American}(x) \wedge \text{Weapon}(y) \wedge \text{Hostile}(z) \wedge \text{Sells}(x, y, z)) \rightarrow \text{Criminal}(x))$$
$$\implies \neg \text{American}(x) \vee \neg \text{Weapon}(y) \vee \neg \text{Hostile}(z) \vee \neg \text{Sells}(x, y, z) \vee \text{Criminal}(x)$$

- Nono ... has some missiles

$$\exists x. (\text{Owns}(\text{Nono}, x) \wedge \text{Missile}(x)) \implies \text{Owns}(\text{Nono}, M_1) \wedge \text{Missile}(M_1)$$

- All of its missiles were sold to it by Colonel West

$$\forall x. ((\text{Missile}(x) \wedge \text{Owns}(\text{Nono}, x)) \rightarrow \text{Sells}(\text{West}, x, \text{Nono}))$$
$$\implies \neg \text{Missile}(x) \vee \neg \text{Owns}(\text{Nono}, x) \vee \text{Sells}(\text{West}, x, \text{Nono})$$

- Missiles are weapons:

$$\forall x. (\text{Missile}(x) \rightarrow \text{Weapon}(x)) \implies \neg \text{Missile}(x) \vee \text{Weapon}(x)$$

- An enemy of America counts as “hostile”:

$$\forall x. (\text{Enemy}(x, \text{America}) \rightarrow \text{Hostile}(x))$$
$$\implies \neg \text{Enemy}(x, \text{America}) \vee \text{Hostile}(x)$$

- West, who is American ...:  $\text{American}(\text{West})$

- The country Nono, an enemy of America ...:

$$\text{Enemy}(\text{Nono}, \text{America})$$

# A (Very-Basic) Forward-Chaining Procedure

```
function FOL-FC-ASK( $KB, \alpha$ ) returns a substitution or false  
  inputs:  $KB$ , the knowledge base, a set of first-order definite clauses  
            $\alpha$ , the query, an atomic sentence  
  local variables:  $new$ , the new sentences inferred on each iteration  
  
  repeat until  $new$  is empty  
     $new \leftarrow \{ \}$   
    for each  $rule$  in  $KB$  do  
       $(p_1 \wedge \dots \wedge p_n \Rightarrow q) \leftarrow \text{STANDARDIZE-VARIABLES}(rule)$   
      for each  $\theta$  such that  $\text{SUBST}(\theta, p_1 \wedge \dots \wedge p_n) = \text{SUBST}(\theta, p'_1 \wedge \dots \wedge p'_n)$   
        for some  $p'_1, \dots, p'_n$  in  $KB$   
           $q' \leftarrow \text{SUBST}(\theta, q)$   
          if  $q'$  does not unify with some sentence already in  $KB$  or  $new$  then  
            add  $q'$  to  $new$   
             $\phi \leftarrow \text{UNIFY}(q', \alpha)$   
            if  $\phi$  is not fail then return  $\phi$   
  add  $new$  to  $KB$   
  return false
```

# Example of Forward Chaining

*American(West), Missile(M<sub>1</sub>), Owns(Nono, M<sub>1</sub>), Enemy(Nono, America)*

$\forall x. (\text{Missile}(x) \rightarrow \text{Weapon}(x))$

$\forall x. ((\text{Missile}(x) \wedge \text{Owns}(\text{Nono}, x)) \rightarrow \text{Sells}(\text{West}, x, \text{Nono}))$

$\forall x. (\text{Enemy}(x, \text{America}) \rightarrow \text{Hostile}(x))$

$\forall x, y, z. ((\text{American}(x) \wedge \text{Weapon}(y) \wedge \text{Hostile}(z) \wedge \text{Sells}(x, y, z)) \rightarrow \text{Criminal}(x))$

American(West)

Missile(M<sub>1</sub>)

Owns(Nono, M<sub>1</sub>)

Enemy(Nono, America)

# Example of Forward Chaining

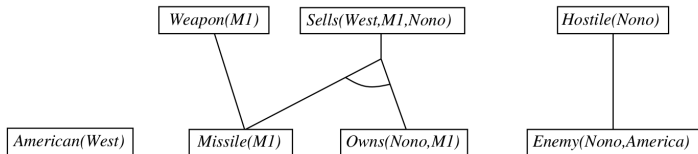
$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$

