Automated Reasoning and Formal Verification
Module I: Automated Reasoning
Ch. 01: **Propositional Satisfiability (SAT)**

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Outline

1. Boolean Logics and SAT
2. Basic SAT-Solving Techniques
   - Generalities
   - Resolution
   - Tableaux
   - DPLL
3. Ordered Binary Decision Diagrams – OBDDs
4. Modern CDCL SAT Solvers
   - Limitations of Chronological Backtracking
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
   - SAT Under Assumptions & Incremental SAT
5. SAT Functionalities: proofs, unsat cores, optimization

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Propositional Logic (aka Boolean Logic)

Basic Definitions

- Propositional formula (aka Boolean formula)
  - $\top, \bot$ are formulas
  - a propositional atom $A_1, A_2, A_3, ...$ is a formula;
  - if φ_1 and φ_2 are formulas, then
    $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$
    are formulas.
- Ex: $\varphi \overset{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4)))$
- *Atoms*(φ): the set $\{A_1, ..., A_N\}$ of atoms occurring in φ.
  - Ex: $Atoms(\varphi) = \{A_1, A_2, A_3, A_4\}$
- Literal: a propositional atom A_i (positive literal) or its negation $\neg A_i$ (negative literal)
  - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- Clause: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee ...)$)
- Cube: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge ...)$)

Semantics of Boolean operators

Truth Table

α	β	$\neg\alpha$	$\alpha\wedge\beta$	$\alpha\vee\beta$	$\alpha\rightarrow\beta$	$\alpha\leftarrow\beta$	$\alpha\leftrightarrow\beta$	$\alpha\oplus\beta$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

English meaning of connectives

English	Logic
A and B	$A \wedge B$
A if B | A when B | A whenever B	$A \leftarrow B$
if A, then B | A implies B | A forces B | A requires B	$A \rightarrow B$
A precisely when B | A if and only if B	$A \leftrightarrow B$
A or B (or both) | A unless B	$A \vee B$ (logical or)
either A or B (but not both)	$A \oplus B$ (exclusive or)

Semantics of Boolean operators (cont.)

Note

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are commutative:

$$\begin{array}{lcl}
(\alpha \wedge \beta) & \iff & (\beta \wedge \alpha) \\
(\alpha \vee \beta) & \iff & (\beta \vee \alpha) \\
(\alpha \leftrightarrow \beta) & \iff & (\beta \leftrightarrow \alpha) \\
(\alpha \oplus \beta) & \iff & (\beta \oplus \alpha)
\end{array}$$

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are associative:

$$\begin{array}{lclcl}
((\alpha \wedge \beta) \wedge \gamma) & \iff & (\alpha \wedge (\beta \wedge \gamma)) & \iff & (\alpha \wedge \beta \wedge \gamma) \\
((\alpha \vee \beta) \vee \gamma) & \iff & (\alpha \vee (\beta \vee \gamma)) & \iff & (\alpha \vee \beta \vee \gamma) \\
((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) & \iff & (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) & \iff & (\alpha \leftrightarrow \beta \leftrightarrow \gamma) \\
((\alpha \oplus \beta) \oplus \gamma) & \iff & (\alpha \oplus (\beta \oplus \gamma)) & \iff & (\alpha \oplus \beta \oplus \gamma)
\end{array}$$

- $\rightarrow$, $\leftarrow$ are neither commutative nor associative:

$$\begin{array}{lcl}
(\alpha \rightarrow \beta) & \not\iff & (\beta \rightarrow \alpha) \\
((\alpha \rightarrow \beta) \rightarrow \gamma) & \not\iff & (\alpha \rightarrow (\beta \rightarrow \gamma))
\end{array}$$

