Fundamentals of Artificial Intelligence
Chapter 03: **Problem Solving as Search**

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Outline

Outline

Problem Solving as Search

One of the dominant approaches to AI problem solving:
formulate a problem/task as search in a state space.

Main Paradigm

1. Goal formulation: define the successful world states
   - Ex: a set of states, a Boolean test function ...
2. Problem formulation:
   - define a representation for states
   - define legal actions and transition functions
3. Search: find a solution by means of a search process
   - solutions are sequences of actions
4. Execution: given the solution, perform the actions

$\Longrightarrow$ Problem-solving agents are (a kind of) goal-based agents

Problem Solving as Search: Example

Example: Traveling in Romania

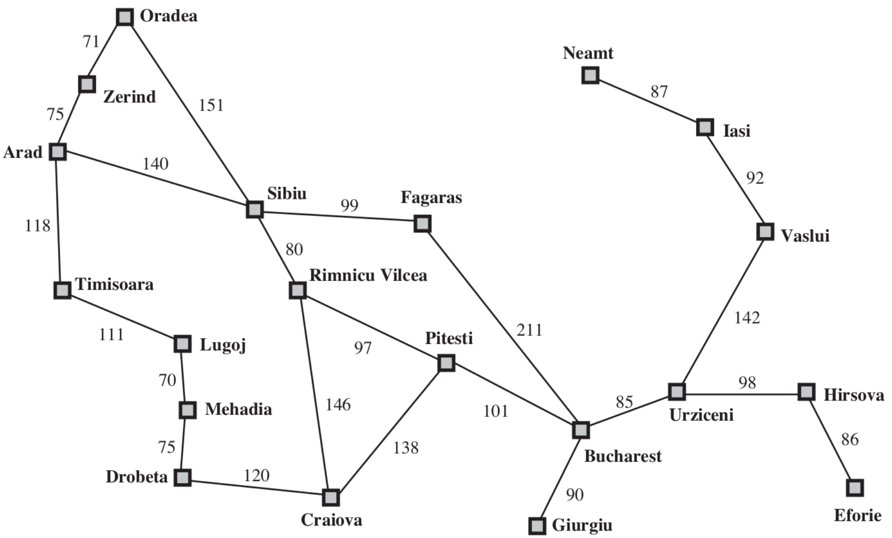
- Informal description: On holiday in Romania; currently in Arad. Flight leaves tomorrow from Bucharest
- Formulate goal: (Be in) Bucharest
- Formulate problem:
  - States: various cities
  - Actions: drive between cities
  - Initial state: Arad
- Search for a solution: sequence of cities from Arad to Bucharest
  - e.g. Arad, Sibiu, Fagaras, Bucharest
  - explore a search tree/graph

Note

The agent is assumed to have no heuristic knowledge about traveling in Romania to exploit.

Problem Solving as Search: Example [cont.]

A simplified road map of part of Romania.



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Problem Solving as Search [cont.]

Assumptions for Problem-solving Agents (this chapter only)

- state representations are atomic

  $\Longrightarrow$ world states are considered as wholes, with no internal structure

  - Ex: Arad, Sibiu, Zerind, Bucharest,...
- the environment is observable

  $\Longrightarrow$ the agent always knows the current state

  - Ex: Romanian cities & roads have signs
- the environment is discrete

  $\Longrightarrow$ at any state there are only finitely many actions to choose from

  - Ex: from Arad, (go to) Sibiu, or Zerind, or Timisoara (see map)
- the environment is known

  $\Longrightarrow$ the agent knows which states are reached by each action

  - ex: the agent has the map
- the environment is deterministic

  $\Longrightarrow$ each action has exactly one outcome

  - Ex: from Arad choose go to Sibiu $\Longrightarrow$ next step in Sibiu

Problem Solving as Search [cont.]

Notes about search

- Search happens inside the agent
  - a planning stage before acting
  - different from searching in the world
- An agent is given a description of what to achieve, not an algorithm to solve it

  $\Longrightarrow$ the only possibility is to search for a solution
- Searching can be computationally very demanding (NP-hard)
- Can be driven with benefits by knowledge of the problem (heuristic knowledge) $\Longrightarrow$ informed/heuristic search

Problem-solving Agent: Schema

function SIMPLE-PROBLEM-SOLVING-AGENT(*percept*) **returns** an action
  persistent: *seq*, an action sequence, initially empty
    state, some description of the current world state
    goal, a goal, initially null
    problem, a problem formulation

  state ← UPDATE-STATE(*state*, *percept*)
  if *seq* is empty **then**
    goal ← FORMULATE-GOAL(*state*)
    problem ← FORMULATE-PROBLEM(*state*, *goal*)
    seq ← SEARCH(*problem*)
    if *seq* = *failure* **then return** a null action
  action ← FIRST(*seq*)
  seq ← REST(*seq*)
  return *action*

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While executing the solution sequence the agent ignores its percepts when choosing an action since it knows in advance what they will be ("open loop system")

Well-defined problems and solutions

Problem Formulation: Components

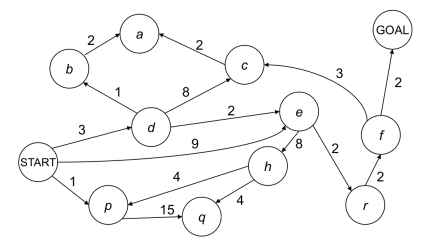
- the initial state the agent starts in
  - Ex: *In*(*Arad*)
- the set of applicable actions available in a state (ACTIONS(S))
  - Ex: if *s* is *In*(*Arad*), then the applicable actions are $\{Go(Sibiu), Go(Timisoara), Go(Zerind)\}$
- a description of what each action does (aka transition model)
  - RESULT(S,A): state resulting from applying action A in state S
  - Ex: RESULT(IN(ARAD), GO(ZERIND)) is IN(ZERIND)
- the goal test determining if a given state is a goal state
  - Explicit (e.g.: $\{In(Bucharest)\}$)
  - Implicit (e.g. (Ex: CHECKMATE(X))
- the path cost function assigns a numeric cost to each path
  - in this chapter: the sum of the costs of the actions along the path

Well-defined problems and solutions [cont.]

State Space, Graphs, Paths, Solutions and Optimal Solutions

Initial state, actions, and transition model implicitly define the state space of the problem

- the state space forms a directed graph (e.g. the Romania map)
  - typically too big to be created explicitly and be stored in full
  - in a state space graph, each state occurs only once
- a path is a sequence of states connected by actions
- a solution is a path from the initial state to a goal state
- an optimal solution is a solution with the lowest path cost



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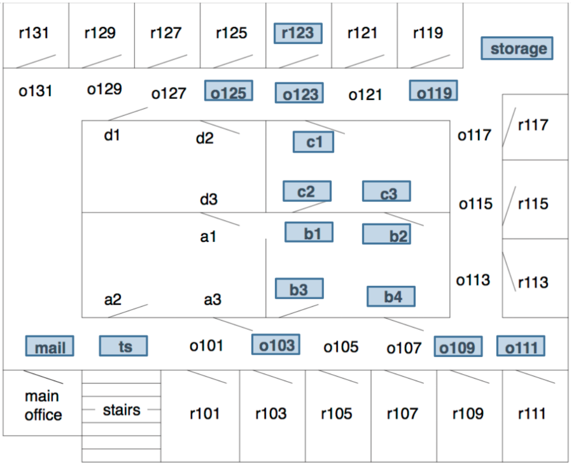
Example: Path finding for a Delivery Robot

Task: move from o103 to r123

- States
- Initial state
- Goal state
- State graph
- Optimal solution



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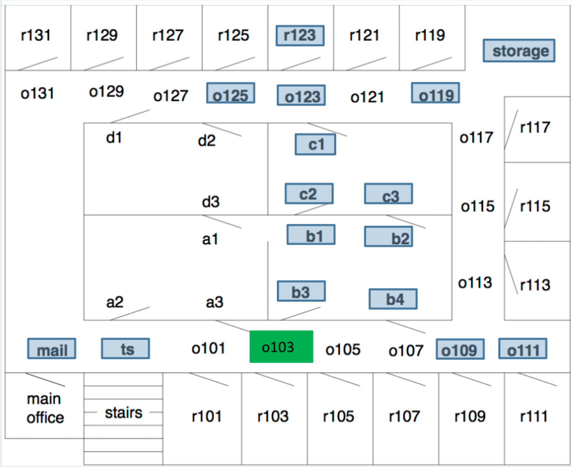

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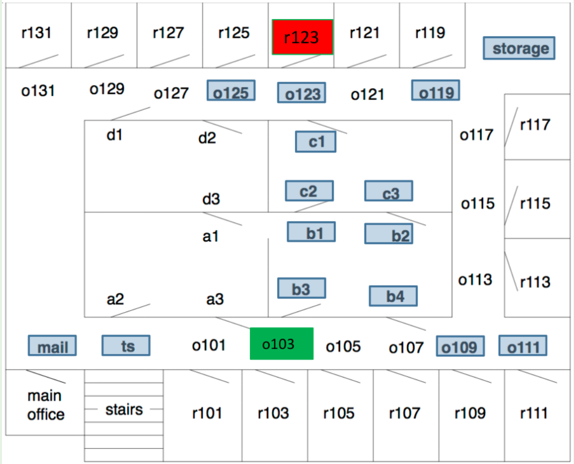

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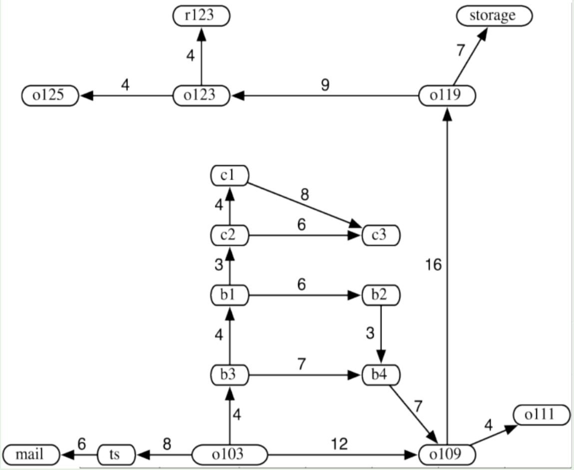

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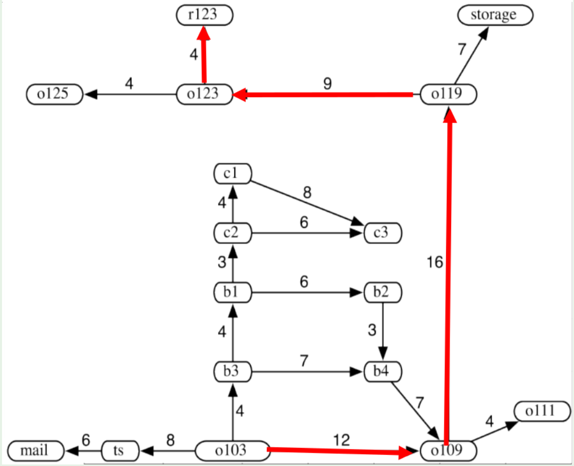

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Well-defined problems and solutions [cont.]

