Fundamentals of Artificial Intelligence
Chapter 14: **Probabilistic Reasoning**

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M.S. Course "Artificial Intelligence Systems", academic year 2020-2021

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Outline

1. Bayesian Networks
2. Constructing Bayesian Networks
3. Exact Inference with Bayesian Networks

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Bayesian Networks

- Bayesian Networks (aka Belief Networks):
  - allow for compact specification of full joint distributions
  - represent explicit conditional dependencies among variables: an arc from X to Y means that X has a direct influence on Y
- Syntax: a directed acyclic graph (DAG):
  - each node represents a random variable (discrete or continuous)
  - directed arcs connect pairs of nodes: $X \rightarrow Y$ (X is a parent of Y)
  - a conditional distribution $\mathbf{P}(X_i | Parents(X_i))$ for each node X_i
- Conditional distribution represented as a conditional probability table (CPT)
  - distribution over X_i for each combination of parent values
- Topology encodes conditional independence assertions:
  - Toothache and Catch conditionally independent given Cavity
  - Tootchache, Catch depend on Cavity
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- No arc $\Longleftrightarrow$ independence

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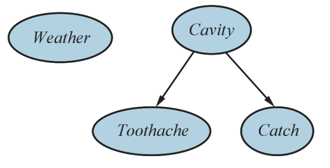

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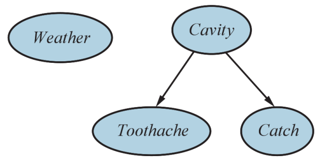

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Example (from Judea Pearl, UCLA)

"*The burglary alarm goes off very likely on burglary and occasionally on earthquakes. John and Mary are neighbors who agreed to call when the alarm goes off. Their reliability is different ...*"

- Variables: Burglary, Earthquake, Alarm, JohnCalls, MaryCalls
- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call
- CPTs:
  - alarm setoff if bunglar in 94% of cases
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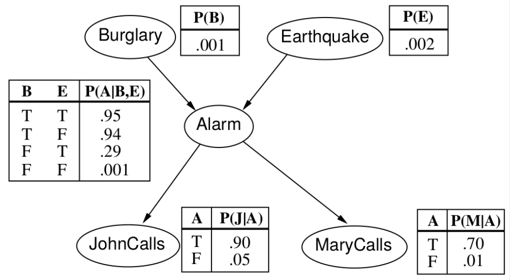

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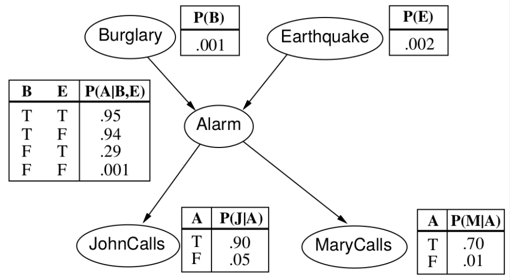

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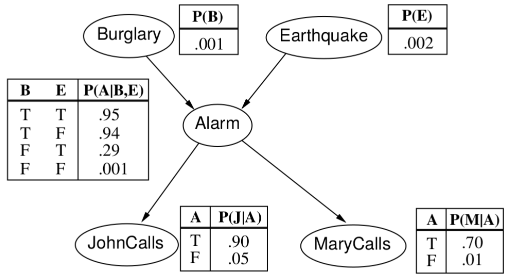

Compactness of Bayesian Networks

- In most domains, it is reasonable to suppose that each random variable is directly influenced by at most k others, for some k.
- A CPT for Boolean X_i with k Boolean parents has
  - 2^k rows for the combinations of parent values
  - each row requires one number p for $P(X_i = true)$ $(P(X_i = false) = 1 - P(X_i = true))$

$\Longrightarrow$ If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

  - a full joint distribution requires $2^n - 1$ numbers
  - linear vs. exponential!
- Ex: for burglary example:
  - $1 + 1 + 4 + 2 + 2 = 10$ numbers vs. $2^5 - 1 = 31$

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Global Semantics of Bayesian Networks

- Global semantics defines the full joint distribution as the product of the local conditional distributions:
  $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
  - if X_i has no parent, then prior probability $\mathbf{P}(X_i)$
- Intuition: order $X_1, ..., X_n$ s.t. $parents(X_i) \prec X_i$ for each i:
  $\mathbf{P}(X_1, ..., X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, ..., X_{i-1})$ // chain rule
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Global Semantics: Example

- $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i | parents(X_i))$
- Ex: “Prob. that both John and Mary call, the alarm sets off but no burglary nor earthquake”