Equivalences with Boolean Operators

$$
\begin{aligned}
\neg\neg\alpha &\iff \alpha\\
(\alpha \lor \beta) &\iff \neg(\neg\alpha \land \neg\beta)\\
\neg(\alpha \lor \beta) &\iff (\neg\alpha \land \neg\beta)\\
(\alpha \land \beta) &\iff \neg(\neg\alpha \lor \neg\beta)\\
\neg(\alpha \land \beta) &\iff (\neg\alpha \lor \neg\beta)\\
(\alpha \rightarrow \beta) &\iff (\neg\alpha \lor \beta)\\
\neg(\alpha \rightarrow \beta) &\iff (\alpha \land \neg\beta)\\
(\alpha \leftarrow \beta) &\iff (\alpha \lor \neg\beta)\\
\neg(\alpha \leftarrow \beta) &\iff (\neg\alpha \land \beta)\\
(\alpha \leftrightarrow \beta) &\iff ((\alpha \rightarrow \beta) \land (\alpha \leftarrow \beta))\\
&\iff ((\neg\alpha \lor \beta) \land (\alpha \lor \neg\beta))\\
\neg(\alpha \leftrightarrow \beta) &\iff (\neg\alpha \leftrightarrow \beta)\\
&\iff (\alpha \leftrightarrow \neg\beta)\\
&\iff ((\alpha \lor \beta) \land (\neg\alpha \lor \neg\beta))\\
(\alpha \oplus \beta) &\iff \neg(\alpha \leftrightarrow \beta)
\end{aligned}
$$

Boolean logic can be expressed in terms of $\{\neg, \land\}$ (or $\{\neg, \lor\}$) only!

Tree & DAG Representations of Formulas

- Formulas can be represented either as trees or as DAGS (Directed Acyclic Graphs)
- DAG representation can be up to exponentially smaller
  - in particular, when ↔'s are involved

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

$$\Downarrow$$

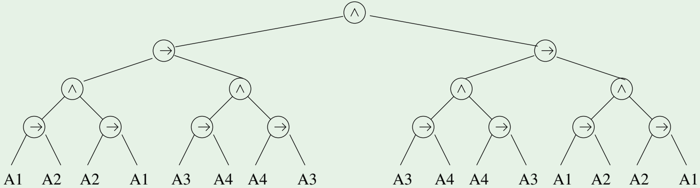
$$(((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge$$
$$((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2)))$$

$$\Downarrow$$

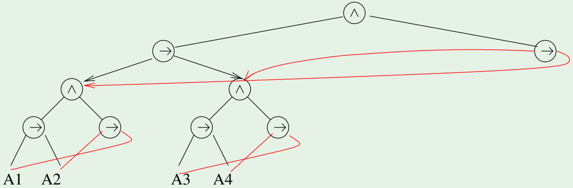
$$(((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge$$
$$(((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1))))$$

Tree & DAG Representations of Formulas: Example

$$(((A_1 \to A_2) \wedge (A_2 \to A_1)) \to ((A_3 \to A_4) \wedge (A_4 \to A_3))) \wedge$$
$$(((A_3 \to A_4) \wedge (A_4 \to A_3)) \to (((A_1 \to A_2) \wedge (A_2 \to A_1))))$$



Tree Representation



DAG Representation

Semantics: Basic Definitions

- Total truth assignment μ for φ:
  $\mu : Atoms(\varphi) \longmapsto \{\top, \bot\}$.
  - represents a possible world or a possible state of the world
- Partial Truth assignment μ for φ:
  $\mu : \mathcal{A} \longmapsto \{\top, \bot\}$, $\mathcal{A} \subset Atoms(\varphi)$.
  - represents 2^k total assignments, k is $\#$ unassigned variables
- Notation: set and formula representations of an assignment
  - μ can be represented as a set of literals:
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies \{A_1, \neg A_2\}$
  - μ can be represented as a formula (cube):
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies (A_1 \wedge \neg A_2)$

Semantics: Basic Definitions [cont.]

- A total truth assignment μ satisfies φ (μ is a model of φ, $\mu \models \varphi$):

$$\begin{array}{l}
\mu \models A_i \iff \mu(A_i) = \top \\
\mu \models \neg\varphi \iff \textit{not}\ \mu \models \varphi \\
\mu \models \alpha \land \beta \iff \mu \models \alpha\ \textit{and}\ \mu \models \beta \\
\mu \models \alpha \lor \beta \iff \mu \models \alpha\ \textit{or}\ \mu \models \beta \\
\mu \models \alpha \rightarrow \beta \iff \ \textit{if}\ \mu \models \alpha,\ \textit{then}\ \mu \models \beta \\
\mu \models \alpha \leftrightarrow \beta \iff \mu \models \alpha\ \textit{iff}\ \mu \models \beta \\
\mu \models \alpha \oplus \beta \iff \mu \models \alpha\ \textit{iff not}\ \mu \models \beta
\end{array}$$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$ (the set of models of φ)
- A partial truth assignment μ satisfies φ iff all total assignments extending μ satisfy φ
  - Ex: $\{A_1\} \models (A_1 \lor A_2))$ because both $\{A_1, A_2\} \models (A_1 \lor A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \lor A_2)$
- φ is satisfiable iff $\mu \models \varphi$ for some μ (i.e. $M(\varphi) \neq \emptyset$)
- α entails β ($\alpha \models \beta$): $\alpha \models \beta$ iff $\mu \models \alpha \Longrightarrow \mu \models \beta$ for all μs
(i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is valid ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for all μs
(i.e., $\mu \in M(\varphi)$ for all μs)