Abstraction

Problem formulations are models of reality (i.e. abstract descriptions)

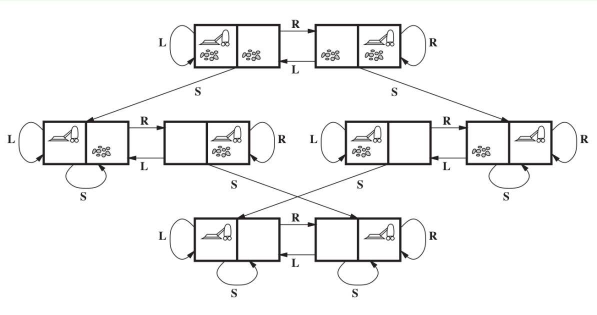
- real world is absurdly complex
  $\Longrightarrow$ state space must be abstracted for problem solving
- lots of details removed because irrelevant to the problem
  - Ex: exact position, "turn steering wheel to the left by 20 degree", ...
- abstraction: the process of removing detail from representations
  - abstract state represents many real states
  - abstract action represents complex combination of real actions
- valid abstraction: can expand any abstract solution into a solution in the detailed world
- useful abstraction: if carrying out each of the actions in the solution is easier than in the original problem

The choice of a good abstraction involves removing as much detail as possible, while retaining validity and ensuring that the abstract actions are easy to carry out.

Outline

Toy Example: Simple Vacuum Cleaner

- States: 2 locations, each $\{clean, dirty\}$: $2 \cdot 2^2 = 8$ states
- Initial State: any
- Actions: $\{Left, Right, Suck\}$
- Transition Model: Left [Right] if A [B], Suck if clean $\Longrightarrow$ no effect
- Goal Test: check if squares are clean
- Path Cost: each step costs 1 $\Longrightarrow$ path cost is # of steps in path



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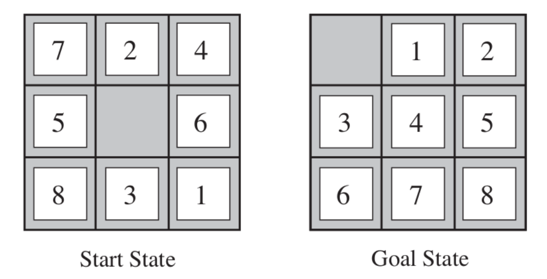
Toy Example: The 8-Puzzle

- States: Integer location of each tile $\Longrightarrow$ $9!/2$ reachable states
- Initial State: any
- Actions: moving $\{Left, Right, Up, Down\}$ the empty space
- Transition Model: empty switched with the tile in target location
- Goal Test: checks state corresponds with goal configuration
- Path Cost: each step costs 1 $\Longrightarrow$ path cost is # of steps in path



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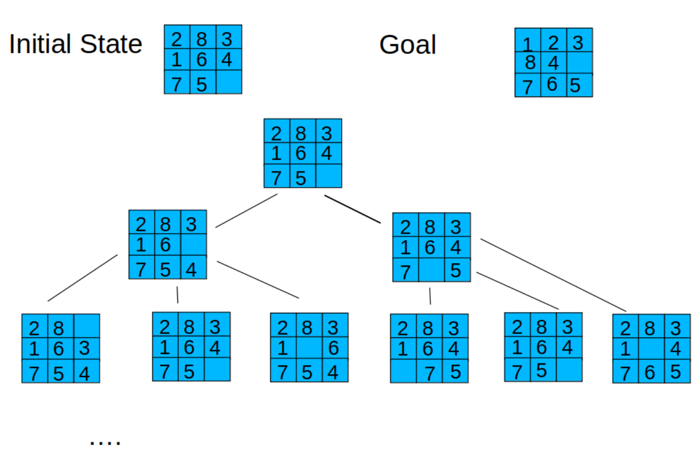
Toy Example: The 8-Puzzle [cont.]



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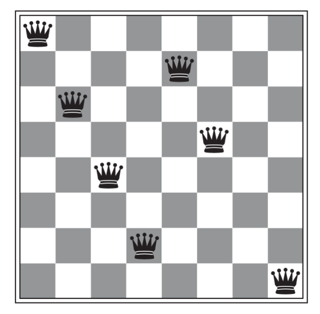
NP-complete: N-Puzzle ($N = k^2 - 1$): $N!/2$ reachable states

Toy Example: 8-Queens Problem

- States: any arrangement of 0 to 8 queens on the board
  $\Longrightarrow$ $64 \cdot 63 \cdot ... \cdot 57 \approx 1.8 \cdot 10^{14}$ possible sequences
- Initial State: no queens on the board
- Actions: add a queen to any empty square
- Transition Model: returns the board with a queen added
- Goal Test: 8 queens on the board, none attacked by other queen
- Path Cost: none



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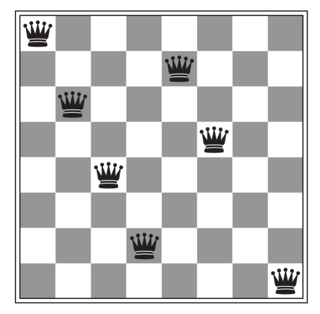

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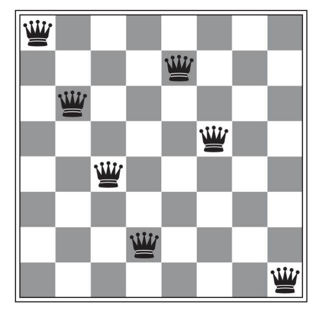
Toy Example: 8-Queens Problem (incremental)

- States: $n \leq 8$ queens on board, one per column in the n leftmost columns, no queen attacking another
  $\Longrightarrow$ 2057 possible sequences
- Actions: Add a queen to any square in the leftmost empty column such that it is not attacked by any other queen.
- ...



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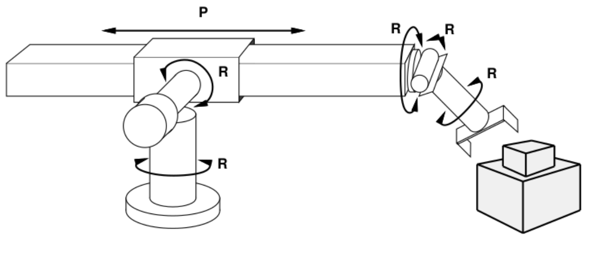
Real-World Example: Robotic Assembly

- States: real-valued coordinates of robot joint angles, and of parts of the object to be assembled
- Initial State: any arm position and object configuration
- Actions: continuous motions of robot joints
- Transition Model: position resulting from motion
- Goal Test: complete assembly (without robot)
- Path Cost: time to execute



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Other Real-World Examples

- Airline travel planning problems
- Touring problems
- VLSI layout problem
- Robot navigation
- Automatic assembly sequencing
- Protein design
- ...

Outline

Searching for Solutions

Search: Generate sequences of actions.

- Expansion: one starts from a state, and applying the operators (or successor function) will generate new states
- Search strategy: at each step, choose which state to expand.
- Search Tree: It represents the expansion of all states starting from the initial state (the root of the tree)
- The leaves of the tree represent either:
  - states to expand
  - solutions
  - dead-ends

Tree Search Algorithms

Tree Search: Basic idea

- Off-line, simulated exploration of state space
  - start from initial state
  - pick one leaf node, and generate its successors (a.k.a. expanding a node) states)
    - set of current leaves called frontier (a.k.a. fringe, open list)
    - strategy for picking leaves critical (search strategy)
  - ends when either a goal state is reached, or no more candidates to expand are available (or time-out/memory-out occur)

function TREE-SEARCH(*problem*) **returns** a solution, or failure
  initialize the frontier using the initial state of *problem*
  loop do
    if the frontier is empty **then return** failure
    choose a leaf node and remove it from the frontier
    if the node contains a goal state **then return** the corresponding solution
    expand the chosen node, adding the resulting nodes to the frontier

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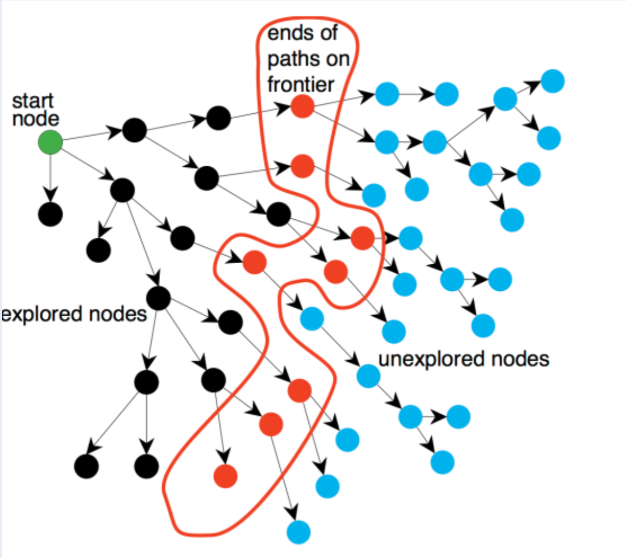
Tree Search Algorithms [cont.]



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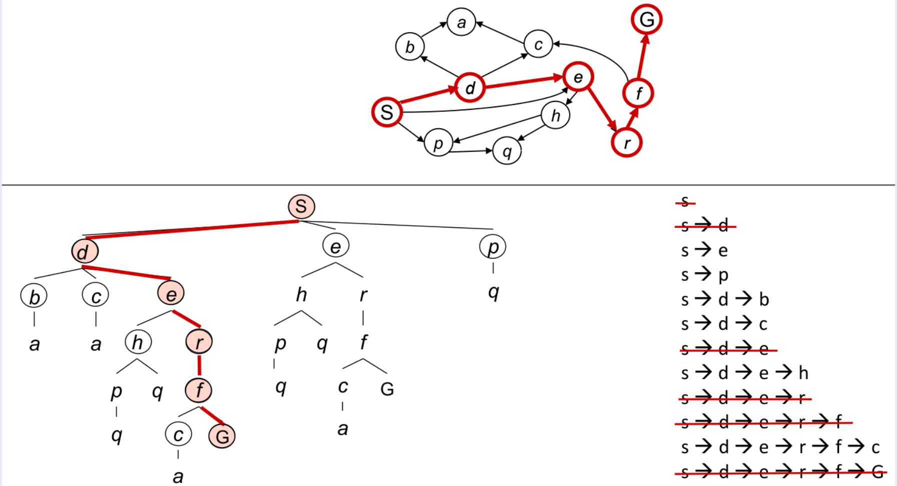
Tree-Search Example



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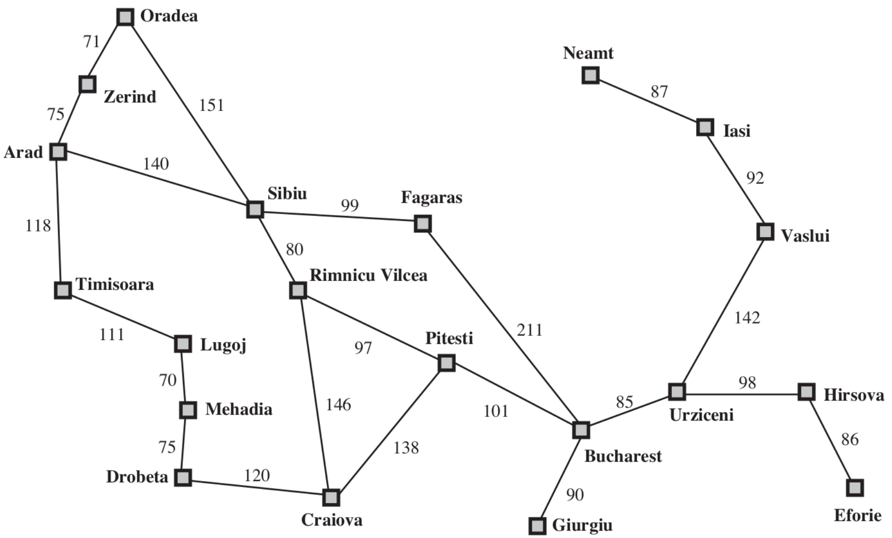
Tree-Search Example: Trip from Arad to Bucharest

A simplified road map of part of Romania.