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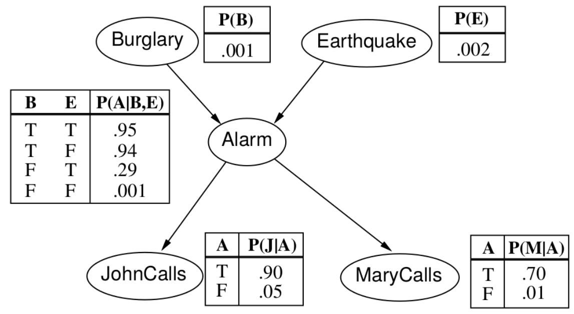

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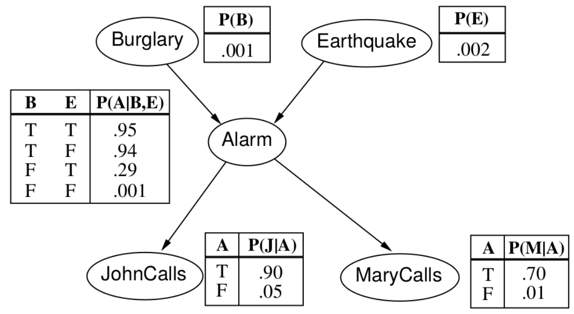

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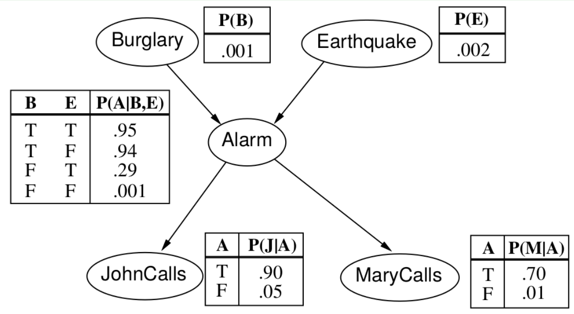

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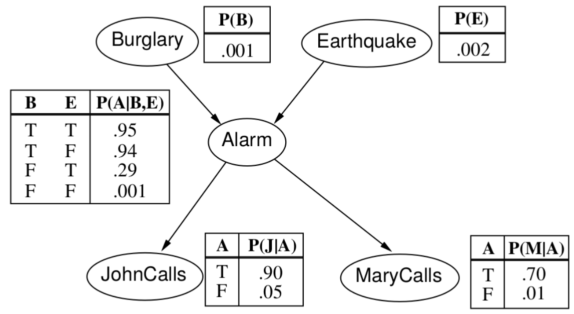

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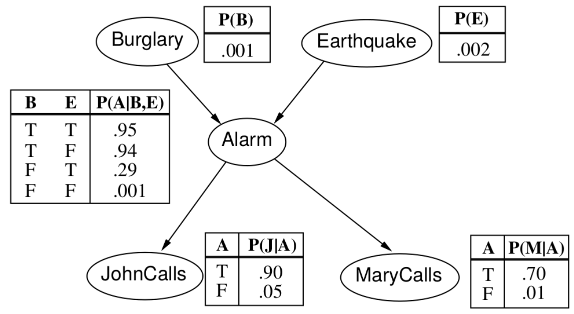

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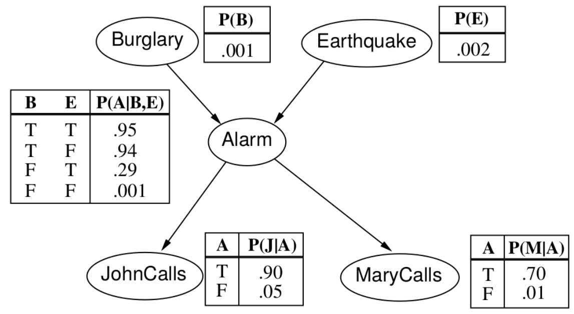

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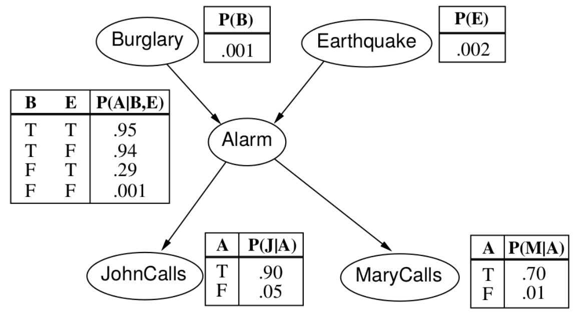

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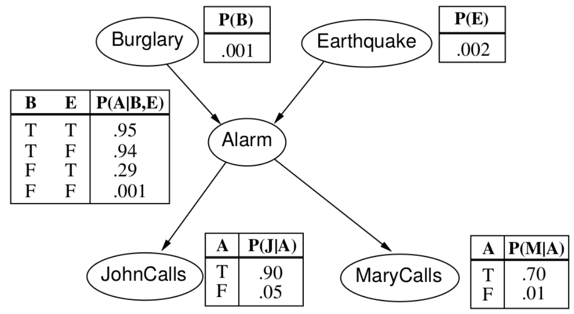

