

# Course Automated Reasoning and Formal Verification

## Module I: Automated Reasoning

### Ch. 02: **Satisfiability Modulo Theories (SMT)**

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- 1 Introduction
  - Basics on First-order Logic
  - What is a Theory?
  - Satisfiability Modulo Theories
  - Motivations and Goals of SMT
- 2 Efficient SMT solving
  - Combining SAT with Theory Solvers
  - Theory Solvers for Theories of Interest
  - SMT for Combinations of Theories
- 3 Beyond Solving: Advanced SMT Functionalities
  - Proofs and Unsatisfiable Cores
  - All-SMT & Predicate Abstraction (hints)
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# First-Order Logic (FOL)

- PL assumes world contains **facts**
  - atomic events
- FOL is **structured**: a world/state includes **objects**, each of which may have **attributes** of its own as well as **relationships** to other objects
- FOL assumes the world contains:
  - **Objects**:  
e.g., **people, houses, numbers, theories, Jim Morrison, colors, basketball games, wars, centuries, ...**
  - **Relations**:  
e.g., **red, round, bogus, prime, tall ...,**  
**brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...**
  - **Functions**:  
e.g., **father of, best friend, one more than, end of, ...**
- Allows to **quantify** on objects
  - ex: **“All man are equal”, “some persons are left-handed”, ...**

# Syntax of FOL: Basic Elements

- **Constant symbols:** KingJohn, 2, UniversityofTrento,...
- **Predicate symbols:** Man(.), Brother(.,.), (. > .), AllDifferent(...),...
  - may have different arities (1,2,3,...)
  - may be **prefix** (e.g. Brother(.,.)) or **infix** (e.g. (. > .))
- **Function symbols:** Sqrt, LeftLeg, MotherOf
  - may have different arities (1,2,3,...)
  - may be **prefix** (e.g. Sqrt(.)) or **infix** (e.g. (. + .))
- **Variable symbols:** x, y, a, b, ...
- **Propositional Connectives:**  $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$
- **Equality:** “=” (also “ $\neq$ ” s.t. “ $a \neq b$ ” shortcut for “ $\neg(a = b)$ ”)
- **Quantifiers:** “ $\forall$ ” (“forall”), “ $\exists$ ” (“exists”, aka “for some”)
- **Punctuation Symbols:** “,”, “(”, “)”

- Constants symbols are 0-ary function symbols
- Propositions are 0-ary predicates  $\implies$  PL subcase of FOL
- **Signature:** the set of predicate, function & constant symbols

# FOL: Syntax

- Terms:

- constant or variable or  $function(term_1, \dots, term_n)$
- ex: KingJohn, x, LeftLeg(Richard),  $(z * \log(2))$
- denote objects in the real world (aka domain)

- Atomic sentences (aka atomic formulas):

- $\top, \perp$
- proposition or  $predicate(term_1, \dots, term_n)$  or  $term_1 = term_2$
- $(Length(LeftLeg(Richard)) > Length(LeftLeg(KingJohn)))$
- denote facts

- Non-atomic sentences/formulas:

- $\neg\alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta, \alpha \leftrightarrow \beta, \alpha \oplus \beta,$   
 $\forall x.\alpha, \exists x.\alpha$  s.t.  $x$  (typically) occurs in  $\alpha$
- Ex:  $\forall y.(Italian(y) \rightarrow President(Mattarella, y))$   
 $\exists x\forall y.President(x, y) \rightarrow \forall y\exists x.President(x, y)$   
 $\forall x.(P(x) \wedge Q(x)) \leftrightarrow ((\forall x.P(x)) \wedge (\forall x.Q(x)))$   
 $\forall x.(((x \geq 0) \wedge (x \leq \pi)) \rightarrow (sin(x) \geq 0))$
- denote (complex) facts

# Truth in FOL: Intuitions

- Sentences are true with respect to a **model**
  - containing a **domain** and an **interpretation**
- The **domain** contains  $\geq 1$  objects (**domain elements**) and relations and functions over them
- An **interpretation** specifies referents for
  - **variables**  $\rightarrow$  objects
  - **constant symbols**  $\rightarrow$  objects
  - **predicate symbols**  $\rightarrow$  relations
  - **function symbols**  $\rightarrow$  functional relations
- An atomic sentence  $P(t_1, \dots, t_n)$  is true in an interpretation iff the objects referred to by  $t_1, \dots, t_n$  are in the relation referred to by  $P$

# FOL: Semantics

## FOL Models (aka possible worlds)

- A model  $\mathcal{M}$  is a pair  $\langle \mathcal{D}, \mathcal{I} \rangle$  ( $\langle$ *domain*, *interpretation* $\rangle$ )
- **Domain  $\mathcal{D}$** : a **non-empty** set of objects (aka **domain elements**)
- **Interpretation  $\mathcal{I}$** : a (non-injective) map on elements of the signature
  - **constant symbols  $\mapsto$  domain elements**:  
a constant symbol  $C$  is mapped into a particular object  $[C]^{\mathcal{I}}$  in  $\mathcal{D}$
  - **predicate symbols  $\mapsto$  domain relations**:  
a  $k$ -ary predicate  $P(\dots)$  is mapped into a subset  $[P]^{\mathcal{I}}$  of  $\mathcal{D}^k$   
(i.e., the set of object tuples satisfying the predicate in this world)
  - **functions symbols  $\mapsto$  domain functions**:  
a  $k$ -ary function  $f$  is mapped into a domain function  $[f]^{\mathcal{I}} : \mathcal{D}^k \mapsto \mathcal{D}$  ( $[f]^{\mathcal{I}}$  must be total)

(we denote by  $[\cdot]^{\mathcal{I}}$  the result of the interpretation  $\mathcal{I}$ )

An **Interpretation  $\mathcal{I}$**  is extended to assign domain values to variables, domain values to terms and truth values to formulas.

# FOL: Semantics [cont.]

## Interpretation of terms

### $\mathcal{I}$ maps terms into domain elements

- Variables are assigned domain values
  - variables  $\mapsto$  domain elements:  
a variable  $x$  is mapped into a particular object  $[x]^{\mathcal{I}}$  in  $\mathcal{D}$
- A term  $f(t_1, \dots, t_k)$  is mapped by  $\mathcal{I}$  into the value  $[f(t_1, \dots, t_k)]^{\mathcal{I}}$  returned by applying the domain function  $[f]^{\mathcal{I}}$ , into which  $f$  is mapped, to the values  $[t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}}$  obtained by applying recursively  $\mathcal{I}$  to the terms  $t_1, \dots, t_k$ :
  - $[f(t_1, \dots, t_k)]^{\mathcal{I}} = [f]^{\mathcal{I}}([t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}})$
  - Ex: if “Me, Mother, Father” are interpreted as usual, then “Mother(Father(Me))” is interpreted as my (paternal) grandmother
  - Ex: if “+, -, ·, 0, 1, 2, 3, 4” are interpreted as usual, then “(3 - 1) · (0 + 2)” is interpreted as 4

# FOL: Semantics [cont.]

## Interpretation of formulas

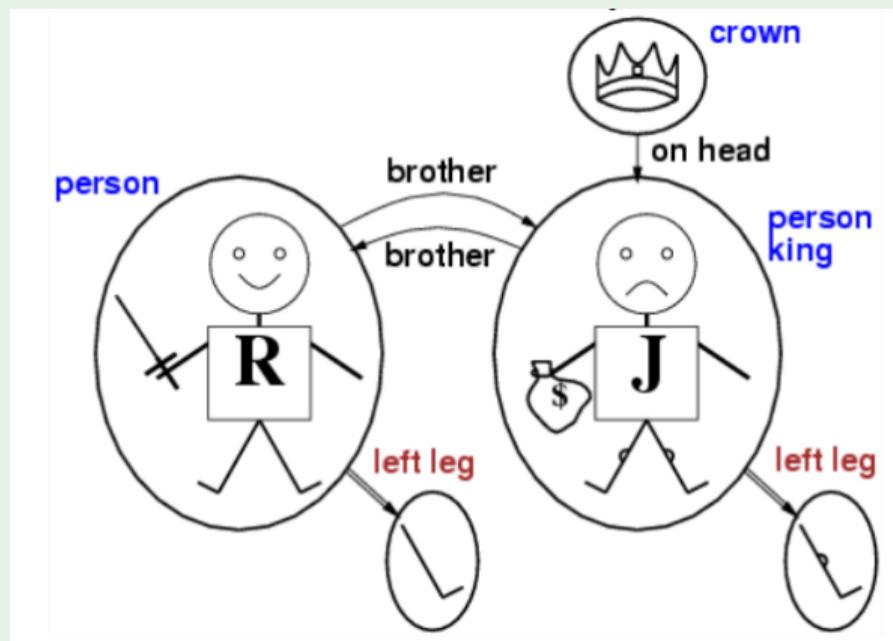
### $\mathcal{I}$ maps formulas into truth values

- An atomic formula  $P(t_1, \dots, t_k)$  is true in  $\mathcal{I}$  iff the objects into which the terms  $t_1, \dots, t_k$  are mapped by  $\mathcal{I}$  comply to the relation into which  $P$  is mapped
  - $[P(t_1, \dots, t_k)]^{\mathcal{I}}$  is true iff  $\langle [t_1]^{\mathcal{I}}, \dots, [t_k]^{\mathcal{I}} \rangle \in [P]^{\mathcal{I}}$
  - Ex: if “Me, Mother, Father, Married” are interpreted as tradition, then “Married(Mother(Me), Father(Me))” is interpreted as true
  - Ex: if “+, -, >, 0, 1, 2, 3, 4” are interpreted as usual, then “ $(4 - 0) > (1 + 2)$ ” is interpreted as true
- An atomic formula  $t_1 = t_2$  is true in  $\mathcal{I}$  iff the terms  $t_1, t_2$  are mapped by  $\mathcal{I}$  into the same domain element
  - $[t_1 = t_2]^{\mathcal{I}}$  is true iff  $[t_1]^{\mathcal{I}}$  same as  $[t_2]^{\mathcal{I}}$
  - Ex: if “Mother” is interpreted as usual, Richard, John are brothers, then “Mother(Richard)=Mother(John)” is interpreted as true
  - Ex: if “+, -, 0, 1, 2, 3, 4” are interpreted as usual, then “ $(4 - 1) = (1 + 2)$ ” is interpreted as true
- $\neg, \wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus$  interpreted by  $\mathcal{I}$  as in PL

# Models for FOL: Example

## Richard Lionheart and John Lackland

- $\mathcal{D}$ : domain at right
- $\mathcal{I}$ : s.t.
  - $[\text{Richard}]^{\mathcal{I}}$ : Richard the Lionheart
  - $[\text{John}]^{\mathcal{I}}$ : evil King John
  - $[\text{Brother}]^{\mathcal{I}}$ : brotherhood
- $[\text{Brother}(\text{Richard}, \text{John})]^{\mathcal{I}}$  is true
- $[\text{LeftLeg}]^{\mathcal{I}}$  maps any individual to his left leg
- ...



(© S. Russell & P. Norvig, Artificial Intelligence: A Modern Approach, III Ed., Pearson)

# Equality

- Equality is a special predicate:  $t_1 = t_2$  is true under a given interpretation if and only if  $t_1$  and  $t_2$  refer to the same object
  - Ex:  $1 = 2$  and  $x * x = x$  are satisfiable (!)
  - Ex:  $2 = 2$  is valid
- Ex: definition of *Siblings* in terms of *Parent*  
 $\forall x, y. (Siblings(x, y) \leftrightarrow [\neg(x = y) \wedge \exists p_1, p_2. (\neg(p_1 = p_2) \wedge Parent(p_1, x) \wedge Parent(p_2, x) \wedge Parent(p_1, y) \wedge Parent(p_2, y))])$ )

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## Traditional Definition (FOL)

Given a FOL signature  $\Sigma$ , a  $\Sigma$ -Theory  $\mathcal{T}$  (hereafter simply “theory”) is a (possibly infinite) set of FOL closed formulas (axioms)

- Typically used to provide some *intended interpretation* to the symbols in the signature  $\Sigma$
- FOL formulas deduces from these axioms via inference rules
- Definition used by logicians,
- Very low practical use in AR & Formal Verification

# Example: A FOL Theory of Positive Integer Numbers (aka “Peano Arithmetic”, $\mathcal{P}$ )

- Signature

- (basic) unary predicate symbol:  $\text{NatNum}$  (“natural number”)
- (basic) unary function symbol:  $S$  (“successor”)
- (basic) constant symbol:  $0$
- (derived) binary function symbols:  $+, *$  (infix)
- (derived) constant symbols:  $1, 2, 3, 4, 5, 6, \dots$

- Axioms

- 1  $\text{NatNum}(0)$
- 2  $\forall x. (\text{NatNum}(x) \rightarrow \text{NatNum}(S(x)))$
- 3  $\forall x. (\text{NatNum}(x) \rightarrow (0 \neq S(x)))$
- 4  $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
- 5  $\forall x. (\text{NatNum}(x) \rightarrow (x = (0 + x)))$
- 6  $\forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow (S(x) + y) = S(x + y))$
- 7  $1 = S(0), 2 = S(1), 3 = S(2), \dots$

- Formulas deduced

- ex:  $\mathcal{P} \vdash \text{NatNum}(25)$
- ex:  $\mathcal{P} \vdash \forall x, y. ((\text{NatNum}(x) \wedge \text{NatNum}(y)) \rightarrow ((x + y) = (y + x)))$

# FOL Theories (cont.)

## SMT Definition

Given a FOL signature  $\Sigma$ , a  $\Sigma$ -Theory  $\mathcal{T}$  (hereafter simply “theory”) is one (or more) model(s) constraining the interpretations of  $\Sigma$

- Provides an **intended interpretation** to the symbols in  $\Sigma$ 
  - constants mapped into domain elements
    - ex: “1” mapped into the number one
  - predicate symbols mapped into relations on domain elements
    - ex: “. < .” mapped into the arithmetical relation “less than”
  - function symbols mapped into functions on domain elements
    - ex: “S(.)” mapped into the arithmetical function “successor of”

These symbols are called **interpreted**

- Compliant with previous definition: **model(s) satisfying all axioms**
- Ad hoc “ $\mathcal{T}$ -aware” decision procedures for reasoning on formulas
- Very effective in practical applications

## Example: Linear Arithmetic on the Integers ( $\mathcal{LIA}$ )

- Domain: integer numbers
- Numerical constants interpreted as **numbers**
  - ex: “1”, “1346231” mapped directly into the corresponding number
- function and predicates interpreted as **arithmetical operations**
  - “+” as addition, “\*” as multiplication, “<” as less-than, . etc.
- **ILP solvers** used to do logical reasoning
  - ex:  $(3x - 2y \leq 3) \wedge (4y - 2z < -7) \models (6x - 2z < -1)$

# Satisfiability, Validity, Entailment (Modulo a Theory $\mathcal{T}$ )

## Definitions

- Idea: **We restrict to models satisfying  $\mathcal{T}$**  (“ $\mathcal{T}$ -models”)
- A formula is **satisfiable in  $\mathcal{T}$**  (aka “ $\varphi$  is  $\mathcal{T}$ -satisfiable”) iff some  $\mathcal{T}$ -model satisfies also  $\varphi$ 
  - ex:  $(x < 3)$  satisfiable in  $\mathcal{LIA}$
  - ex:  $(1 = 2)$  **not** satisfiable in  $\mathcal{LIA}$ !
- A formula  $\varphi$  is **valid in  $\mathcal{T}$**  (aka “ $\varphi$  is  $\mathcal{T}$ -valid” or “ $\models_{\mathcal{T}} \varphi$ ”) iff all  $\mathcal{T}$ -models satisfy also  $\varphi$ 
  - ex:  $(x < 3) \rightarrow (x < 4)$  valid in  $\mathcal{LIA}$
- A formula  $\varphi$  **entails  $\psi$  in  $\mathcal{T}$**  (aka “ $\varphi$   $\mathcal{T}$ -entails  $\psi$ ” or “ $\varphi \models_{\mathcal{T}} \psi$ ”) iff all  $\mathcal{T}$ -models satisfying  $\varphi$  satisfy also  $\psi$ 
  - ex:  $(x < 3) \models_{\mathcal{LIA}} (x < 4)$

## Properties

- $\varphi$  is  $\mathcal{T}$ -valid iff  $\neg\varphi$  is  $\mathcal{T}$ -unsatisfiable
  - $\varphi \models_{\mathcal{T}} \psi$  iff  $\varphi \rightarrow \psi$  is  $\mathcal{T}$ -valid
- $\implies \varphi \models_{\mathcal{T}} \psi$  iff  $\varphi \wedge \neg\psi$   $\mathcal{T}$ -unsatisfiable

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# Satisfiability Modulo Theories (SMT( $\mathcal{T}$ ))

## Satisfiability Modulo Theories (SMT( $\mathcal{T}$ ))

The problem of deciding the satisfiability of (typically quantifier-free) formulas in some decidable first-order theory  $\mathcal{T}$

- $\mathcal{T}$  can also be a **combination of theories**  $\bigcup_i \mathcal{T}_i$ .

# SMT( $\mathcal{T}$ ): Theories of Interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions ( $\mathcal{EUF}$ ):  
 $((x = y) \wedge (y = f(z))) \rightarrow (g(x) = g(f(z)))$
- Difference logic ( $\mathcal{DL}$ ):  $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
- UTVPI ( $\mathcal{UTVPI}$ ):  $((x = y) \wedge (y - z \leq 4)) \rightarrow (x + z \leq 6)$
- Linear arithmetic over the rationals ( $\mathcal{LRA}$ ):  
 $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
- Linear arithmetic over the integers ( $\mathcal{LIA}$ ):  $(x = x_l + 2^{16}x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
- Arrays ( $\mathcal{AR}$ ):  $(i = j) \vee \text{read}(\text{write}(a, i, e), j) = \text{read}(a, j)$
- Bit vectors ( $\mathcal{BV}$ ):  $x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \lll w_{[8]}[3 : 0]$
- Non-Linear arithmetic over the reals ( $\mathcal{NLA}(\mathbb{R})$ ):  
 $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = b + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
- ...

# Satisfiability Modulo Theories (SMT( $\mathcal{T}$ )): Example

## Example: SMT( $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ )

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- involves arithmetical, arrays, and uninterpreted function/predicate symbols, plus Boolean operators
  - Is it satisfiable?
  - No:

$$\begin{array}{ll} \implies_{\mathcal{LIA}} & \varphi \\ \implies_{\mathcal{EUF}} & (d = 0) \\ \implies_{\text{Bool}} & (f(d) = f(0)) \\ \implies_{\mathcal{LIA}} & (\text{read}(\text{write}(V, i, x), i + d) = x + 1) \\ \implies_{\mathcal{LIA}} & (\text{read}(\text{write}(V, i, x), i) = x + 1) \\ \implies_{\mathcal{LIA}} & \neg(\text{read}(\text{write}(V, i, x), i) = x) \\ \implies_{\mathcal{AR}} & \perp \end{array}$$

# SMT and SMT solvers

## Common fact about SMT problems from various applications

SMT requires capabilities for **heavy Boolean reasoning** combined with capabilities for **reasoning in expressive decidable F.O. theories**

- SAT alone not expressive enough
- standard automated theorem proving inadequate (e.g., arithmetic)
- may involve also numerical computation (e.g., simplex)

## Modern SMT solvers

- combine **SAT solvers** with  $\mathcal{T}$ -specific **decision procedures** (**theory solvers** or  **$\mathcal{T}$ -solvers**)
  - contributions from SAT, Automated Theorem Proving (ATP), formal verification (FV) and operational research (OR)

## Notational remark (1): most/all examples in $\mathcal{LRA}$

For better readability, in most/all the examples of this presentation we will use the theory of linear arithmetic on rational numbers ( $\mathcal{LRA}$ ) because of its intuitive semantics. E.g.:

$$(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$$

Nevertheless, analogous examples can be built with all other theories of interest.

## Notational remark (2): “constants” vs. “variables”

- Consider, e.g., the formula:  
 $(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$
- How do we call  $A_1, A_2$ ?:
  - (a) Boolean/propositional **variables**?
  - (b) uninterpreted **0-ary predicates**?
- How do we call  $x_1, x_2, x_3$ ?:
  - (a) domain **variables**?
  - (b) uninterpreted Skolem **constants/0-ary uninterpreted functions**?
- Hint:
  - (a) typically used in SAT, CSP and OR communities
  - (b) typically used in logic & ATP communities

Hereafter we call  $A_1, A_2$  “Boolean/propositional **variables**” and  $x_1, x_2, x_3$  “domain **variables**” (logic purists, please forgive me!)

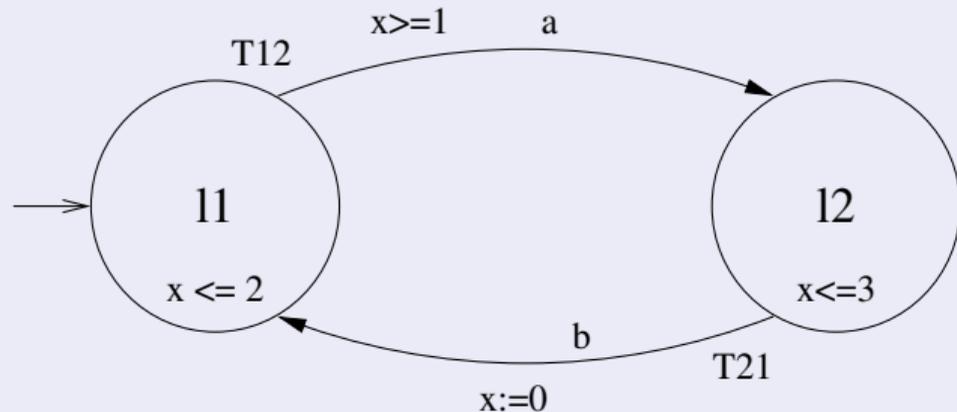
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# Some Motivating Applications

Interest in SMT triggered by some real-world applications

- Verification of Hybrid & Timed Systems
- Verification of RTL Circuit Designs & of Microcode
- SW Verification
- Planning with Resources
- Temporal reasoning
- Scheduling
- Compiler optimization
- ...

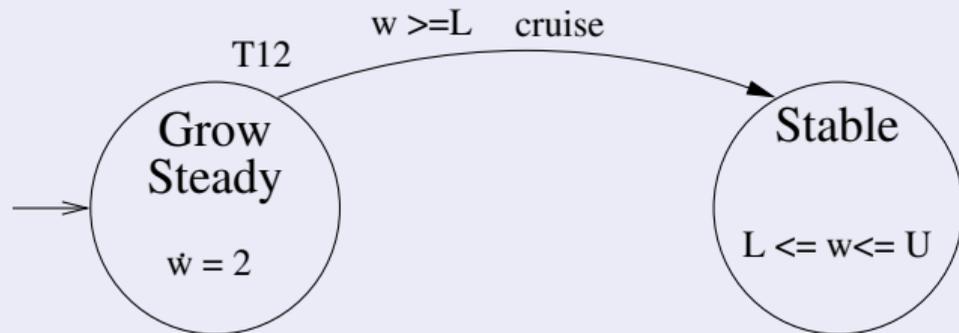
# Verification of Timed Systems



- Model checking of Timed Systems [6, 35, 58], ...
- Timed Automata encoded into  $\mathcal{T}$ -formulas:
  - discrete information (locations, transitions, events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences ( $t_3 - x_3 \leq 2$ ), equalities ( $x_4 = x_3$ ) and linear constraints ( $t_8 - x_8 = t_2 - x_2$ ) on  $\mathbb{Q}$

⇒ SMT on  $\mathcal{DL}(\mathbb{Q})$  or  $\mathcal{LRA}$  required

# Verification of Hybrid Systems ...

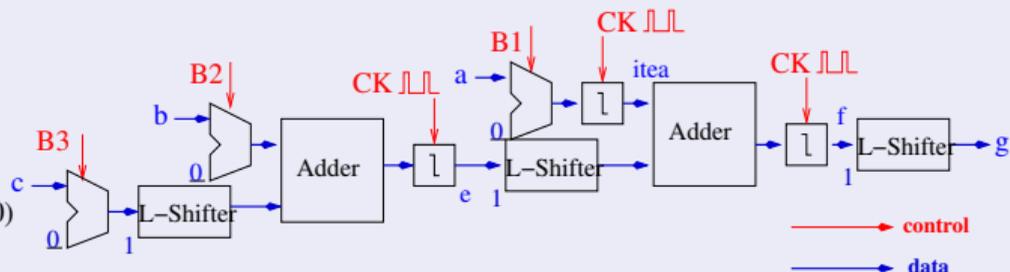


- **Model checking of Hybrid Systems** [5],...
- **Hybrid Automata** encoded into  $\mathcal{L}$ -formulas:
  - **discrete information** (locs, trans., events) with Boolean vars.
  - **timed information** (clocks, elapsed time) with differences ( $t_3 - x_3 \leq 2$ ), equalities ( $x_4 = x_3$ ) and linear constraints ( $t_8 - x_8 = t_2 - x_2$ ) on  $\mathbb{Q}$
  - **Evolution of Physical Variables** (e.g., speed, pressure) with linear ( $\omega_4 = 2\omega_3$ ) and non-linear constraints ( $P_1 V_1 = 4T_1$ ) on  $\mathbb{Q}$
- **Undecidable** under simple hypotheses!

$\Rightarrow$  SMT on  $\mathcal{DL}(\mathbb{Q})$ ,  $\mathcal{LRA}$  or  $\mathcal{NLA}(\mathbb{R})$  required

# Verification of HW circuit designs & microcode

$g = 2 * f$   
 $f = itea + 2 * e$   
 $itea' = ITE(B1; a; 0)$   
 $e' = ITE(B2; b; 0) + 2 * ite(B3; c; 0)$



- SAT/SMT-based **Model Checking & Equiv. Checking** of RTL designs, **symbolic simulation** of  $\mu$ -code [25, 22, 42]
  - **Control paths** handled by Boolean reasoning
  - **Data paths** information abstracted into theory-specific terms
    - **words** (bit-vectors, integers,  $\mathcal{EUF}$  vars, ... ):  $\underline{a}[31 : 0]$ ,  $a$
    - **word operations**: ( $\mathcal{BV}$ ,  $\mathcal{EUF}$ ,  $\mathcal{AR}$ ,  $\mathcal{LIA}$ ,  $\mathcal{NLA}(\mathbb{Z})$ ) operators  
 $x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \ll w_{[8]}[3 : 0]$ , ( $a = a_L + 2^{16}a_H$ ), ( $m_1 = store(m_0, l_0, v_0)$ ),  
 ...
  - Trades **heavy Boolean reasoning** ( $\approx 2^{64}$  factors) with  **$\mathcal{T}$ -solving**
- $\Rightarrow$  SMT on  $\mathcal{BV}$ ,  $\mathcal{EUF}$ ,  $\mathcal{AR}$ , modulo- $\mathcal{LIA}$  [ $\mathcal{NLA}(\mathbb{Z})$ ] required

# Verification of SW systems

```
...  
10. i = 0;  
11. acc = 0.0;  
12. while (i < dim) {  
13.   acc += V[i];  
14.   i++;  
15. }  
...
```

```
....  
 $\wedge (pc = 10 \rightarrow ((i' = 0) \wedge (pc' = 11)))$   
 $\wedge (pc = 11 \rightarrow ((acc' = 0.0) \wedge (pc' = 12)))$   
 $\wedge (pc = 12 \rightarrow ((i < dim) \rightarrow \wedge (pc' = 13)))$   
 $\wedge (pc = 12 \rightarrow (\neg(i < dim) \rightarrow \wedge (pc' = 16)))$   
 $\wedge (pc = 13 \rightarrow ((acc' = acc + read(V, i)) \wedge (pc' = 14)))$   
 $\wedge (pc = 14 \rightarrow (i' = i + 1) \wedge (pc' = 15))$   
 $\wedge (pc = 15 \rightarrow (pc' = 12))$   
...
```

- Verification of SW code

- BMC, K-induction, Check of proof obligations, interpolation-based model checking, symbolic simulation, concolic testing, ...