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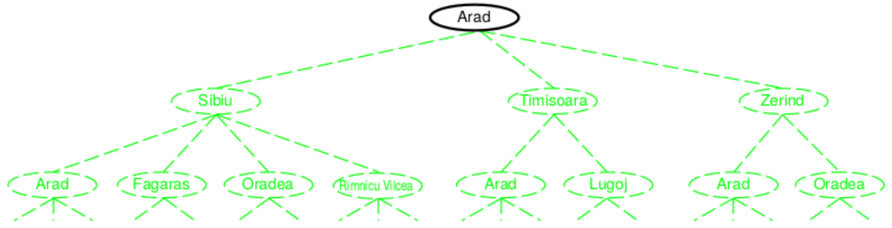
Tree-Search Example: Trip from Arad to Bucharest

Expanding the search tree

- Initial state: $\{Arad\}$
- Expand initial state $\Longrightarrow$ $\{Sibiu, Timisoara, Zerind\}$
- Pick&expand Sibiu $\Longrightarrow$ $\{Arad, Fagaras, Oradea, Rimnicu\ Vicea\}$
- ...



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Beware: Arad $\mapsto$ Sibiu $\mapsto$ Arad (repeated state $\Longrightarrow$ loopy path!)

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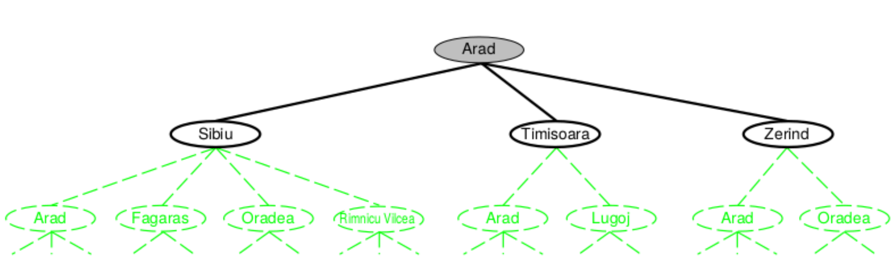

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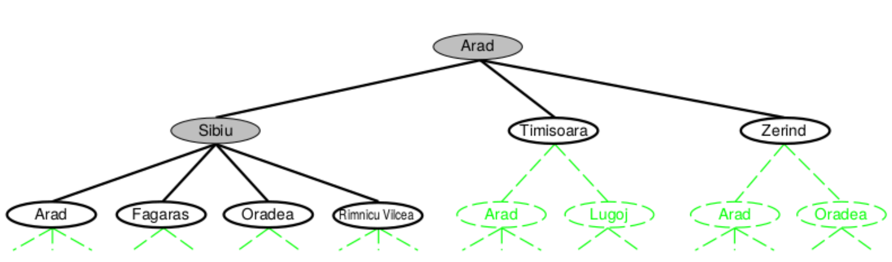

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Repeated states & Redundant Paths

- redundant paths occur when there is more than one way to get from one state to another

  $\Longrightarrow$ same state explored more than once

- Failure to detect repeated states can:
  - cause infinite loops
  - turn linear problem into exponential

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Moral: Algorithms that forget their history are doomed to repeat it!

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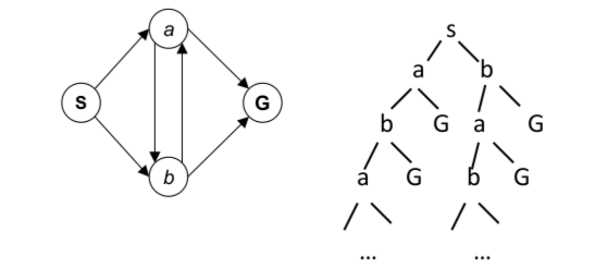

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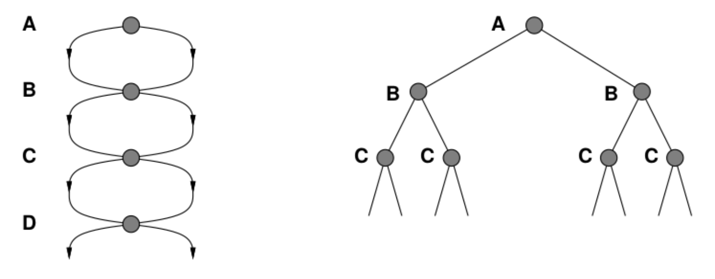

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Graph Search Algorithms

Graph Search: Basic idea

- add a data structure which remembers every expanded node
  - a.k.a. explored set or closed list
  - typically a hash table (access $O(1)$)
- do not expand a node if it already occurs in explored set

function GRAPH-SEARCH(*problem*) **returns** a solution, or failure
  initialize the frontier using the initial state of *problem*
  initialize the explored set to be empty
  loop do
    if the frontier is empty **then return** failure
    choose a leaf node and remove it from the frontier
    if the node contains a goal state **then return** the corresponding solution
    add the node to the explored set
    expand the chosen node, adding the resulting nodes to the frontier
      only if not in the frontier or explored set

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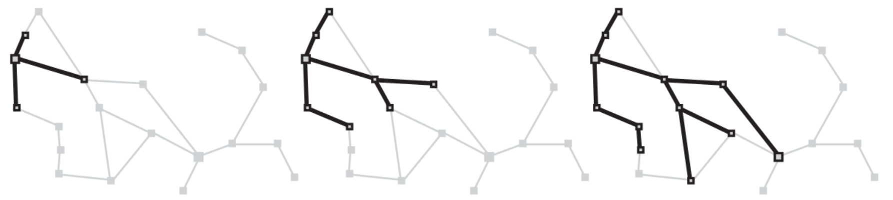
Graph Search Algorithms: Example

Graph search on the Romania trip problem

- (at each stage each path extended by one step)
- two states become dead-end



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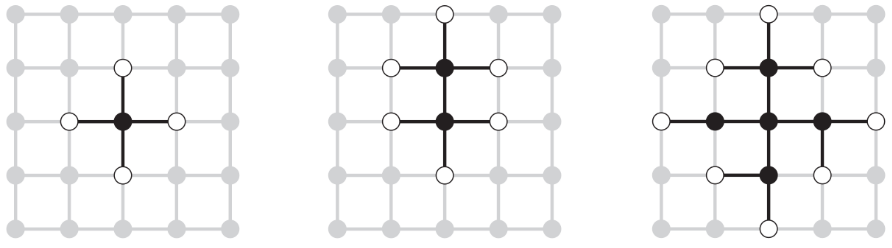
Graph Search Algorithms: Example

Separation Property of graph search: the frontier separates the state-space graph into the explored region and the unexplored region

Graph search on a rectangular-grid problem



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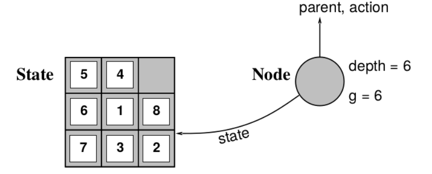
Implementation: States vs. Nodes

- A state is a representation of a physical configuration
- A node is a data structure constituting part of a search tree
  - includes fields: state, parent, action, path cost $g(x)$

$\Longrightarrow$ node $\neq$ state

- It is easy to compute a child node from its parent



function CHILD-NODE(*problem*, *parent*, *action*) **returns** a node
  return a node with
    STATE = *problem*.RESULT(*parent*.STATE, *action*),
    PARENT = *parent*, ACTION = *action*,
    PATH-COST = *parent*.PATH-COST + *problem*.STEP-COST(*parent*.STATE, *action*)

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Implementation: Frontier and Explored

Frontier/Fringe

- Implemented as a Queue:
  - First-in-First-Out, FIFO (aka "queue"): $O(1)$ access
  - Last-in-First-Out, LIFO (aka "stack"): $O(1)$ access
  - Best-First-out (aka "priority queue"): $O(log(n))$ access
- Two primitives:
  - ISEMPTY(QUEUE): returns true iff there are no more elements
  - POP(QUEUE): removes and returns the first element of the queue
  - INSERT(ELEMENT,QUEUE): inserts an element into queue

Explored

- Implemented as a Hash Table: $O(1)$ access
- Three primitives:
  - ISTHERE(ELEMENT,HASH): returns true iff element is in the hash
  - INSERT(ELEMENT,HASH): inserts element into hash
- Choice of hash function critical for efficiency

Uninformed vs. Informed Search Strategies

Strategies: Two possibilities

- Uninformed strategies (a.k.a. blind strategies)
  - do not use any domain knowledge
  - apply rules arbitrarily and do an exhaustive search strategy

  $\Longrightarrow$ impractical for some complex problems.
- Informed strategies
  - use domain knowledge
  - apply rules following heuristics (driven by domain knowledge)

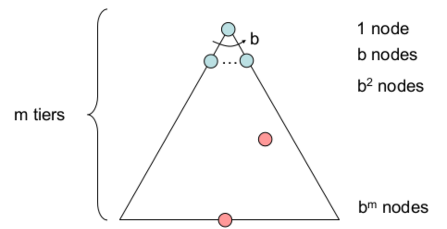
  $\Longrightarrow$ practical for many complex problems.