P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

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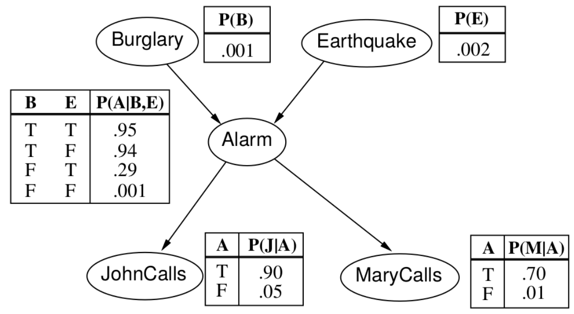

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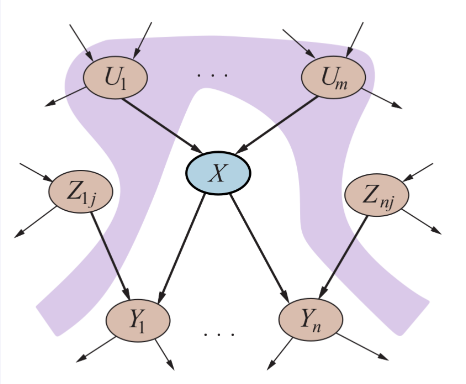
Exercises

- Compute:
  - The probability that John calls and Mary does not, the alarm is not set off with a burglar entering during an earthquake
  - The probability that John calls and Mary does not, given a burglar entering the house
  - The probability of an earthquake given the fact that John has called
  - ...

Local Semantics

- Local Semantics: each node is conditionally independent of its nondescendants given its parents:
  $\mathbf{P}(X|U_1, .., U_m, Z_{1j}, ..., Z_{nj}) = \mathbf{P}(X|U_1, .., U_m)$, for each X
- Theorem: Local semantics holds iff global semantics holds:
  $\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i|parents(X_i))$



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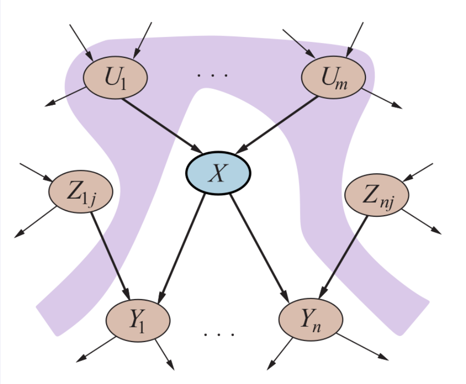

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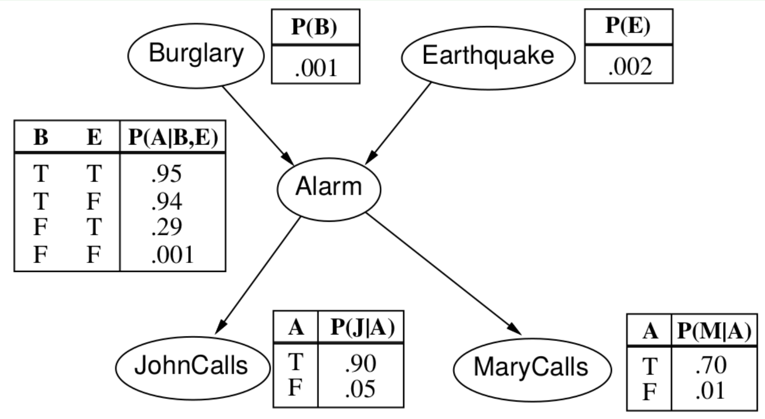
Local Semantics: Example

Ex: JohnCalls is independent of Burglary, Earthquake, and MaryCalls given the value of Alarm

$$\mathbf{P}(JohnCalls \mid Alarm, Burglary, Earthquake, MaryCalls) = \mathbf{P}(JohnCalls \mid Alarm)$$



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

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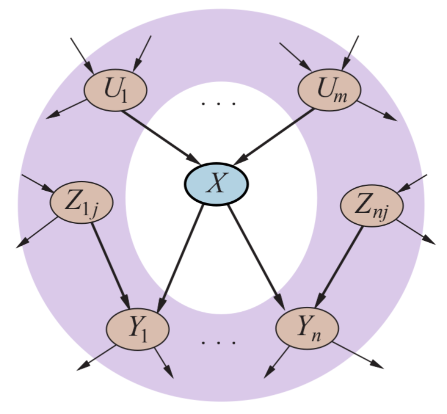
Independence Property: Markov Blanket

In an B.N., each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents:

$\mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj},W_1,...,W_k) =$
$\mathbf{P}(X|U_1,..,U_m,Y_1,..,Y_n,Z_{1j},...,Z_{nj})$, for each X