⇒ SMT on  $BV$ ,  $\mathcal{EUF}$ ,  $\mathcal{AR}$ , (modulo-) $\mathcal{LIA}$  [ $\mathcal{NCA}(\mathbb{Z})$ ] required

## Planning with Resources [80]

- SAT-bases planning augmented with numerical constraints
- Straightforward to encode into into  $SMT(\mathcal{LRA})$

### Example (sketch) [80]

```
(Deliver)                 $\wedge$  // goal
(MaxLoad)                 $\wedge$  // load constraint
(MaxFuel)                 $\wedge$  // fuel constraint
(Move  $\rightarrow$  MinFuel)    $\wedge$  // move requires fuel
(Move  $\rightarrow$  Deliver)    $\wedge$  // move implies delivery
(GoodTrip  $\rightarrow$  Deliver)  $\wedge$  // a good trip requires
(GoodTrip  $\rightarrow$  AllLoaded)  $\wedge$  // a full delivery
-----
(MaxLoad  $\rightarrow$  (load  $\leq$  30))  $\wedge$  // load limit
(MaxFuel  $\rightarrow$  (fuel  $\leq$  15))  $\wedge$  // fuel limit
(MinFuel  $\rightarrow$  (fuel  $\geq$  7 + 0.5load))  $\wedge$  // fuel constraint
(AllLoaded  $\rightarrow$  (load = 45)) //
```

## (Disjunctive) Temporal Reasoning & Scheduling [77, 2]

- Temporal reasoning problems encoded as disjunctions of difference constraints

$$\begin{aligned} & ((x_1 - x_2 \leq 6) \quad \vee \quad (x_3 - x_4 \leq -2)) \quad \wedge \\ & ((x_2 - x_3 \leq -2) \quad \vee \quad (x_4 - x_5 \leq 5)) \quad \wedge \\ & ((x_2 - x_1 \leq 4) \quad \vee \quad (x_3 - x_7 \leq -6)) \quad \wedge \\ & \dots \end{aligned}$$

- Scheduling problems encoded as disjunctions of difference constraints

$$\begin{aligned} & ((start_1 + Duration_1 \leq start_2) \quad \vee \quad (start_2 + Duration_2 \leq start_1)) \quad \wedge \\ & ((start_1 + Duration_1 \leq start_3) \quad \vee \quad (start_3 + Duration_3 \leq start_1)) \quad \wedge \\ & \dots \end{aligned}$$

⇒ Straightforward to encode into SMT( $\mathcal{DL}$ )

# Goal

Provide an overview of standard “lazy” SMT:

- foundations
- SMT-solving techniques
- beyond solving: advanced SMT functionalities
- ongoing research

We do **not** cover related approaches like:

- Eager SAT encodings
- Rewrite-based approaches

We refer to [70, 10] for an overview and references.

- 1 Introduction
  - Basics on First-order Logic
  - What is a Theory?
  - Satisfiability Modulo Theories
  - Motivations and Goals of SMT
- 2 **Efficient SMT solving**
  - Combining SAT with Theory Solvers
  - Theory Solvers for Theories of Interest
  - SMT for Combinations of Theories
- 3 Beyond Solving: Advanced SMT Functionalities
  - Proofs and Unsatisfiable Cores
  - All-SMT & Predicate Abstraction (hints)
  - SMT with Optimization (Optimization Modulo Theories)

- 1 Introduction
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  - SMT with Optimization (Optimization Modulo Theories)

# Modern “lazy” SMT( $\mathcal{T}$ ) solvers

A prominent “lazy” approach [45, 2, 80, 3, 8, 35] (aka “DPLL( $\mathcal{T}$ )”)

- a **CDCL SAT solver** is used to enumerate truth assignments  $\mu_i$  for (the Boolean abstraction  $\varphi^D$  of) the input formula  $\varphi$ 
  - the Boolean abstraction  $\varphi^D$  of  $\varphi$  maps theory atoms in  $\varphi$  into fresh Boolean variables
- a theory-specific solver  **$\mathcal{T}$ -solver** checks the  $\mathcal{T}$ -satisfiability of the **set of  $\mathcal{T}$ -literals** corresponding to each assignment

- Built on top of modern SAT CDCL solvers
  - benefit for free from all modern CDCL techniques (e.g., **Boolean preprocessing, backjumping & learning, restarts,...**)
  - benefit for free from all state-of-the-art data structures and implementation tricks (e.g., **two-watched literals,...**)
- Many techniques to maximize the benefits of integration [70, 10]
- Many lazy SMT tools available ( **Barcelogic, CVC5, MathSAT, OpenSMT, Yices, Z3, ...** )

# Basic schema: example

$\varphi =$

$$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$$

$$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$$

$$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$$

$$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$$

$$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$$

$$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$$

$$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$$

*true, false*

$\varphi^p =$

$$\neg B_1 \vee A_1$$

$$\neg A_2 \vee B_2$$

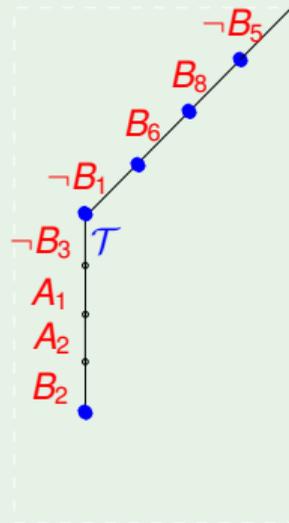
$$B_3 \vee A_2$$

$$\neg B_4 \vee \neg B_5 \vee \neg A_1$$

$$A_1 \vee B_3$$

$$B_6 \vee B_7 \vee \neg A_1$$

$$A_1 \vee B_8 \vee A_2$$



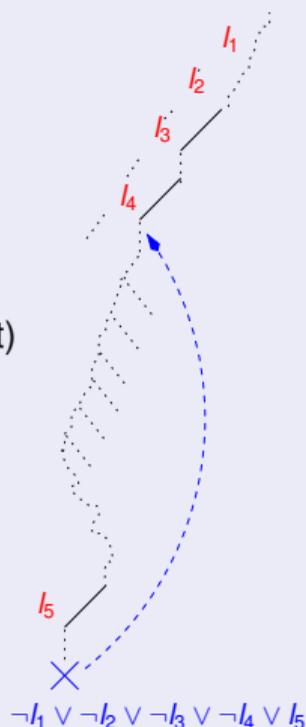
$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$

$\implies$  unsatisfiable in  $\mathcal{LR}\mathcal{A} \implies$  backtrack

# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning [50, 80, 3, 8, 35]

- Similar to Boolean backjumping & learning
- important property of  $\mathcal{T}$ -solver:
  - **extraction of  $\mathcal{T}$ -conflict sets**: if  $\mu$  is  $\mathcal{T}$ -unsatisfiable, then  $\mathcal{T}$ -solver( $\mu$ ) returns the subset  $\eta$  of  $\mu$  causing the  $\mathcal{T}$ -unsatisfiability of  $\mu$  ( $\mathcal{T}$ -conflict set)
- If so, the  **$\mathcal{T}$ -conflict clause**  $C := \neg\eta$  is used to drive the backjumping & learning mechanism of the SAT solver  
 $\implies$  lots of search saved
- **the less redundant is  $\eta$ , the more search is saved**

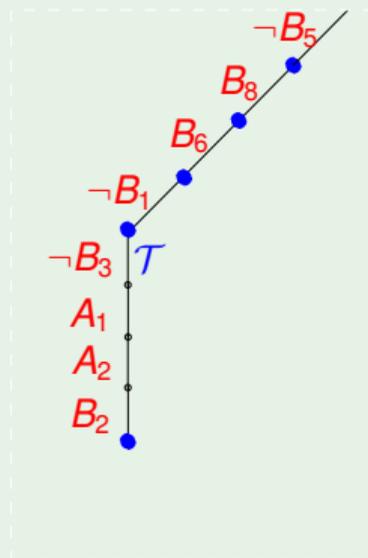


# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

true, false



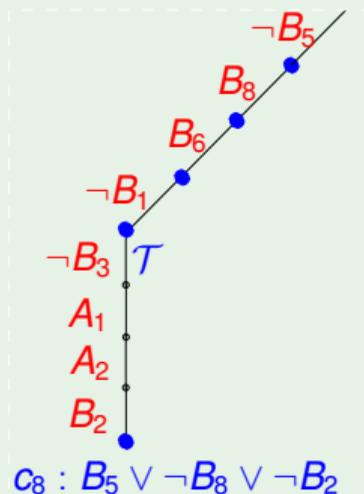
$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$

# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

true, false



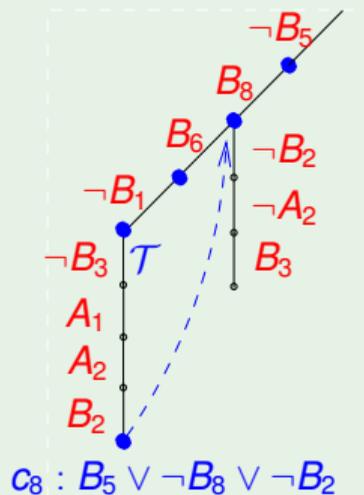
$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$

# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \end{array}$$

$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$

*true, false*



$$\begin{array}{l} \mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\} \\ \mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \\ \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\} \\ \eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\} \\ \eta^p = \{\neg B_5, B_8, B_2\} \end{array}$$



# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning: example (2)

$\varphi =$

$$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$$

$$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$$

$$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$$

$$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$$

$$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$$

$$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$$

$$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$$

$$c'_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$$

$$c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots$$

true, false

$\varphi^p =$

$$\neg B_1 \vee A_1$$

$$\neg A_2 \vee B_2$$

$$B_3 \vee A_2$$

$$\neg B_4 \vee \neg B_5 \vee \neg A_1$$

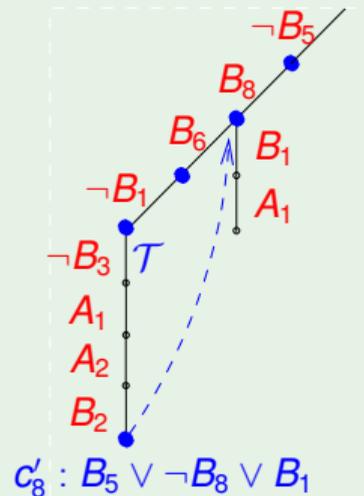
$$A_1 \vee B_3$$

$$B_6 \vee B_7 \vee \neg A_1$$

$$A_1 \vee B_8 \vee A_2$$

$$B_5 \vee \neg B_8 \vee B_1$$

$$B_5 \vee \neg B_8 \vee \neg B_2$$



$c_8$ : theory conflicting clause

$$\frac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_2} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2} \quad (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3} \quad (\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}} \quad (B_3)$$

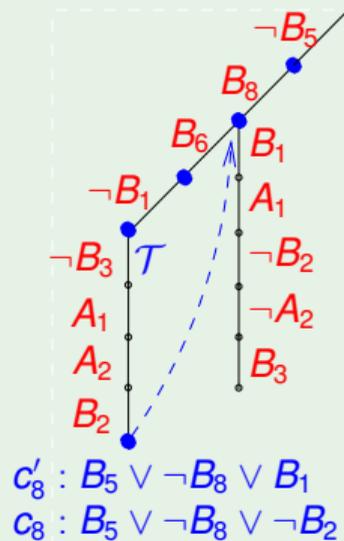
$$\frac{B_5 \vee \neg B_8 \vee B_3}{B_5 \vee \neg B_8 \vee B_1}$$

$c'_8$ : mixed Boolean+theory conflict clause

# $\mathcal{T}$ -Backjumping & $\mathcal{T}$ -learning: example (2)

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \\ c_8' : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \\ c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \dots \\ \text{true, false} \end{array}$$

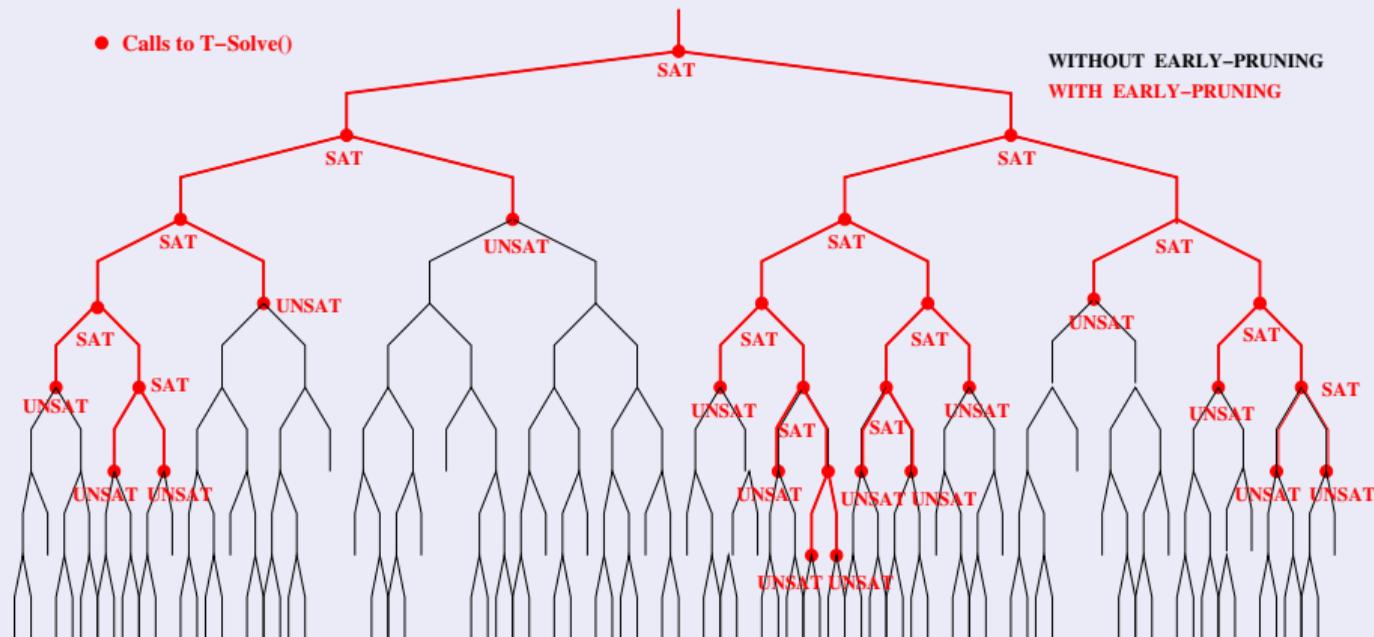
$$\begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \\ B_5 \vee \neg B_8 \vee B_1 \\ B_5 \vee \neg B_8 \vee \neg B_2 \end{array}$$



$$\begin{array}{c} c_8: \text{theory conflicting clause} \\ \frac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2} \quad (B_2) \quad \overbrace{B_3 \vee A_2}^{c_3} \quad (\neg A_2) \\ \frac{B_5 \vee \neg B_8 \vee B_3}{B_5 \vee \neg B_8 \vee B_1} \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_T} \quad (B_3) \\ \underbrace{B_5 \vee \neg B_8 \vee B_1} \\ c_8': \text{mixed Boolean+theory conflict clause} \end{array}$$

# Early Pruning [45, 2, 80]

- Introduce a  $\mathcal{T}$ -satisfiability test on **intermediate assignments**:  
if  $\mathcal{T}$ -solver returns UNSAT, the procedure backtracks.
  - benefit: prunes drastically the Boolean search
  - Drawback: possibly **many useless calls to  $\mathcal{T}$ -solver**



## Early pruning: example

$$\begin{aligned}\varphi = & \{ \neg(2v_2 - v_3 > 2) \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee (2v_1 - 4v_5 > 3) \} \wedge \\ & \{ (3v_1 - 2v_2 \leq 3) \vee A_2 \} \wedge \\ & \{ \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (3v_1 - 2v_2 \leq 3) \} \wedge \\ & \{ (v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \} . \\ \varphi^p = & \{ \neg B_1 \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee B_2 \} \wedge \\ & \{ B_3 \vee A_2 \} \wedge \\ & \{ \neg B_4 \vee \neg B_5 \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee B_3 \} \wedge \\ & \{ B_6 \vee B_7 \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee B_8 \vee A_2 \} .\end{aligned}$$

- Suppose it is built the intermediate assignment:

$$\mu'^p = \neg B_1 \wedge \neg A_2 \wedge B_3 \wedge \neg B_5.$$

corresponding to the following set of  $\mathcal{T}$ -literals

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

- If  $\mathcal{T}$ -solver is invoked on  $\mu'$ , then it returns UNSAT, and DPLL backtracks **without exploring any extension of  $\mu'$** .

## Early Pruning [45, 2, 80] (cont.)

- Different strategies for interleaving Boolean search steps and  $\mathcal{T}$ -solver calls
  - **Eager E.P.** [80, 11, 78, 44]: invoke  $\mathcal{T}$ -solver every time a new  $\mathcal{T}$ -atom is added to the assignment (unit propagations included)
  - **Selective E.P.**: Do not call  $\mathcal{T}$ -solver if the have been added only literals which hardly cause any  $\mathcal{T}$ -conflict with the previous assignment (e.g., Boolean literals, disequalities  $(x - y \neq 3)$ ,  $\mathcal{T}$ -literals introducing new variables  $(x - z = 3)$ )
  - **Weakened E.P.**: for intermediate checks only, use **weaker** but faster versions of  $\mathcal{T}$ -solver (e.g., check  $\mu$  on  $\mathbb{R}$  rather than on  $\mathbb{Z}$ ):  $\{(x - y \leq 4), (z - x \leq -6), (z = y), (3x + 2y - 3z = 4)\}$

# Early pruning: remark

## Incrementality & Backtrackability of $\mathcal{T}$ -solvers

- With early pruning, lots of **incremental calls to  $\mathcal{T}$ -solver**:

|   |                            |  |                          |
|---|----------------------------|--|--------------------------|
| $\mathcal{T}\text{-solver}(\mu_1)$                                  | $\Rightarrow \text{Sat}$   | Undo $\mu_4, \mu_3, \mu_2$                                 |                          |
| $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$                       | $\Rightarrow \text{Sat}$   | $\mathcal{T}\text{-solver}(\mu_1 \cup \mu'_2)$             | $\Rightarrow \text{Sat}$ |
| $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2 \cup \mu_3)$            | $\Rightarrow \text{Sat}$   | $\mathcal{T}\text{-solver}(\mu_1 \cup \mu'_2 \cup \mu'_3)$ | $\Rightarrow \text{Sat}$ |
| $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2 \cup \mu_3 \cup \mu_4)$ | $\Rightarrow \text{Unsat}$ | ...  |                          |

$\Rightarrow$  Desirable features of  $\mathcal{T}$ -solvers:

- **incrementality**:  $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$  reuses computation of  $\mathcal{T}\text{-solver}(\mu_1)$  without restarting from scratch
- **backtrackability (resettability)**:  $\mathcal{T}\text{-solver}$  can efficiently undo steps and return to a previous status on the stack

$\Rightarrow$   $\mathcal{T}$ -solver requires a **stack-based interface**

## $\mathcal{T}$ -Propagation [2, 3, 44]

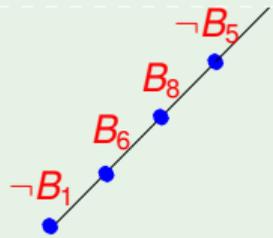
- strictly related to early pruning
- important property of  $\mathcal{T}$ -solver:
  - **$\mathcal{T}$ -deduction**: when a partial assignment  $\mu$  is  $\mathcal{T}$ -satisfiable,  $\mathcal{T}$ -solver may be able to return also an assignment  $\eta$  to some unassigned atom occurring in  $\varphi$  s.t.  $\mu \models_{\mathcal{T}} \eta$ .
- If so:
  - the literal  $\eta$  is then unit-propagated;
  - optionally, a  **$\mathcal{T}$ -deduction clause**  $C := \neg\mu' \vee \eta$  can be learned,  $\mu'$  being the subset of  $\mu$  which caused the deduction ( $\mu' \models_{\mathcal{T}} \eta$ )
  - **lazy explanation**: compute  $C$  only if needed for conflict analysis

⇒ may prune drastically the search

Both  $\mathcal{T}$ -deduction clauses and  $\mathcal{T}$ -conflict clauses are called  **$\mathcal{T}$ -lemmas** since they are valid in  $\mathcal{T}$

# $\mathcal{T}$ -propagation: example

$$\begin{array}{l} \varphi = \\ c_1 : \neg(2v_2 - v_3 > 2) \vee A_1 \\ c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1) \\ c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2 \\ c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1 \\ c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3) \\ c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \\ c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \end{array} \quad \begin{array}{l} \varphi^p = \\ \neg B_1 \vee A_1 \\ \neg A_2 \vee B_2 \\ B_3 \vee A_2 \\ \neg B_4 \vee \neg B_5 \vee \neg A_1 \\ A_1 \vee B_3 \\ B_6 \vee B_7 \vee \neg A_1 \\ A_1 \vee B_8 \vee A_2 \end{array}$$



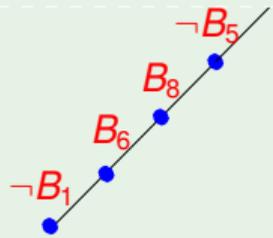
*true, false*

$$\begin{aligned} \mu^p &= \{\neg B_5, B_8, B_6, \neg B_1\} \\ \mu &= \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2)\} \\ &\models_{\mathcal{LR}, \mathcal{A}} \underbrace{\neg(3v_1 - 2v_2 \leq 3)}_{\neg B_3} \end{aligned}$$

$\Rightarrow$  propagate  $\neg B_3$  [and learn the deduction clause  $B_5 \vee B_1 \vee \neg B_3$ ]

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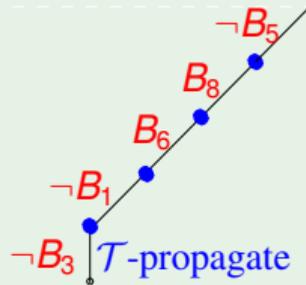
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## Pure-literal filtering [80, 3, 17]

### Property

If we have non-Boolean  $\mathcal{T}$ -atoms occurring only positively [negatively] in the original formula  $\varphi$  (learned clauses are not considered), we can drop every negative [positive] occurrence of them from the assignment to be checked by  $\mathcal{T}$ -solver (and from the  $\mathcal{T}$ -deducible ones).

- increases the chances of finding a model
- reduces the effort for the  $\mathcal{T}$ -solver
- eliminates unnecessary “nasty” negated literals (e.g. negative equalities like  $\neg(3v_1 - 9v_2 = 3)$  in  $\mathcal{LIA}$  force splitting:  $(3v_1 - 9v_2 > 3) \vee (3v_1 - 9v_2 < 3)$ ).
- may weaken the effect of early pruning.

## Pure literal filtering: example

$$\begin{aligned}\varphi = & \{ \neg(2v_2 - v_3 > 2) \vee A_1 \} \wedge \\ & \{ \neg A_2 \vee (2v_1 - 4v_5 > 3) \} \wedge \\ & \{ (3v_1 - 2v_2 \leq 3) \vee A_2 \} \wedge \\ & \{ \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq -2) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (3v_1 - 2v_2 \leq 3) \} \wedge \\ & \{ (v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1 \} \wedge \\ & \{ A_1 \vee (v_3 = 3v_5 + 4) \vee A_2 \} \wedge \\ & \{ (2v_2 - v_3 > 2) \vee \neg(3v_1 - 2v_2 \leq 3) \vee (3v_1 - v_3 \leq -2) \} \text{ learned}\end{aligned}$$

$$\mu' = \{ \neg(2v_2 - v_3 > 2), \neg A_2, (3v_1 - 2v_2 \leq 3), \neg A_1, (v_3 = 3v_5 + 4), (3v_1 - v_3 \leq -2) \}.$$

$\implies$  Sat:  $v_1 = v_2 = v_3 = 0, v_5 = -4/3$  is a solution

### Note

- $(3v_1 - v_3 \leq -2)$  “filtered out” from  $\mu'$  because it occurs only negatively in the original formula  $\varphi$
- $\mu' \cup \{(3v_1 - v_3 \leq -2)\}$  is  $\mathcal{LRA}$ -unsatisfiable

## Preprocessing atoms [45, 50, 4]

### Source of inefficiency:

Semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other]

⇒ they may be assigned different [resp. identical] truth values.

⇒ lots of redundant unsatisfiable assignment generated

### Solution

Rewrite a priori trivially-equivalent atoms/literals into the same atom/literal.

## Preprocessing atoms (cont.)

- **Sorting:**  $(v_1 + v_2 \leq v_3 + 1), (v_2 + v_1 \leq v_3 + 1), (v_1 + v_2 - 1 \leq v_3) \implies (v_1 + v_2 - v_3 \leq 1)$ ;
- **Rewriting dual operators:**  
 $(v_1 < v_2), (v_1 \geq v_2) \implies (v_1 < v_2), \neg(v_1 < v_2)$
- **Exploiting associativity:**  
 $(v_1 + (v_2 + v_3) = 1), ((v_1 + v_2) + v_3) = 1 \implies (v_1 + v_2 + v_3 = 1)$ ;
- **Factoring**  $(v_1 + 2.0v_2 \leq 4.0), (-2.0v_1 - 4.0v_2 \geq -8.0), \implies (0.25v_1 + 0.5v_2 \leq 1.0)$ ;
- **Exploiting properties of  $\mathcal{T}$ :**  
 $(v_1 \leq 3), (v_1 < 4) \implies (v_1 \leq 3)$  if  $v_1 \in \mathbb{Z}$ ;
- ...

## Static Learning [2, 4]

- Often possible to quickly detect a priori short and “obviously unsatisfiable” pairs or triplets of literals occurring in  $\varphi$ .
    - mutual exclusion  $\{x = 0, x = 1\}$ ,
    - congruence  $\{(x_1 = y_1), (x_2 = y_2), \neg(f(x_1, x_2) = f(y_1, y_2))\}$ ,
    - transitivity  $\{(x - y = 2), (y - z \leq 4), \neg(x - z \leq 7)\}$ ,
    - substitution  $\{(x = y), (2x - 3z \leq 3), \neg(2y - 3z \leq 3)\}$
    - ...
  - Preprocessing step: detect these literals and add blocking clauses to the input formula:  
(e.g.,  $\neg(x = 0) \vee \neg(x = 1)$ )
- ⇒ **No assignment including one such group of literals is ever generated:** as soon as all but one literals are assigned, the remaining one is immediately assigned false by unit-propagation.

## Other optimization techniques

- $\mathcal{T}$ -deduced-literal filtering
- Ghost-literal filtering
- $\mathcal{T}$ -solver layering
- $\mathcal{T}$ -solver clustering
- ...

(see [70, 10] for an overview)

# Other SAT-solving techniques for SMT?

## Frequently-asked question:

Are CDCL SAT solvers the only suitable Boolean Engines for SMT?

## Some previous attempts:

- Ordered Binary Decision Diagrams (OBDDs) [81, 60, 1]
- Stochastic Local Search [49]

CDCL based currently much more efficient.

# SMT formulas = “partially-invisible” SAT formulas

An SMT problem  $\varphi$  from the perspective of a SAT solver:

- a “partially-invisible” Boolean CNF formula  $\varphi^p \wedge \tau^p$ :
  - $\varphi^p$ : the Boolean abstraction of the input formula  $\varphi$
  - $\tau^p$ : (the Boolean abstraction of) the set  $\tau$  of all  $\mathcal{T}$ -lemmas on atoms in  $\varphi$ .

$\varphi$   $\mathcal{T}$ -satisfiable iff  $\varphi^p \wedge \tau^p$  satisfiable.