$\forall x.(Missile(x) \rightarrow Weapon(x))$

$\forall x.((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x.(Enemy(x, America) \rightarrow Hostile(x))$

$\forall x, y, z.((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow Criminal(x))$



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# Example of Forward Chaining

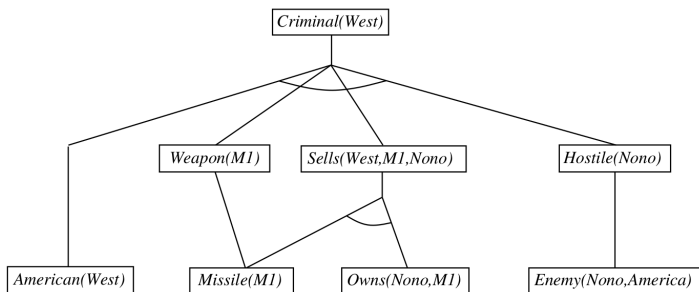
$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$

$\forall x.(Missile(x) \rightarrow Weapon(x))$

$\forall x.((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x.(Enemy(x, America) \rightarrow Hostile(x))$

$\forall x, y, z.((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow Criminal(x))$



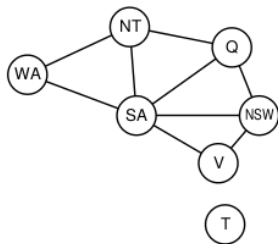
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# Properties of Forward Chaining

- **Sound**: every inference is just an application of GMP
- **Complete** (for definite KBs): answers every query entailed by KB
- if  $KB \models \alpha$ , it always terminates
- if  $KB \not\models \alpha$ , may not terminate (**Semi-decidable**)
- Solves always Datalog queries in time:  $O(p \cdot n^k)$ , s.t.  
*p = #predicates, n = #number constants, k = maximum arity*
- Improvement: no need to match a rule on iteration k if a premise wasn't added on iteration k-1  
⇒ match each rule whose premise contains a newly added literal
- Matching can be expensive
  - matching conjunctive premises against known facts is NP-hard
- **Forward chaining is widely used in deductive databases and expert systems**

# Hard Matching Example

- **Colorable()** is inferred iff the CSP has solution  
⇒ NP-Hard



$$\begin{aligned} & Diff(wa, nt) \wedge Diff(wa, sa) \wedge \\ & Diff(nt, q) Diff(nt, sa) \wedge \\ & Diff(q, nsw) \wedge Diff(q, sa) \wedge \\ & Diff(nsw, v) \wedge Diff(nsw, sa) \wedge \\ & Diff(v, sa) \Rightarrow Colorable() \end{aligned}$$

$$\begin{aligned} & Diff(Red, Blue) \quad Diff(Red, Green) \\ & Diff(Green, Red) \quad Diff(Green, Blue) \\ & Diff(Blue, Red) \quad Diff(Blue, Green) \end{aligned}$$

# A (Very-Basic) Backward-Chaining Procedure

```
function FOL-BC-ASK(KB, goals,  $\theta$ ) returns a set of substitutions
inputs: KB, a knowledge base
           goals, a list of conjuncts forming a query ( $\theta$  already applied)
            $\theta$ , the current substitution, initially the empty substitution  $\{ \}$ 
local variables: answers, a set of substitutions, initially empty

if goals is empty then return  $\{ \theta \}$ 
 $q' \leftarrow \text{SUBST}(\theta, \text{FIRST}(\text{goals}))$ 
for each sentence r in KB
    where  $\text{STANDARDIZE-APART}(r) = (p_1 \wedge \dots \wedge p_n \Rightarrow q)$ 
    and  $\theta' \leftarrow \text{UNIFY}(q, q')$  succeeds
     $\text{new\_goals} \leftarrow [p_1, \dots, p_n | \text{REST}(\text{goals})]$ 
     $\text{answers} \leftarrow \text{FOL-BC-ASK}(\text{KB}, \text{new\_goals}, \text{COMPOSE}(\theta', \theta)) \cup \text{answers}$ 
return answers
```