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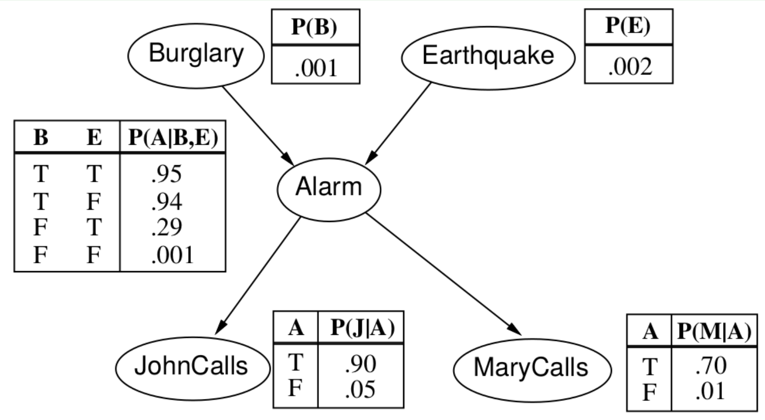
Markov Blanket: Example

Ex: Burglary is independent of JohnCalls and MaryCalls, given Alarm and Earthquake

$$\mathbf{P}(Burglary | Alarm, Earthquake, JohnCalls, MaryCalls) = \mathbf{P}(Burglary | Alarm, Earthquake)$$



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Exercise

Verify numerically the two previous examples:

- Local Semantics
- Markov Blanket

Outline

Constructing Bayesian Networks

Building the graph

Given a set of random variables

1. Choose an ordering $\{X_1, ..., X_n\}$
   - in principle, any ordering will work (but some may cause blowups)
   - general rule: follow causality, $X \prec Y$ if $X \in causes(Y)$
2. For i=1 to n do
   1. add X_i to the network
   2. as $Parents(X_i)$, choose a subset of $\{X_1, ..., X_{i-1}\}$ s.t. $P(X_i|Parents(X_i)) = P(X_i|X_1, ..., X_{i-1})$

$\Longrightarrow$ Guarantees the global semantics by construction

$$\mathbf{P}(X_1, ..., X_N) = \prod_{i=1} \mathbf{P}(X_i|parents(X_i))$$

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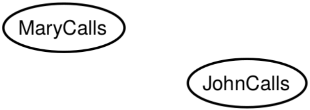

$P(J|M) = P(J)$?

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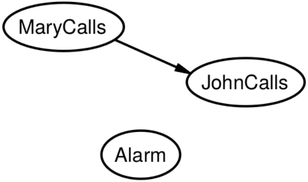

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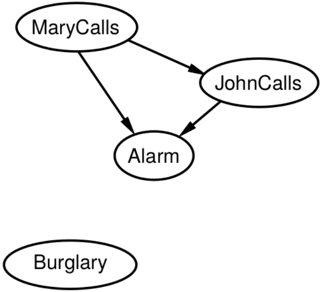

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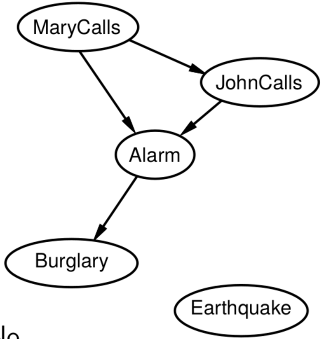

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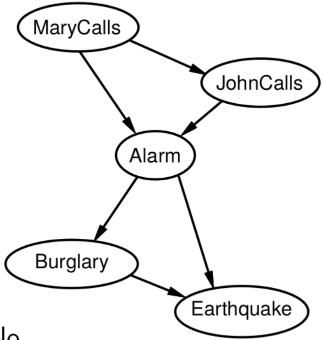

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Constructing Bayesian Networks: Example [cont.]

- In non-causal directions
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  - assessing conditional probabilities is hard
  - typically networks less compact
- Ex: $1+2+4+2+4=13$ numbers needed (rather than 10)
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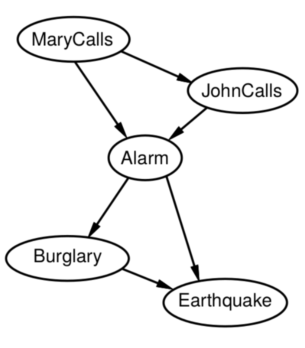

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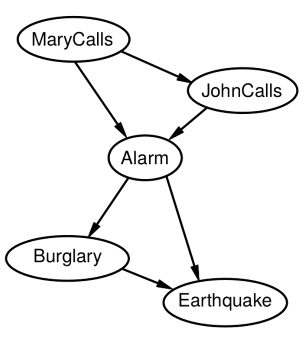

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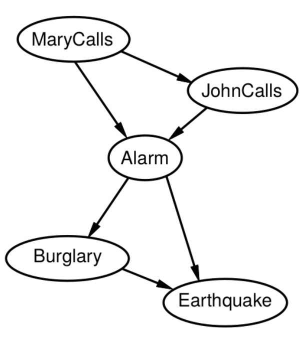