- the SAT solver:
  - “sees” only  $\varphi^p$
  - finds  $\mu^p$  s.t.  $\mu^p \models \varphi^p$
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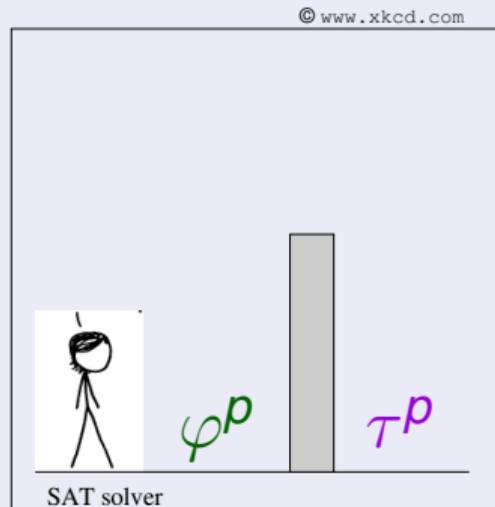
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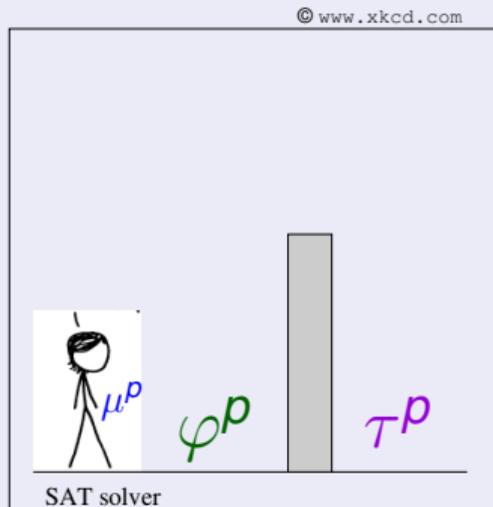
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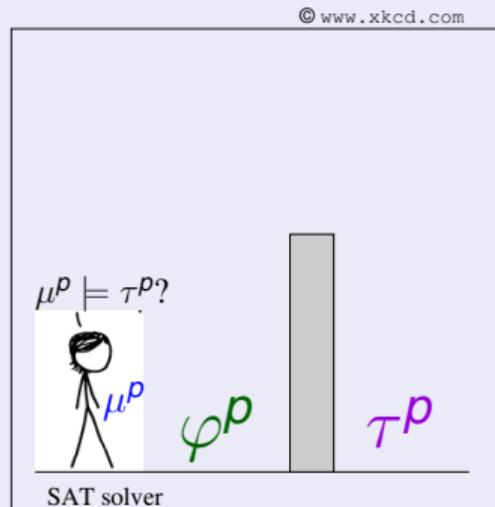
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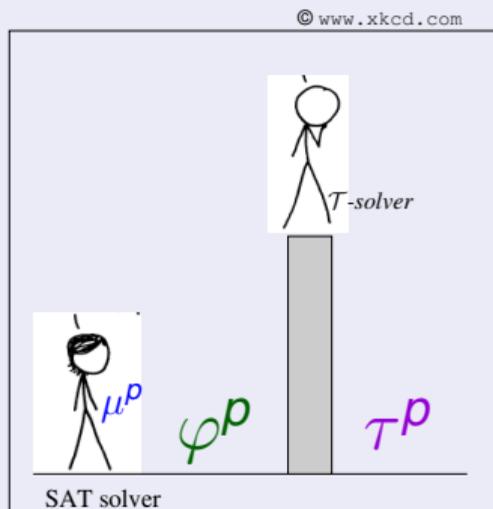
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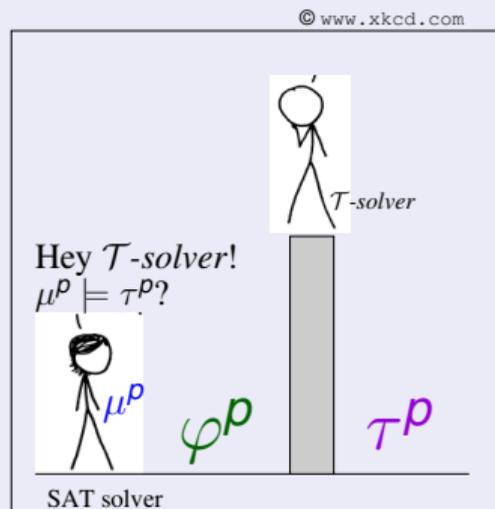
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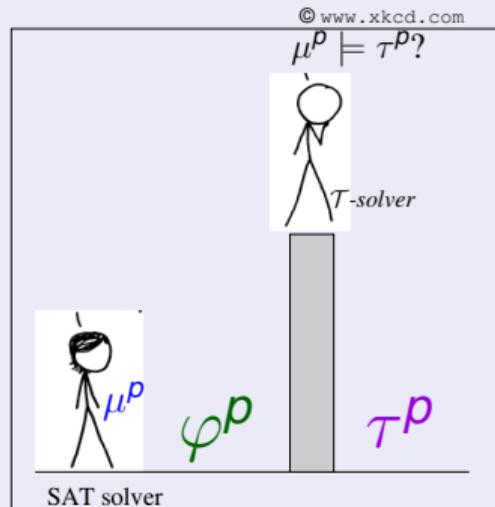
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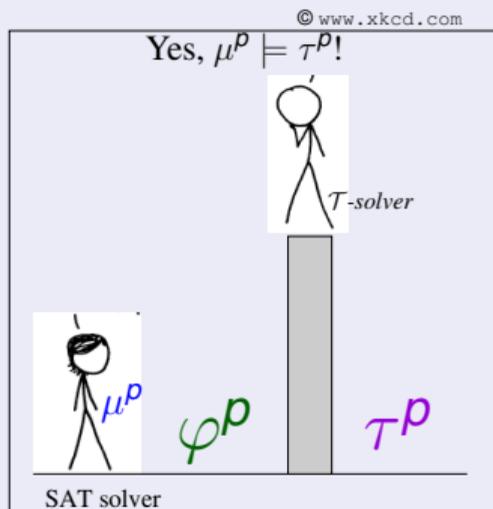
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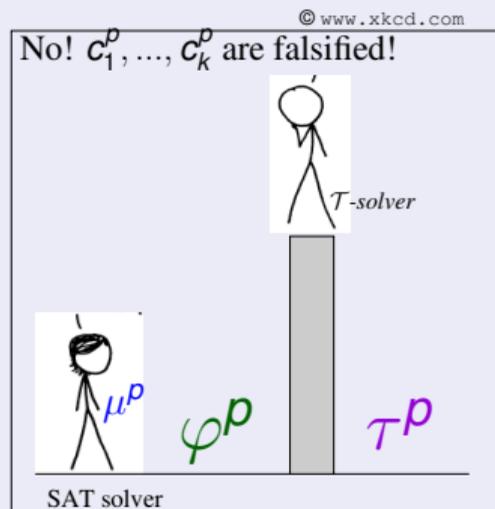
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# Example

$\varphi$  :  
 $c_1$  :  $\{A_1\}$   
 $c_2$  :  $\{\neg A_1 \vee (x - z > 4)\}$   
 $c_3$  :  $\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$   
 $c_4$  :  $\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$   
 $c_5$  :  $\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$   
 $c_6$  :  $\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$   
 $c_7$  :  $\{A_3 \vee \neg(x + y = 0) \vee A_2\}$   
 $c_8$  :  $\{\neg A_3 \vee (z + y = 2)\}$   
 $\tau$  : (all possible  $\mathcal{T}$ -lemmas on the  $\mathcal{T}$ -atoms of  $\varphi$ )  
 $c_9$  :  $\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$   
 $c_{10}$  :  $\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$   
 $c_{11}$  :  $\{(x - z > 4) \vee (x - y \leq 3) \vee (y - z \leq 1)\}$   
 $c_{12}$  :  $\{\neg(x - z > 4) \vee \neg(x + y = 1) \vee \neg(z + y = 2)\}$   
 $c_{13}$  :  $\{\neg(x - z > 4) \vee \neg(x + y = 0) \vee \neg(z + y = 2)\}$   
 ...  
 $\mu_1^p$  :  $\{A_1, B_1, \neg A_2, A_3, \neg A_4, \neg A_5, \neg B_6, B_5, B_3, B_4, B_7, \neg B_2\}$   
 $\mu_1$  :  $\{\underline{(x - z > 4)}, \neg(x + y = 0), \underline{(x + y = 1)}, \underline{(x - y \leq 3)}, \underline{(y - z \leq 1)}, \underline{(z + y = 2)}, \neg(y \geq 1)\}$

satisfies  $\varphi^p$ , but violates both  $c_{10}$  and  $c_{12}$  in  $\tau^p$ .

$\varphi^p$  :  
 $c_1$  :  $\{A_1\}$   
 $c_2$  :  $\{\neg A_1 \vee B_1\}$   
 $c_3$  :  $\{\neg A_3 \vee A_1 \vee B_2\}$   
 $c_4$  :  $\{\neg A_2 \vee \neg B_1 \vee \neg A_1\}$   
 $c_5$  :  $\{B_3 \vee \neg A_4 \vee A_5\}$   
 $c_6$  :  $\{\neg B_4 \vee B_5 \vee \neg A_5\}$   
 $c_7$  :  $\{A_3 \vee \neg B_6 \vee A_2\}$   
 $c_8$  :  $\{\neg A_3 \vee B_7\}$   
 $\tau^p$  :  
 $c_9$  :  $\{\neg B_6 \vee \neg B_5\}$   
 $c_{10}$  :  $\{\neg B_1 \vee \neg B_3 \vee \neg B_4\}$   
 $c_{11}$  :  $\{B_1 \vee B_3 \vee B_4\}$   
 $c_{12}$  :  $\{\neg B_1 \vee \neg B_5 \vee \neg B_7\}$   
 $c_{13}$  :  $\{\neg B_1 \vee \neg B_6 \vee \neg B_7\}$   
 ...

# Exercise

Consider the following formula in the theory  $\mathcal{EUF}$ .

$$\begin{aligned}\varphi = & \{(f(x) = f(f(y))) \vee A_2\} \wedge \\ & \{\neg(h(x, f(y)) = h(g(x), y)) \vee \neg(h(x, g(z) = h(f(x), y))) \vee \neg A_1\} \wedge \\ & \{A_1 \vee (h(x, y) = h(y, x))\} \wedge \\ & \{\underline{x = f(x)} \vee A_3 \vee \neg A_1\} \wedge \\ & \{\neg(w(x) = g(f(y))) \vee A_1\} \wedge \\ & \{\underline{\neg A_2} \vee (w(g(x)) = w(f(x)))\} \wedge \\ & \{A_1 \vee \underline{y = g(z)} \vee A_2\}\end{aligned}$$

and consider the partial truth assignment  $\mu$  given by the underlined literals above:

$$\{\neg(w(x) = g(f(y))), \neg A_2, \neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\}.$$

- 1 Does (the Boolean abstraction of)  $\mu$  propositionally satisfy (the Boolean abstraction of)  $\varphi$ ?
- 2 Is  $\mu$  satisfiable in  $\mathcal{EUF}$ ?
  - 1 If no, find a minimal conflict set for  $\mu$  and the corresponding conflict clause  $C$ .
  - 2 If yes, show one unassigned literal which can be deduced from  $\mu$ , and show the corresponding deduction clause  $C$ .

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  - SMT with Optimization (Optimization Modulo Theories)

## Summary: desirable properties for $\mathcal{T}$ -solver

- Correctness & Completeness: be correct & complete
- Time efficiency: be fast
- Incrementality & backtrackability:  $\mathcal{T}\text{-solver}(\mu_1 \cup \mu_2)$  reuses computation of  $\mathcal{T}\text{-solver}(\mu_1)$
- Diagnosis capabilities:  $\mathcal{T}\text{-solver}$  able to produce conflict sets
- Deduction capabilities:  $\mathcal{T}\text{-solver}$  able to deduce assignments

# $\mathcal{T}$ -solvers for Equality and Uninterpreted Functions ( $\mathcal{EUF}$ )

- Typically used as a “core”  $\mathcal{T}$ -solver
- $\mathcal{EUF}$  polynomial:  $O(n \cdot \log(n))$
- Fully incremental and backtrackable (stack-based)
- Uses a congruence closure data structures (**E-Graphs**) [39, 64, 34],
  - based on the Union-Find data-structure for equivalence classes
- Supports efficient  $\mathcal{T}$ -propagation
  - Exhaustive for positive equalities
  - Incomplete for disequalities
- Supports Lazy explanations and conflict generation
  - However, minimality not guaranteed
- Supports efficient extensions  
(e.g., Integer offsets, Bit-vector slicing and concatenation)

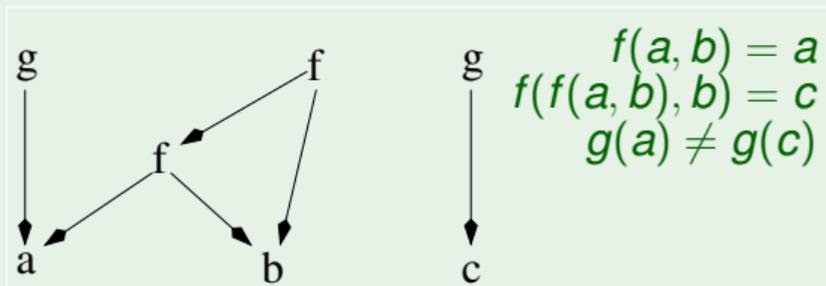
## $\mathcal{T}$ -solvers for $\mathcal{EUF}$ : Example

Idea (sketch), based on [E-Graphs](#):

given the set of terms occurring in the formula represented as nodes in a DAG (aka [term bank](#)):

- if  $(t = s)$ , then merge the eq. classes of  $t$  and  $s$ 
  - e.g. use the [union-find](#) data structure
- if  $\forall i \in 1 \dots k, t_i$  and  $s_i$  pairwise belong to the same eq. classes, then merge the eq. classes of  $f(t_1, \dots, t_k)$  and  $f(s_1, \dots, s_k)$
- if  $(t \neq s)$  and  $t$  and  $s$  belong to the same eq. class, then conflict

E-Graph:



Example borrowed from [39].

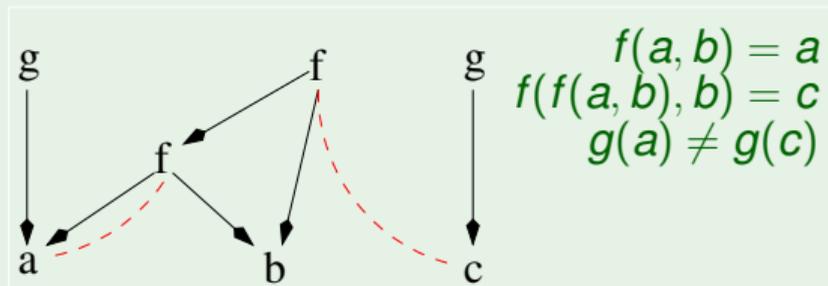
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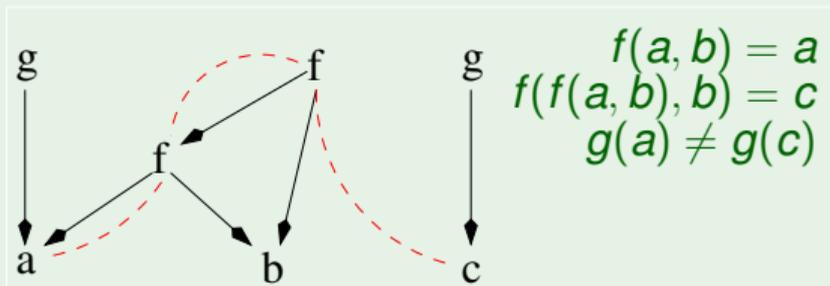
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E-Graph:



Example borrowed from [39].

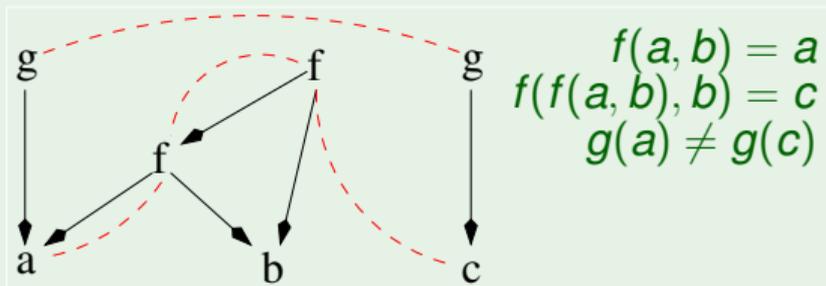
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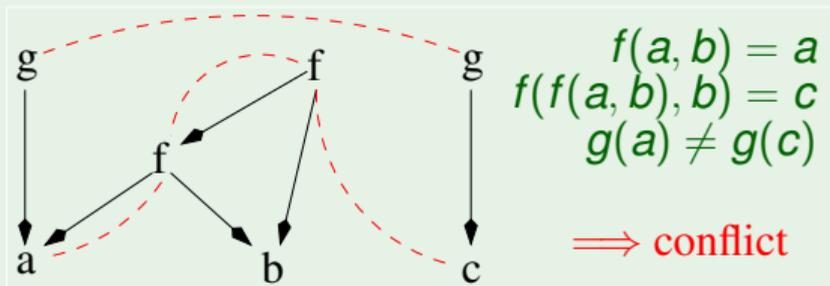
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Idea (sketch), based on **E-Graphs**:

given the set of terms occurring in the formula represented as nodes in a DAG (aka **term bank**):

- if  $(t = s)$ , then merge the eq. classes of  $t$  and  $s$ 
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- if  $(t \neq s)$  and  $t$  and  $s$  belong to the same eq. class, then conflict

**E-Graph:**



Example borrowed from [39].

# Exercises

For each of the following sets of  $\mathcal{EUF}$ -literals

- 1 build the E-graph
- 2 say if the set is  $\mathcal{EUF}$ -satisfiable or not
- 3 if  $\mathcal{EUF}$ -unsatisfiable, return a  $\mathcal{EUF}$ -conflict set

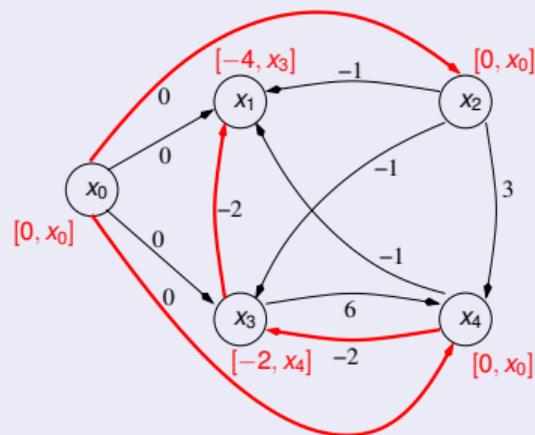
Sets of  $\mathcal{EUF}$ -literals:

- (a)  $\{g(a) = b, h(b) = a, h(a) = c, g(h(b)) = c, h(h(g(a))) \neq c\}$
- (b)  $\{g(a) = b, h(b) = a, h(a) = c, f(h(g(a)), g(h(b))) \neq f(a, b)\}$
- (c) [... invent your own sets of  $\mathcal{EUF}$ -literals ]

# $\mathcal{T}$ -solvers for Difference logic ( $\mathcal{DL}$ )

- $\mathcal{DL}$  polynomial:  $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 33]
- Ex:

$$\{(x_1 - x_2 \leq -1), (x_1 - x_4 \leq -1), (x_1 - x_3 \leq -2), (x_2 - x_1 \leq 2), \\ (x_3 - x_4 \leq -2), (x_3 - x_2 \leq -1), (x_4 - x_2 \leq 3), (x_4 - x_3 \leq 6)\}$$

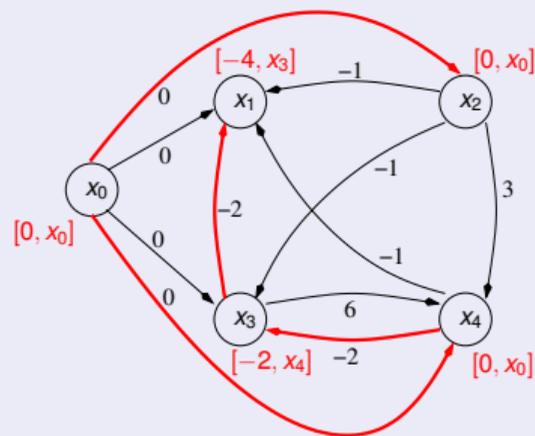


$\Rightarrow$  Sat

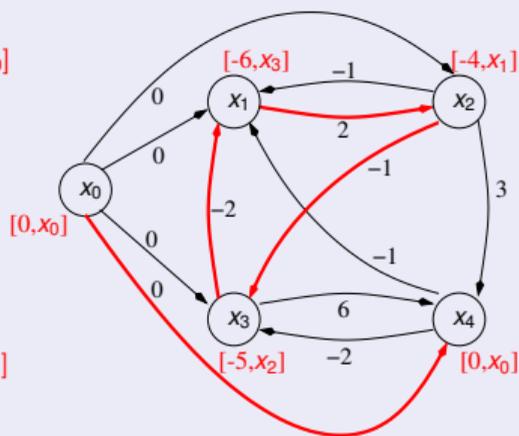
# T-solvers for Difference logic ( $\mathcal{DL}$ )

- $\mathcal{DL}$  polynomial:  $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 33]
- Ex:

$$\{(x_1 - x_2 \leq -1), (x_1 - x_4 \leq -1), (x_1 - x_3 \leq -2), (x_2 - x_1 \leq 2), \\ (x_3 - x_4 \leq -2), (x_3 - x_2 \leq -1), (x_4 - x_2 \leq 3), (x_4 - x_3 \leq 6)\}$$



$\Rightarrow$  Sat



$\Rightarrow$  Unsat

# Exercises

For each of the following sets of  $\mathcal{DL}$ -literals

- 1 build the minimum-path graph
- 2 say if the set is  $\mathcal{DL}$ -satisfiable or not
- 3 if  $\mathcal{DL}$ -unsatisfiable, return a  $\mathcal{DL}$ -conflict set

Sets of  $\mathcal{DL}$ -literals:

$$(a) \left\{ \begin{array}{l} (x_1 - x_3 \leq 5), (x_1 - x_2 \leq -2), (x_1 - x_4 \leq -3), (x_2 - x_4 \leq 0), (x_3 - x_5 \leq -2), \\ (x_4 - x_5 \leq 8), (x_5 - x_1 \leq -4) \end{array} \right\}$$

$$(b) \left\{ \begin{array}{l} (x_1 - x_2 \leq 3), (x_1 - x_4 \leq -1), (x_1 - x_5 \leq 4), (x_2 - x_3 \leq 0), (x_2 - x_5 \leq -1) \\ (x_3 - x_1 \leq 1), (x_3 - x_2 \leq 2), (x_4 - x_3 \leq 1), (x_5 - x_3 \leq -2), (x_5 - x_4 \leq -5) \end{array} \right\}$$

(c) [... invent your own sets of  $\mathcal{DL}$ -literals ]

## $\mathcal{T}$ -solvers for Linear arithmetic over the rationals ( $\mathcal{LRA}$ )

- EX:  $\{(s_1 - s_2 \leq 5.2), (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0), \neg(s_1 = s_0)\}$
- $\mathcal{LRA}$  polynomial
- variants of the simplex LP algorithm [41]
- [41] allows for detecting conflict sets & performing  $\mathcal{T}$ -propagation
- strict inequalities  $t < 0$  rewritten as  $t + \epsilon \leq 0$ ,  $\epsilon$  treated symbolically

$$\begin{array}{c} \mathcal{B} \\ \left[ \begin{array}{c} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{array} \right] \end{array} = \begin{array}{c} \left[ \begin{array}{ccc} \dots & A_{1j} & \dots \\ & \vdots & \\ A_{i1} & \dots & A_{ij} & \dots & A_{iM} \\ & \vdots & & & \\ \dots & A_{Nj} & \dots \end{array} \right] \end{array} \begin{array}{c} \mathcal{N} \\ \left[ \begin{array}{c} x_{N+1} \\ \vdots \\ x_j \\ \vdots \\ x_{N+M} \end{array} \right] \end{array} ;$$

Invariant:  $\beta(x_j) \in [l_j, u_j] \forall x_j \in \mathcal{N}$

## Remark: infinite precision arithmetic

In order to avoid incorrect results due to numerical errors and to overflows, all  $\mathcal{T}$ -solvers for  $\mathcal{LRA}$ ,  $\mathcal{LIA}$  and their subtheories which are based on numerical algorithms must be implemented on top of infinite-precision-arithmetic software packages.

# $\mathcal{T}$ -solvers for Linear arithmetic over the integers ( $\mathcal{LIA}$ )

- EX:  $\{(x := x_l + 2^{16}x_h), (x \geq 0), (x \leq 2^{16} - 1)\}$
- $\mathcal{LIA}$  NP-complete
- combination of many techniques: simplex, branch&bound, cutting planes, ... [41, 47]

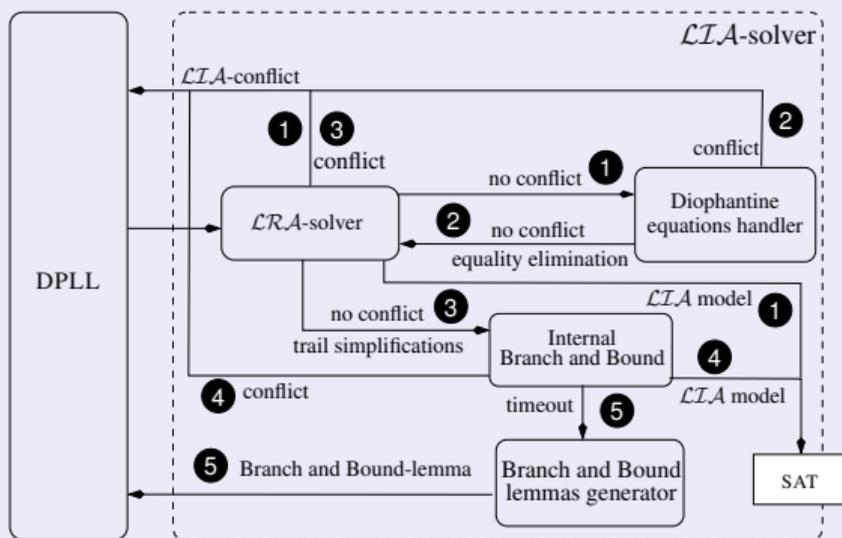


Figure courtesy of A. Griggio [47]

# $\mathcal{T}$ -solvers for Arrays ( $\mathcal{AR}$ )

- EX:  $(write(A, i, v) = write(B, i, w)) \wedge \neg(v = w)$
- NP-complete
- congruence closure ( $\mathcal{EUF}$ ) plus on-the-fly instantiation of array's axioms:

$$\forall a. \forall i. \forall e. (read(write(a, i, e), i) = e), \quad (1)$$

$$\forall a. \forall i. \forall j. \forall e. ((i \neq j) \rightarrow read(write(a, i, e), j) = read(a, j)), \quad (2)$$

$$\forall a. \forall b. (\forall i. (read(a, i) = read(b, i)) \rightarrow (a = b)). \quad (3)$$

- EX:

$$\text{Input : } (write(A, i, v) = write(B, i, w)) \wedge \neg(v = w)$$

$$\text{inst. (1) : } (read(write(A, i, v), i) = v) \\ (read(write(B, i, w), i) = w)$$

$$\models_{\mathcal{EUF}} (v = w)$$

$$\models_{Bool} \perp$$

- many strategies discussed in the literature (e.g., [39, 46, 20, 38])

# Exercises

For each of the following sets of  $\mathcal{AR}$ -literals, try to infer  $\perp$  by instantiating the  $\mathcal{AR}$  axioms and applying  $\mathcal{EUF}$ -inference and  $\mathcal{BOOL}$ -inference.