Properties & Results

Property

φ is valid iff $\neg\varphi$ is not satisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

Equivalence and Equi-Satisfiability

- α and β are equivalent iff, for every μ, $\mu \models \alpha$ iff $\mu \models \beta$ (i.e., if $M(\alpha) = M(\beta)$)
- α and β are equi-satisfiable iff exists μ_1 s.t. $\mu_1 \models \alpha$ iff exists μ_2 s.t. $\mu_2 \models \beta$ (i.e., if $M(\alpha) \neq \emptyset$ iff $M(\beta) \neq \emptyset$)
- α, β equivalent
  $\Downarrow \quad \not\Uparrow$
  α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
  $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α: $\beta \stackrel{\text{def}}{=} T(\alpha)$:
  - T is validity-preserving [resp. satisfiability-preserving] iff $T(\alpha)$ and α are equivalent [resp. equi-satisfiable]

Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\begin{array}{lcl} \exists v.\varphi & := & \varphi|_{v=\bot} \vee \varphi|_{v=\top} \\ \forall v.\varphi & := & \varphi|_{v=\bot} \wedge \varphi|_{v=\top} \end{array}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \ldots, w_n).\varphi \ := \ \exists w_1 \ldots \exists w_n.\varphi$

- Intuition:
  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \bot\}$ s.t. $\mu \cup \{v := \mathit{truthvalue}\} \models \varphi$
  - $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \bot\}$, $\mu \cup \{v := \mathit{truthvalue}\} \models \varphi$
- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) \ = \ a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

Complexity

NP-Completeness of SAT

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is NP-complete

$\Longrightarrow$ The most important logical problems (validity, inference, entailment, equivalence, ...) can be straightforwardly reduced to (un)satisfiability, and are thus (co)NP-complete.

$$\Downarrow$$

No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
  - φ occurs positively in φ;
  - if $\neg\varphi_1$ occurs positively [negatively] in φ,
    then φ_1 occurs negatively [positively] in φ
  - if $\varphi_1 \land \varphi_2$ or $\varphi_1 \lor \varphi_2$ occur positively [negatively] in φ,
    then φ_1 and φ_2 occur positively [negatively] in φ;
  - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ,
    then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
  - if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ,
    then φ_1 and φ_2 occur positively and negatively in φ;

Negative Normal Form (NNF)

- φ is in Negative normal form iff it is given only by the recursive applications of $\wedge, \vee$ to literals.
- every φ can be reduced into NNF:

  (i) substituting all $\rightarrow$'s and $\leftrightarrow$'s:

$$\begin{array}{lcl} \alpha \rightarrow \beta & \Longrightarrow & \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta & \Longrightarrow & (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta) \end{array}$$

  (ii) pushing down negations recursively:

$$\begin{array}{lcl} \neg(\alpha \wedge \beta) & \Longrightarrow & \neg\alpha \vee \neg\beta \\ \neg(\alpha \vee \beta) & \Longrightarrow & \neg\alpha \wedge \neg\beta \\ \neg\neg\alpha & \Longrightarrow & \alpha \end{array}$$

- Every non-atomic subformula in $\mathit{NNF}(\varphi)$ occurs with positive polarity only

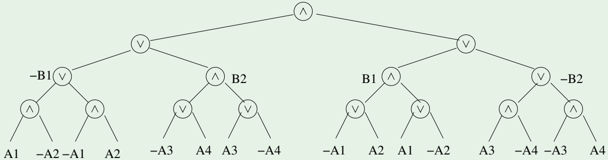
  $\Longrightarrow$ a subformula ψ occurring with both polarities in φ is encoded as both $\mathit{NNF}(\psi)$ and $\mathit{NNF}(\neg\psi)$
- The reduction is linear if a DAG representation is used.
- Preserves the equivalence of formulas.