# Backward Chaining: Example

$American(West), Missile(M_1), Owns(Nono, M_1), Enemy(Nono, America)$   
 $\forall x, y, z. ((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow$   
 $Criminal(x))$

$\forall x. (Missile(x) \rightarrow Weapon(x))$

$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$

$Criminal(West)$

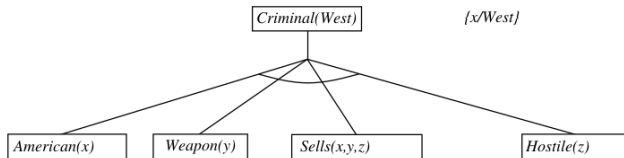
# Backward Chaining: Example

$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$   
 $\forall x, y, z. ((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow$   
 $Criminal(x))$

$\forall x. (Missile(x) \rightarrow Weapon(x))$

$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$



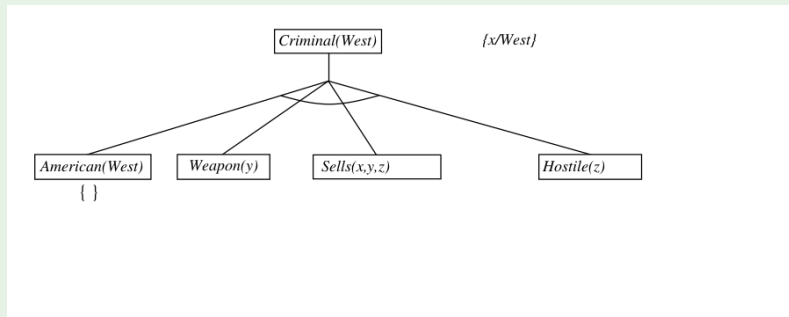
# Backward Chaining: Example

*American(West)*, *Missile(M<sub>1</sub>)*, *Owns(Nono, M<sub>1</sub>)*, *Enemy(Nono, America)*  
 $\forall x, y, z. ((\textit{American}(x) \wedge \textit{Weapon}(y) \wedge \textit{Hostile}(z) \wedge \textit{Sells}(x, y, z)) \rightarrow \textit{Criminal}(x))$

$\forall x. (\textit{Missile}(x) \rightarrow \textit{Weapon}(x))$

$\forall x. ((\textit{Missile}(x) \wedge \textit{Owns}(\textit{Nono}, x)) \rightarrow \textit{Sells}(\textit{West}, x, \textit{Nono}))$

$\forall x. (\textit{Enemy}(x, \textit{America}) \rightarrow \textit{Hostile}(x))$



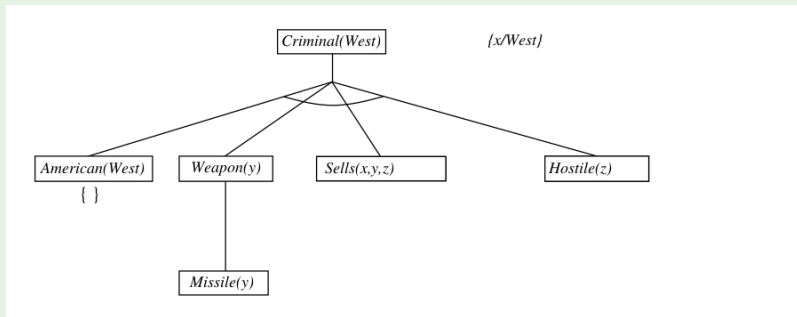
# Backward Chaining: Example

$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$   
 $\forall x, y, z. ((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow$   
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$\forall x. (Missile(x) \rightarrow Weapon(x))$