Sets of  $\mathcal{AR}$ -literals:

- (a)  $\{ A_1 = \text{Write}(A_0, i, x), A_2 = \text{Write}(A_0, j, y), (i = j), \neg(k = i), \neg(\text{Read}(A_2, k) = \text{Read}(A_1, k)) \}$
- (b)  $\{ A_1 = \text{Write}(A_0, i, x), A_2 = \text{Write}(A_0, i, y), \neg(x = y), (A_1 = A_2) \}$
- (c)  $\{ A_1 = \text{Write}(A_0, i, x), \text{Read}(A_0, i) = \text{Read}(A_1, i), \neg(A_1 = A_0) \}$
- (d) [... invent your own sets of  $\mathcal{AR}$ -literals ]

# $\mathcal{T}$ -solvers for Bit vectors ( $\mathcal{BV}$ )

## Bit vectors ( $\mathcal{BV}$ )

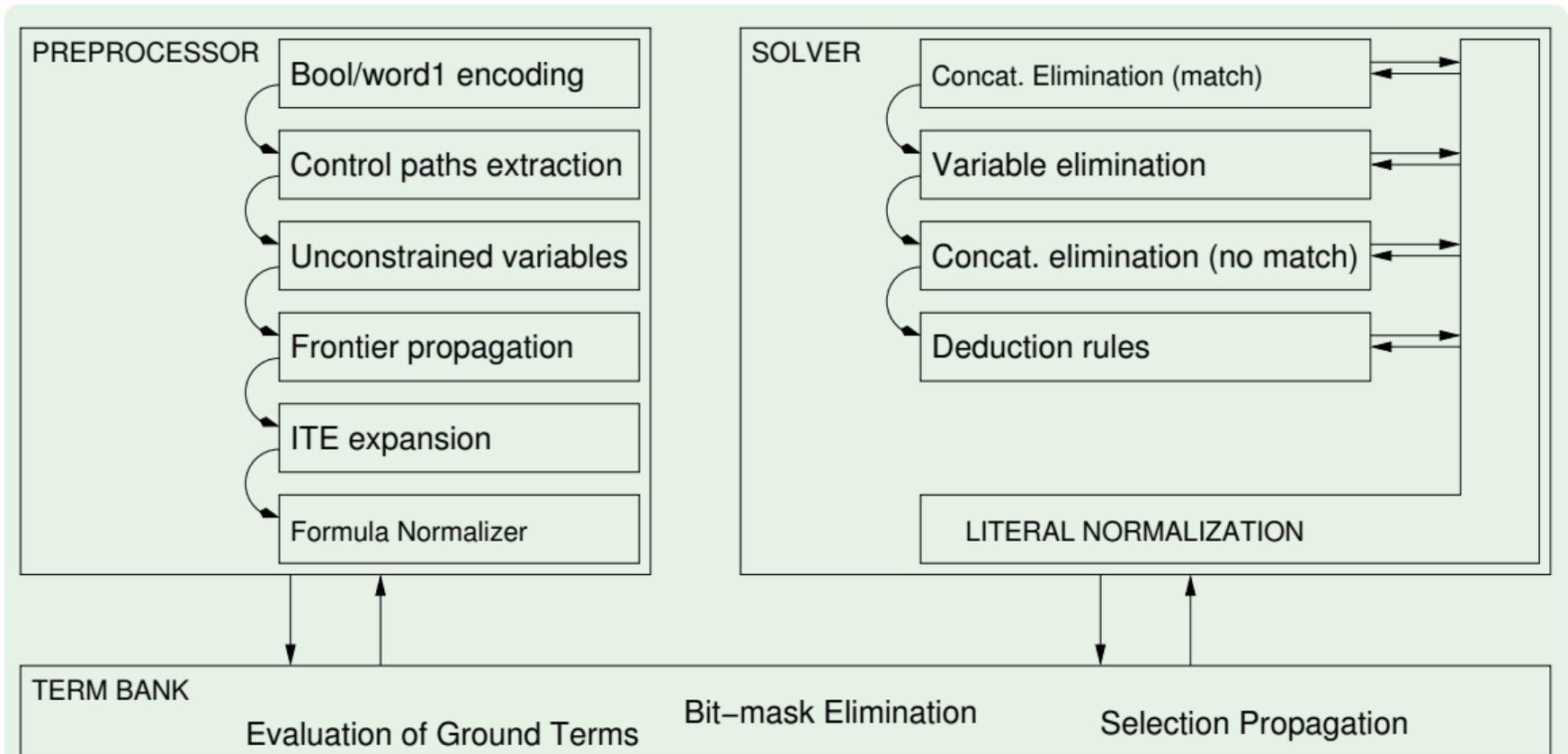
- EX:  $\{(x_{[16]}[15 : 0] = (y_{[16]}[15 : 8] :: z_{[16]}[7 : 0]) \ll w_{[16]}[3 : 0]), \dots\}$
- NP-hard
- involve complex word-level operations: word partition/concat, modulo- $2^N$  arithmetic, shifts, bitwise-operations, multiplexers, ...
- $\mathcal{T}$ -solving: combination of rewriting & simplification techniques with either:
  - final encoding into  $\mathcal{LIA}$  [19, 22]
  - final encoding into SAT (**lazy bit-blasting**) [25, 43, 21, 42]

## Eager approach

Most solvers use an **eager** approach for  $\mathcal{BV}$  (e.g., [21]):

- Heavy preprocessing, based on rewriting rules
- bit-blasting

# $\mathcal{T}$ -solvers for Bit vectors ( $BV$ ) [cont.]



Example borrowed from [22]

# $\mathcal{T}$ -solvers for Bit vectors ( $\mathcal{BV}$ ) [cont.]

## Lazy bit-blasting

- Two nested SAT solvers

- bit-blast each  $\mathcal{BV}$  atom  $\psi_i$

$$\implies \Phi \stackrel{\text{def}}{=} \bigwedge_i (A_i \leftrightarrow \text{BB}(\psi_i)),$$

$A_i$  fresh variables labeling  $\mathcal{BV}$ -atoms  $\psi_i$  in  $\varphi$

$\implies \varphi$   $\mathcal{BV}$ -satisfiable iff  $\varphi^p \wedge \Phi$  satisfiable

- Exploit SAT under assumptions

- let  $\mu^p$  an assignment for  $\varphi^p$ , s.t.  $\mu^p \stackrel{\text{def}}{=} \{[\neg]A_1, \dots, [\neg]A_n\}$

- $\mathcal{T}$ -solver for  $\mathcal{BV}$ :  $\text{SAT}_{\text{assumption}}(\Phi, \mu^p)$

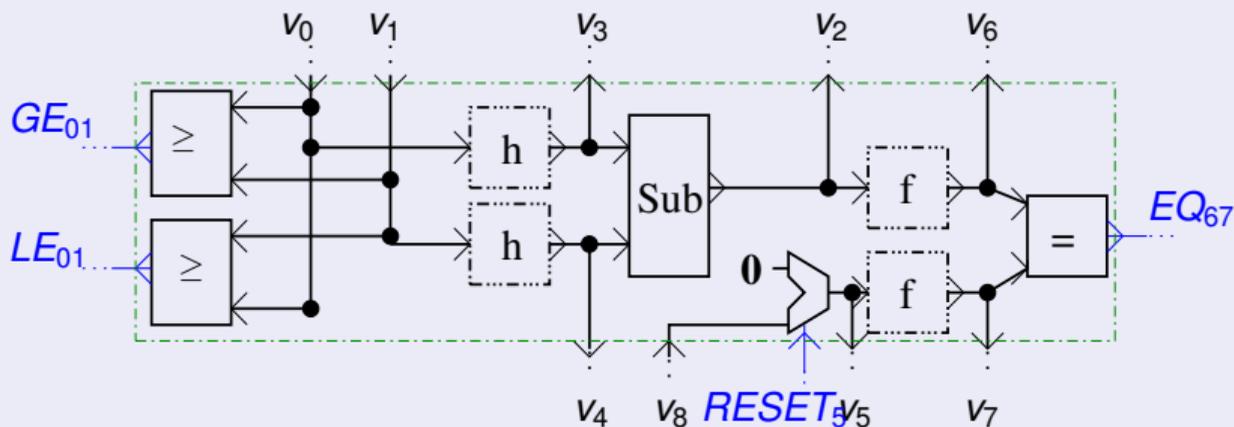
- If UNSAT, generate the **unsat core**  $\eta^p \subseteq \mu^p$

$\implies \neg\eta^p$  used as blocking clause

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# SMT for combined theories: $SMT(\cup_i T_i)$

Problem: Many problems can be expressed as SMT problems only in combination of theories



$\cup_i T_i - SMT(\cup_i T_i)$

$\mathcal{L}IA$ :  $(GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge$

$\mathcal{E}UF$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge$

$\mathcal{L}IA$ :  $(v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

$\mathcal{E}UF$  or  $\mathcal{L}IA$ :  $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge$

$\mathcal{E}UF$ :  $(v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{E}UF$  or  $\mathcal{L}IA$ :  $(EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \dots$

## SMT for combined theories: $\text{SMT}(\mathcal{T}_1 \cup \mathcal{T}_2)$

- Combined theories may be much harder to decide [Pratt'77]
- Solvers have to be combined
- Standard approach for combining  $\mathcal{T}_i$ -solvers:  
(deterministic) Nelson-Oppen/Shostak (N.O.) [61, 63, 75]
  - based on deduction and exchange of equalities on shared variables
  - combined  $\mathcal{T}_i$ -solvers integrated with a SAT tool
- SMT-specific approaches: Delayed Theory Combination [15, 14] and Model-Based Theory Combination [36]
  - based on Boolean search on equalities on shared variables
  - $\mathcal{T}_i$ -solvers integrated directly with a SAT tool

# Background: Pure Formulas

Consider two theories  $\mathcal{T}_1, \mathcal{T}_2$  with equality and disjoint signatures  $\Sigma_1, \Sigma_2$

- W.l.o.g. we assume all input formulas  $\phi \in \mathcal{T}_1 \cup \mathcal{T}_2$  are **pure**.
  - A formula  $\phi$  is **pure** iff every atom in  $\phi$  is  $i$ -pure for some  $i \in \{1, 2\}$ .
  - An atom/literal  $\psi$  in  $\phi$  is  $i$ -**pure** if only  $=$ , variables and symbols from  $\Sigma_i$  can occur in  $\psi$

## Purification:

Maps a formula into an equisatisfiable pure formula by labeling terms with fresh variables

$$(f(\underbrace{x + 3y}_w) = g(\underbrace{2x - y}_t)) \quad [not\ pure]$$

$$\Downarrow$$
$$(w = x + 3y) \wedge (t = 2x - y) \wedge (f(w) = g(t)) \quad [pure]$$

## Exercise

- Purify the following  $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

# Background: Interface equalities

## Interface variables & equalities

- A variable  $v$  occurring in a pure formula  $\phi$  is an **interface variable** iff it occurs in both 1-pure and 2-pure atoms of  $\phi$ .
- An equality  $(v_i = v_j)$  is an **interface equality** for  $\phi$  iff  $v_i, v_j$  are interface variables for  $\phi$ .
- We denote the interface equality  $v_i = v_j$  by “ $e_{ij}$ ”

## Example:

$$\begin{aligned} \mathcal{LIA} : & \quad (GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge \\ \mathcal{EUF} : & \quad (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge \\ \mathcal{LIA} : & \quad (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ \mathcal{EUF} \text{ or } \mathcal{LIA} : & \quad (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \\ \mathcal{EUF} : & \quad (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{EUF} \text{ or } \mathcal{LIA} : & \quad (EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \dots \end{aligned}$$

$v_0, v_1, v_2, v_3, v_4, v_5$  are interface variables,  $v_6, v_7, v_8$  are not  
 $\implies (v_0 = v_1)$  is an interface equality,  $(v_0 = v_6)$  is not.

# Background: Stably-infinite & Convex Theories

## Stably-infinite Theories

A  $\Sigma$ -theory  $\mathcal{T}$  is **stably-infinite** iff every quantifier-free  $\mathcal{T}$ -satisfiable formula is satisfiable in an infinite model of  $\mathcal{T}$ .

- $\mathcal{EUF}$ ,  $\mathcal{DL}$ ,  $\mathcal{LRA}$ ,  $\mathcal{LIA}$  are stably-infinite
- (fixed-width) bit-vector theories are not stably-infinite

Intuition: a variable can be given an infinite amount of distinct values

## Convex Theories

A  $\Sigma$ -theory  $\mathcal{T}$  is **convex** iff, for every collection  $l_1, \dots, l_k, l', l''$  of literals in  $\mathcal{T}$  s.t.  $l', l''$  are in the form  $(x = y)$ ,  $x, y$  being variables, we have that:

$$\{l_1, \dots, l_k\} \models_{\mathcal{T}} (l' \vee l'') \iff \{l_1, \dots, l_k\} \models_{\mathcal{T}} l' \text{ or } \{l_1, \dots, l_k\} \models_{\mathcal{T}} l''$$

- $\mathcal{EUF}$ ,  $\mathcal{DL}$ ,  $\mathcal{LRA}$  are convex
- $\mathcal{LIA}$  is not convex:
  - $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \models ((v = v_0) \vee (v = v_1))$ ,
  - $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \not\models (v = v_0)$
  - $\{(v_0 = 0), (v_1 = 1), (v \geq 0), (v \leq v_1)\} \not\models (v = v_1)$

Intuition: non-convexity produces “case splits”

# SMT( $\bigcup_i \mathcal{T}_i$ ) via “classic” Nelson-Oppen

## Main Problem

- Given  $\mu \stackrel{\text{def}}{=} \bigcup_i \mu_i$  s.t. each  $\mu_i$  contains i-pure literals
  - distinct  $\mathcal{T}_i$ -solver can be invoked separately on each  $\mu_i$ ...
  - ...producing distinct  $\mathcal{T}_i$ -specific models  $\mathcal{M}_i$
- If the theories share no constant, function or predicate symbol  $\implies$  trivial
- Problem: One predicate (nearly) always shared between distinct theories  $\mathcal{T}_i$ : equality “=”

$\implies$  all models must agree on interface equalities:

$$\mathcal{M}_i \models_{\mathcal{T}_i} (v_k = v_l) \text{ iff } \mathcal{M}_j \models_{\mathcal{T}_j} (v_k = v_l),$$

for every pair of shared variables  $v_k, v_l$

## Main idea

Combine two or more  $\mathcal{T}_i$ -solvers into one  $(\bigcup_i \mathcal{T}_i)$ -solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 76]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities,  $e_{ij}$ s)
- important improvements and evolutions [68, 7, 39]

## Schema of N.O. combination of $\mathcal{T}$ -solvers: $\text{no}(\mathcal{T}_1, \mathcal{T}_2)$

For  $i \in \{1, 2\}$ , let  $\mathcal{T}_i$  be a stably infinite theory admitting a satisfiability  $\mathcal{T}_i$ -solver, and  $\mu_i$  a set of  $i$ -pure literals.

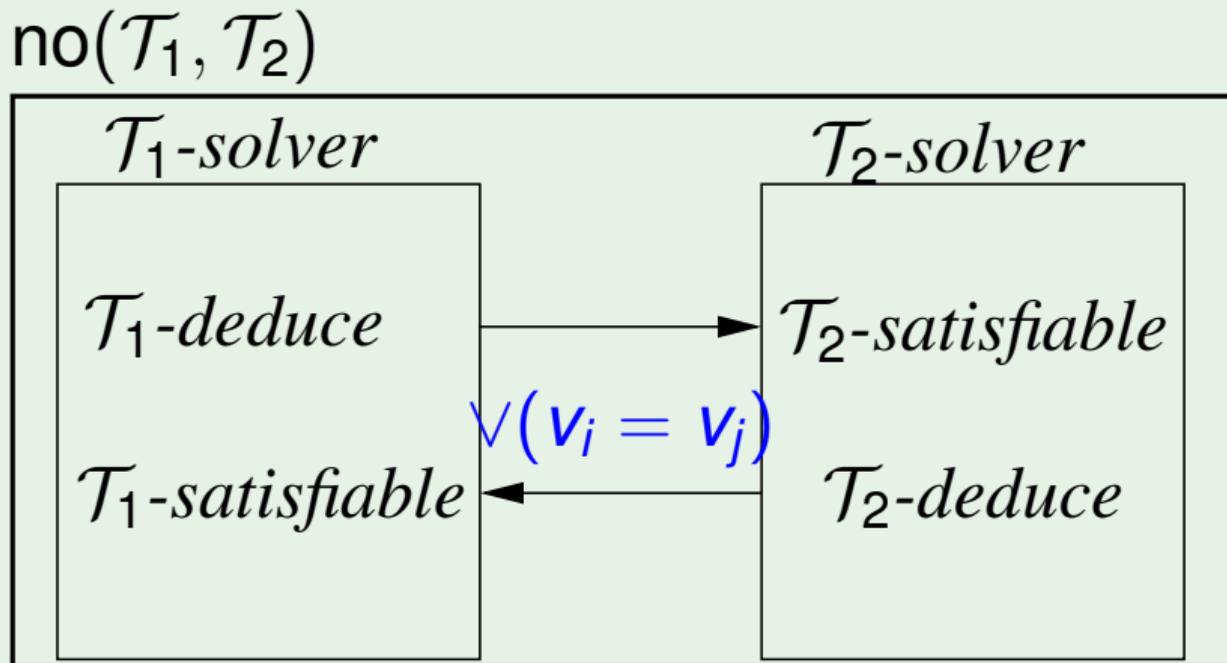
We want to decide the  $\mathcal{T}_1 \cup \mathcal{T}_2$ -satisfiability of  $\mu_1 \cup \mu_2$

- each  $\mathcal{T}_i$ -solver, in turn
  - checks the  $\mathcal{T}_i$ -satisfiability of  $\mu_i$ ,
  - deduces all the (disjunctions of) interface equalities which derive from  $\mu_i$
  - passes them to  $\mathcal{T}_j$ -solve,  $j \neq i$ , which adds them to  $\mu_j$

until either:

- one  $\mathcal{T}_i$ -solver detects unsatisfiability ( $\mu_1 \cup \mu_2$  is  $\mathcal{T}_1 \cup \mathcal{T}_2$ -unsat)
- no more deductions are possible ( $\mu_1 \cup \mu_2$  is  $\mathcal{T}_1 \cup \mathcal{T}_2$ -sat)
- disjunctions of literals (due to non-convexity) force case-splitting

# Schema of N.O. combination of T-solvers: $\text{no}(\mathcal{T}_1, \mathcal{T}_2)$



## N.O. Example (Convex Theory)

$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{LRA}$ :  $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

$\text{Both}$ :  $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .

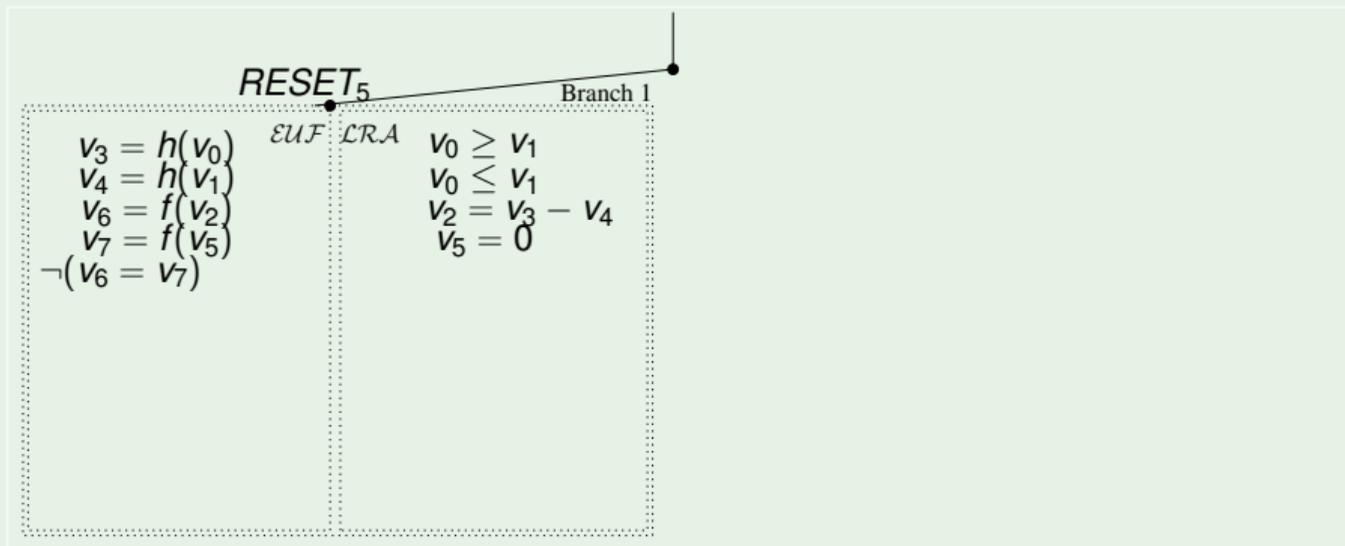


# N.O. Example (Convex Theory)

$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

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*Both*:  $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .

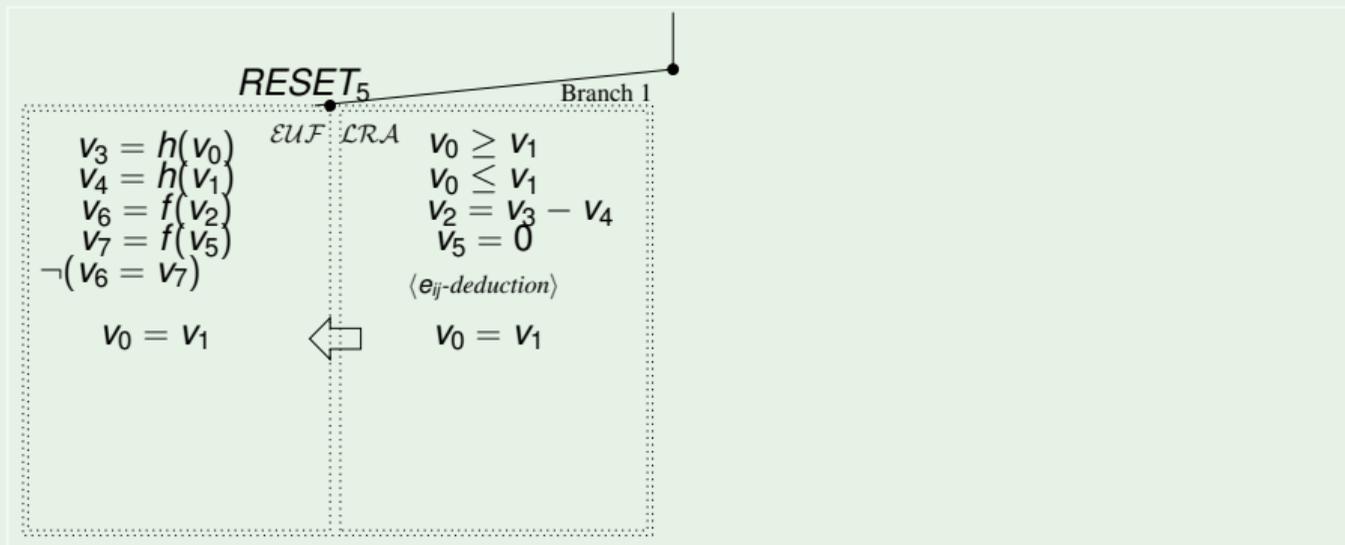


# N.O. Example (Convex Theory)

$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

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*Both*:  $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .

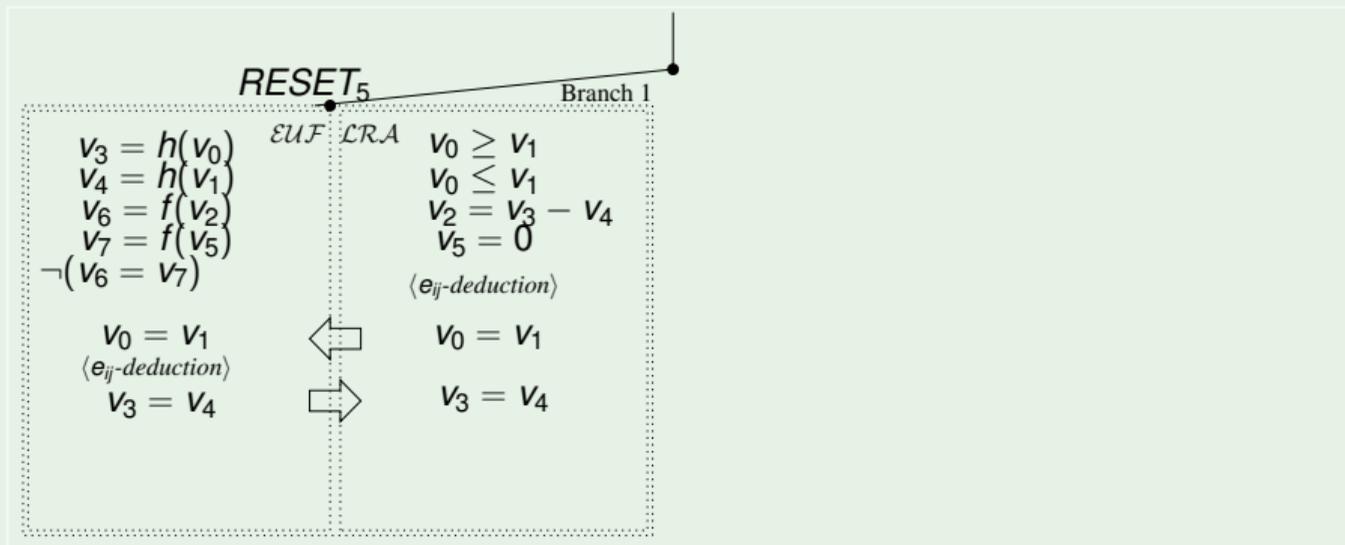


# N.O. Example (Convex Theory)

$$\mathcal{EUF} : (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

$$\mathcal{LRA} : (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$$

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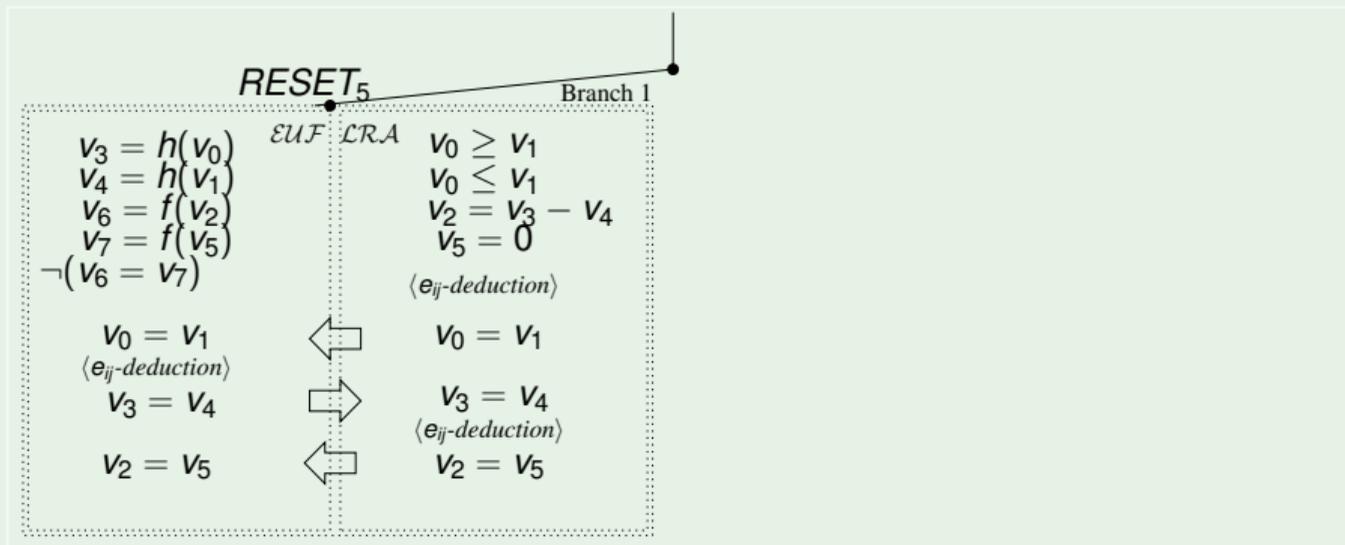


# N.O. Example (Convex Theory)

$$\mathcal{EUF} : (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

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$$\text{Both} : (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$$