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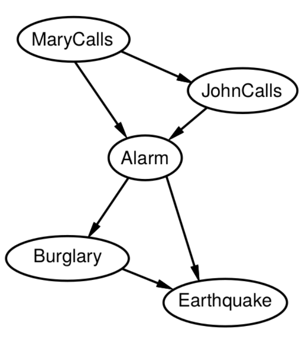

Building Conditional Probability Tables, CPTs

- Problem: CPT grow exponentially with number of parents
- If the causes don't interact: use a Noisy-OR distribution
  - let parents $U_1, ..., U_k$ include all causes (can add leak node)
  - add Independent failure probability q_i for each cause U_i

  $\Longrightarrow P(\neg X | U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = \prod_{i=1}^{j} q_i$
  - number of parameters linear in number of parents!
- Ex: $q_{Cold} = 0.6, \; q_{Flu} = 0.2, \; q_{Malaria} = 0.1$:

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$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	**0.0**	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network

Outline

1. Bayesian Networks
2. Constructing Bayesian Networks
3. Exact Inference with Bayesian Networks

Exact inference in Bayesian Networks

- Given:
  - X: the query variable (we assume one for simplicity)
  - **E/e**: the set of evidence variables $\{E_1, ..., E_m\}$ and of evidence values $\{e_1, ..., e_m\}$
  - **Y/y**: the set of unknown variables (aka hidden variables) $\{Y_1, ..., Y_l\}$ and unknown values $\{y_1, ..., y_l\}$

  $\Longrightarrow$ $\mathbf{X} = X \cup \mathbf{E} \cup \mathbf{Y}$

  A typical query asks for the posterior probability distribution: **P**(X | **E**=**e**) (also written **P**(X | **e**))
- Ex: $P(Burglar|JohnCalls = true, MaryCalls = true)$
  - query: $Burglar$
  - evidence variables: $\mathbf{E} = \{JohnCalls, MaryCalls\}$
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Inference by Enumeration

- We defined a procedure for the task as:
  $\mathbf{P}(X|\mathbf{e}) = \alpha\mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$

$\Longrightarrow$ $\mathbf{P}(X, \mathbf{e}, \mathbf{y})$ can be rewritten as product of prior and conditional probabilities according to the Bayesian Network

  - then apply factorization and simplify algebraically when possible

- Ex:
  $\mathbf{P}(B|j, m) =$
  $\alpha \sum_e \sum_a \mathbf{P}(B, e, a, j, m) =$
  $\alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B, e)P(j|a)P(m|a) =$
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- Recursive depth-first enumeration:
  $O(n)$ space, $O(2^n)$ time with n propositional variables
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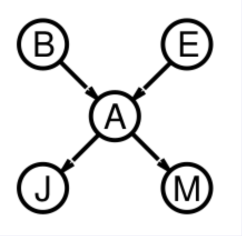

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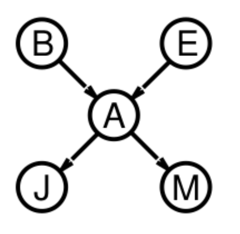

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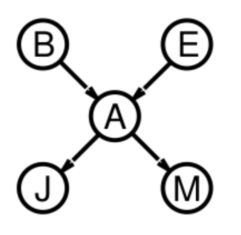

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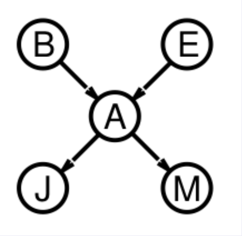
  $\alpha\sum_e\sum_a \mathbf{P}(B, e, a, j, m) =$
  $\alpha\sum_e\sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B, e)P(j|a)P(m|a) =$
  $\alpha\mathbf{P}(B)\sum_e P(e)\sum_a \mathbf{P}(a|B, e)P(j|a)P(m|a)$

$\Longrightarrow$ $P(b|j, m) =$
$\alpha P(b)\sum_e P(e)\sum_a P(a|b, e)P(j|a)P(m|a)$



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- Recursive depth-first enumeration:
  $O(n)$ space, $O(2^n)$ time with n propositional variables
- Enumeration is inefficient: repeated computation

Enumeration Algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X computes $\mathbf{P}(X \mid \mathbf{e})$
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* $\mathbf{Y}$ = *hidden variables* */

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$) computes P(xi, **Y**, **e**) (single probability value)
      where $\mathbf{e}_{x_i}$ is $\mathbf{e}$ extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