$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$



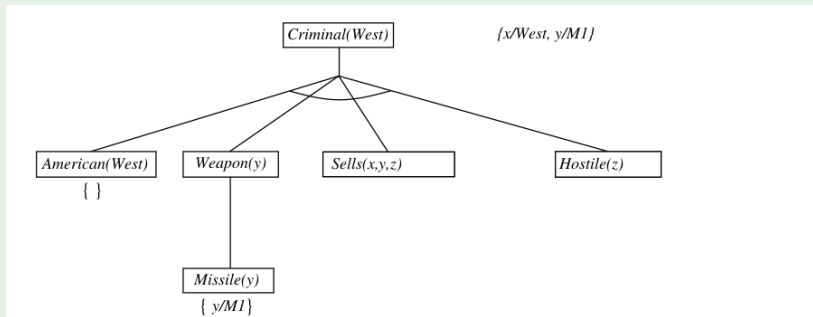
# Backward Chaining: Example

$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$   
 $\forall x, y, z. ((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow$   
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$\forall x. (Missile(x) \rightarrow Weapon(x))$

$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$



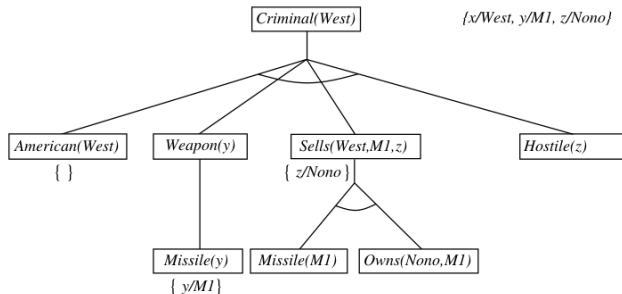
# Backward Chaining: Example

$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$   
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$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$



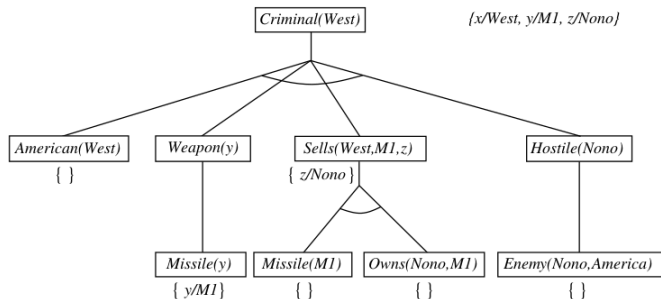
# Backward Chaining: Example

$American(West)$ ,  $Missile(M_1)$ ,  $Owns(Nono, M_1)$ ,  $Enemy(Nono, America)$   
 $\forall x, y, z. ((American(x) \wedge Weapon(y) \wedge Hostile(z) \wedge Sells(x, y, z)) \rightarrow Criminal(x))$

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$\forall x. ((Missile(x) \wedge Owns(Nono, x)) \rightarrow Sells(West, x, Nono))$

$\forall x. (Enemy(x, America) \rightarrow Hostile(x))$



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# Properties of Backward Chaining

- Depth-first recursive proof search: space is linear in size of proof
- **Incomplete** due to infinite loops
  - e.g.,  $P(x) \rightarrow P(x) \implies P(c), P(c), P(c)\dots$  (easy to fix)
  - e.g.,  $Q(f(x)) \rightarrow Q(x) \implies Q(c), Q(f(c)), Q(f(f(c))))), \dots$
- **Inefficient** due to repeated subgoals
  - fix using caching of previous results  $\implies$  need extra space!
- Widely used for **logic programming** (e.g. **prolog**)



# Outline

- 1 Propositional vs. First-Order Inference
- 2 Unification and Lifting
- 3 Forward & Backward Chaining for Definite FOL KBs
- 4 Resolution for General FOL KBs**

# Conjunctive Normal Form (CNF)

- A FOL formula  $\varphi$  is in **Conjunctive normal form** iff it is a conjunction of disjunctions of quantifier-free literal:

$$\bigwedge_{i=1}^L \bigvee_{j=1}^{K_i} l_{ji}$$

- the disjunctions of literals  $\bigvee_{j=1}^{K_i} l_{ji}$  are called **clauses**
- every literal a quantifier-free atom or its negation
- free variables implicitly universally quantified
- Easier to handle: list of lists of literals.  
 $\implies$  no reasoning on the recursive structure of the formula
- Ex:  $\neg \text{Missile}(x) \vee \neg \text{Owns}(\text{Nono}, x) \vee \text{Sells}(\text{West}, x, \text{Nono})$