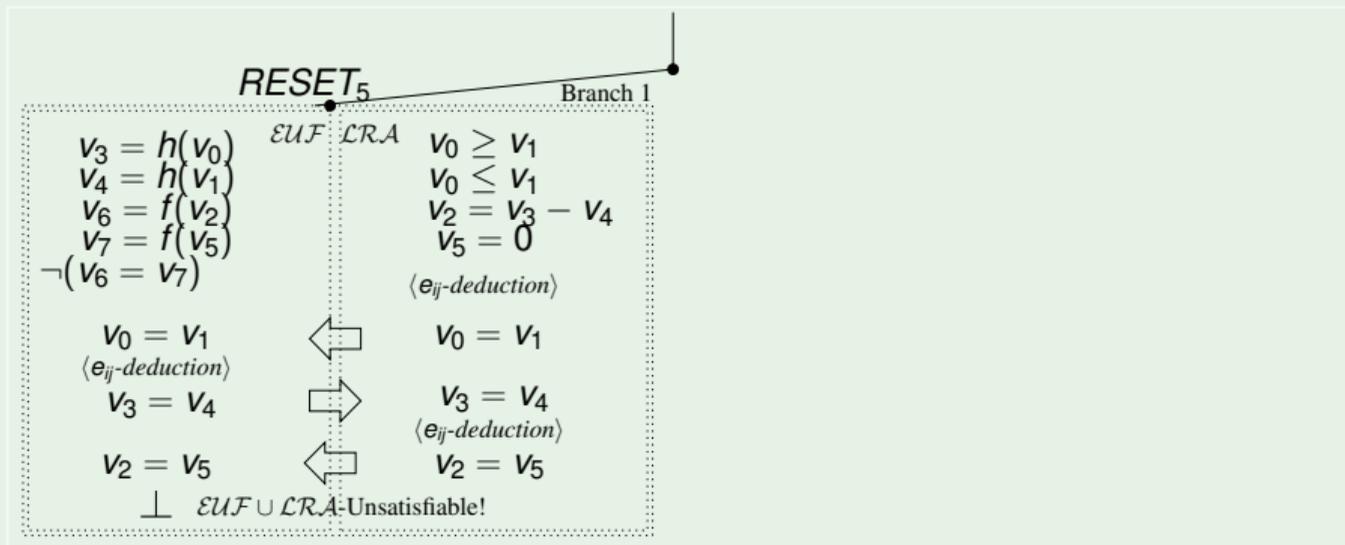


# N.O. Example (Convex Theory)

$$\mathcal{EUF}: (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$$

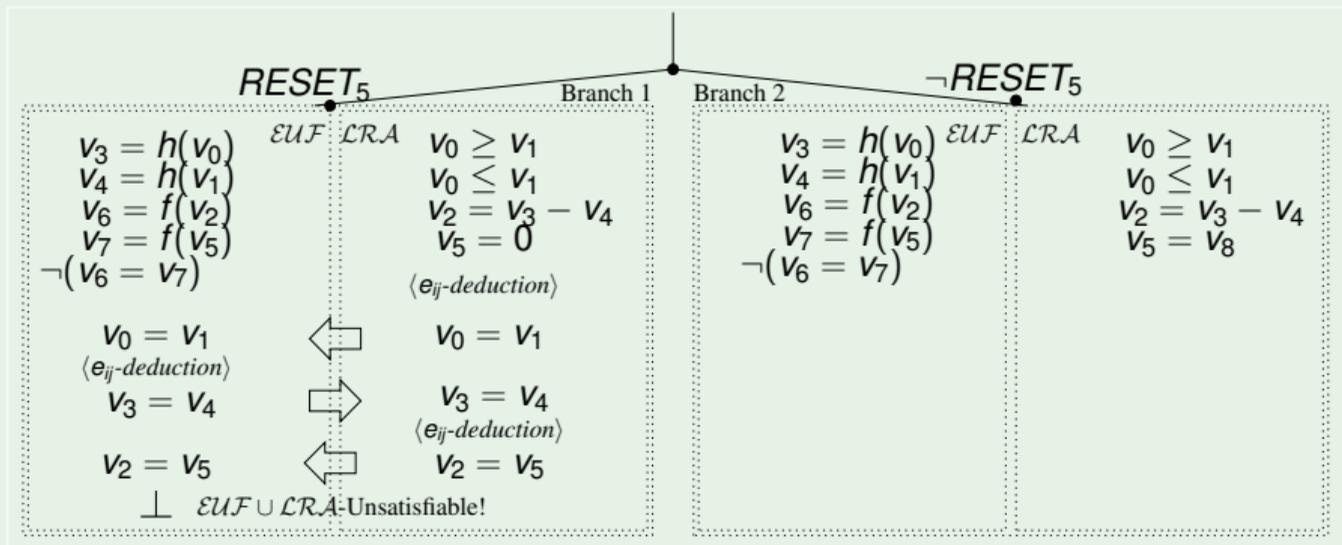
$$\mathcal{LRA}: (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$$

$$\text{Both}: (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$$



# N.O. Example (Convex Theory)

$\mathcal{EUF} : (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$   
 $\mathcal{LRA} : (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$   
 $\text{Both} : (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$

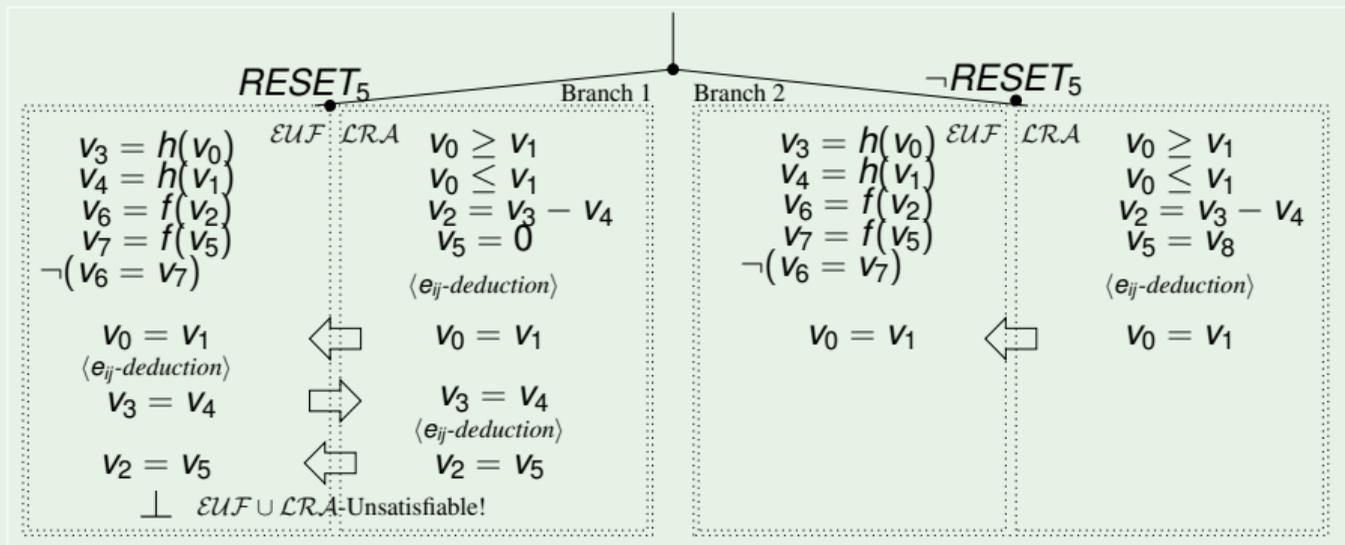


# N.O. Example (Convex Theory)

$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

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Both:  $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .

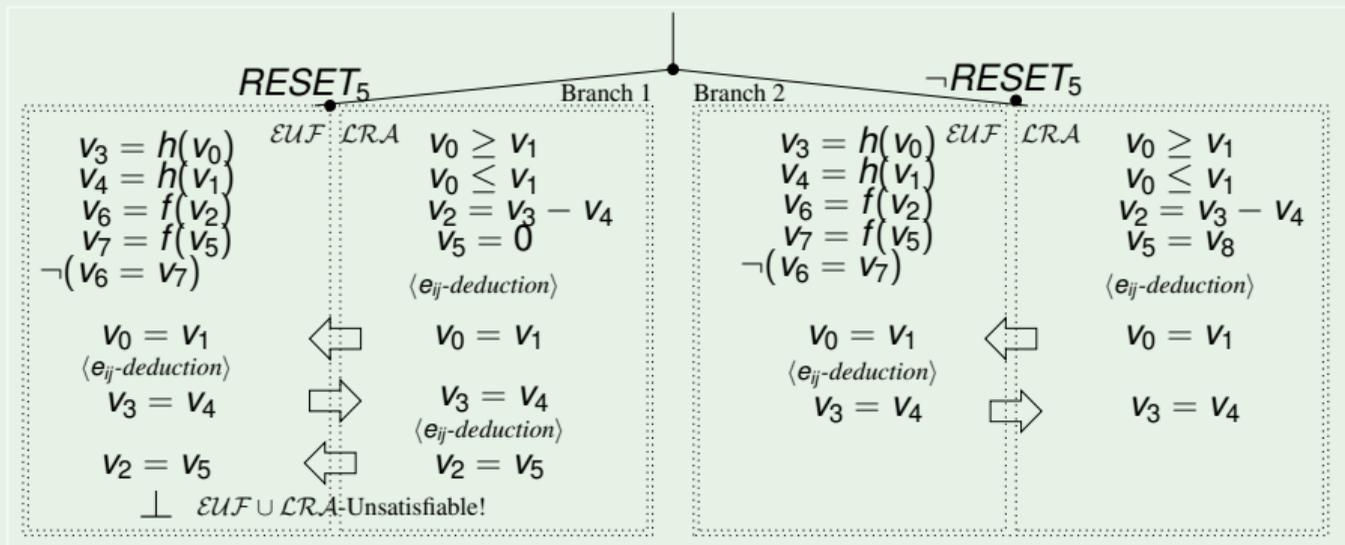


# N.O. Example (Convex Theory)

$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{LRA}$ :  $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

Both:  $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .

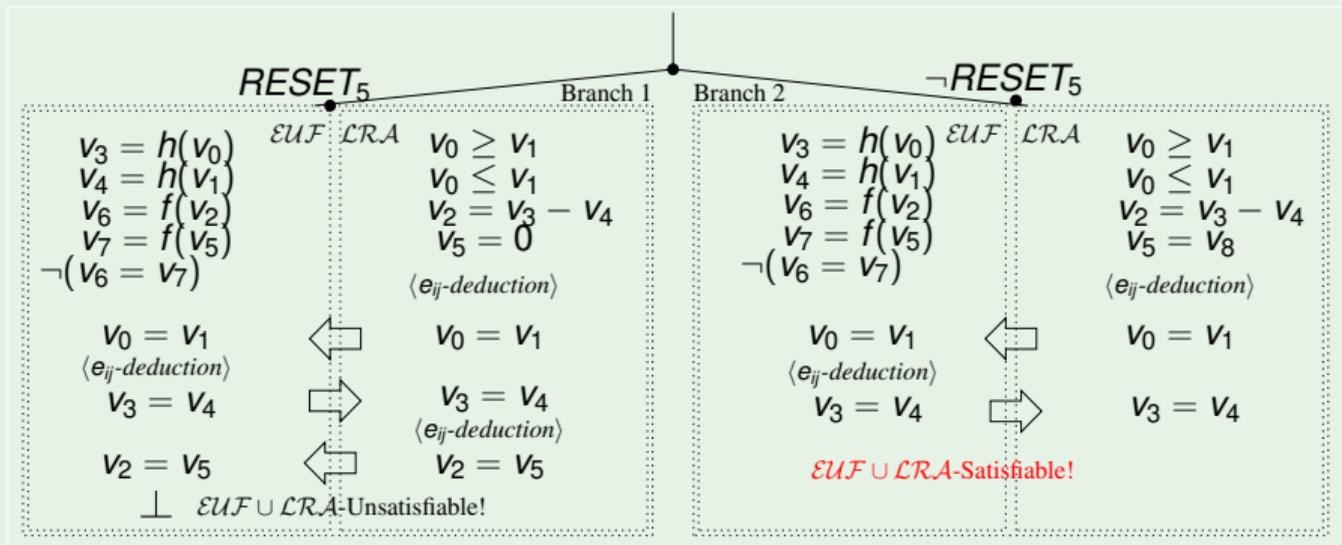


# N.O. Example (Convex Theory)

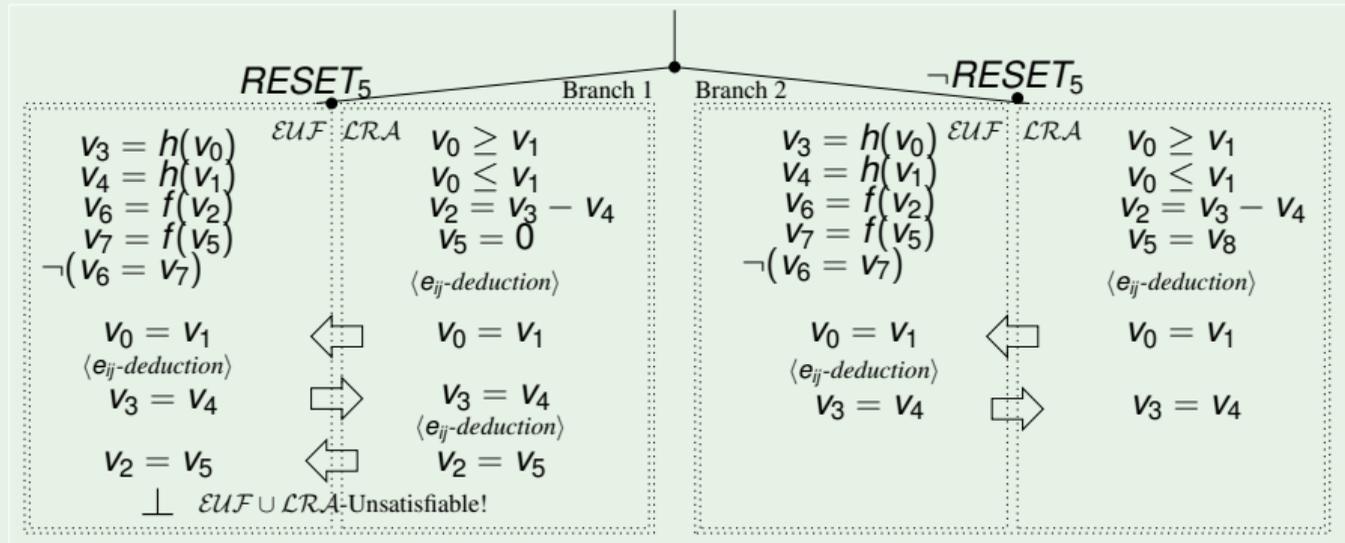
$\mathcal{EUF}$ :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{LRA}$ :  $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$

Both:  $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$ .



# N.O.: example (convex theory) [cont.]



$EU\mathcal{F}$ -conflict :  $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_5)) \rightarrow \perp$   
 $LRA$ -deduction :  $((v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$   
 $EU\mathcal{F}$ -deduction :  $((v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$   
 $LRA$ -deduction :  $((v_0 \geq v_1) \wedge (v_0 \leq v_1)) \rightarrow (v_0 = v_1)$   
 $\Rightarrow$   
 $EU\mathcal{F} \cup LRA$ -conflict :  $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 \geq v_1)) \rightarrow \perp$

For the previous N.O. example:

- write the (minimal) clauses corresponding to each  $e_{ij}$ -deduction
- find the final conflict clauses by resolving the  $e_{ij}$ -deduction clauses

## N.O.: example (non-convex theory)

$\mu_{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

$\mu_{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

## N.O.: example (non-convex theory)

$\mu_{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$

$\mu_{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$



$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $\theta_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$



$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

~~$v_1 = v_3$~~

*( $\theta_{ij}$ -deduction)*

$$v_5 = v_6$$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

*( $e_{ij}$ -deduction)*

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

*( $e_{ij}$ -deduction)*

$$v_5 = v_6$$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

*( $e_{ij}$ -deduction)*

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

*( $e_{ij}$ -deduction)*

$$v_5 = v_6$$

$$v_2 = v_3$$

$\perp$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

*( $e_{ij}$ -deduction)*

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

*( $e_{ij}$ -deduction)*

$$v_5 = v_6$$

$$\begin{array}{ll} v_2 = v_3 & v_2 = v_4 \\ \perp & \perp \end{array}$$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

*( $e_{ij}$ -deduction)*

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

*( $e_{ij}$ -deduction)*

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

*( $e_{ij}$ -deduction)*

$$v_5 = v_6$$

$$v_1 = v_4$$

SAT!

$$v_2 = v_3$$

$\perp$

$$v_2 = v_4$$

$\perp$

# N.O.: example (non-convex theory)

$\mu\mathcal{LIA}$

$$\begin{array}{ll} v_1 \geq 0 & v_5 = v_4 - 1 \\ v_1 \leq 1 & v_3 = 0 \\ v_2 \geq v_6 & v_4 = 1 \\ v_2 \leq v_6 + 1 & \end{array}$$

$\langle e_{ij}\text{-deduction} \rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

$\langle e_{ij}\text{-deduction} \rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu\mathcal{EUF}$

$$\begin{array}{l} \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}$$

$$v_1 = v_3$$

$\langle e_{ij}\text{-deduction} \rangle$

$$v_5 = v_6$$

$$v_2 = v_3$$

$\perp$

$$v_1 = v_4$$

SAT!

3  $e_{ij}$ -deductions,

3 branches

$$v_2 = v_4$$

$\perp$

# SMT( $\bigcup_i \mathcal{T}_i$ ) via “classic” Nelson-Oppen

## Main idea

Combine two or more  $\mathcal{T}_i$ -solvers into one ( $\bigcup_i \mathcal{T}_i$ )-solver via **Nelson-Oppen/Shostak (N.O.) combination procedure** [62, 76]

- based on the deduction and exchange of equalities between shared variables/terms (**interface equalities,  $e_{ij}$ s**)
- important improvements and evolutions [68, 7, 39]
- drawbacks [23, 24]:
  - require (possibly expensive) deduction capabilities from  $\mathcal{T}_i$ -solvers
  - [ with non-convex theories ] case-splits forced by the deduction of disjunctions of  $e_{ij}$ 's
  - generate (typically long) ( $\bigcup_i \mathcal{T}_i$ )-lemmas, without interface equalities  
 $\implies$  **no backjumping & learning from  $e_{ij}$ -reasoning**

## Exercise (cont.)

- Purify the following  $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- Solve it via Nelson-Oppen procedure

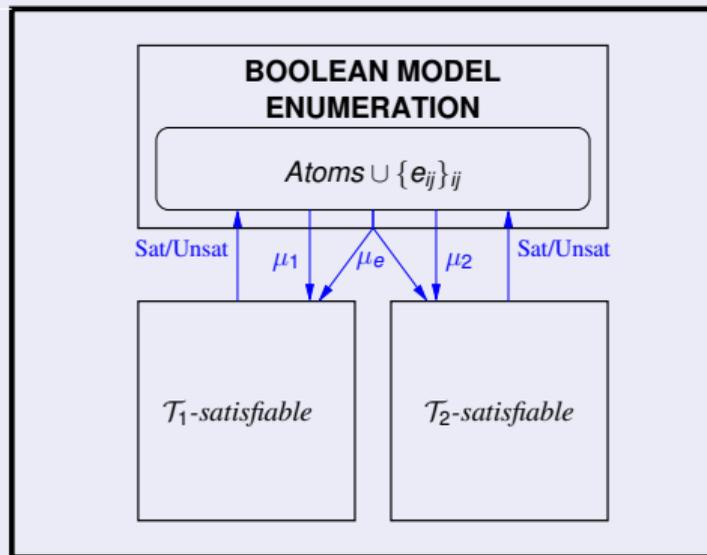
# SMT( $\bigcup_i \mathcal{T}_i$ ) via Delayed Theory Combination (DTC)

## Main idea

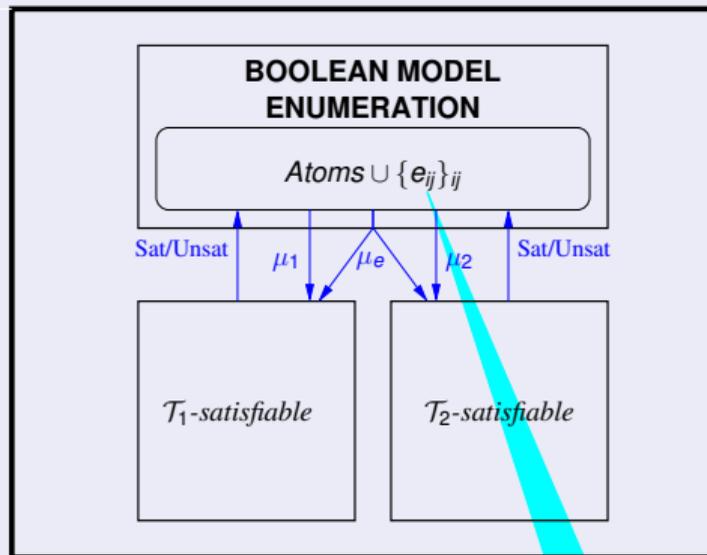
Delegate to the CDCL SAT solver part/most of the (possibly very expensive) reasoning effort on interface equalities previously due to the  $\mathcal{T}_i$ -solvers ( $e_{ij}$ -deduction, case-split). [15, 16, 24]

- based on Boolean reasoning on interface equalities via CDCL (plus  $\mathcal{T}$ -propagation)
- important improvements and evolutions [36, 9]
- feature wrt N.O. [23, 24]
  - do not require (possibly expensive) deduction capabilities from  $\mathcal{T}_i$ -solvers
  - with non-convex theories, case-splits on  $e_{ij}$ 's handled by SAT solver
  - generate  $\mathcal{T}_i$ -lemmas with interface equalities  
 $\implies$  **backjumping & learning from  $e_{ij}$ -reasoning**

# DTC: Basic schema



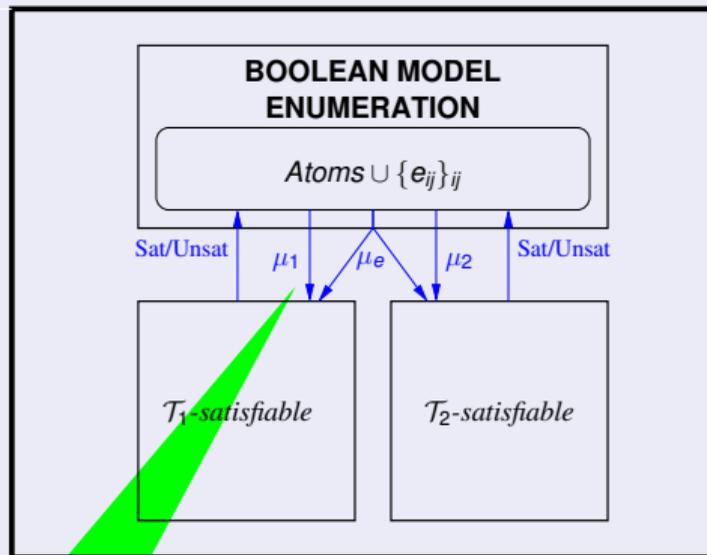
# DTC: Basic schema



The boolean solver assigns values not only to atoms in  $Atoms(\phi)$ , but also to interface equalities  $\{(v_i = v_j)\}_{ij}$ :

$$\mu = \mu_1 \cup \mu_2 \cup \mu_e, \quad \mu_e := \{[\neg](v_i = v_j) \mid v_i, v_j \in \mu_1 \cup \mu_2\}$$

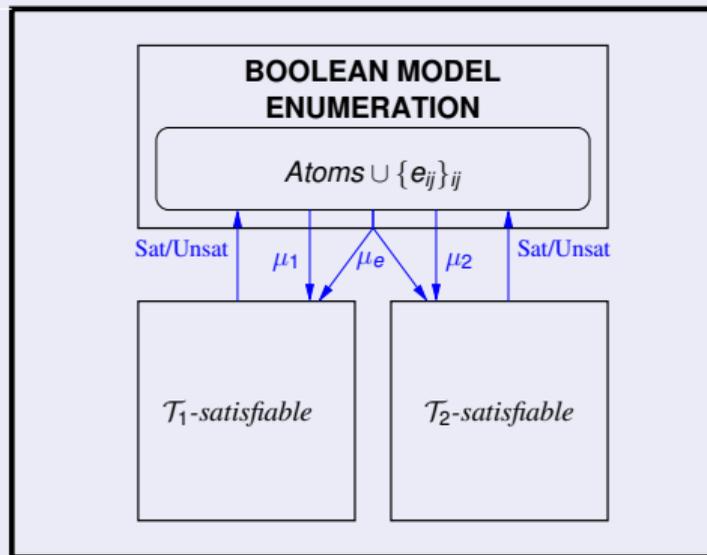
# DTC: Basic schema



Each  $\mathcal{T}_i$ -solver interacts only with the boolean solver

- receives  $\mu'_i := \mu_i \cup \mu_e$  from Bool
- checks the  $\mathcal{T}_i$ -satisfiability of  $\mu'_i$

# DTC: Basic schema



...until either:

- some  $\mu$  propositionally satisfies  $\phi$  and both  $\mu'_i := \mu_i \cup \mu_e$  are  $\mathcal{T}_i$ -consistent  
 $\implies (\phi \text{ is } \mathcal{T}_1 \cup \mathcal{T}_2\text{-sat})$
- no more assignment  $\mu$  are available  
 $\implies (\phi \text{ is } \mathcal{T}_1 \cup \mathcal{T}_2\text{-unsat})$

## DTC: enhanced schema

- **CDCL-based assignment enumeration** on  $Atoms(\phi) \cup \{e_{ij}\}_{ij}$ ,  
⇒ benefits of state-of-the-art SAT techniques
- **Early pruning**: invoke the  $\mathcal{T}_i$ -solver's before every Boolean decision  
⇒ total assignments generated only when strictly necessary
- **Branching**: branching on  $e_{ij}$ 's postponed  
⇒ Boolean search on  $e_{ij}$ 's performed only when strictly necessary
- **Theory-Backjumping & Learning**:  $e_{ij}$ 's are involved in conflicts  
⇒  $e_{ij}$ 's can be assigned by unit propagation
- **Theory-deduction & learning**: if  $\mathcal{T}_i$ -solver deduces unassigned literals  $I$  on  $Atoms(\phi) \cup \{e_{ij}\}_{ij}$ 
  - $I$  is passed back to the Boolean solver, which unit-propagates it
  - the deduction  $\mu' \models I$  is learned as a clause  $\mu' \rightarrow I$  (deduction clause)
- ...

# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l} \mu_{\mathcal{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \left. \vphantom{\begin{array}{l} \mu_{\mathcal{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array}} \right\} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l} \mu_{\mathcal{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{\mathcal{LIA}}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

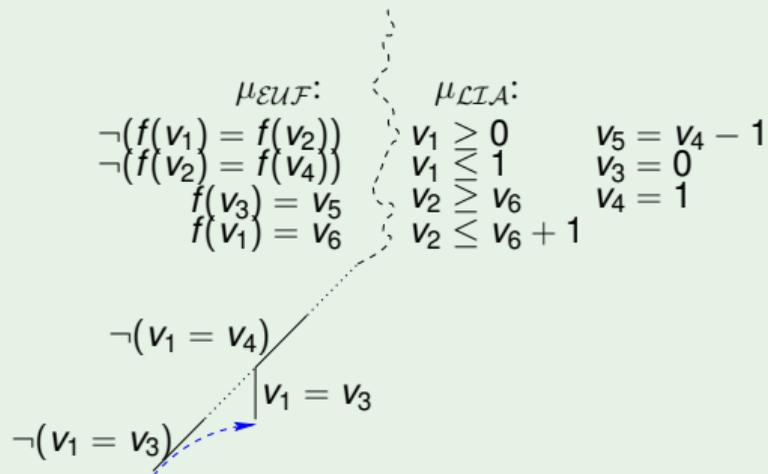
$$\neg(v_1 = v_4)$$

$$\neg(v_1 = v_3)$$

$\mathcal{LIA}$ -unsat,  $C_{13}$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



$$G_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)

|                         |                        |                 |
|-------------------------|------------------------|-----------------|
| $\mu_{\mathcal{EUF}}:$  | $\mu_{\mathcal{LIA}}:$ |                 |
| $\neg(f(v_1) = f(v_2))$ | $v_1 \geq 0$           | $v_5 = v_4 - 1$ |
| $\neg(f(v_2) = f(v_4))$ | $v_1 \leq 1$           | $v_3 = 0$       |
| $f(v_3) = v_5$          | $v_2 \geq v_6$         | $v_4 = 1$       |
| $f(v_1) = v_6$          | $v_2 \leq v_6 + 1$     |                 |

$$\neg(v_1 = v_4)$$

$$\neg(v_1 = v_3)$$

$$\neg(v_5 = v_6)$$

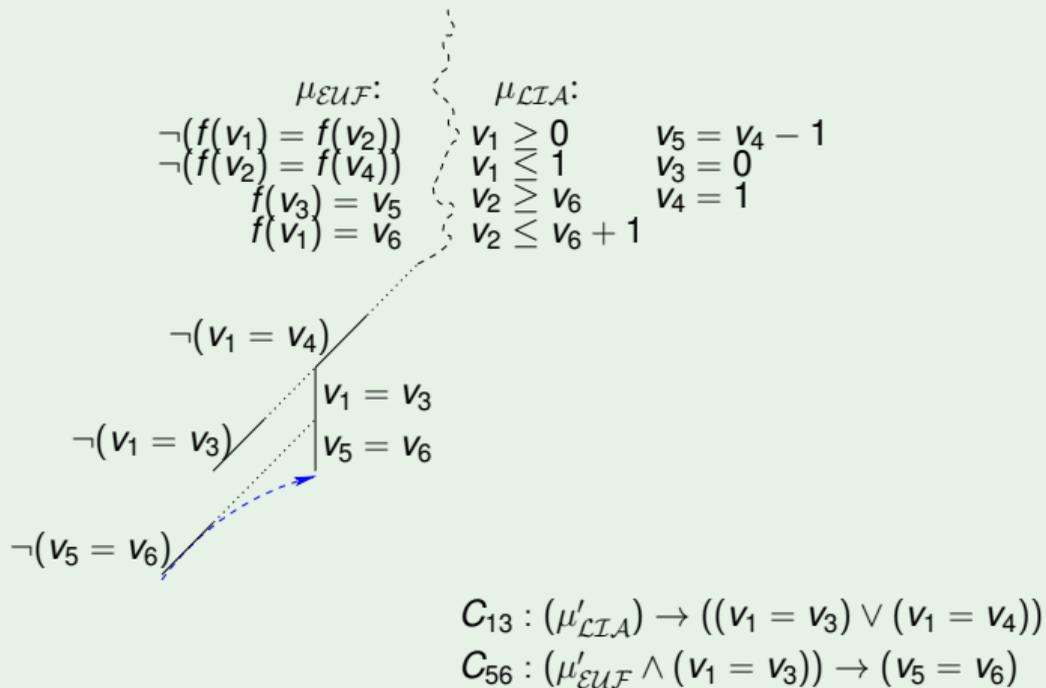
$$v_1 = v_3$$

$\mathcal{EUF}$ -unsat,  $C_{56}$

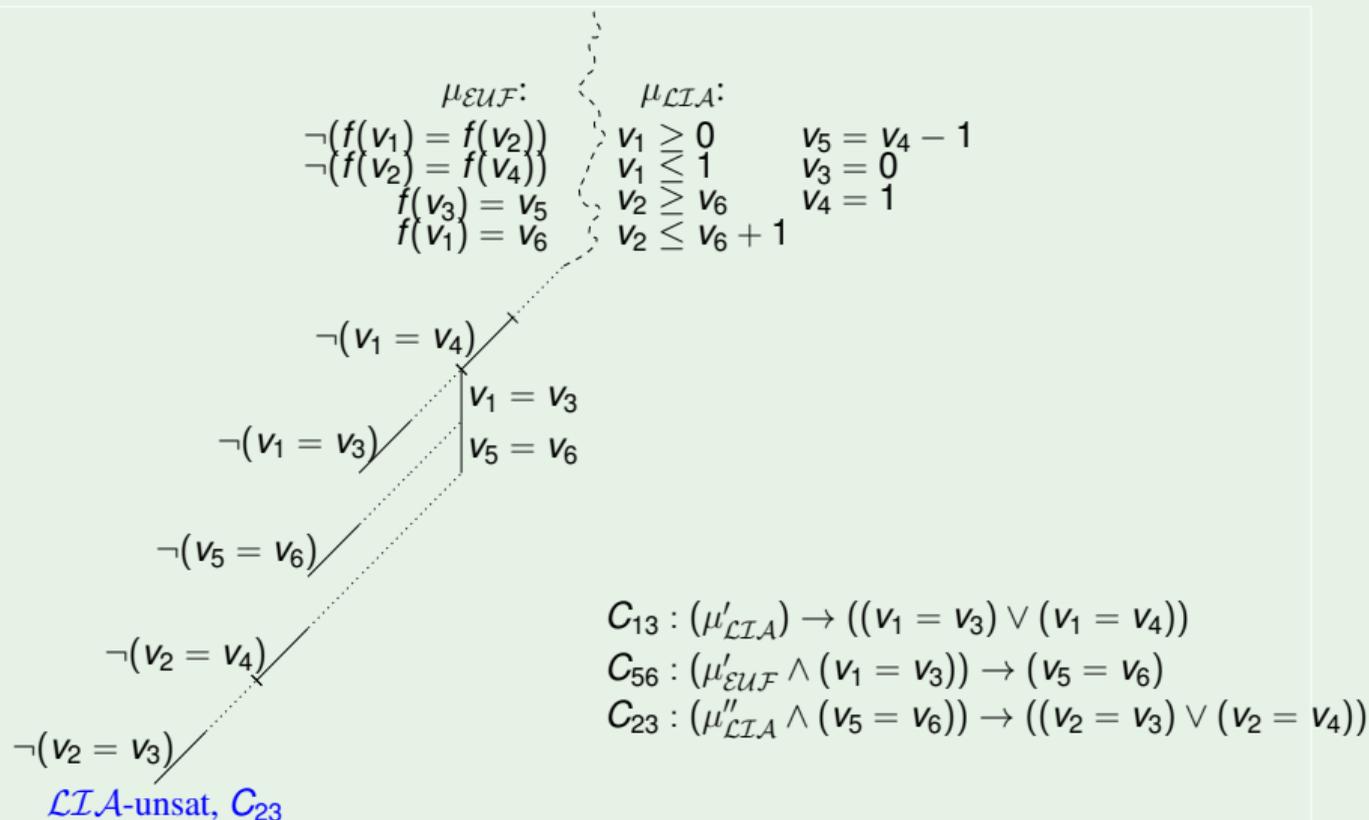
$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

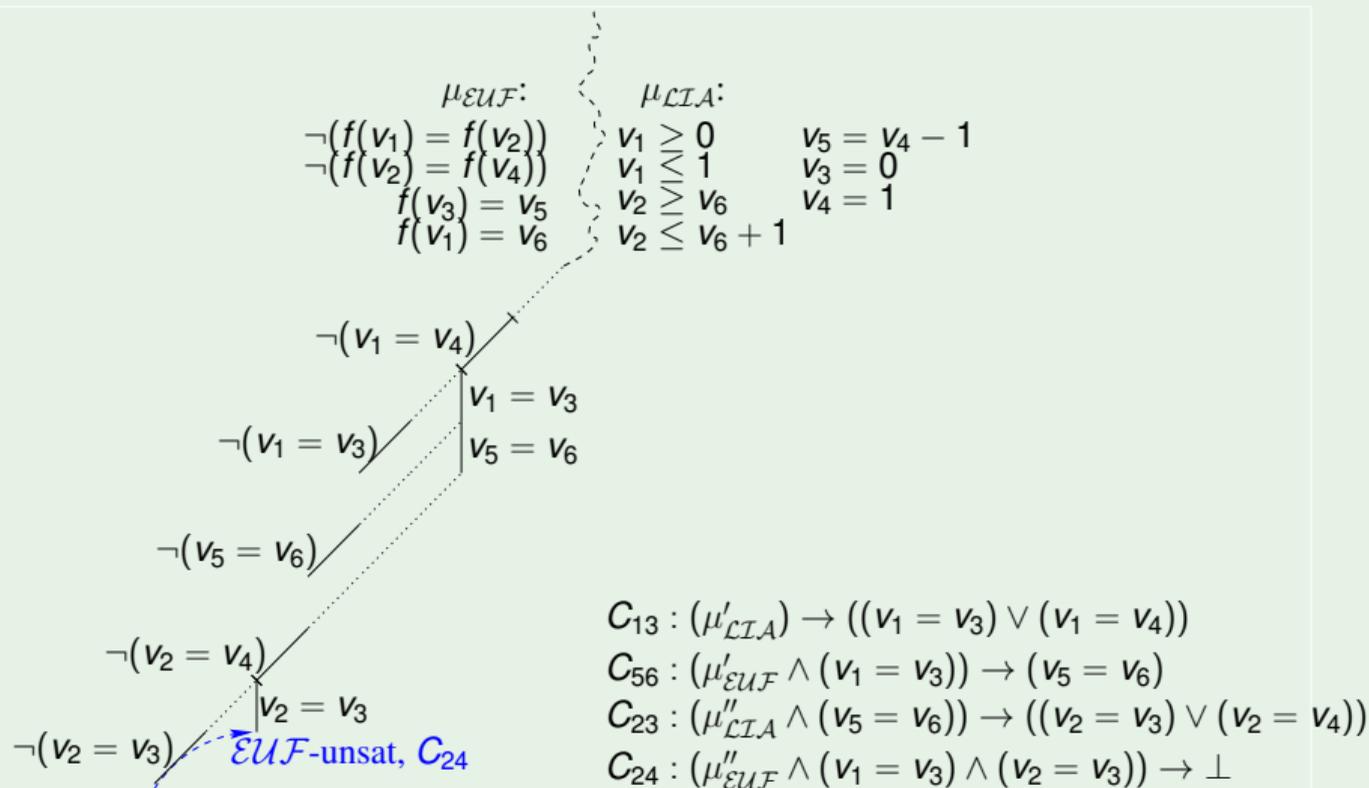
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



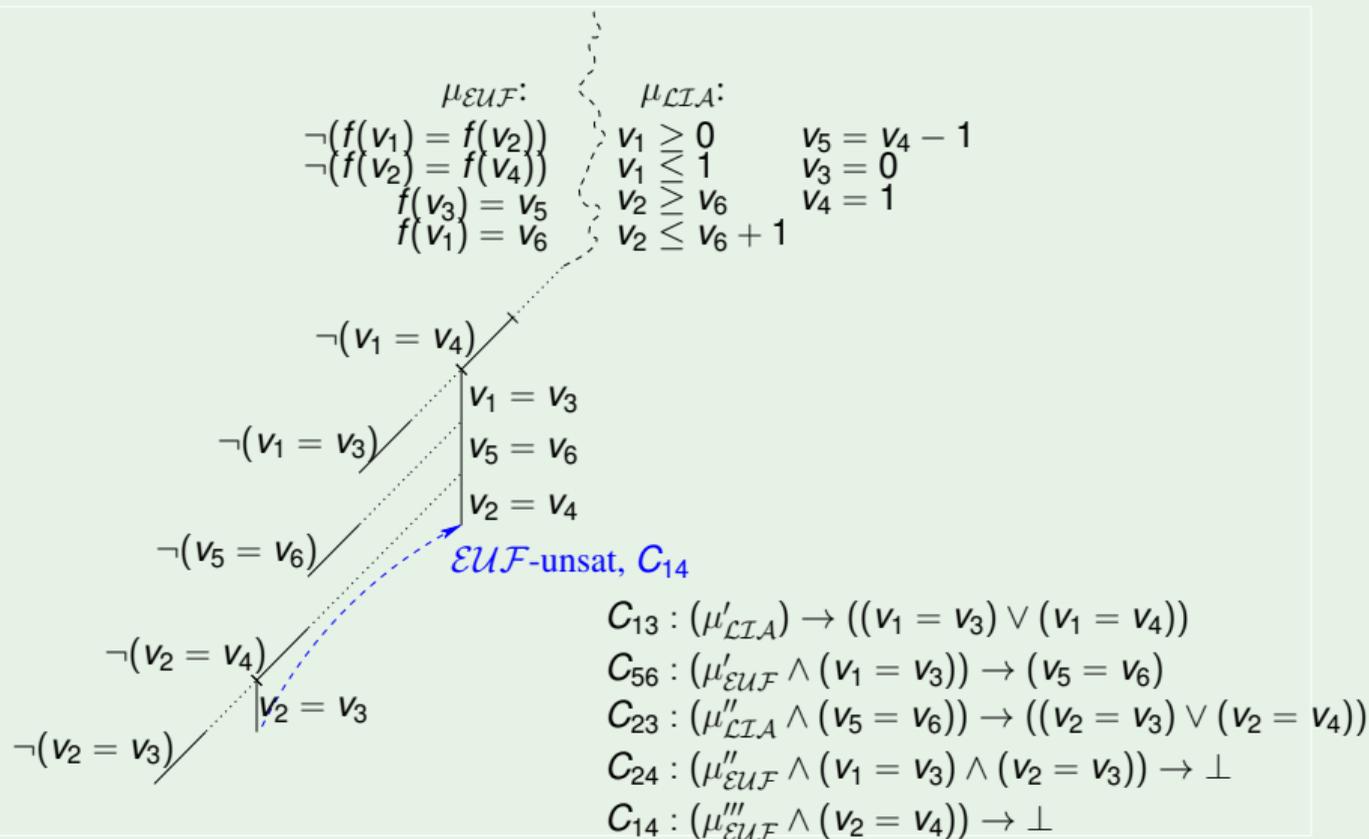
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



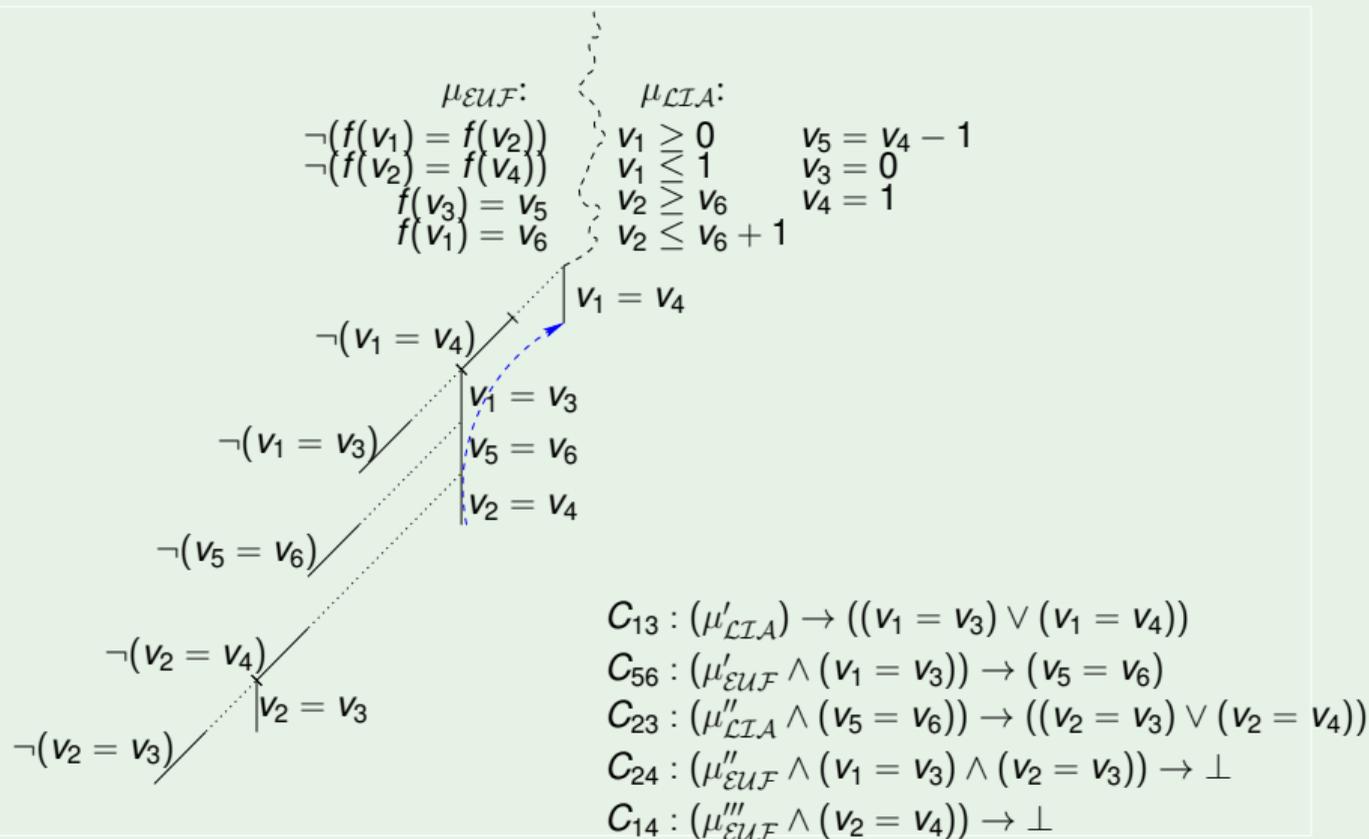
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



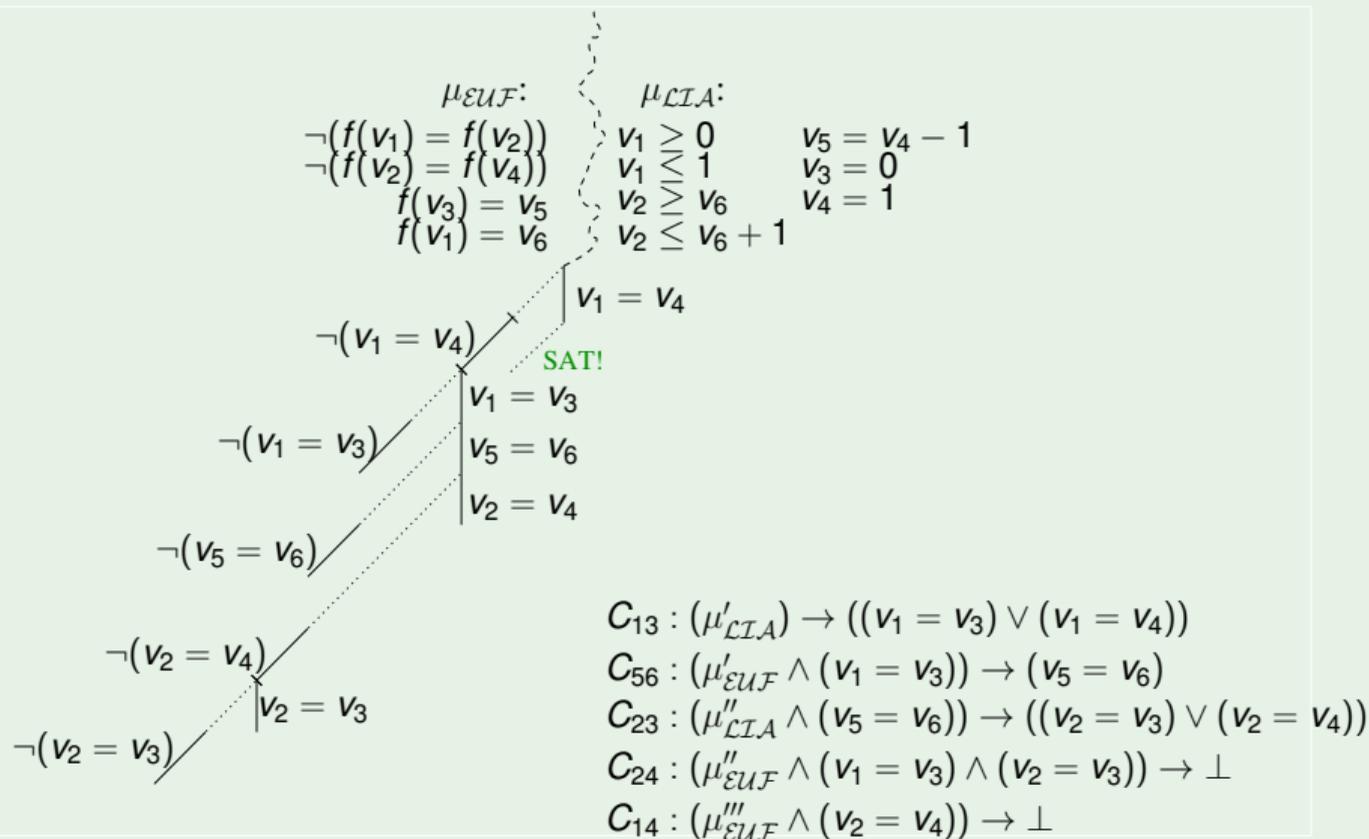
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



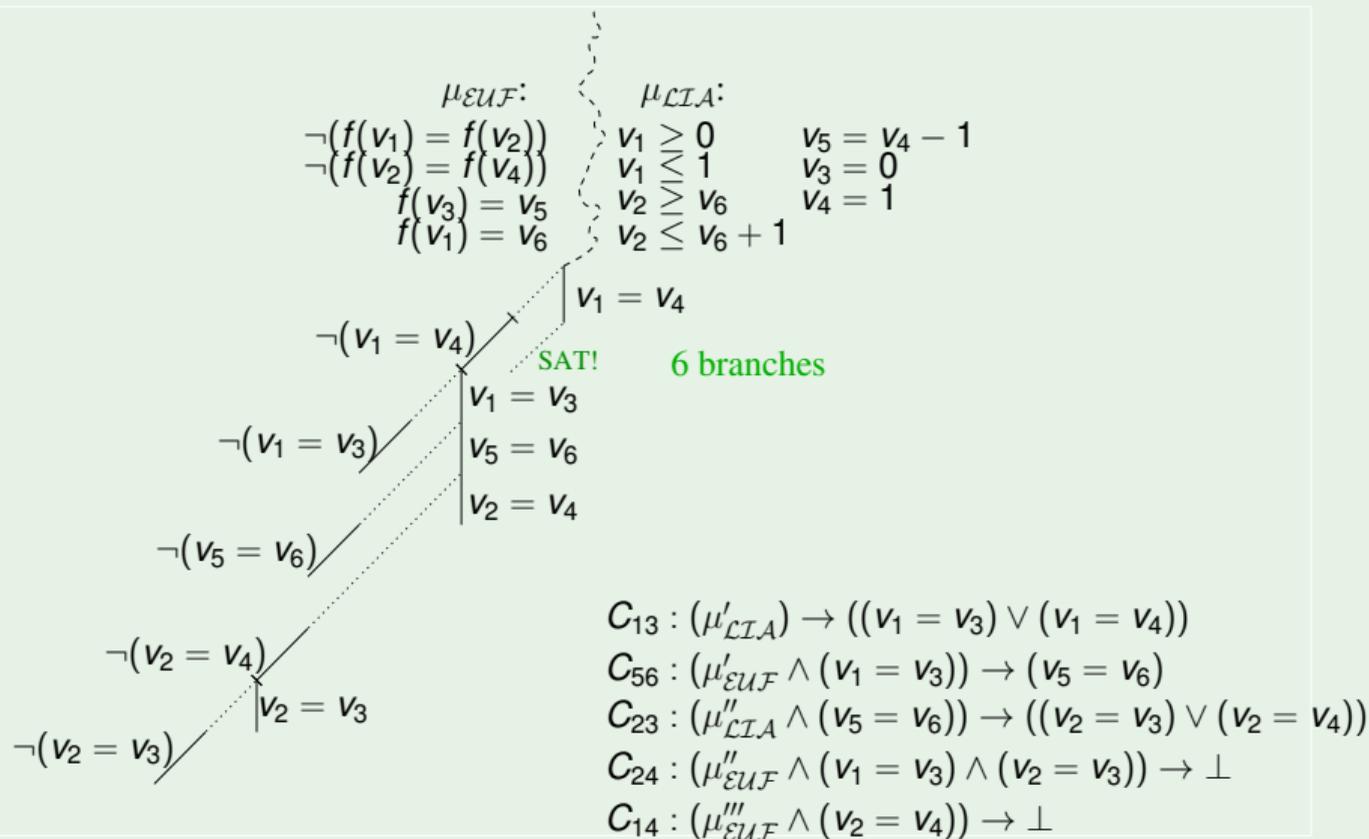
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



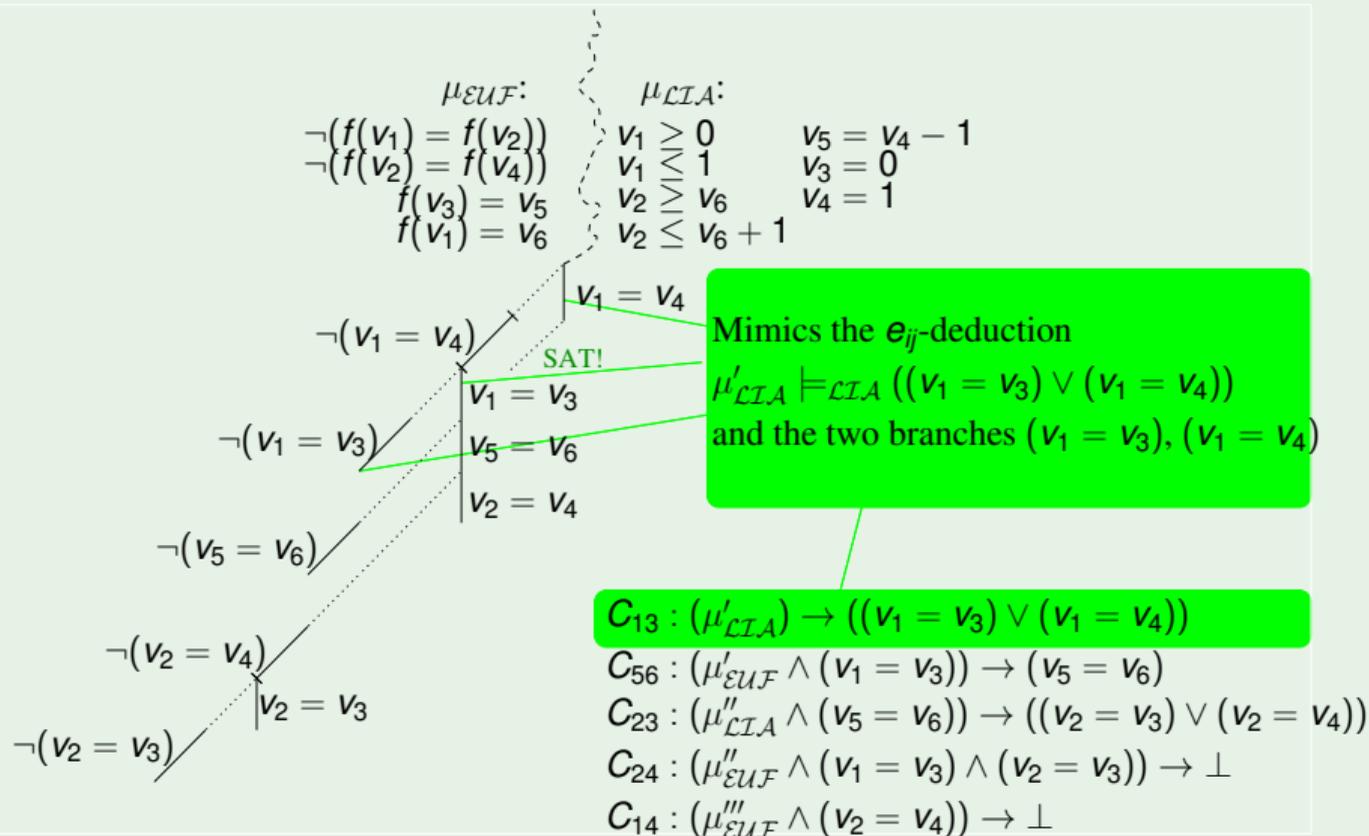
# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



# DTC: example w.out $\mathcal{T}$ -prop. (non-convex theory)



# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l} \mu_{\text{EUF}}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{\text{LIA}}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l} \mu_{EUF}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$

$\mathcal{LIA}$ -deduce  $(v_1 = v_4) \vee (v_1 = v_3)$ ,  $C_{13}$

$$C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l} \mu_{EUF}: \\ \neg(f(v_1) = f(v_2)) \\ \neg(f(v_2) = f(v_4)) \\ f(v_3) = v_5 \\ f(v_1) = v_6 \end{array} \quad \begin{array}{l} \mu_{LIA}: \\ v_1 \geq 0 \\ v_1 \leq 1 \\ v_2 \geq v_6 \\ v_2 \leq v_6 + 1 \end{array} \quad \begin{array}{l} v_5 = v_4 - 1 \\ v_3 = 0 \\ v_4 = 1 \end{array}$$
  
$$\begin{array}{l} \neg(v_1 = v_4) \\ v_1 = v_3 \end{array}$$

$$C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{\mathcal{EUF}}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{\mathcal{LIA}}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1
 \end{array}$$

$$\begin{array}{l}
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6
 \end{array}
 \left| \begin{array}{l}
 \mathcal{EUF}\text{-deduce } (v_5 = v_6), C_{56}
 \end{array} \right.$$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{EUF}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{LIA}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1 \\
 \\
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6 \quad \text{LIA-deduce } (v_2 = v_4) \vee (v_2 = v_3), C_{23}
 \end{array}$$

$$\begin{array}{l}
 C_{13}: (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\
 C_{56}: (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\
 C_{23}: (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))
 \end{array}$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

$$\begin{array}{l}
 \mu_{\mathcal{EUF}}: \\
 \neg(f(v_1) = f(v_2)) \\
 \neg(f(v_2) = f(v_4)) \\
 f(v_3) = v_5 \\
 f(v_1) = v_6 \\
 \\
 \mu_{\mathcal{LIA}}: \\
 v_1 \geq 0 \\
 v_1 \leq 1 \\
 v_2 \geq v_6 \\
 v_2 \leq v_6 + 1 \\
 \\
 v_5 = v_4 - 1 \\
 v_3 = 0 \\
 v_4 = 1
 \end{array}$$

$$\begin{array}{l}
 \neg(v_1 = v_4) \\
 v_1 = v_3 \\
 v_5 = v_6 \\
 \neg(v_2 = v_4) \\
 v_2 = v_3
 \end{array}$$