Outline

Evaluating Search Strategies

- Strategies are evaluated along the following dimensions:
  - completeness: does it always find a solution if one exists?
  - time complexity: how many steps to find a solution?
  - space complexity: how much memory is needed?
  - optimality: does it always find a least-cost solution?
- Time and space complexity are measured in terms of
  - b: maximum branching factor of the search tree
  - d: depth of the least-cost solution
  - m: maximum depth of the state space (may be $+\infty$)

$\Longrightarrow$ # nodes: $1 + b + b^2 + ... + b^m = O(b^m)$



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Uninformed Search Strategies

Uninformed strategies

Use only the information available in the problem definition

- Different uninformed search stategies
  - Breadth-first search
  - Uniform-cost search
  - Depth-first search
  - Depth-limited search & Iterative-deepening search
- Defined by the access strategy of the frontier/fringe (i.e. the order of node expansion)

Outline

Breadth-First Search Strategy (BFS)

Breadth-First Search

- Idea: Expand first the shallowest unexpanded nodes
- Implementation: frontier/fringe implemented as a FIFO queue

  $\Longrightarrow$ novel successors pushed to the end of the queue

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Breadth-First Search Strategy (BFS)

Breadth-First Search

- Idea: Expand first the shallowest unexpanded nodes
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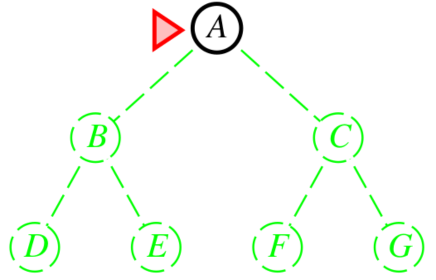

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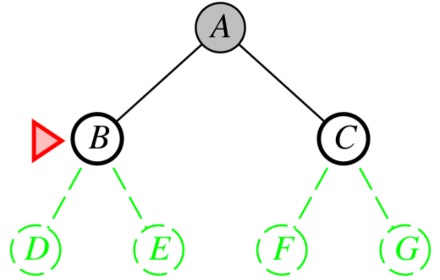

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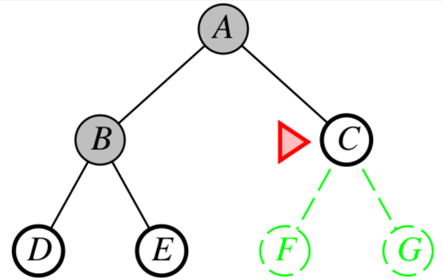

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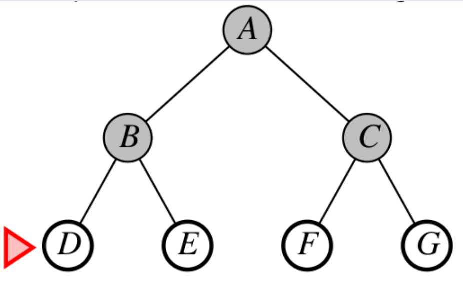

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Breadth-First Search Strategy (BFS) [cont.]

BFS, Graph version (Tree version without “explored”)

function BREADTH-FIRST-SEARCH(*problem*) **returns** a solution, or failure

```
node ← a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
frontier ← a FIFO queue with node as the only element
explored ← an empty set
loop do
    if EMPTY?(frontier) then return failure
    node ← POP(frontier)   /* chooses the shallowest node in frontier */
    add node.STATE to explored
    for each action in problem.ACTIONS(node.STATE) do
        child ← CHILD-NODE(problem, node, action)
        if child.STATE is not in explored or frontier then
            if problem.GOAL-TEST(child.STATE) then return SOLUTION(child)
            frontier ← INSERT(child, frontier)
```

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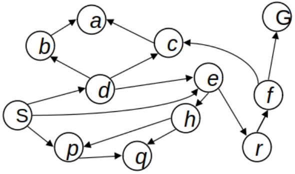
Note: the goal test is applied to each node when it is generated, rather than when it is selected for expansion
$\Longrightarrow$ solution detected 1 layer earlier

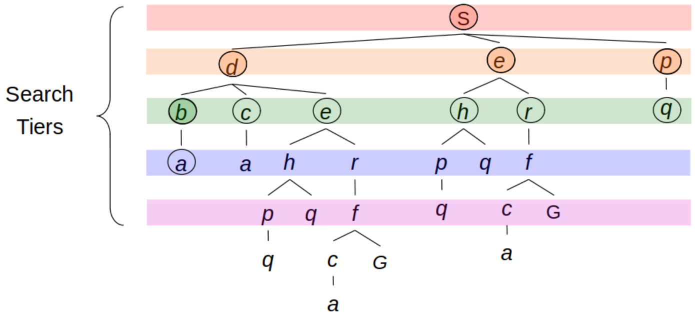
Breadth-First Search: Tiers

State space is explored by tiers (tree version)

Strategy: expand a shallowest node first

Implementation: Fringe is a FIFO queue





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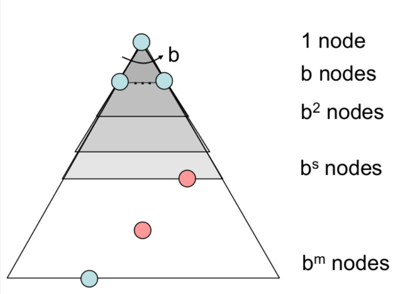
Breadth-First Search (BFS): Properties

d: depth of shallowest solution

- How many steps?
  - processes all nodes above shallowest solution
    $\Longrightarrow$ takes $O(b^d)$ time
- How much memory?
  - max frontier size: b^d nodes
    $\Longrightarrow$ $O(b^d)$ memory size
- Is it complete?
  - if solution exists, b^d finite
    $\Longrightarrow$ Yes
- Is it optimal?
  - if and only if all costs are 1
    $\Longrightarrow$ shallowest solution



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Memory requirement is a major problem for breadth-first search

Breadth-First Search (BFS): Time and Memory

- Assume:
  - 1 million nodes generated per second
  - 1 node requires 1000 bytes of storage
  - branching factor $b = 10$

Depth	Nodes	Time	Memory
2	110	.11 milliseconds	107 kilobytes
4	11,110	11 milliseconds	10.6 megabytes
6	10^6	1.1 seconds	1 gigabyte
8	10^8	2 minutes	103 gigabytes
10	10^{10}	3 hours	10 terabytes
12	10^{12}	13 days	1 petabyte
14	10^{14}	3.5 years	99 petabytes
16	10^{16}	350 years	10 exabytes

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Memory requirements bigger problem for BFS than execution time

Outline

Uniform-Cost Search Strategy (UCS)

Uniform-Cost Search

- Idea: Expand first the node with lowest path cost $g(n)$
- Implementation: frontier/fringe implemented as a priority queue ordered by $g()$

  $\Longrightarrow$ novel nearest successors pushed to the top of the queue
- similar to BFS if step costs are all equal

(Courtesy of Michela Milano, UniBO)

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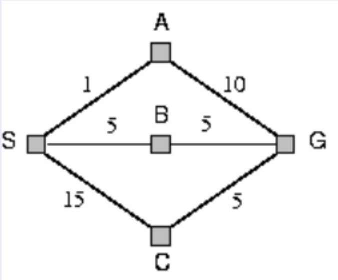

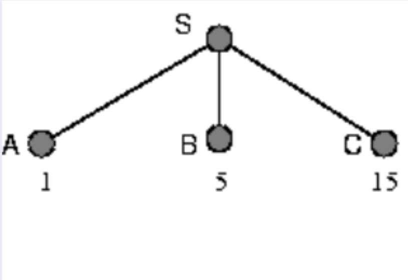

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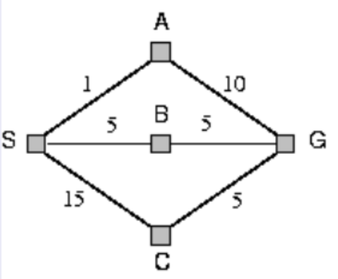

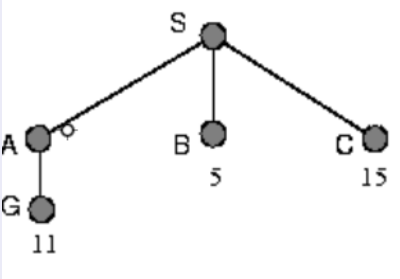

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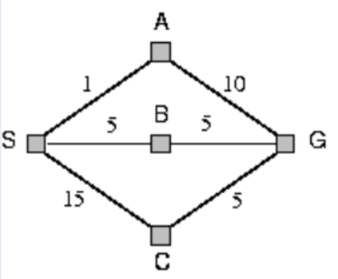

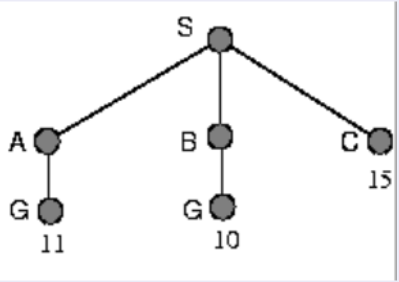

(Courtesy of Michela Milano, UniBO)

Uniform-Cost Search Strategy (UCS) [cont.]

UCS, Graph version (Tree version without “explored”)

```
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
  node ← a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  frontier ← a priority queue ordered by PATH-COST, with node as the only element
  explored ← an empty set
  loop do
     if EMPTY?(frontier) then return failure
     node ← POP(frontier)   /* chooses the lowest-cost node in frontier */
     if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
     add node.STATE to explored
     for each action in problem.ACTIONS(node.STATE) do
        child ← CHILD-NODE(problem, node, action)
        if child.STATE is not in explored or frontier then
           frontier ← INSERT(child, frontier)
        else if child.STATE is in frontier with higher PATH-COST then
           replace that frontier node with child
```

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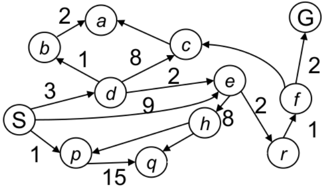
- apply the goal test to a node when it is selected for expansion rather than when it is first generated
- replace in the frontier a node with same state but worse path cost

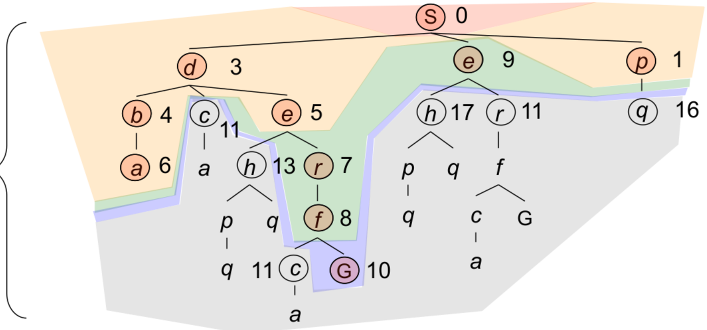
$\Longrightarrow$ avoid generating suboptimal paths

Uniform-Cost Search

Strategy: expand a cheapest node first:

Fringe is a priority queue (priority: cumulative cost)





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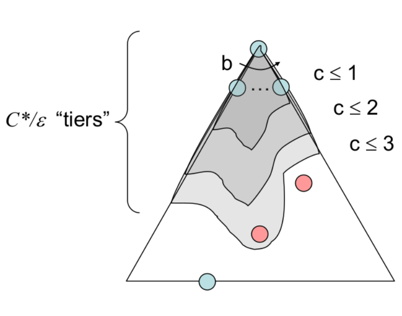
Uniform-Cost Search (UCS): Properties

C^*: cost of cheapest solution; ϵ: minimum arc cost
$\Longrightarrow 1 + \lfloor C^*/\epsilon \rfloor$ "effective depth"

- How many steps?
  - processes all nodes costing less than cheapest solution $\Longrightarrow$ takes $O(b^{1+\lfloor C^*/\epsilon \rfloor})$ time
- How much memory?
  - max frontier size: $b^{1+\lfloor C^*/\epsilon \rfloor}$ $\Longrightarrow$ $O(b^{1+\lfloor C^*/\epsilon \rfloor})$ memory size
- Is it complete?
  - if solution exists, finite cost $\Longrightarrow$ Yes
- Is it optimal?
  - Yes



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Memory requirement is a major problem also for uniform-cost search

Outline

Depth-First Search Strategy (DFS)

Depth-First Search

- Idea: Expand first the deepest unexpanded nodes
- Implementation: frontier/fringe implemented as a LIFO queue (aka stack)