# FOL CNF Conversion $CNF(\varphi)$

Every FOL formula  $\varphi$  can be reduced into CNF:

1 Eliminate implications and biconditionals:

$$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

2 Push inwards negations recursively:

$$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$$

$$\neg\neg\alpha \implies \alpha$$

$$\neg\forall x.\alpha \implies \exists x.\neg\alpha$$

$$\neg\exists x.\alpha \implies \forall x.\neg\alpha$$

$\implies$  Negation normal form: negations only in front of atomic formulae

$\implies$  quantifiers occur under the scope of no negation symbol

3 Standardize variables: each quantifier should use a different var

$$(\forall x.\exists y.\alpha) \wedge \exists y.\beta \wedge \forall x.\gamma \implies (\forall x_1.\exists y_1.\alpha) \wedge \exists y_1.\beta \wedge \forall x_1.\gamma \{y/y_1\} \wedge \forall x_1.\gamma \{x/x_1\}$$

## 4 Skolemize (a generalization of EI):

Each existential variable is replaced by a fresh **Skolem function** applied to the enclosing universally-quantified variables

$$\begin{aligned} \exists y. \alpha &\implies \alpha\{y/c\} \\ \forall x. (\dots \exists y. \alpha \dots) &\implies \forall x. (\dots \alpha\{y/F_1(x)\} \dots) \\ \forall x_1 x_2. (\dots \exists y. \alpha \dots) &\implies \forall x_1 x_2. (\dots \alpha\{y/F_1(x_1, x_2)\} \dots) \\ \exists y_1 \forall x_1 x_2 \exists y_2 \forall x_3 \exists y_3. \alpha &\implies \forall x_1 x_2 x_3. \\ &\quad \alpha\{y_1/c, y_2/F_1(x_1, x_2), y_3/F_2(x_1, x_2, x_3)\} \end{aligned}$$

Ex:  $\forall x \exists y. \text{Father}(x, y) \implies \forall x. \text{Father}(x, s(x))$

( $s(x)$  implicitly means "son of  $x$ " although  $s()$  is a fresh function)

## FOL CNF Conversion $CNF(\varphi)$ [cont.]

5 Drop universal quantifiers:

$$\forall x_1 \dots x_k. \alpha \implies \alpha$$

$\implies$  free variables implicitly universally quantified

6 CNF-ize propositionally (see Ch.08):

either apply recursively the DeMorgan's Rule:

$$(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$$

or rename subformulas and add definitions:

$$(\alpha \wedge \beta) \vee \gamma \implies (B \vee \gamma) \wedge CNF(B \leftrightarrow (\alpha \wedge \beta))$$

Preserves satisfiability:  $M(\varphi) \neq \emptyset$  iff  $M(CNF(\varphi)) \neq \emptyset$

## Conversion to CNF: Example

Consider: “Everyone who loves all animals is loved by someone”

$$\forall x.([\forall y.(Animal(y) \rightarrow Loves(x, y))] \rightarrow [\exists y.Loves(y, x)])$$

- 1 Eliminate implications and biconditionals:

$$\forall x.(\neg[\forall y.(\neg Animal(y) \vee Loves(x, y))] \vee [\exists y.Loves(y, x)])$$

- 2 Push inwards negations recursively:

$$\forall x.([\exists y.\neg(\neg Animal(y) \vee Loves(x, y))] \vee [\exists y.Loves(y, x)])$$

$$\forall x.([\exists y.(\neg\neg Animal(y) \wedge \neg Loves(x, y))] \vee [\exists y.Loves(y, x)])$$

$$\forall x.([\exists y.(Animal(y) \wedge \neg Loves(x, y))] \vee [\exists y.Loves(y, x)])$$

- 3 Standardize variables:

$$\forall x.([\exists y.(Animal(y) \wedge \neg Loves(x, y))] \vee [\exists z.Loves(z, x)])$$

- 4 Skolemize:

$$\forall x.([Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee [Loves(G(x), x)])$$

(F(x): “an animal unloved by x”; G(x): “someone who loves x”)