---

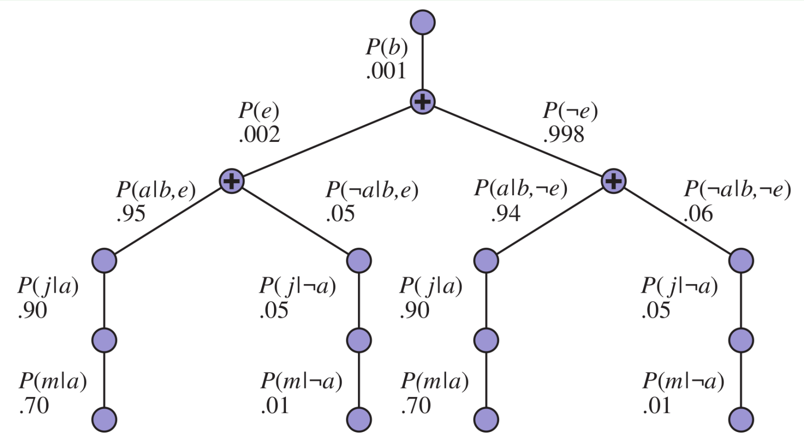
function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$) X or evidence var
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$) hidden var
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Inference by Enumeration: Example

$P(\ \ b|j,m) = \alpha P(\ \ b) \sum_e P(e) \sum_a P(a|\ \ b,e) P(j|a) P(m|a) = \alpha \cdot 0.00059224$

$P(\neg b|j,m) = \alpha P(\neg b) \sum_e P(e) \sum_a P(a|\neg b,e) P(j|a) P(m|a) = \alpha \cdot 0.0014919$

$\Longrightarrow \mathbf{P}(B|j,m) = \alpha \cdot \langle 0.00059224, 0.0014919 \rangle = [normal.] \approx \langle 0.284, 0.716 \rangle$



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Repeated computation: $P(j|a)P(m|a)$ & $P(j|\neg a)P(m|\neg a)$ for each value of e

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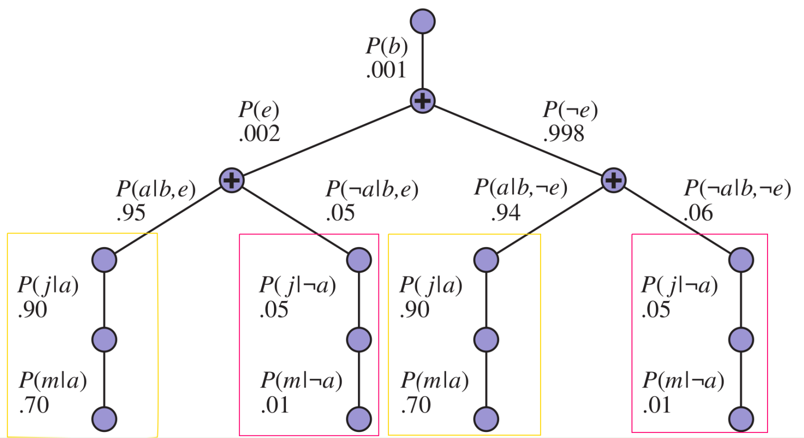

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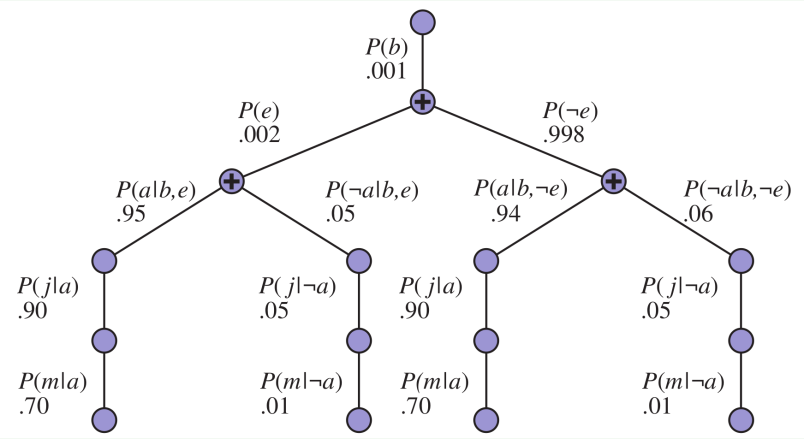

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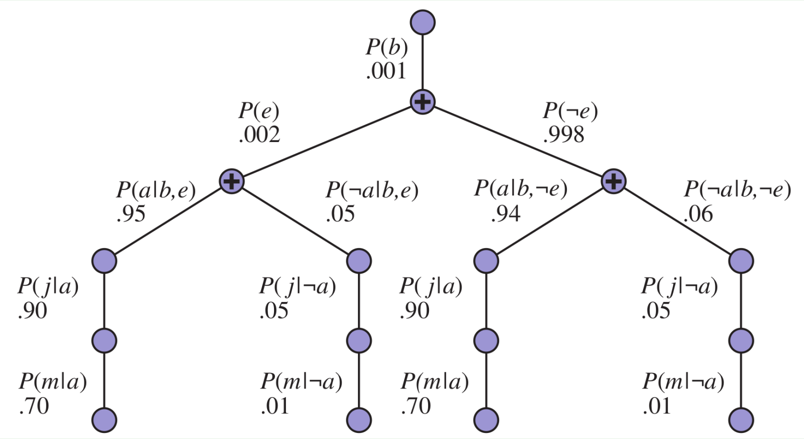