  $\Longrightarrow$ novel successors pushed to the top of the stack

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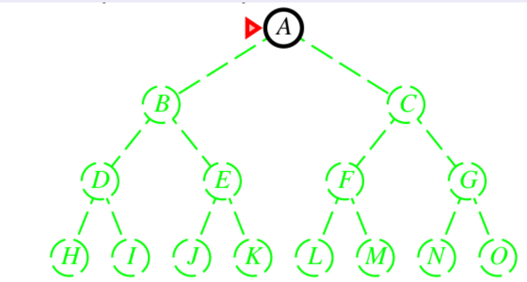

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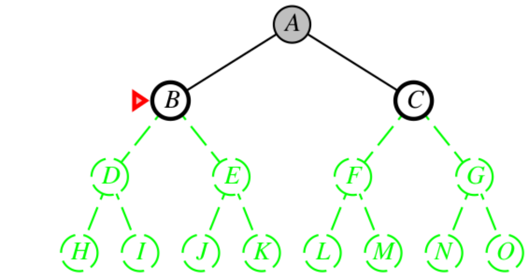

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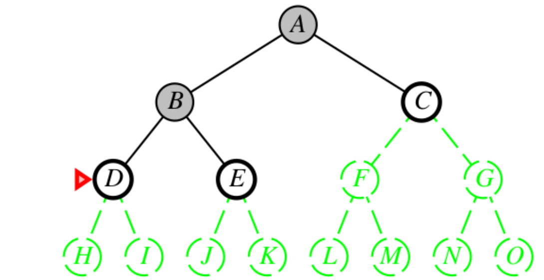

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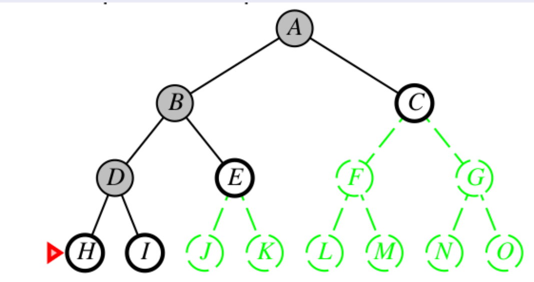

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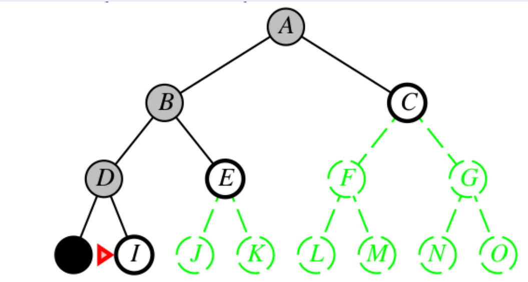

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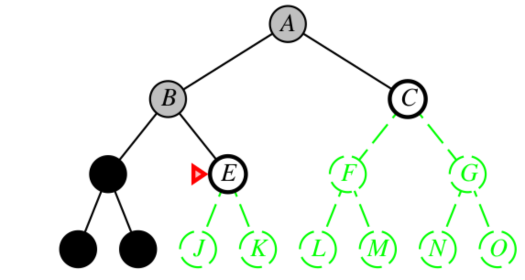

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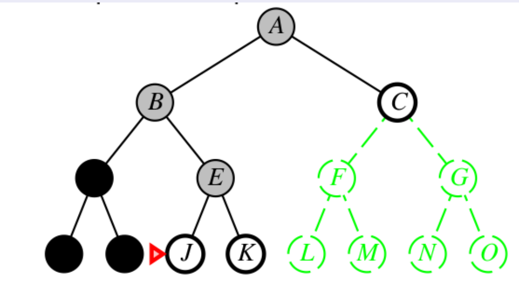

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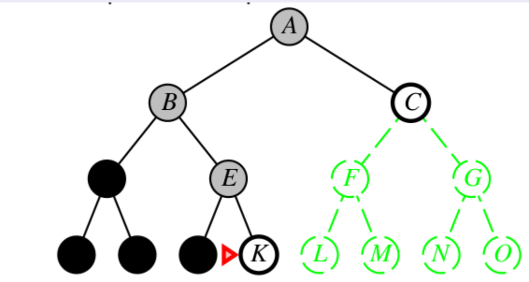

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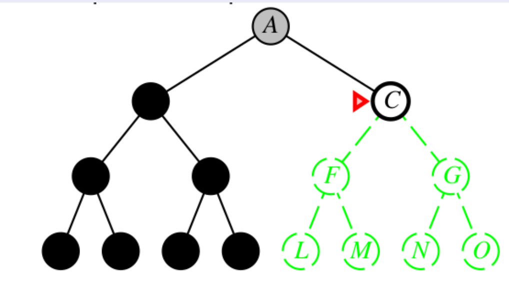

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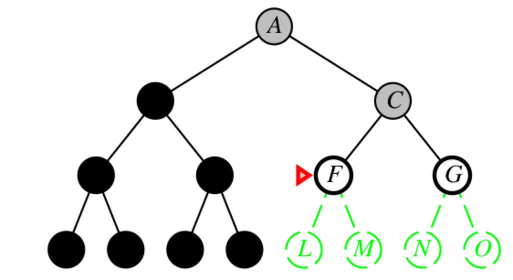

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Depth-First Search

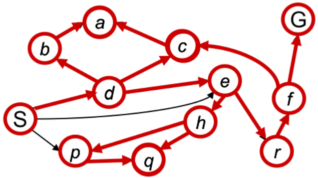
DFS on a Graph

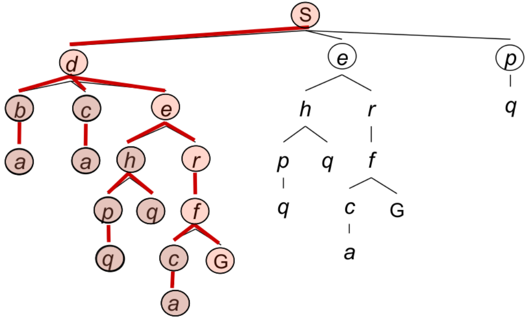
Similar to BFS, using a LIFO access for frontier/fringe rather than FIFO.

Depth-First Search

Strategy: expand a deepest node first

Implementation: Fringe is a LIFO stack





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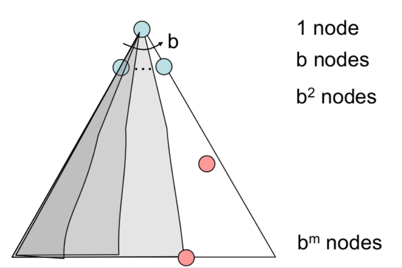
Depth-First Search (DFS): Properties

- How many steps?
  - could process the whole tree! $\Longrightarrow$ if m finite, takes $O(b^m)$ time
- How much memory?
  - only siblings on path to root $\Longrightarrow$ $O(bm)$ memory size
- Is it complete?
  - if infinite state space: no
  - if finite state space:
    - graph version: yes
    - tree version: only if we prevent loops
- Is it optimal?
  - No, regardless of depth/cost $\Longrightarrow$ "leftmost" solution



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Memory requirement much better than BFS: $O(bm)$ vs. $O(b^d)$!
$\Longrightarrow$ typically preferred to BFS

A Variant of DFS: Backtracking Search

Backtracking Search

- Idea: only one successor is generated at the time
  - each partially-expanded node remembers which successor to generate next
  - generate a successor by modifying the current state description, rather than copying it firs
  - Applied in CSP, SAT/SMT and Logic Programming

$\Longrightarrow$ only $O(m)$ memory is needed rather than $O(bm)$

Outline

Depth-Limited Search (DLS) Strategy

Depth-Limited Search (DLS)

- Idea: depth-first search with depth limit l
  - i.e., nodes at depth l treated as having no successors
  - DFS is DLS with $l = +\infty$
- solves the infinite-path problem of DFS
  $\Longrightarrow$ allows DFS deal with infinite-state spaces
- useful also if maximum-depth is known by domain knowledge
  - e.g., if maximum node distance in a graph (diameter) is known
  - Ex: Romania trip: 9 steps
- Drawbacks (d: depth of the shallowest goal):
  - if $d > l \Longrightarrow$ incomplete
  - if $d < l \Longrightarrow$ takes $O(b^l)$ instead of $O(b^d)$ steps

Depth-Limited Search (DLS) Strategy [cont.]

Recursive DLS

```
function DEPTH-LIMITED-SEARCH(problem, limit) returns a solution, or failure/cutoff
  return RECURSIVE-DLS(MAKE-NODE(problem.INITIAL-STATE), problem, limit)

function RECURSIVE-DLS(node, problem, limit) returns a solution, or failure/cutoff
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  else if limit = 0 then return cutoff
  else
     cutoff_occurred? ← false
     for each action in problem.ACTIONS(node.STATE) do
        child ← CHILD-NODE(problem, node, action)
        result ← RECURSIVE-DLS(child, problem, limit − 1)
        if result = cutoff then cutoff_occurred? ← true
        else if result ≠ failure then return result
     if cutoff_occurred? then return cutoff else return failure
```

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Iterative-Deepening Search Strategy (IDS)

Iterative-Deepening Search

- Idea: call iteratively DLS for increasing depths $l = 0, 1, 2, 3...$
- combines the advantages of breadth- and depth-first strategies
  - complete (like BFS)
  - takes $O(b^d)$ steps (like BFS and DFS)
  - requires $O(bd)$ memory (like DFS)
  - explores a single branch at a time (like DFS)
  - optimal only if step cost = 1
    - optimal variants exist: iterative-lengthening search (see AIMA)
- The favorite search strategy when the search space is very large and depth is not known

function ITERATIVE-DEEPENING-SEARCH($problem$) **returns** a solution, or failure
  for $depth$ = 0 **to** ∞ **do**
    $result \leftarrow$ DEPTH-LIMITED-SEARCH($problem$, $depth$)
    if $result \neq$ cutoff **then return** $result$

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Iterative-Deepening Search: Example

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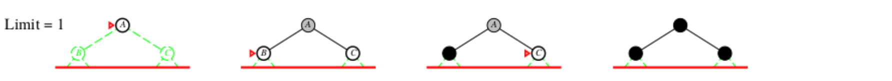
Iterative-Deepening Search: Example



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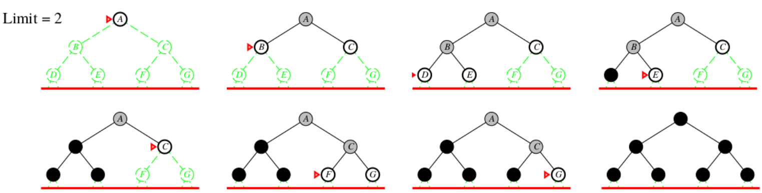

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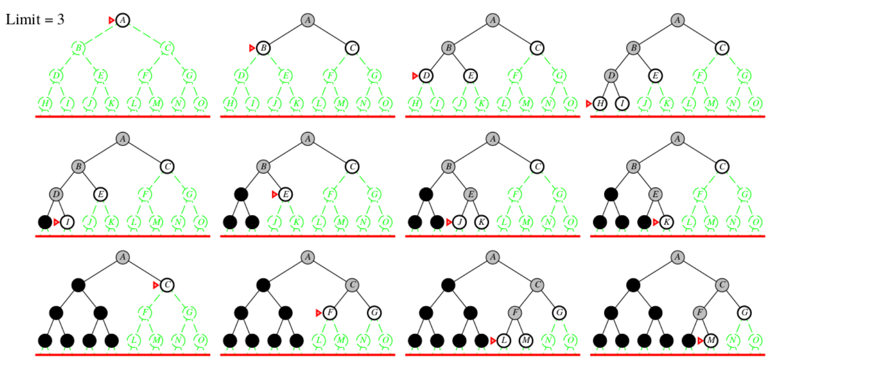

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Iterative-Deepening Search Strategy (IDS) [cont.]