- 5 Drop universal quantifiers::

$$[Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee [Loves(G(x), x)]$$

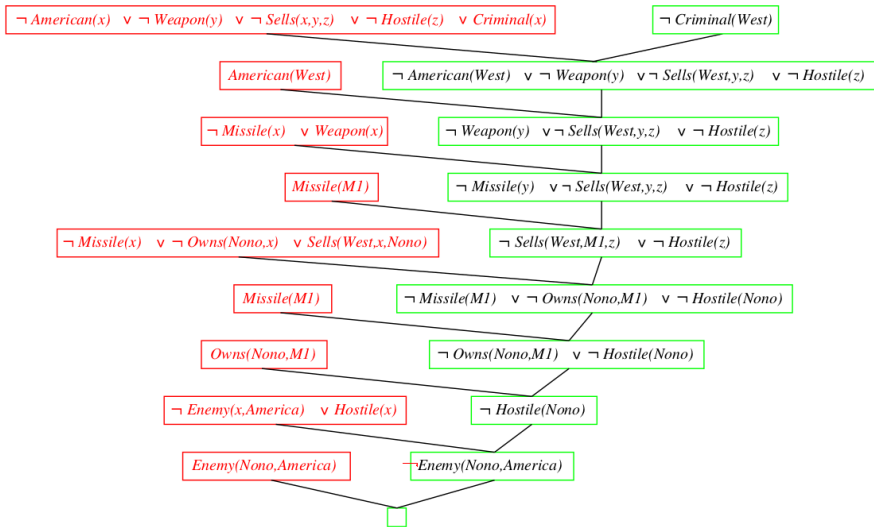
- 6 CNF-ize propositionally:

$$(Animal(F(x)) \vee Loves(G(x), x)) \wedge (\neg Loves(x, F(x)) \vee Loves(G(x), x))$$

# Resolution

- FOL resolution rule, let  $\theta \stackrel{\text{def}}{=} \text{Unify}(l_i, \neg m_j)$ , s.t.  $l_i\theta = \neg m_j\theta$ :
$$\frac{(l_1 \vee \dots \vee l_k) \quad (m_1 \vee \dots \vee m_n)}{(l_1 \vee \dots \vee l_{i-1} \vee l_{i+1} \vee \dots \vee l_k \vee m_1 \vee \dots \vee m_{j-1} \vee m_{j+1} \vee \dots \vee m_n)\theta}$$
$$\frac{\text{Man}(\text{Socrates}) \quad (\neg \text{Man}(x) \vee \text{Mortal}(x))}{\text{Mortal}(\text{Socrates})} \quad \text{s.t. } \theta \stackrel{\text{def}}{=} \{x/\text{Socrates}\}$$
- Ex:
- To prove that  $KB \models \alpha$  in FOL:
  - convert  $KB \wedge \neg\alpha$  to CNF
  - apply repeatedly resolution rule to  $\text{CNF}(KB \wedge \neg\alpha)$  until
    - the empty clause is generated  $\implies KB \models \alpha$
    - no more resolution step is applicable  $\implies KB \not\models \alpha$
    - resource (time, memory) exhausted  $\implies ??$
  - Hint: apply resolution first to unit clauses (unit resolution)
    - unit resolution alone complete for definite clauses
- Complete:
  - If there is a substitution  $\theta$  such that  $KB \models \theta\alpha$ , then it will return  $\theta$
  - If there is no such  $\theta$ , then the procedure may not terminate
- Many strategies and tools available

# Example: Resolution with Definite Clauses







## Dealing with Term Equalities [hints.]

To deal with equality formulas ( $t_1 = t_2$ )

- Combine resolution with Equal-term substitution rule

- Ex:

$$(4 \geq 3) \frac{(S(x) = x + 1) \quad (\neg(y \geq z) \vee (S(y) \geq S(z)))}{(\neg(y \geq z) \vee (y + 1 \geq z + 1))} \\ 4 + 1 \geq 3 + 1$$

- Ad-hoc rules rule for equality: **Demodulation** and **Paramodulation** (not covered here)

# Exercise

Consider the FOL formalization of the Wumpus World in Ch. 08

- 1 CNF-ize the KB
- 2 Perform the inference steps via resolution & equal-term substitution