$\mathcal{EUF}$ -unsat,  $C_{24}$

$$\begin{array}{l}
 C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\
 C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\
 C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4)) \\
 C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp
 \end{array}$$

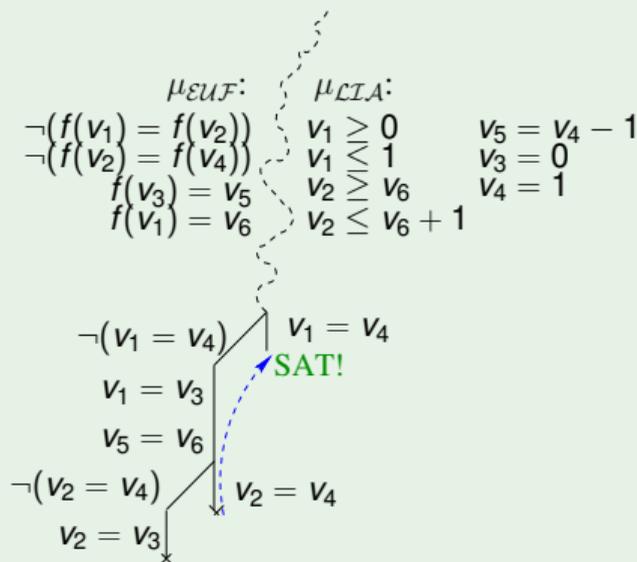
# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)

|                         |                    |                 |
|-------------------------|--------------------|-----------------|
| $\mu_{EUF}:$            | $\mu_{LIA}:$       |                 |
| $\neg(f(v_1) = f(v_2))$ | $v_1 \geq 0$       | $v_5 = v_4 - 1$ |
| $\neg(f(v_2) = f(v_4))$ | $v_1 \leq 1$       | $v_3 = 0$       |
| $f(v_3) = v_5$          | $v_2 \geq v_6$     | $v_4 = 1$       |
| $f(v_1) = v_6$          | $v_2 \leq v_6 + 1$ |                 |

$$\begin{array}{l} \neg(v_1 = v_4) \\ v_1 = v_3 \\ v_5 = v_6 \\ \neg(v_2 = v_4) \\ v_2 = v_3 \end{array} \quad \begin{array}{l} v_2 = v_4 \\ \text{EUF-unsat, } C_{14} \end{array}$$

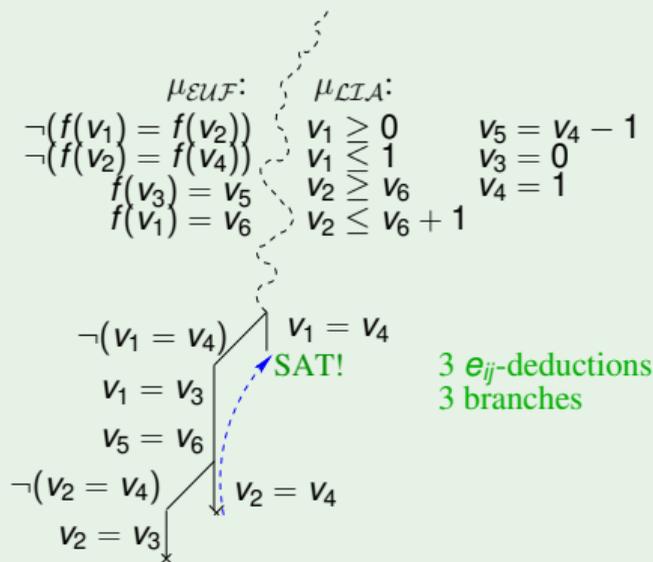
$$\begin{aligned} C_{13} &: (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4)) \\ C_{56} &: (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6) \\ C_{23} &: (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4)) \\ C_{24} &: (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp \\ C_{14} &: (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp \end{aligned}$$

# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)



- $C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$   
 $C_{56} : (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$   
 $C_{23} : (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$   
 $C_{24} : (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp$   
 $C_{14} : (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp$

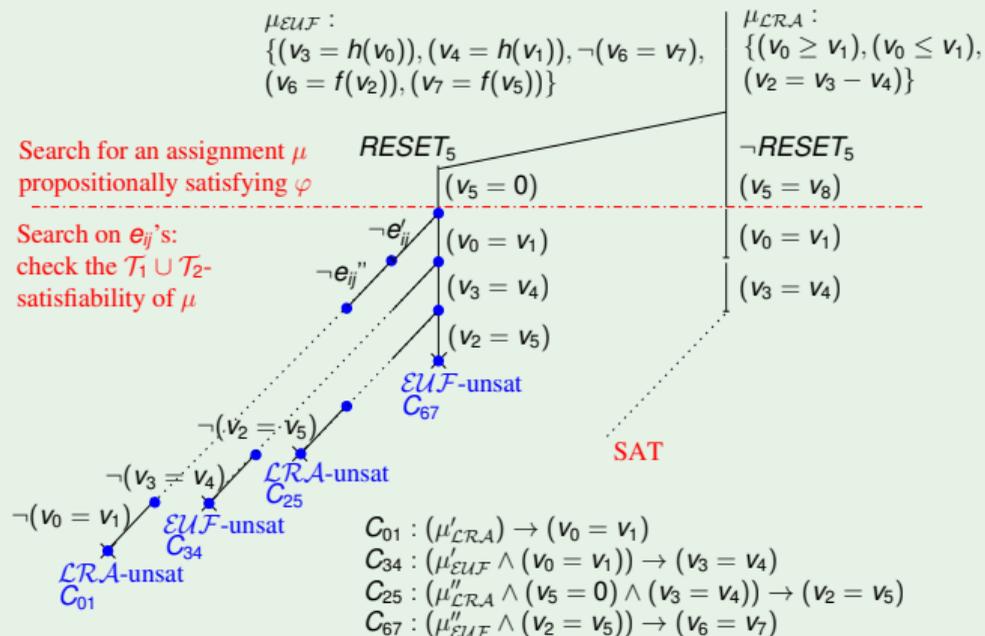
# DTC: example with $\mathcal{T}$ -prop. (non-convex theory)



- $C_{13} : (\mu'_{LIA}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$   
 $C_{56} : (\mu'_{EUF} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$   
 $C_{23} : (\mu''_{LIA} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$   
 $C_{24} : (\mu''_{EUF} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \perp$   
 $C_{14} : (\mu'''_{EUF} \wedge (v_2 = v_4)) \rightarrow \perp$

# DTC: example without $\mathcal{T}$ -propagation (convex theory)

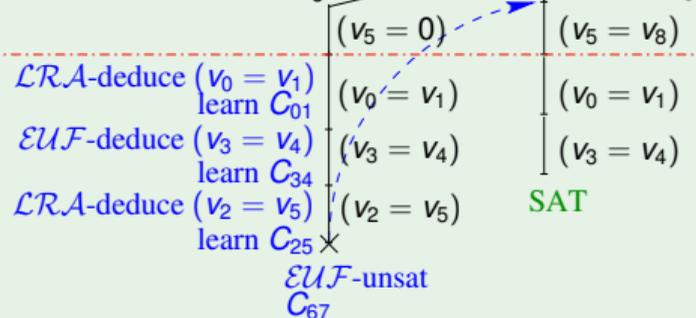
$$\begin{aligned} \mathcal{EUF} : & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA} : & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge \\ \text{Both} : & (\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{aligned}$$



# DTC: example with $\mathcal{T}$ -propagation (convex theory)

$\mathcal{EUF}$  :  $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$   
 $\mathcal{LRA}$  :  $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (\text{RESET}_5 \rightarrow (v_5 = 0)) \wedge$   
 $\text{Both}$  :  $(\neg \text{RESET}_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$

$\mu_{\mathcal{EUF}}$  :  
 $\{(v_3 = h(v_0)), (v_4 = h(v_1)), \neg(v_6 = v_7),$   
 $(v_6 = f(v_2)), (v_7 = f(v_5))\}$   
 $\mu_{\mathcal{LRA}}$  :  
 $\{(v_0 \geq v_1), (v_0 \leq v_1),$   
 $(v_2 = v_3 - v_4)\}$



$C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1)$   
 $C_{34} : (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$   
 $C_{25} : (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$   
 $C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)$

# DTC + Model-based heuristic (aka Model-Based Theory Combination) [36]

- Initially, no interface equalities generated
- When a model is found, check against all the possible interface equalities
  - If  $\mathcal{T}_1$  and  $\mathcal{T}_2$  agree on the implied equalities, then return SAT
  - Otherwise, branch on equalities implied by  $\mathcal{T}_1$ -model but not by  $\mathcal{T}_2$ -model
- “Optimistic” approach, similar to axiom instantiation

# Exercises

For each of the previous DTC examples:

- write the (minimal) clauses corresponding to each  $e_{ij}$ -deduction (as clauses rather than as implications)
- compute the conflict-analysis steps leading to the backjumping steps in the figures.

For each of the previous DTC examples, draw the case in which the  $\mathcal{EUF}$ -solver has deduction capabilities and the  $\mathcal{LRA}$ -solver (resp. the  $\mathcal{LIA}$ -solver) does not.

## Exercise (cont.)

- Purify the following  $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$ -formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge \\ ((f(d) = f(0)) \rightarrow (\text{read}(\text{write}(V, i, x), i + d) = x + 1))$$

- Solve it via the DTC procedure
  - assuming no  $\mathcal{T}$ -deduction capabilities of the  $\mathcal{T}$ -solvers
  - assuming  $\mathcal{T}$ -deduction capabilities of the  $\mathcal{T}$ -solvers

- 1 Introduction
  - Basics on First-order Logic
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  - Satisfiability Modulo Theories
  - Motivations and Goals of SMT
- 2 Efficient SMT solving
  - Combining SAT with Theory Solvers
  - Theory Solvers for Theories of Interest
  - SMT for Combinations of Theories
- 3 **Beyond Solving: Advanced SMT Functionalities**
  - Proofs and Unsatisfiable Cores
  - All-SMT & Predicate Abstraction (hints)
  - SMT with Optimization (Optimization Modulo Theories)

# Beyond Solving: advanced SAT & SMT functionalities

Advanced SMT functionalities (very important in FV):

- Building **proofs of  $\mathcal{T}$ -unsatisfiability**
- Extracting  **$\mathcal{T}$ -unsatisfiable Cores**
- [ Computing **Craig interpolants** ]
- Performing **All-SMT and Predicate Abstraction**
- Deciding/optimizing **SMT problems with costs**

# Outline

- 1 Introduction
  - Basics on First-order Logic
  - What is a Theory?
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# Building (Resolution) Proofs of $\mathcal{T}$ -Unsatisfiability

## Resolution proof of $\mathcal{T}$ -unsatisfiability

Very similar to building proofs with plain SAT:

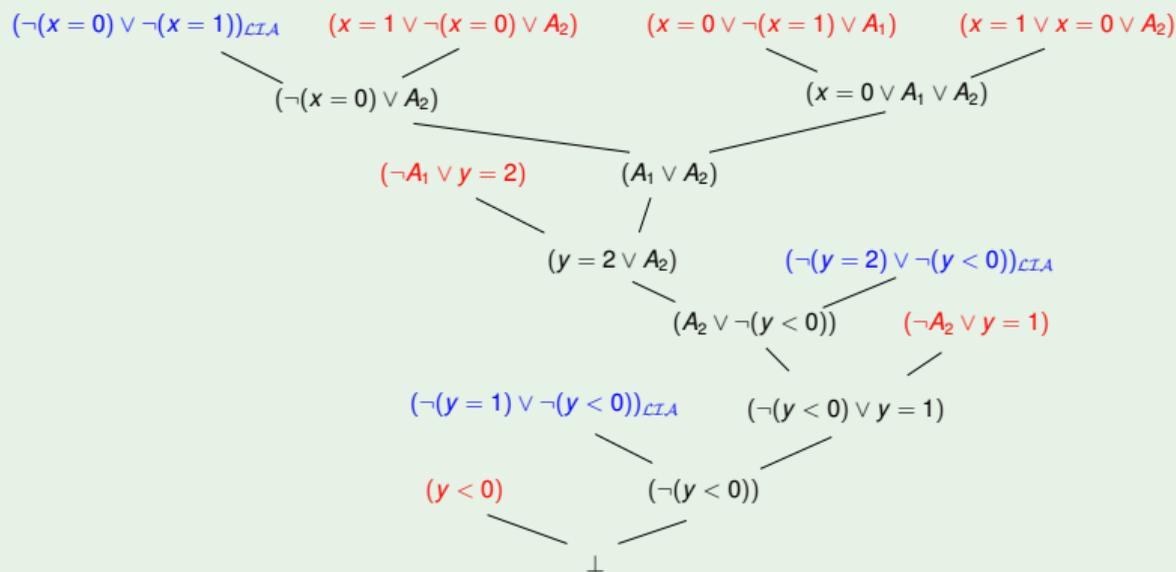
- resolution proofs whose leaves are original clauses and  $\mathcal{T}$ -lemmas returned by the  $\mathcal{T}$ -solver (i.e.,  $\mathcal{T}$ -conflict and  $\mathcal{T}$ -deduction clauses)
- built by backward traversal of implication graphs, as in CDCL SAT
- Sub-proofs of  $\mathcal{T}$ -lemmas can be built in some  $\mathcal{T}$ -specific deduction framework if requested

Important for:

- certifying  $\mathcal{T}$ -unsatisfiability results
- computing unsatisfiable cores
- computing interpolants

# Building Proofs of $\mathcal{T}$ -Unsatisfiability: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge \\ (\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$



relevant original clauses, irrelevant original clauses,  $\mathcal{T}$ -lemmas

## Example: proof on non-strict $\mathcal{LRA}$ inequalities

- A proof of unsatisfiability for a set of non-strict  $\mathcal{LRA}$  inequalities can be obtained by building a linear combination (with positive coefficients) of such inequalities, each time eliminating one or more variables, until you get a contradictory inequality on constant values.
- Example:

$$\varphi \stackrel{\text{def}}{=} (0 \leq x_1 - 3x_2 + 1), (0 \leq x_1 + x_2), (0 \leq x_3 - 2x_1 - 3), (0 \leq 1 - 2x_3).$$

A proof of unsatisfiability  $P$  for  $\varphi$  is the following:

$$\frac{\frac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\text{COMB } (0 \leq 4x_1 + 1) \text{ with coefficients } 1 \text{ and } 3} \quad \frac{(0 \leq x_3 - 2x_1 - 3) \quad (0 \leq 1 - 2x_3)}{\text{COMB } (0 \leq -4x_1 - 5) \text{ with coefficients } 2 \text{ and } 1}}{\text{COMB } (0 \leq -4) \text{ with coefficients } 1 \text{ and } 1}$$

- It is possible to produce such proof from an unsatisfiable tableau in Simplex procedure for  $\mathcal{LRA}$  [29, 31]
- It is straightforward to produce such proof from a negative cycle in the graph-based procedure for  $\mathcal{DL}$  [29, 31]

# Exercises

- Given the following  $\mathcal{LRA}$ -unsatisfiable sets of  $\mathcal{LRA}$ -inequalities, produce a proof of unsatisfiability by means of the technique described in the previous slides.

$$\bullet \left\{ \begin{array}{l} (0 \leq +2x_1 - 2x_2 \quad +6x_4 + 1) \\ (0 \leq -x_1 + 2x_2 \quad -x_4 - 2) \\ (0 \leq \quad -2x_2 + 2x_3 \quad + 3) \\ (0 \leq \quad \quad -x_3 - x_4 - 2) \\ (0 \leq \quad +2x_2 \quad +x_4 + 2) \end{array} \right\}$$

- [create your own  $\mathcal{LRA}$ -unsatisfiable set of  $\mathcal{LRA}$ -inequalities ]
- Given the following  $\mathcal{DL}$ -unsatisfiable sets of  $\mathcal{DL}$ -inequalities, produce a proof of unsatisfiability by means of the technique described in the previous slides.
  - the  $\mathcal{DL}$  example in the  $\mathcal{T}$ -solver section and its negative cycle
  - [create your own  $\mathcal{DL}$ -unsatisfiable set of  $\mathcal{LRA}$ -inequalities ]

# Extraction of $\mathcal{T}$ -unsatisfiable cores

## The problem

Given a  $\mathcal{T}$ -unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum)  $\mathcal{T}$ -unsatisfiable subset ( $\mathcal{T}$ -unsatisfiable core)

- Wide literature in SAT
- Some implementations, very few literature for SMT [28, 56]
- We recognize three approaches:
  - **Proof-based** approach (CVC4, MathSAT):  
byproduct of finding a resolution proof
  - **Assumption-based** approach (Yices):  
use extra variables labeling clauses, as in the plain Boolean case
  - **Lemma-Lifting** approach [28] :  
use an external (possibly-optimized) Boolean unsat-core extractor

# The proof-based approach to $\mathcal{T}$ -unsat cores

Idea (adapted from [82])

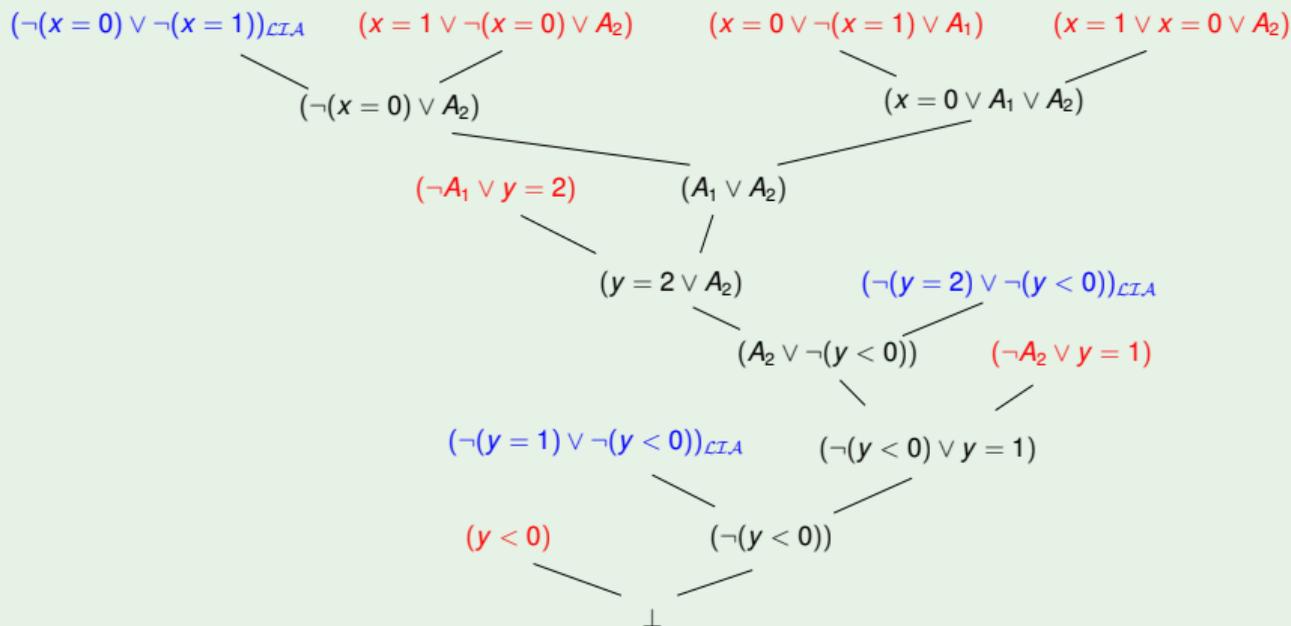
Unsatisfiable core of  $\varphi$ :

- in SAT: the set of leaf clauses of a resolution proof of unsatisfiability of  $\varphi$
- in  $\text{SMT}(\mathcal{T})$ : the set of leaf clauses of a resolution proof of  $\mathcal{T}$ -unsatisfiability of  $\varphi$ , minus the  $\mathcal{T}$ -lemmas

# The proof-based approach to $\mathcal{T}$ -unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$

$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$



# The Assumption-based approach to $\mathcal{T}$ -unsat cores

Idea (adapted from [57])

Let  $\varphi$  be  $\bigwedge_{i=1}^n C_i$  s.t.  $\varphi$  unsatisfiable.

- 1 each clause  $C_i$  in  $\varphi$  is substituted by  $\neg S_i \vee C_i$ , s.t.  $S_i$  fresh “selector” variable
- 2 the resulting formula is checked for **satisfiability under the assumption of all  $S_i$ 's**
- 3 final conflict clause at dec. level 0:  $\bigvee_j \neg S_j$   
 $\implies \{C_j\}_j$  is the unsat core

Extends straightforwardly to  $\text{SMT}(\mathcal{T})$ .

## The assumption-based approach to $\mathcal{T}$ -unsat cores: Example

$$\begin{aligned} & (\mathcal{S}_1 \rightarrow (x = 0 \vee \neg(x = 1) \vee A_1)) \wedge (\mathcal{S}_2 \rightarrow (x = 0 \vee x = 1 \vee A_2)) \wedge \\ & \quad (\mathcal{S}_3 \rightarrow (\neg(x = 0) \vee x = 1 \vee A_2)) \wedge (\mathcal{S}_4 \rightarrow (\neg A_2 \vee y = 1)) \wedge \\ & \quad (\mathcal{S}_5 \rightarrow (\neg A_1 \vee x + y > 3)) \wedge (\mathcal{S}_6 \rightarrow y < 0) \wedge \\ & \quad (\mathcal{S}_7 \rightarrow (A_2 \vee x - y = 4)) \wedge (\mathcal{S}_8 \rightarrow (y = 2 \vee \neg A_1)) \wedge (\mathcal{S}_9 \rightarrow x \geq 0) \end{aligned}$$

Conflict analysis (Yices 1.0.6) returns:

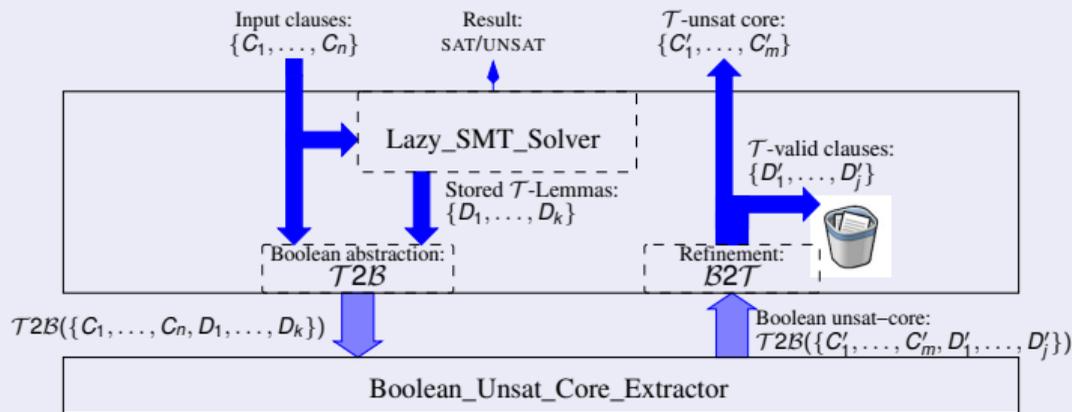
$$\neg \mathcal{S}_1 \vee \neg \mathcal{S}_2 \vee \neg \mathcal{S}_3 \vee \neg \mathcal{S}_4 \vee \neg \mathcal{S}_6 \vee \neg \mathcal{S}_7 \vee \neg \mathcal{S}_8,$$

corresponding to the unsat core in red.

# The lemma-lifting approach to $\mathcal{T}$ -unsat cores

Idea [28, 32]

- (i) The  $\mathcal{T}$ -lemmas  $D_i$  are valid in  $\mathcal{T}$
- (ii) The conjunction of  $\varphi$  with all the  $\mathcal{T}$ -lemmas  $D_1, \dots, D_k$  is propositionally unsatisfiable:  
 $\mathcal{T}2\mathcal{B}(\varphi \wedge \bigwedge_{i=1}^n D_i) \models \perp$ .



- interfaces with an external Boolean Unsat-core Extractor

⇒ benefits for free of all state-of-the-art size-reduction techniques

## The lemma-lifting approach to $\mathcal{T}$ -unsat cores (cont.)

```
<SatValue, Clause_set>  $\mathcal{T}$ -Unsat_Core(Clause_set  $\varphi$ ) {  
  //  $\varphi$  is  $\{C_1, \dots, C_n\}$   
  if (Lazy_SMT_Solver( $\varphi$ ) == SAT)  
    then return <SAT,  $\emptyset$ >;  
  //  $D_1, \dots, D_k$  are the  $\mathcal{T}$ -lemmas stored by Lazy_SMT_Solver  
   $\psi^p$  = Boolean_Core_Extractor( $\mathcal{T}2\mathcal{B}(\{C_1, \dots, C_n, D_1, \dots, D_k\})$ );  
  //  $\psi^p$  is  $\mathcal{T}2\mathcal{B}(\{C'_1, \dots, C'_m, D'_1, \dots, D'_j\})$ ;  
  return <UNSAT,  $\{C'_1, \dots, C'_m\}$ >;  
}
```

# The lemma-lifting approach to $\mathcal{T}$ -unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge \\ (\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$

1 The SMT solver generates the following set of  $\mathcal{LIA}$ -lemmas:

$$\{(\neg(x = 1) \vee \neg(x = 0)), (\neg(y = 2) \vee \neg(y < 0)), (\neg(y = 1) \vee \neg(y < 0))\}.$$

2 The following formula is passed to the external Boolean core extractor

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (B_6 \vee \neg A_1) \wedge B_7 \wedge \\ (\neg B_1 \vee \neg B_0) \wedge (\neg B_6 \vee \neg B_4) \wedge (\neg B_2 \vee \neg B_4)$$

which returns the unsat core in red.

3 The unsat-core is mapped back, the three  $\mathcal{T}$ -lemmas are removed  
 $\implies$  the final  $\mathcal{T}$ -unsat core (in red above).

## Exercise

Consider the following set of clauses  $\varphi$  in  $\mathcal{EUF}$ .

$$\left\{ \begin{array}{l} (\neg(x = y) \vee (f(x) = f(y))), \\ (\neg(x = y) \vee \neg(f(x) = f(y))), \\ ((x = y) \vee (f(x) = f(y))), \\ ((x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

Find a minimal  $\mathcal{EUF}$ -unsatisfiable core.

- 1 Introduction
  - Basics on First-order Logic
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  - Satisfiability Modulo Theories
  - Motivations and Goals of SMT
- 2 Efficient SMT solving
  - Combining SAT with Theory Solvers
  - Theory Solvers for Theories of Interest
  - SMT for Combinations of Theories
- 3 **Beyond Solving: Advanced SMT Functionalities**
  - Proofs and Unsatisfiable Cores
  - **All-SMT & Predicate Abstraction (hints)**
  - SMT with Optimization (Optimization Modulo Theories)

## All-SAT/All-SMT (hints)

- **All-SAT**: enumerate all truth assignments satisfying  $\varphi$
- **All-SMT**: enumerate all  $\mathcal{T}$ -satisfiable truth assignments propositionally satisfying  $\varphi$
- **All-SMT over an “important” subset of atoms  $\Gamma \stackrel{\text{def}}{=} \{\gamma_i\}_i$** : enumerate all assignments over  $\Gamma$  which can be extended to  $\mathcal{T}$ -satisfiable truth assignments propositionally satisfying  $\varphi$   
 $\implies$  can compute **predicate abstraction**
- **Algorithms**:
  - **BCLT** [53]  
each time a  $\mathcal{T}$ -satisfiable assignment  $\{l_1, \dots, l_n\}$  is found, perform conflict-driven backjumping as if the restricted clause  $(\bigvee_i \neg l_i) \downarrow \Gamma$  belonged to the clause set
  - **MathSAT/NuSMV** [26]  
As above, plus the Boolean search of the SMT solver is driven by an OBDD.