Remark: Why "only" $O(b^d)$ steps?

- may seem wasteful since states are generated multiple times...
- ... however, only a small fraction of nodes are multiply generated
- number of repeatedly-generated nodes decreases exponentially with number of repetitions
  - depth 1 (b nodes): repeated d times
  - depth 2 (b^2 nodes): repeated $d-1$ times
  - ...
  - depth d (b^d nodes): repeated 1 time

$\Longrightarrow$ The total number of generated nodes is:

$$N(IDS) = (d)b^1 + (d-1)b^2 + ... + (1)b^d = O(b^d)$$
$$N(BFS) = b^1 + b^2 + ... + b^d = O(b^d)$$

- Ex: with $b = 10$ and $d = 5$:

$$N(IDS) = 50 + 400 + 3,000 + 20,000 + 100,000 = 123,000$$
$$N(BFS) = 10 + 100 + 1,000 + 10,000 + 100,000 = 111,110$$

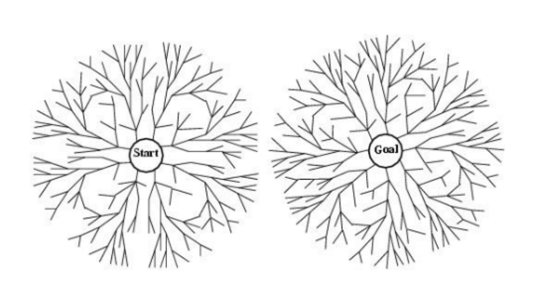
$\Longrightarrow$ not significantly worse than BFS

Bidirectional Search [hints]

- Idea: Two simultaneous searches:
  - forward: from start node
  - backward: from goal node

    checking if the node belongs to the other frontier before expansion
- Rationale: $b^{d/2} + b^{d/2} << b^d$

  $\Longrightarrow$ number of steps and memory consumption are $\approx 2b^{d/2}$
- backward search can be tricky in some cases (e.g. 8-queens)



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Uninformed Search Strategies: Comparison

Evaluation of tree-search strategies

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening	Bidirectional (if applicable)
Complete?	Yes^a	Yes^{a,b}	No	No	Yes^a	Yes^{a,d}
Time	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(b^m)$	$O(b^\ell)$	$O(b^d)$	$O(b^{d/2})$
Space	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(bm)$	$O(b\ell)$	$O(bd)$	$O(b^{d/2})$
Optimal?	Yes^c	Yes	No	No	Yes^c	Yes^{c,d}

^{a}: complete if b is finite
^{b}: complete if step costs $\geq \epsilon$ for some positive ϵ
^{c}: optimal if step costs are all identical
^{d}: if both directions use breadth-first search

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For graph searches, the main differences are:

- depth-first search is complete for finite-state spaces
- space & time complexities are bounded by the state space size

Outline

Informed Search Strategies

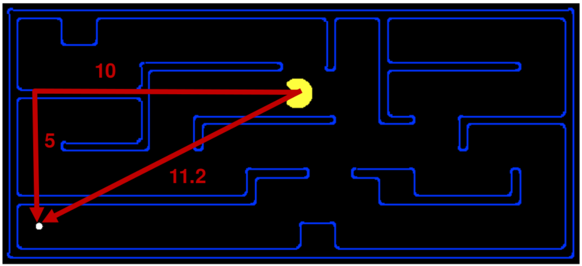
Some general principles

- The intelligence of a system cannot be measured only in terms of search capacity, but in the ability to use knowledge about the problem to reduce/mitigate the combinatorial explosion
- If the system has some control on the order in which candidate solutions are generated, then it is useful to use this order so that actual solutions have a high chance to appear earlier
- Intelligence, for a system with limited processing capacity, is the wise choice of what to do next

Heuristic search and heuristic functions

Heuristic search and heuristic functions

- Idea: don't ignore the goal when selecting nodes
- Intuition: often there is extra knowledge that can be used to guide the search towards the goal: heuristics
- A heuristic is
  - a function $h(n)$ that estimates how close a state n is to a goal
  - designed for a particular search problem
  - Ex Manhattan distance, Euclidean distance for pathing



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Best-first Search Strategies

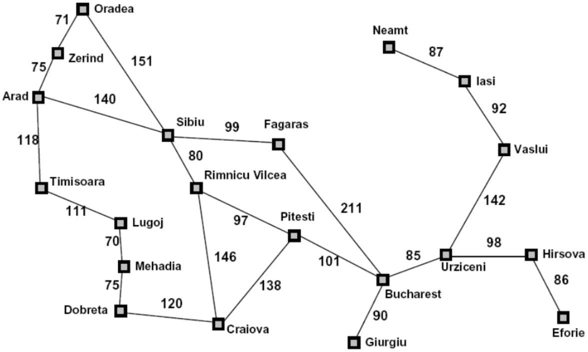
General approach of informed search: Best-first search

- Best-first search: node selected for expansion based on an evaluation function $f(n)$
  - represent a cost estimate $\Longrightarrow$ choose node which appears best
  - implemented like uniform-cost search, with f instead of g

  $\Longrightarrow$ the frontier is a priority queue sorted in decreasing order of $h(n)$
  - both tree-based and graph-based versions
  - most often f includes a heuristic function h(n)
- Heuristic function $h(n) \in \mathbb{R}^+$: estimated cost of the cheapest path from the state at node n to a goal state
  - $h(n) \geq 0 \; \forall n$
  - If G is goal, then $h(G) = 0$
  - implements extra domain knowledge
  - depends only on state, not on node (e.g., independent on paths)
- Main strategies:
  - Greedy search
  - A^* search

Example: Straight-Line Distance $h_{SLD}(n)$

- $h(n) \stackrel{\text{def}}{=} h_{SLD}(n)$: straight-line distance heuristic
  - different from actual minimum-path dinstance
  - cannot be computed from the problem description itself



Straight−line distance to Bucharest

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Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
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Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

h(x)

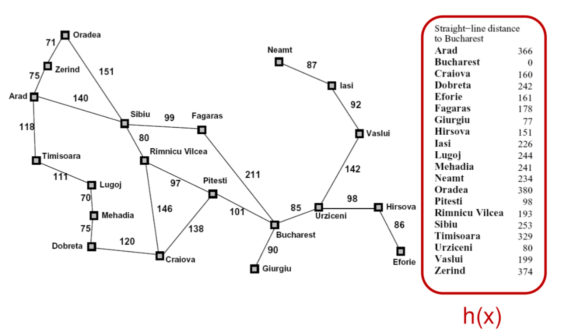
Outline

Greedy Best-First Search Strategy (GBFS)

Greedy Best-First Search (aka Greedy Search)

- Idea: Expand first the node n with lowest estimate cost to the closest goal, $h(n)$
- Implementation: same as uniform-cost search, with $g(n) \stackrel{\text{def}}{=} h(n)$
  $\Longrightarrow$ expands the node that appears to be closest to goal
- both tree and graph versions

Greedy Best-First Search Strategy: Example



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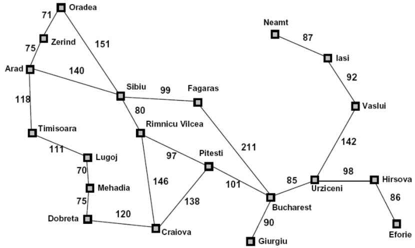

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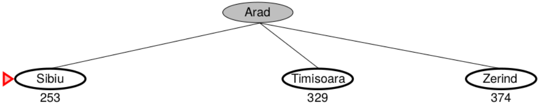
h(x)



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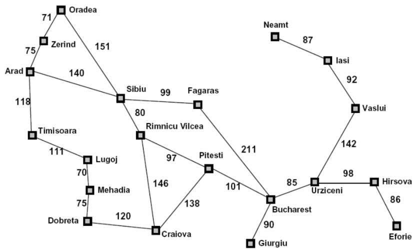
Greedy Best-First Search Strategy: Example



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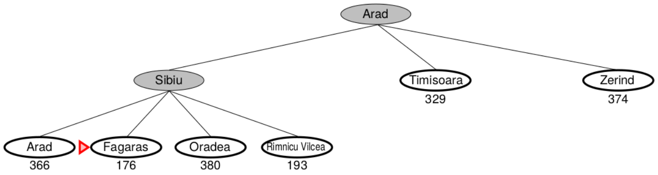
h(x)



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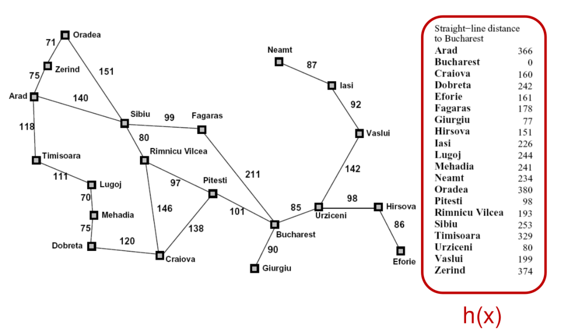
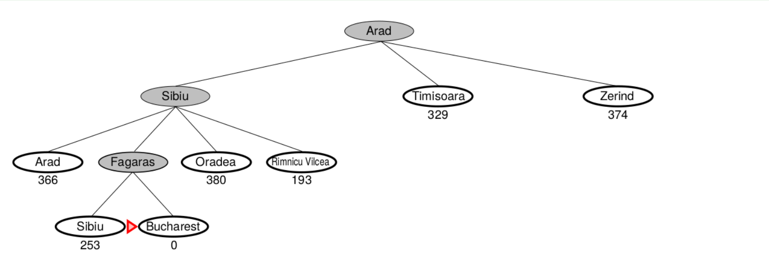
Greedy Best-First Search Strategy: Example





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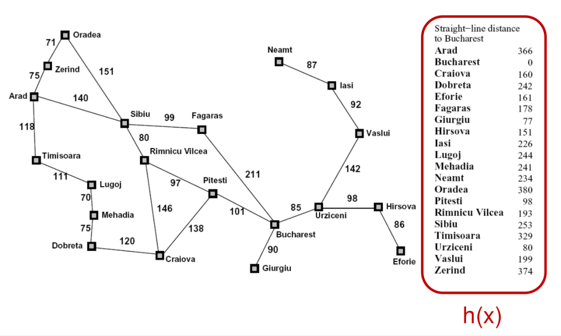

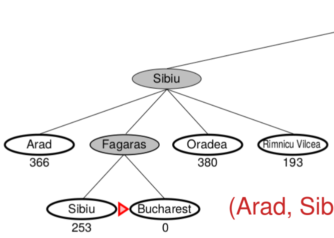

(Arad, Sibiu, Fagaras, Pitesti)