NNF: Example

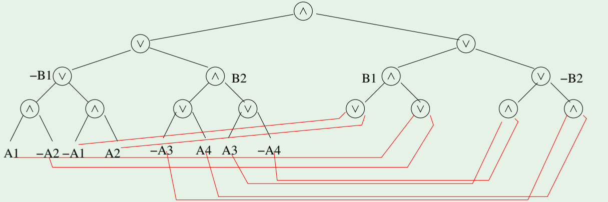
$$\begin{gathered}
(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\
\Downarrow \\
((((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))\wedge \\
(((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))) \\
\Downarrow \\
((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))\wedge \\
(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))) \\
\Downarrow \\
((((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))\wedge \\
(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))))
\end{gathered}$$

NNF: Example [cont.]

Note



Tree Representation



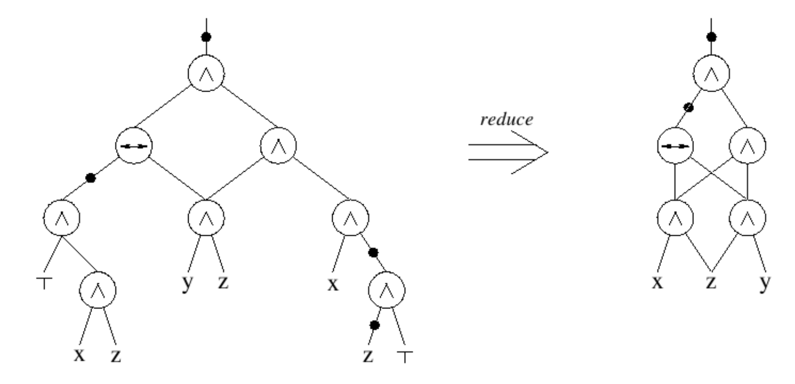
DAG Representation

For each non-literal subformula φ, φ and $\neg\varphi$ have different representations
$\Longrightarrow$ they are not shared.

Optimized polynomial representations

And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations:
  $\{\land, \leftrightarrow, \neg\}$-only, negations on arcs, sorting of subformulae, lifting of $\neg$'s over $\leftrightarrow$'s,...



Conjunctive Normal Form (CNF)

- φ is in Conjunctive normal form iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
- Easier to handle: list of lists of literals.
  $\Longrightarrow$ no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,
  (i) expanding implications and equivalences:
  $$\begin{array}{lcl} \alpha \to \beta & \Longrightarrow & \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta & \Longrightarrow & (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta) \end{array}$$
  (ii) pushing down negations recursively:
  $$\begin{array}{lcl} \neg(\alpha \wedge \beta) & \Longrightarrow & \neg\alpha \vee \neg\beta \\ \neg(\alpha \vee \beta) & \Longrightarrow & \neg\alpha \wedge \neg\beta \\ \neg\neg\alpha & \Longrightarrow & \alpha \end{array}$$
  $\Longrightarrow$ Negation Normal Form, NNF (see previous slides)
  (iii) applying recursively the DeMorgan's Rule: $(\alpha \wedge \beta) \vee \gamma \quad \Longrightarrow \quad (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
- Resulting formula worst-case exponential:
  - ex: $||\mathrm{CNF}(\bigvee_{i=1}^{N}(l_{i1} \wedge l_{i2})|| = ||(l_{11} \vee l_{21} \vee ... \vee l_{N1}) \wedge (l_{12} \vee l_{21} \vee ... \vee l_{N1}) \wedge ... \wedge (l_{12} \vee l_{22} \vee ... \vee l_{N2})|| = 2^N$
- $Atoms(CNF(\varphi)) = Atoms(\varphi)$
- $CNF(\varphi)$ is equivalent to φ.
- Rarely used in practice.