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Exercises

1. Consider the probabilistic Wumpus World of previous chapter
   (a) Describe it as a Bayesian network
   (b) Compute the query $P(P_{1,3}|b^*, p^*)$ via enumeration
   (c) Compare the result with that of the example in Ch. 13

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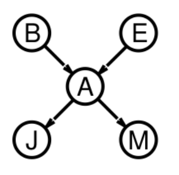

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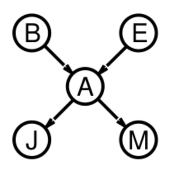

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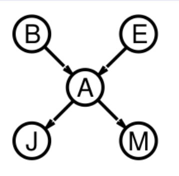

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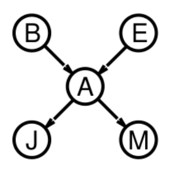

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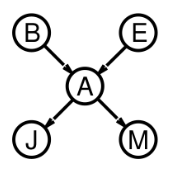

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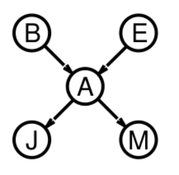

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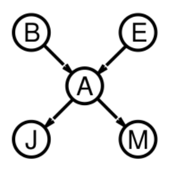

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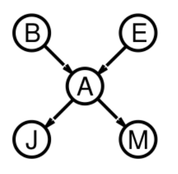

- “$\times$” is the pointwise product (see later)
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- $\mathbf{f}_{\bar{A}...}(.)$: summation over the values of A...

Inference by Variable Elimination

- Variable elimination:
  - carry out summations right-to-left (i.e., bottom-up in the tree)
  - store intermediate results (factors) to avoid recomputation
- Ex: $P(B|j,m)$

$$= \alpha \overbrace{P(B)}^{B} \sum_e \overbrace{P(e)}^{E} \sum_a \overbrace{\mathbf{P}(a|B,e)}^{A} \overbrace{P(j|a)}^{J} \overbrace{P(m|a)}^{M}$$
$$= \alpha P(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e) P(j|a) \times \mathbf{f}_M(A)$$
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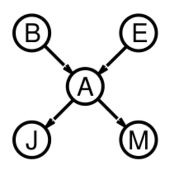

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Variable Elimination: Basic Operations

- Factor summation: $\mathbf{f}_3(X_1, ..., X_j) = \mathbf{f}_1(X_1, ..., X_j) + \mathbf{f}_2(X_1, ..., X_j)$
  - standard matrix summation:

$$\begin{bmatrix} a_{11} & a_{21} & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n1} & \dots \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & \dots \\ \dots & \dots & \dots \\ b_{n1} & b_{n1} & \dots \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots \\ \dots & \dots & \dots \\ a_{n1}+b_{n1} & a_{n1}+b_{n1} & \dots \end{bmatrix}$$

  - must have the same argument variables
- Pointwise product: Multiply the array elements for the same variable values
- Ex:

$$\mathbf{f}_J(A) \times \mathbf{f}_M(A) = \begin{bmatrix} P(j|\ \ a) \\ P(j|\neg a) \end{bmatrix} \times \begin{bmatrix} P(m|\ \ a) \\ P(m|\neg a) \end{bmatrix} = \begin{bmatrix} P(j|\ \ a)P(m|\ \ a) \\ P(j|\neg a)P(m|\neg a) \end{bmatrix}$$

- General case:
  $\mathbf{f}_3(X_1, ..., X_j, Y_1, ..., Y_k, Z_1, ..., Z_l) =$
  $\mathbf{f}_1(X_1, ..., X_j, Y_1, ..., Y_k) \times \mathbf{f}_2(Y_1, ..., Y_k, Z_1, ..., Z_l)$
  - union of arguments
  - values: $\mathbf{f}_3(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{f}_1(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}_2(\mathbf{y}, \mathbf{z})$
  - matrix size: $\mathbf{f}_1 : 2^{j+k}$, $\mathbf{f}_1 : 2^{k+l}$, $\mathbf{f}_3 : 2^{j+k+l}$

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Variable Elimination: Basic Operations

- $\mathbf{f}_3(A,B,C) = \mathbf{f}_1(A,B) \times \mathbf{f}_2(B,C)$
- Summing out one variable:
  $f(B,C) = \sum_a f_3(A,B,C) = f_3(a,B,C) + f_3(\neg a,B,C) =$
  $\begin{bmatrix} 0.06 & 0.24 \\ 0.42 & 0.28 \end{bmatrix} + \begin{bmatrix} 0.18 & 0.72 \\ 0.06 & 0.04 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.96 \\ 0.48 & 0.32 \end{bmatrix}$