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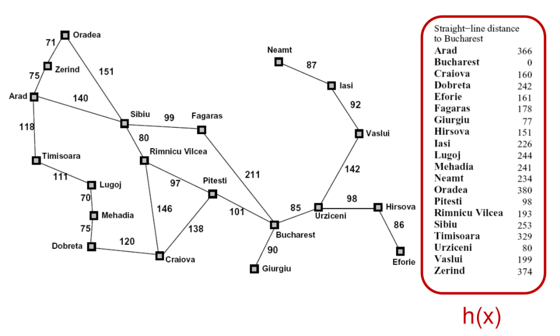
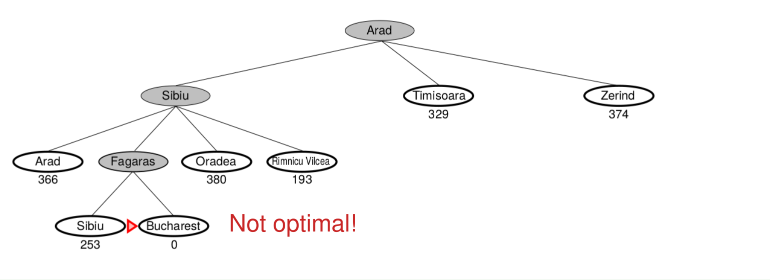
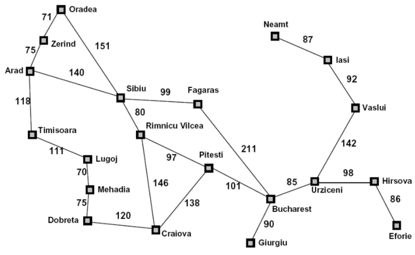
Greedy Best-First Search Strategy: Example





Greedy Best-First Search Strategy: Example



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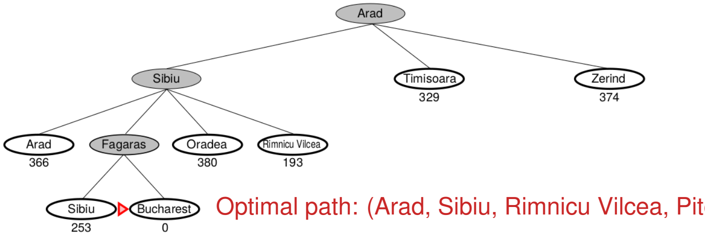
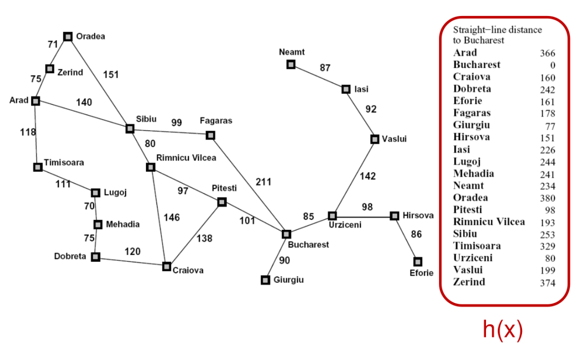
h(x)



Optimal path: (Arad, Sibiu, Rimnicu Vilcea, Pitesti)

Greedy Best-First Search: (Non-)Optimality

- Greedy best-first search is not optimal
  - it is not guaranteed to find the best solution
  - it is not guaranteed to find the best path toward a solution
- picks the node with minimum (estimated) distance to goal, regardless the cost to reach it
  - Ex: when in Sibiu, it picks Fagaras rather than Rimnicu Vicea



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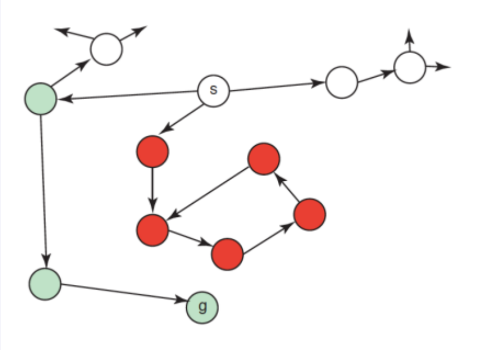
Greedy Best-First Search: (In-)Completeness

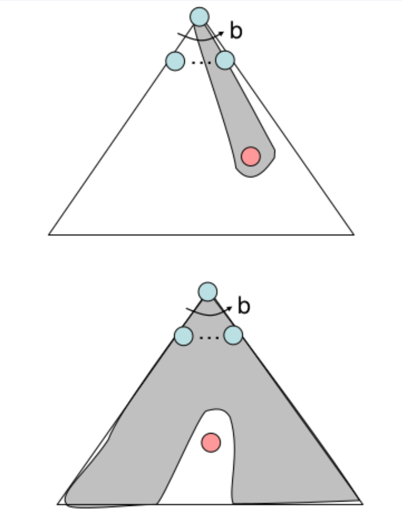
- Tree-based Greedy best-first search is not complete
  - may lead to infinite loops
- Graph-based version complete (if state space finite)
- substantially same completeness issues as DFS



(Courtesy of Maria Simi, UniPI)

Greedy Best-First Search (GBFS): Properties

- How many steps?
  - in worst cases may explore all states
    $\Longrightarrow$ takes $O(b^d)$ time
    if good heuristics:
    $\Longrightarrow$ may give good improvements
- How much memory?
  - max frontier size: b^d
    $\Longrightarrow$ $O(b^d)$ memory size
- Is it complete?
  - tree: no
    graph: yes if space finite
- Is it optimal?
  - No



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Memory requirement is a major problem also for GBFS

Outline

A^* Search Strategy

A^* Search

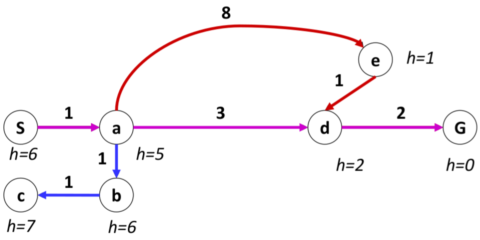
- Best-known form of best-first search
- Idea: avoid expanding paths that are already expensive
- Combine Uniform-Cost and Greedy search: $f(n) = g(n) + h(n)$
  - $g(n)$: cost so far to reach n
  - $h(n)$: estimated cost to goal from n
  - $f(n)$: estimated total cost of path through n to goal

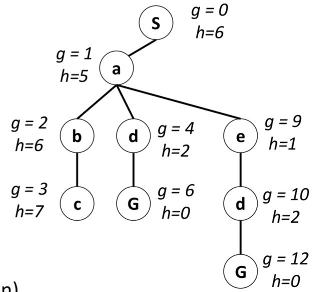
$\Longrightarrow$ Expand first the node n with lowest estimated cost of the cheapest solution through n

- Implementation: same as uniform-cost search, with $g(n) + h(n)$ instead of $g(n)$

A^* Search Strategy: Example

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)





- A* Search orders by the sum: f(n) = g(n) + h(n)

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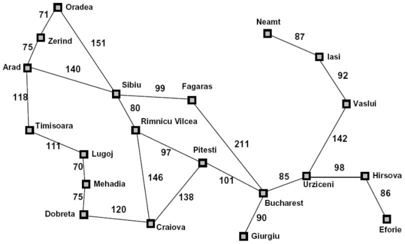
A^* Search Strategy: Example



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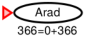
h(x)



Optimal path!

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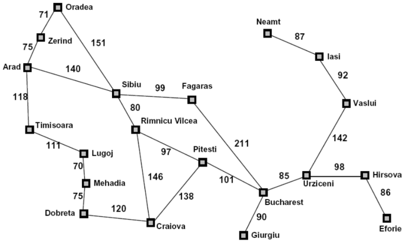
A^* Search Strategy: Example



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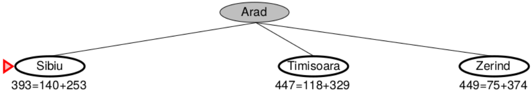
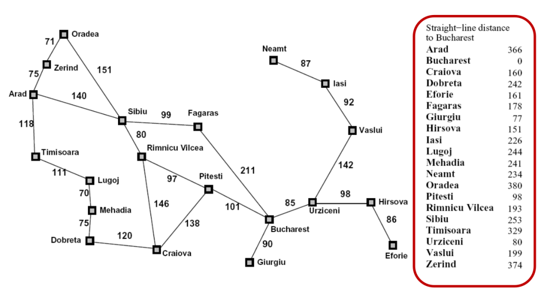
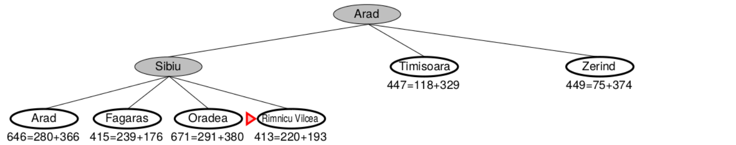


Optimal path!

A^* Search Strategy: Example



h(x)



Optimal path!

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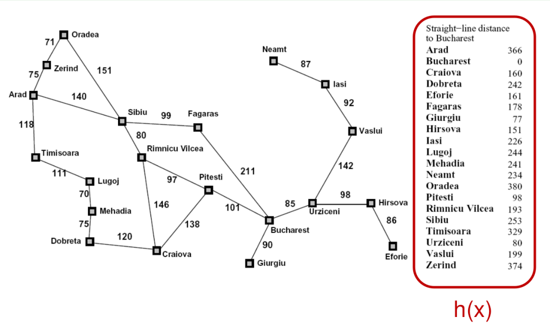
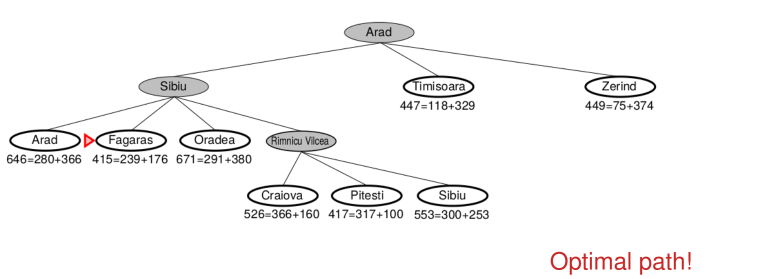
A^* Search Strategy: Example





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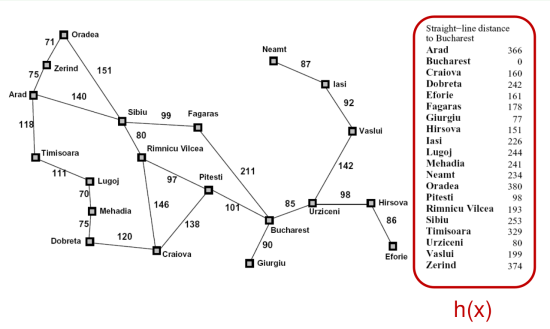
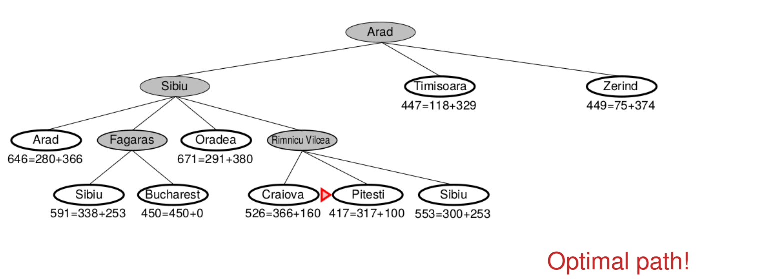
A^* Search Strategy: Example





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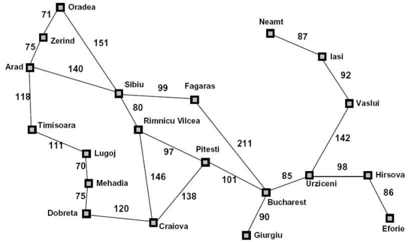

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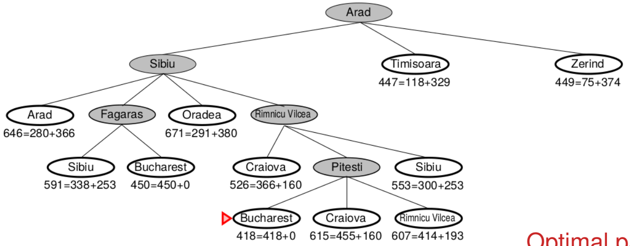


Optimal path!