Labeling CNF conversion $CNF_{label}(\varphi)$

Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Tseitin's CNF-ization)

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

  $$\begin{array}{lcl} \varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j)) \\ \varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j)) \\ \varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j)) \end{array}$$

  l_i, l_j being literals and B being a "new" variable.
- Worst-case linear!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$ is equi-satisfiable (but not equivalent) to φ.
  - moreover: $\exists B_1, ..., B_k . CNF_{label}(\varphi)$ equivalent to φ, s.t. $B_1, ..., B_k$ all fresh variables introduced
- Much more used than classic conversion in practice

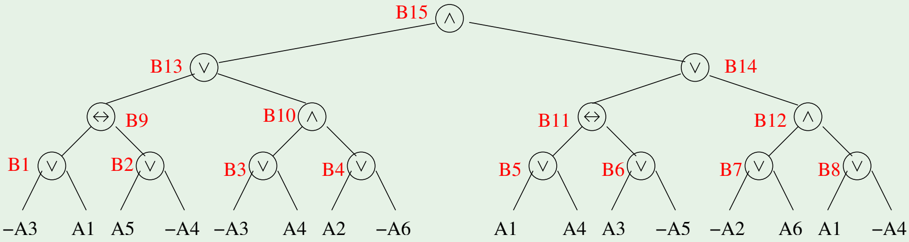
Remark

With $CNF_{label}(\varphi)$, it is not a good idea to convert φ into $NNF(\varphi)$ upfront. Why?

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF Conversion CNF_{label} – Example



$$
\begin{array}{lcl}
CNF(B_1 \leftrightarrow (\neg A_3 \lor A_1))\land & & (\neg B_1 \lor \neg A_3 \lor A_1) \land (B_1 \lor A_3) \land (B_1 \lor \neg A_1)\land \\
\ldots\land & & \ldots\land \\
CNF(B_8 \leftrightarrow (A_1 \lor \neg A_4))\land & & (\neg B_8 \lor A_1 \lor \neg A_4) \land (B_8 \lor \neg A_1) \land (B_8 \lor A_4)\land \\
CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2))\land & & (\neg B_9 \lor \neg B_1 \lor B_2) \land (\neg B_9 \lor B_1 \lor \neg B_2)\land \\
 & & (B_9 \lor B_1 \lor B_2) \land (B_9 \lor \neg B_1 \lor \neg B_2)\land \\
\ldots\land & = & \ldots\land \\
CNF(B_{12} \leftrightarrow (B_7 \land B_8))\land & & (B_{12} \lor \neg B_7 \lor \neg B_8) \land (\neg B_{12} \lor B_7) \land (\neg B_{12} \lor B_8)\land \\
CNF(B_{13} \leftrightarrow (B_9 \lor B_{10}))\land & & (\neg B_{13} \lor B_9 \lor B_{10}) \land (B_{13} \lor \neg B_9) \land (B_{13} \lor \neg B_{10})\land \\
CNF(B_{14} \leftrightarrow (B_{11} \lor B_{12}))\land & & (\neg B_{14} \lor B_{11} \lor B_{12}) \land (B_{14} \lor \neg B_{11}) \land (B_{14} \lor \neg B_{12})\land \\
CNF(B_{15} \leftrightarrow (B_{13} \land B_{14}))\land & & (B_{15} \lor \neg B_{13} \lor \neg B_{14}) \land (\neg B_{15} \lor B_{13}) \land (\neg B_{15} \lor B_{14})\land \\
B_{15} & & B_{15}
\end{array}
$$

Improved Labeling CNF conversion CNF_{label}

Polarity-based Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Plaisted&Greenbaum CNF-ization)

- As in the previous case, applying instead the rules:

$$\begin{array}{lllll}
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \vee l_j)) & \textit{if } (l_i \vee l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF((l_i \vee l_j) \rightarrow B) & \textit{if } (l_i \vee l_j) \textit{ neg.} \\
\varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \wedge l_j)) & \textit{if } (l_i \wedge l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\ CNF((l_i \wedge l_j) \rightarrow B) & \textit{if } (l_i \wedge l_j) \textit{ neg.} \\
\varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \leftrightarrow l_j)) & \textit{if } (l_i \leftrightarrow l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\ CNF((l_i \leftrightarrow l_j) \rightarrow B) & \textit{if } (l_i \leftrightarrow l_j) \textit{ neg.}
\end{array}$$