A	B	$\mathbf{f}_1(A,B)$	B	C	$\mathbf{f}_2(B,C)$	A	B	C	$\mathbf{f}_3(A,B,C)$
T	T	.3	T	T	.2	T	T	T	$.3 \times .2 = .06$
T	F	.7	T	F	.8	T	T	F	$.3 \times .8 = .24$
F	T	.9	F	T	.6	T	F	T	$.7 \times .6 = .42$
F	F	.1	F	F	.4	T	F	F	$.7 \times .4 = .28$
						F	T	T	$.9 \times .2 = .18$
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Variable Elimination Algorithm

```
function ELIMINATION-ASK(X, e, bn) returns a distribution over X
   inputs: X, the query variable
           e, observed values for variables E
           bn, a Bayesian network specifying joint distribution P(X_1, ..., X_n)

   factors ← [ ]
   for each var in ORDER(bn.VARS) do
       factors ← [MAKE-FACTOR(var, e)|factors]
       if var is a hidden variable then factors ← SUM-OUT(var, factors)
   return NORMALIZE(POINTWISE-PRODUCT(factors))
```


- Efficiency depends on variable ordering ORDER(...)
- Efficiency improvements:
  - factor out of summations factors not depending on sum variable
  - remove irrelevant variables

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Factor Out Constant Factors

- If $f_1, ..., f_i$ do not depend on X, then move them out of a $\sum_x (...)$:
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  $\mathbf{P}(J|b) = \alpha \sum_e \sum_a \sum_m \mathbf{P}(J, m, b, e, a) =$
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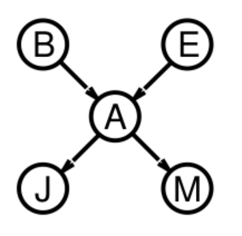

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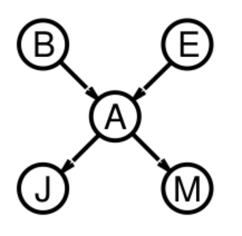

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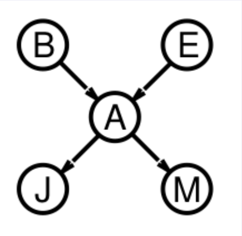

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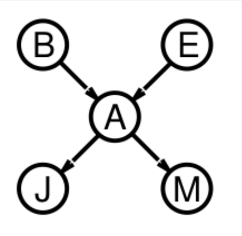

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- Ex: $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
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$\implies$ *MaryCalls* is irrelevant

- Related to backward-chaining with Horn clauses

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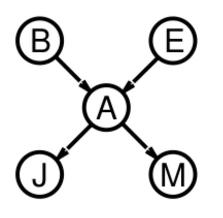

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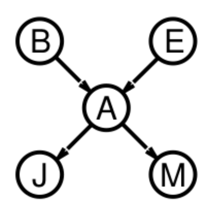

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Remove Irrelevant Variables

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  - $\sum_y P(y|...) = 1 \Longrightarrow$ can be dropped
- Ex: $\mathbf{P}(JohnCalls|Burglary = true)$:
  $\mathbf{P}(J|b) = ... =$
  $\alpha P(b) \sum_e P(e) \sum_a P(a|b,e) \mathbf{P}(J|a) \overbrace{\sum_m P(m|a)}^{1}$
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- Ex: $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and $Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$

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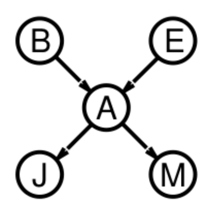

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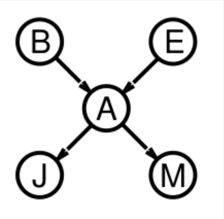

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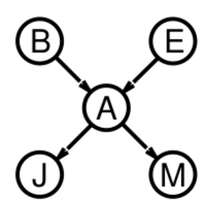

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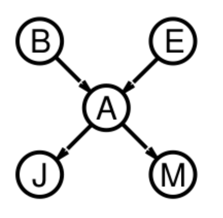

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Exercises

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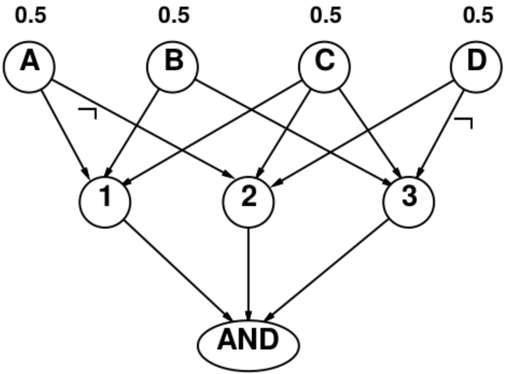

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"That's all Folks!"