A^* Search: Admissible and Consistent Heuristics

Admissible heuristics $h(n)$

- $h(n)$ is admissible (aka optimistic) iff it never overestimates the cost to reach the goal:
  - $h(n) \leq h^*(n)$ where $h^*(n)$ is the true cost from n
  - ex: the straight-line distance $h_{SDL}()$ to Bucharest

Consistent heuristics $h(n)$

- $h(n)$ is consistent (aka monotonic) iff, for every successor n' of n generated by any action a with step cost $c(n, a, n')$,
  $h(n) \leq c(n, a, n') + h(n')$
  - verifies the triangular inequality
  - ex: the straight-line distance $h_{SDL}()$ to Bucharest

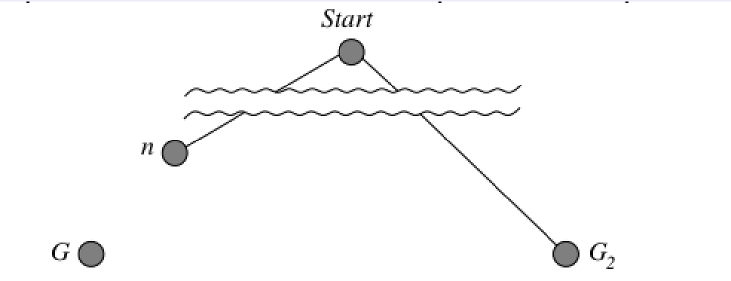
A^* Tree Search: Optimality

If $h(n)$ is admissible, then A^* tree search is optimal

- Suppose some sub-optimal goal G_2 is in the frontier queue.
- n unexpanded node on a shortest path to an optimal goal G.
- then:

$$\begin{aligned} f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\ &> g(G) && \text{since } G_2 \text{ sub-optimal} \\ &\geq f(n) && \text{since } h \text{ is admissible} \end{aligned}$$

$\Longrightarrow$ A^* will not pick G_2 from the frontier queue before n



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A^* Graph Search: Properties

Properties

1. if $h(n)$ is consistent, then $h(n)$ is admissible (straightforward)
2. If h(n) is consistent, then $f(n)$ is non-decreasing along any path:
   - $f(n') = g(n') + h(n') = g(n) + c(n, a, n') + h(n') \geq g(n) + h(n) = f(n)$
3. If (Graph) A^* selects a node n from the frontier, then the optimal path to that node has been found
   - if not so, there would be a node n' in the frontier on the optimal path to n (because of the graph separation property)
   - since f is non-decreasing along any path, $f(n') \leq f(n)$
   - since n' is on the optimal path to n, $f(n') < f(n)$

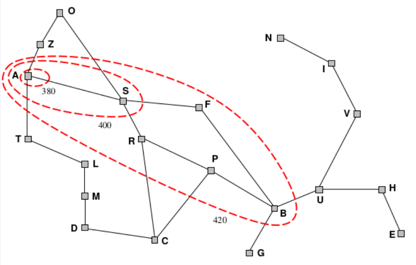
   $\Longrightarrow$ n' would have been selected before n $\square$

$\Longrightarrow$ A^* (graph search) expands nodes in non-decreasing order of f

A^* Graph Search: Optimality

If $h(n)$ is consistent, then A^* graph search is optimal

- A^* expands nodes in order of non-decreasing f value
- Gradually adds "f-contours" of nodes (as BFS adds layers)
  - contour i has all nodes with $f = f_i$, s.t. $f_i < f_{i+1}$
  - cannot expand contour f_{i+1} until contour f_i is fully expanded
- If C^* is the cost of the optimal solution path
  1. A^* expands all nodes s.t. $f(n) < C^*$
  2. A^* might expand some of the nodes on "goal contour" s.t. $f(n) = C^*$ before selecting a goal node.
  3. A^* does not expand nodes s.t. $f(n) > C^*$ (pruning)



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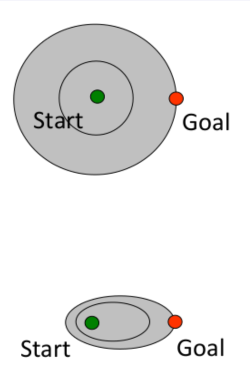
UCS vs A^* Contours

Intuition

- UCS expands equally in all "directions"
- A^* expands mainly toward the goal



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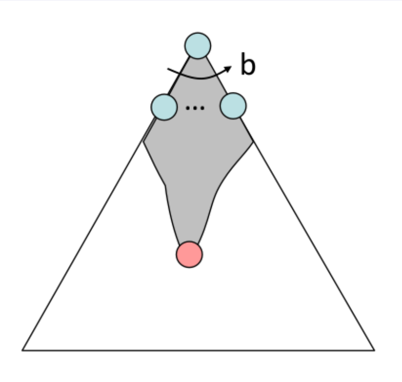
A^* Search: Completeness

If all step costs exceed some finite ϵ and b is finite,
then there are only finitely many nodes n s.t. $f(n) \leq C^*$
$\Longrightarrow$ A^* is complete.

A^* Search: Properties

Let $\epsilon \stackrel{\text{def}}{=} (h^+ - h)/h^*$ (relative error)
b^ϵ: effective branching factor

- How many steps?
  - takes $O((b^\epsilon)^d)$ time
    if good heuristics, may give dramatic improvements
- How much memory?
  - Keeps all nodes in memory
    $\Longrightarrow O(b^d)$ memory size
    (like UCS)
- Is it complete?
  - yes
- Is it optimal?
  - yes



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Memory requirement is a major problem also for A^*

Memory-bounded Heuristic Search (hints)

Some solutions to A^* space problems
(maintain completeness and optimality)

- Iterative-deepening A^* (IDA^*)
  - here cutoff information is the f-cost (g+h) instead of depth
- Recursive best-first search(RBFS)
  - attempts to mimic standard best-first search with linear space
- (simple) Memory-bounded A^* ((S)MA^*)
  - drop the worst-leaf node when memory is full

Outline

Admissible Heuristics

Main problem

What is the best admissible/consistent heuristic?

Dominance of Admissible Heuristics

Dominance

Let $h_1(n), h_2(n)$ admissible heuristics.

- $h_2(n)$ dominates $h_1(n)$ iff $h_2(n) \geq h_1(n)$ for all n.

$\Longrightarrow$ $h_2(n)$ is better for search

- is nearer to $h^*(n)$

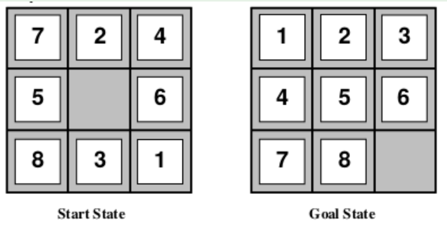
Let $h_1(n), h_2(n)$ admissible heuristics. Let $h_{12} \stackrel{\text{def}}{=} max(h_1(n), h_2(n))$.

- h_{12} is also admissible
- h_{12} dominates both $h_1(n), h_2(n)$

Admissible Heuristics: Example

Ex: Heuristics for the 8-puzzle

- $h_1(n)$: number of misplaced tiles
- $h_2(n)$: total Manhattan distance over all tiles
  - (i.e., # of squares from desired location of each tile)
- $h_1(S)$? 6
- $h_2(S)$? 4+0+3+3+1+0+2+1 = 14
- $h^*(S)$? 26
- both $h_1(n), h_2(n)$ admissible ($\leq$ number of actual steps to solve)
- $h_2(n)$ dominates $h_1(n)$



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Quality of Heuristics

Effective branching factor

- Effective branching factor b^*: the branching factor that a uniform tree of depth d would have in order to contain N+1 nodes

$$N + 1 = 1 + b^* + (b^*)^2 + ... + (b^*)^d$$

  N being the number of nodes generated by the A^* search
- ex: if d=5 and $N = 52$, then $b^* = 1.92$
- experimental measure of b^* is fairly constant for hard problems
  $\Longrightarrow$ can provide a good guide to the heuristic's overall usefulness
- Ideal value of b^* is 1

Admissible Heuristics: Example [cont.]

Average performances on 100 random samples of 8-puzzle

	Search Cost (nodes generated)			Effective Branching Factor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	–	539	113	–	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

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$\Longrightarrow$ Dramatic performance improvement

Admissible Heuristics from Relaxed Problems

Idea: Admissible heuristics can be derived from the exact solution cost of a relaxed version of the problem

- Relaxed 8-puzzle: a tile can move from any tile to any other tile
  $\Longrightarrow$ $h_1(n)$ gives the shortest solution
- Relaxed 8-puzzle: a tile can move to any adjacent square
  $\Longrightarrow$ $h_2(n)$ gives the shortest solution

- The relaxed problem adds edges to the state space
  $\Longrightarrow$ any optimal solution in the original problem is also a solution in the relaxed problem
  $\Longrightarrow$ the cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- the derived heuristic is an exact cost for the relaxed problem
  $\Longrightarrow$ must obey the triangular inequality
  $\Longrightarrow$ consistent

Inferring Automatically Admissible Heuristics

Idea: If a problem definition is written down in a formal language, it is possible to construct relaxed problems automatically

Example

- 8-puzzle actions:

  a tile can move from square A to square B if
  A is horizontally or vertically adjacent to B, and
  B is blank

- we can generate three relaxed problems by removing one or both of the conditions

  (*a*) a tile can move from square A to square B if A is adjacent to B

  (*b*) a tile can move from square A to square B if B is blank

  (*c*) a tile can move from square A to square B

$\Longrightarrow$ (a) corresponds to $h_2(n)$, (c) corresponds to $h_1(n)$,

The tool ABSolver can generate such heuristics automatically.

Learning Admissible Heuristics

- Another way to find an admissible heuristic is through learning from experience:
  - Experience = solving lots of 8-puzzles
  - An inductive learning algorithm can be used to predict costs for other states that arise during search