# Predicate Abstraction

## Predicate abstraction

if  $\varphi(\mathbf{v})$  is a SMT formula over the domain variables  $\mathbf{v} \stackrel{\text{def}}{=} \{v_j\}_j$ ,  $\{\gamma_i\}_i$  is a set of “relevant” predicates over  $\mathbf{v}$ , and  $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$  a set of fresh Boolean labels, then:

$$\begin{aligned} & \text{PredAbs}_{\mathbf{P}}(\varphi) \\ \stackrel{\text{def}}{=} & \exists \mathbf{v}. ( \varphi(\mathbf{v}) \wedge \bigwedge_i P_i \leftrightarrow \gamma_i(\mathbf{v}) ) \\ = & \bigvee \left\{ \mu \mid \begin{array}{l} \mu \text{ truth assignment on } \mathbf{P} \\ \text{s.t. } \mu \wedge \varphi \wedge \bigwedge_i (P_i \leftrightarrow \gamma_i) \text{ is } \mathcal{T}\text{-satisfiable} \end{array} \right\} \end{aligned}$$

- projection of  $\varphi$  over (the Boolean abstraction of) the set  $\{\gamma_i\}_i$ .
- important step in FV: extracts finite-state abstractions from a infinite state space

# Predicate Abstraction: example

$$\varphi \stackrel{\text{def}}{=} (v_1 + v_2 > 12)$$

$$\gamma_1 \stackrel{\text{def}}{=} (v_1 + v_2 = 2)$$

$$\gamma_2 \stackrel{\text{def}}{=} (v_1 - v_2 < 10)$$

↓

$$\begin{aligned} \text{PreAbs}(\varphi)_{\{P_1, P_2\}} &\stackrel{\text{def}}{=} \exists v_1 v_2 . \left( \begin{array}{l} (v_1 + v_2 > 12) \\ (P_1 \leftrightarrow (v_1 + v_2 = 2)) \\ (P_2 \leftrightarrow (v_1 - v_2 < 10)) \end{array} \wedge \right) \\ &= (\neg P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \\ &= \neg P_1. \end{aligned}$$

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# Optimization Modulo Theories: General Case

Ingredients:  $\langle \varphi, cost \rangle$

- a **SMT formula**  $\varphi$  in some background theory  $\mathcal{T} = \mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ 
  - $\bigcup_i \mathcal{T}_i$  may be empty
  - $\mathcal{T}_{\preceq}$  has a predicate  $\preceq$  representing a **total order**
- a  $\mathcal{T}_{\preceq}$ -**variable/term** “*cost*” occurring in  $\varphi$

Optimization Modulo  $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$  (OMT( $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ ))

The problem of finding a model  $\mathcal{M}$  for  $\varphi$  whose value of *cost* is minimum according to  $\preceq$ .

- maximization is dual

## Note

The cost term can be rewritten as a variable

$$\langle \varphi, term \rangle \implies \langle \varphi \wedge (cost = term), cost \rangle, \quad cost \text{ fresh}$$

# Optimization Modulo Theories with $\mathcal{L}\mathcal{A}$ $\mathcal{L}\mathcal{R}\mathcal{A}$ costs

## Ingredients

- an SMT formula  $\varphi$  on  $\mathcal{L}\mathcal{A} \cup \mathcal{T}$   $\mathcal{L}\mathcal{R}\mathcal{A} \cup \mathcal{T}$ 
  - $\mathcal{L}\mathcal{A}$  can be  $\mathcal{L}\mathcal{R}\mathcal{A}$ ,  $\mathcal{L}\mathcal{I}\mathcal{A}$  or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$ , possibly empty
  - $\mathcal{L}\mathcal{A}\mathcal{L}\mathcal{R}\mathcal{A}$  and  $\mathcal{T}_i$  Nelson-Oppen theories (i.e. signature-disjoint infinite-domain theories)
- a  $\mathcal{L}\mathcal{A}$   $\mathcal{L}\mathcal{R}\mathcal{A}$  variable [term] “cost” occurring in  $\varphi$
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t.  $\text{lb} \leq \text{cost} < \text{ub}$  (lb, ub may be  $\mp\infty$ )

## Optimization Modulo Theories with $\mathcal{L}\mathcal{A}$ $\mathcal{L}\mathcal{R}\mathcal{A}$ costs (OMT( $\mathcal{L}\mathcal{A} \cup \mathcal{T}$ ) OMT( $\mathcal{L}\mathcal{R}\mathcal{A} \cup \mathcal{T}$ ))

Find a model for  $\varphi$  whose value of *cost* is minimum.

- maximization dual

We first restrict to the case  $\mathcal{L}\mathcal{A} = \mathcal{L}\mathcal{R}\mathcal{A}$  and  $\bigcup_i \mathcal{T}_i = \{\}$  (OMT( $\mathcal{L}\mathcal{R}\mathcal{A}$ )).

# Solving OMT( $\mathcal{LRA}$ ) [71, 72]

## General idea

Combine standard SMT and LP minimization techniques.

## Offline Schema

- Minimizer: based on the Simplex  $\mathcal{LRA}$ -solver by [40]
  - Handles strict inequalities
- Search Strategies:
  - Linear-Search strategy
  - Mixed Linear/Binary strategy

# A toy example (linear search)

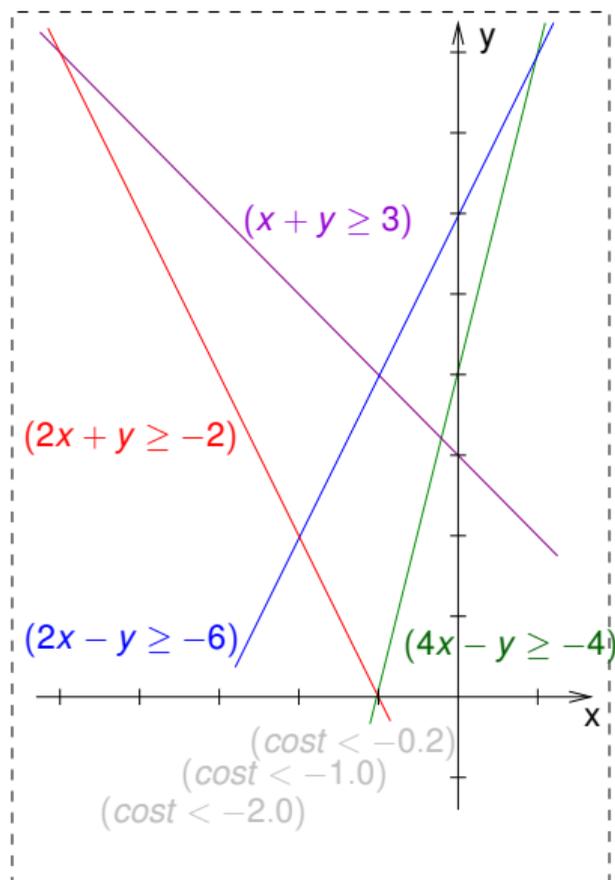
[w. pure-literal filt.  $\implies$  partial assignments]

- OMT( $\mathcal{LRA}$ ) problem:

$$\begin{aligned} \varphi &\stackrel{\text{def}}{=} (\neg A_1 \vee (2x + y \geq -2)) \\ &\wedge (A_1 \vee (x + y \geq 3)) \\ &\wedge (\neg A_2 \vee (4x - y \geq -4)) \\ &\wedge (A_2 \vee (2x - y \geq -6)) \\ &\wedge (\text{cost} < -0.2) \\ &\wedge (\text{cost} < -1.0) \\ &\wedge (\text{cost} < -2.0) \end{aligned}$$

$$\text{cost} \stackrel{\text{def}}{=} x$$

$$\bullet \mu = \left\{ \begin{array}{l} A_1, \neg A_1, A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$$



# A toy example (linear search)

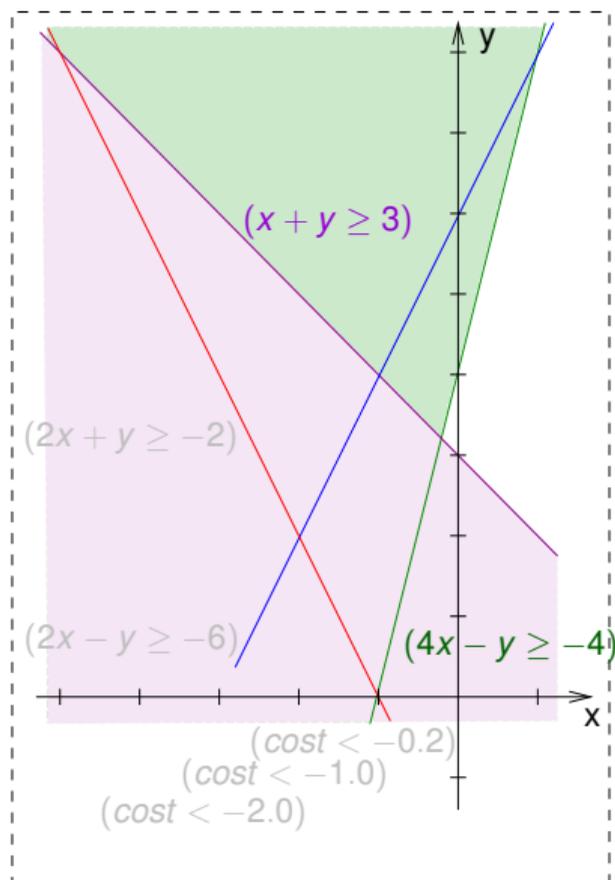
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$$\text{cost} \stackrel{\text{def}}{=} x$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$   
 $\implies \text{SAT}, \min = -0.2$



# A toy example (linear search)

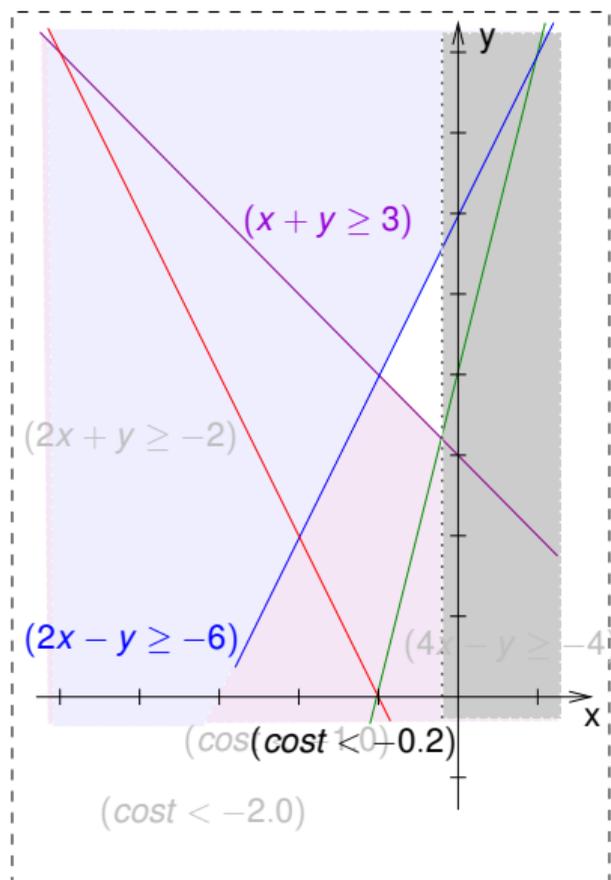
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$$\text{cost} \stackrel{\text{def}}{=} x$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$   
 $\implies \text{SAT}, \min = -1.0$



# A toy example (linear search)

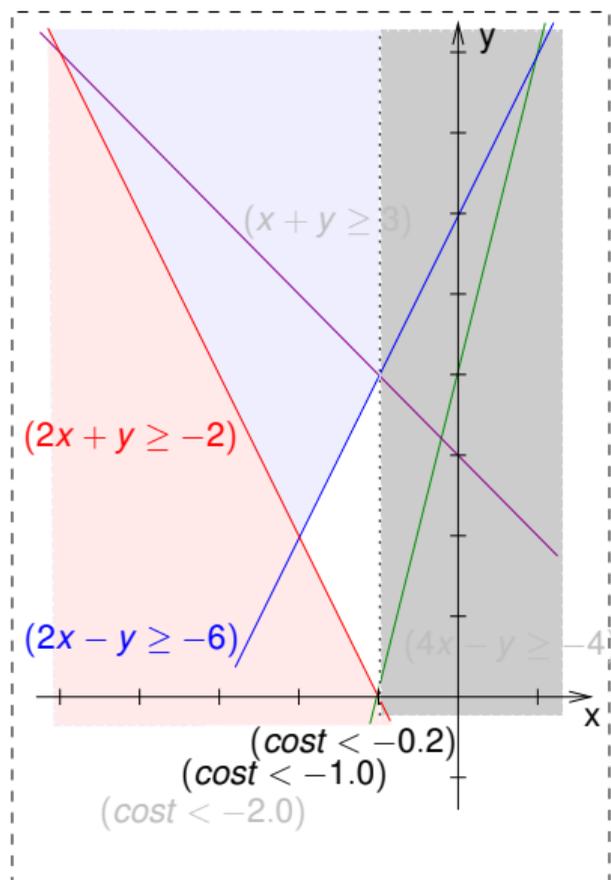
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- OMT( $\mathcal{LRA}$ ) problem:

$$\begin{aligned} \varphi &\stackrel{\text{def}}{=} (\neg A_1 \vee (2x + y \geq -2)) \\ &\wedge (A_1 \vee (x + y \geq 3)) \\ &\wedge (\neg A_2 \vee (4x - y \geq -4)) \\ &\wedge (A_2 \vee (2x - y \geq -6)) \\ &\wedge (\text{cost} < -0.2) \\ &\wedge (\text{cost} < -1.0) \\ &\wedge (\text{cost} < -2.0) \end{aligned}$$

$$\text{cost} \stackrel{\text{def}}{=} x$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$   
 $\implies \text{SAT}, \min = -2.0$





# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$

$l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{ \neg(cost < lb), (cost < ub) \};$

**while** ( $l < u$ ) **do**



# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$

**while** ( $l < u$ ) **do**

**if** (BinSearchMode()) **then** // Binary-search Mode

**else** // Linear-search Mode



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**while** ( $l < u$ ) **do**

**if** (BinSearchMode()) **then** // Binary-search Mode

**else** // Linear-search Mode

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$



# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb$ ;  $u \leftarrow ub$ ;  $\mathcal{M} \leftarrow \emptyset$ ;  $\varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\}$ ;

**while** ( $l < u$ ) **do**

**if** (BinSearchMode()) **then** // Binary-search Mode

**else** // Linear-search Mode

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi)$ ;

**if** ( $res = \text{SAT}$ ) **then**

$\langle \mathcal{M}, u \rangle \leftarrow \text{LRA-Solver.Minimize}(cost, \mu)$ ;

$\varphi \leftarrow \varphi \cup \{(cost < u)\}$ ;

**else** { $res = \text{UNSAT}$ }



# Offline Schema: Mixed Linear/Binary-Search Strategy

```
Input:  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb$ ;  $u \leftarrow ub$ ;  $\mathcal{M} \leftarrow \emptyset$ ;  $\varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\}$ ;  
while ( $l < u$ ) do  
  if (BinSearchMode()) then // Binary-search Mode  
  else // Linear-search Mode  
     $\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi)$ ;  
    if ( $res = \text{SAT}$ ) then  
      else { $res = \text{UNSAT}$ }  
       $l \leftarrow u$ ;  
return  $\langle \mathcal{M}, u \rangle$ 
```



# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$

**while** ( $l < u$ ) **do**

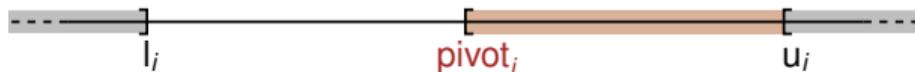
**if** (BinSearchMode()) **then** // Binary-search Mode

$pivot \leftarrow \text{ComputePivot}(l, u);$

$\varphi \leftarrow \varphi \cup \{(cost < pivot)\};$

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$

**else** // Linear-search Mode



# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$

**while** ( $l < u$ ) **do**

**if** (BinSearchMode()) **then** // Binary-search Mode

    pivot  $\leftarrow$  ComputePivot( $l, u$ );

$\varphi \leftarrow \varphi \cup \{(cost < pivot)\};$

$\langle res, \mu \rangle \leftarrow$  SMT.IncrementalSolve( $\varphi$ );

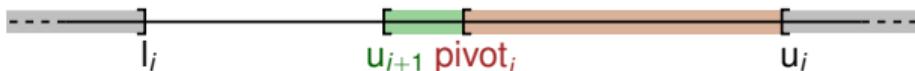
**else** // Linear-search Mode

**if** ( $res = SAT$ ) **then**

$\langle \mathcal{M}, u \rangle \leftarrow$   $\mathcal{LRA}$ -Solver.Minimize( $cost, \mu$ );

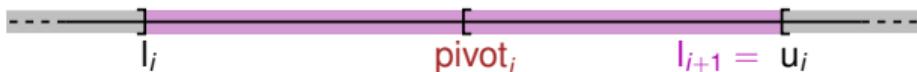
$\varphi \leftarrow \varphi \cup \{(cost < u)\};$

**else** { $res = UNSAT$ }



# Offline Schema: Mixed Linear/Binary-Search Strategy

```
Input:  $\langle \varphi, cost, lb, ub \rangle$  // lb can be  $-\infty$ , ub can be  $+\infty$   
 $l \leftarrow lb; u \leftarrow ub; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\};$   
while ( $l < u$ ) do  
  if (BinSearchMode()) then // Binary-search Mode  
     $pivot \leftarrow \text{ComputePivot}(l, u);$   
     $\varphi \leftarrow \varphi \cup \{(cost < pivot)\};$   
     $\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi);$   
  else // Linear-search Mode  
    if ( $res = \text{SAT}$ ) then  
      return  $\langle \mathcal{M}, u \rangle$   
    else { $res = \text{UNSAT}$ }  
      if  $((cost < pivot) \notin \text{SMT.ExtractUnsatCore}(\varphi))$  then  
         $l \leftarrow u;$   
      else
```



# Offline Schema: Mixed Linear/Binary-Search Strategy

**Input:**  $\langle \varphi, cost, lb, ub \rangle$  //  $lb$  can be  $-\infty$ ,  $ub$  can be  $+\infty$   
 $l \leftarrow lb$ ;  $u \leftarrow ub$ ;  $\mathcal{M} \leftarrow \emptyset$ ;  $\varphi \leftarrow \varphi \cup \{\neg(cost < lb), (cost < ub)\}$ ;

**while** ( $l < u$ ) **do**

**if** (BinSearchMode()) **then** // Binary-search Mode

$pivot \leftarrow \text{ComputePivot}(l, u)$ ;

$\varphi \leftarrow \varphi \cup \{(cost < pivot)\}$ ;

$\langle res, \mu \rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi)$ ;

**else** // Linear-search Mode

**if** ( $res = \text{SAT}$ ) **then**

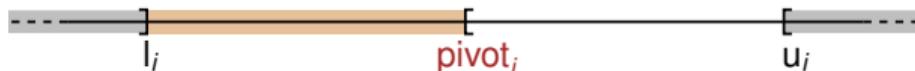
**else** { $res = \text{UNSAT}$ }

**if** ( $(cost < pivot) \notin \text{SMT.ExtractUnsatCore}(\varphi)$ ) **then**

**else**

$l \leftarrow pivot$ ;

$\varphi \leftarrow (\varphi \setminus \{(cost < pivot)\}) \cup \{\neg(cost < pivot)\}$ ;



# OMT with Independent Objectives (aka Boxed OMT) [55, 74]

The problem:  $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$  [55]

Given  $\langle \varphi, \mathcal{C} \rangle$  s.t.:

- $\varphi$  is the input formula
- $\mathcal{C} \stackrel{\text{def}}{=} \{cost_1, \dots, cost_k\}$  is a set of  $\mathcal{LA}$ -terms on variables in  $\varphi$ ,

$\langle \varphi, \mathcal{C} \rangle$  is the problem of finding a set of independent  $\mathcal{LA}$ -models  $\mathcal{M}_1, \dots, \mathcal{M}_k$  s.t. s.t. each  $\mathcal{M}_i$  makes  $cost_i$  minimum.

## Notes

- derives from SW verification problems [55]
- equivalent to  $k$  independent problems  $\langle \varphi, cost_1 \rangle, \dots, \langle \varphi, cost_k \rangle$
- intuition: share search effort for the different objectives
- generalizes to  $OMT(\mathcal{LA} \cup \mathcal{T})$  straightforwardly

# OMT with Multiple Objectives [55, 13, 74]

## Solution

- Intuition: when a  $\mathcal{T}$ -satisfiable satisfying assignment  $\mu$  is found,

```
  foreach  $cost_i$ 
```

```
     $min_i := \min\{min_i, \mathcal{T}solver.minimize(\mu, cost_i)\};$ 
```

```
  learn  $\bigvee_i (cost_i < min_i);$  //  $(cost_i < -\infty) \equiv \perp$ 
```

```
  proceed until UNSAT;
```

- Notice:

- for each  $\mu$ , guaranteed improvement of at least one  $min_i$
- in practice, for each  $\mu$ , multiple  $cost_i$  minima are improved

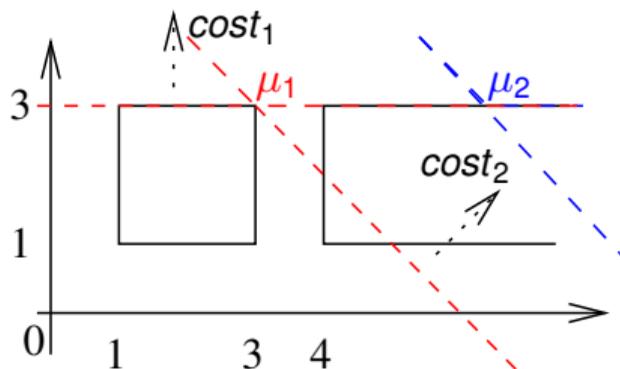
- Implemented improvements:

(a) drop previous clauses  $\bigvee_i (cost_i < min_i)$

(b)  $(cost_i < min_i)$  pushed in  $\mu$  first: if  $\mathcal{T}$ -unsatisfiable, skip minimization

(c) learn  $\neg(cost_i < min_i) \vee (cost_i < min_i^{old})$ , s.t.  $min_i^{old}$  previous  $min_i$   
 $\implies$  reuse previously-learned clauses like  $\neg(cost_i < min_i^{old}) \vee C$

# Boxed OMT: Example [55, 74]



$$\begin{aligned} \varphi &= (1 \leq y) \wedge (y \leq 3) \wedge (((1 \leq x) \wedge (x \leq 3)) \vee (x \geq 4)) \\ &\wedge (cost_1 = -y) \wedge (cost_2 = -x - y) \end{aligned}$$

$$\begin{aligned} \mu_1 &= \{(1 \leq y), (y \leq 3), (1 \leq x), (x \leq 3)\} \implies \text{SAT} \implies [-3, -6] \\ &\implies \text{learn} \quad \{(cost_1 < -3) \vee (cost_2 < -6)\} \end{aligned}$$

$$\begin{aligned} \mu_2 &= \{(1 \leq y), (y \leq 3), (x \geq 4)\} \implies \text{SAT} \implies [-3, -\infty] \\ &\implies \text{learn} \quad \{(cost_1 < -3)\} \\ &\implies \text{UNSAT} \end{aligned}$$

# OMT with Lexicographic Combination of Objectives [13]

## The problem

Find one optimal model  $\mathcal{M}$  minimizing  $\underline{c} \stackrel{\text{def}}{=} cost_1, cost_2, \dots, cost_k$  lexicographically.

## Solution

- Intuition:

- {*minimize cost<sub>1</sub>*}

- when UNSAT*

- {*substitute unit clause (cost<sub>1</sub> < min<sub>1</sub>) with (cost<sub>1</sub> = min<sub>1</sub>)*}

- {*minimize cost<sub>2</sub>*}

- ...

- improvement:

- each time UNSAT is found, add  $\bigwedge_i (cost_i \leq \mathcal{M}_i(cost_i))$  to  $\varphi$

# Optimization problems encoded into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ I

## SMT with Pseudo-Boolean Constraints & Weighted MaxSMT

$$\text{OMT} + \text{PB} : \quad \sum_j w_j \cdot A_j, \quad w_i > 0 \quad // (\sum_j \text{ite}(A_j, w_j, 0))$$

$\Downarrow$

$$\begin{aligned} & \sum_j x_j, \quad x_j \text{ fresh} \\ \text{s.t.} \quad & \dots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

$$\text{MaxSMT} : \quad \langle \varphi_h, \bigwedge_j \psi_j \rangle \quad \text{s.t. } \psi_j \text{ soft}, \quad w_j = \text{weight}(\psi_j), \quad w_i > 0$$

$\Downarrow$

$$\begin{aligned} & \text{minimize } \sum_j x_j, \quad x_j, A_j \text{ fresh} \\ & \varphi_h \wedge \bigwedge_j (A_j \vee \psi_j) \wedge \bigwedge_j (\neg A_j \vee (x_j = w_j)) \wedge (A_j \vee (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

## Remark: range constraints “ $(x_j \geq 0) \wedge (x_j \leq w_j)$ ”

$$\begin{aligned} OMT + PB : \quad & \sum_j w_j \cdot A_j, w_i > 0 \quad //(\sum_j \text{ite}(A_j, w_j, 0)) \\ & \downarrow \\ & \sum_j x_j, x_j \text{ fresh} \\ \text{s.t.} \quad & \dots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{aligned}$$

Range constraints “ $(x_j \geq 0) \wedge (x_j \leq w_j)$ ” logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound **only after all  $A_i$ 's are assigned** :

Ex:  $w_1 = 4, w_2 = 7, w_i > 0 \forall i > 2, \sum_{i=1}^{10000} x_i < 10, A_1 = A_2 = \top, A_i = * \forall i > 2.$

- With range constraints, the SMT solver detects the violation as soon as the assigned  $A_i$ 's violate a bound  
 $\implies$  drastic pruning of the search
- same for weighted MaxSMT

## Optimization problems encoded into $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$ II

### OMT with Min-Max [Max-Min] optimization

Given  $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$ , find a solution which minimizes the maximum value among  $\{cost_1, \dots, cost_k\}$ . (Max-Min dual.)

- Frequent in some applications (e.g. [72, 79])

$\Rightarrow$  encode into  $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$  problem  $\{\varphi \wedge \bigwedge_i (cost_i \leq cost), cost\}$  s.t.  $cost$  fresh.

### OMT with linear combinations of costs

Given  $\langle \varphi, \{cost_1, \dots, cost_k\} \rangle$  and a set of weights  $\{w_1, \dots, w_k\}$ , find a solution which minimizes  $\sum_i w_i \cdot cost_i$ .

$\Rightarrow$  encode into  $\text{OMT}(\mathcal{L}\mathcal{A} \cup \mathcal{T})$  problem  $\{\varphi \wedge (cost = \sum_i w_i \cdot cost_i), cost\}$  s.t.  $cost$  fresh.

These objectives can be composed with other  $\text{OMT}(\mathcal{L}\mathcal{A})$  objectives.

## Other OMT Functionalities [hints]

### Incremental interface [13, 74]

Allows for pushing/popping sub-formulas into a stack, and then run OMT incrementally over them, reusing previous search.

- useful in some applications (e.g., BMC with parametric systems)
- straightforward variant of incremental SAT and SMT solvers

## Some OMT tools

- **BCLT** [66, 54]  
<https://www.cs.upc.edu/~oliveras/bclt-main.html>
- **OPTIMATHSAT** [71, 72, 74, 73], on top of MATHSAT [27]  
<https://optimathsat.disi.unitn.it>
- **SYMBA** [55], on top of Z3 [37]  
<https://bitbucket.org/arieg/symba/src>
- **$\nu$ Z** [13, 12], on top of Z3 [37]  
<https://z3.codeplex.com>

- survey papers:

- Roberto Sebastiani: "Lazy Satisfiability Modulo Theories".  
Journal on Satisfiability, Boolean Modeling and Computation, JSAT. Vol. 3, 2007. Pag 141–224, ©IOS Press.
- Clark Barrett, Roberto Sebastiani, Sanjit Seshia, Cesare Tinelli "Satisfiability Modulo Theories".  
Part II, Chapter 33, The Handbook of Satisfiability, II ed. 2021. ©IOS press.
- Leonardo de Moura and Nikolaj Bjørner. "Satisfiability modulo theories: introduction and applications".  
Communications of the ACM, 54 (9), 2011. ©ACM press.

- web links:

- The SMT library SMT-LIB: <https://goedel.cs.uiowa.edu/smtlib/>
- The SMT Competition SMT-COMP: <https://www.smtcomp.org/>
- The SAT/SMT Schools <https://satassociation.org/sat-smt-school.html>

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## Disclaimer

The list of references above is by no means intended to be all-inclusive. I apologize both with the authors and with the readers for all the relevant works which are not cited here.