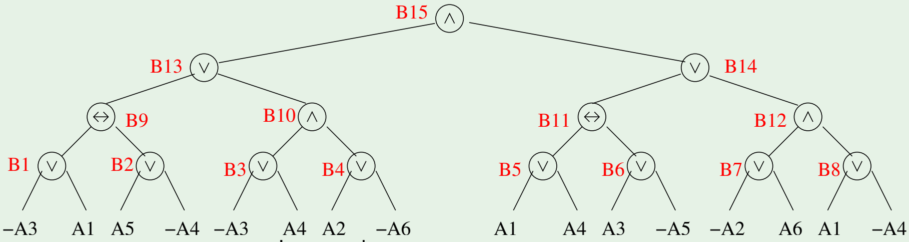
- Smaller in size:

$$\begin{array}{ll}
CNF(B \rightarrow (l_i \vee l_j)) & = (\neg B \vee l_i \vee l_j) \\
CNF(((l_i \vee l_j) \rightarrow B)) & = (\neg l_i \vee B) \wedge (\neg l_j \vee B)
\end{array}$$

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i \neg l_j)$
$CNF(B \rightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j)$
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Improved Labeling CNF conversion CNF_{label} – example



Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \quad \wedge$

$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \quad \wedge$

$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \quad \wedge$

$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \quad \wedge$

$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \quad \wedge$

B_{15}

Improved

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_8 \rightarrow (A_1 \vee \neg A_4)) \quad \wedge$

$CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_{12} \rightarrow (B_7 \wedge B_8)) \quad \wedge$

$CNF(B_{13} \rightarrow (B_9 \vee B_{10})) \quad \wedge$

$CNF(B_{14} \rightarrow (B_{11} \vee B_{12})) \quad \wedge$

$CNF(B_{15} \rightarrow (B_{13} \wedge B_{14})) \quad \wedge$

B_{15}

Labeling CNF conversion CNF_{label} – further improvements

- Do not apply CNF_{label} when not necessary:
  (e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \Longrightarrow CNF_{label}(\varphi_1) \wedge \varphi_2$, if φ_2 already in CNF)
- Apply DeMorgan's rules where it is more effective:
  (e.g., $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \Longrightarrow CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$)
- Exploit the associativity of $\wedge$'s and $\vee$'s:
  $\ldots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \ldots \Longrightarrow \ldots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3))\ldots$
- Before applying CNF_{label}, rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Exercises

1. Consider the following Boolean formula φ:
   $\neg(((\neg A_1 \rightarrow \ A_2) \wedge (\neg A_3 \rightarrow \ A_4)) \vee ((\ A_5 \rightarrow \ A_6) \wedge (\ A_7 \rightarrow \neg A_8)))$
   Compute the Negative Normal Form of φ

2. Consider the following Boolean formula φ:
   $((\neg A_1 \wedge \ A_2) \vee (\ A_7 \wedge \ A_4) \vee (\neg A_3 \wedge \neg A_2) \vee (\ A_5 \wedge \neg A_4))$
   1. Produce the CNF formula $CNF(\varphi)$.
   2. Produce the CNF formula $CNF_{label}(\varphi)$.
   3. Produce the CNF formula $CNF_{label}(\varphi)$ (improved version)

   Do the same for the formula in Exercise (1)

3. Do the same as with Exercises (1) and (2), with the following formulas:
   - $(((\ A_1 \oplus \ A_2) \oplus \ A_3) \oplus \ A_4)$
   - $\neg((\neg A_1 \vee \ A_2) \wedge (\ A_7 \vee \ A_4) \wedge (\neg A_3 \vee \neg A_2) \wedge (\ A_5 \vee \neg A_4))$

4. For the same formulas as in Exercises (1), (2) and (3), compute $CNF_{label}(NNF(\varphi))$, and compare the result with $CNF_{label}(\varphi)$. What do you notice?

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Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
  - AI, formal verification, circuit synthesis, operational research,....
- Important in AI: $KB \models \alpha$: entail fact α from knowledge base KB (aka Model Checking: $M(KB) \subseteq M(\alpha)$)
  - typically $|KB| \gg |\alpha|$
- All propositional reasoning tasks reduced to satisfiability (SAT)
  - $KB \models \alpha \Longrightarrow$ SAT($KB \wedge \neg\alpha$)=false
  - input formula CNF-ized and fed to a SAT solver
- Current SAT solvers dramatically efficient:
  - handle industrial problems with $10^6 - 10^7$ variables & clauses!
  - used as backend engines in a variety of systems

Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|Atoms(\varphi)|}$ lines.
- Never used in practice.

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The Resolution Rule

- Resolution: deduction of a new clause from a pair of clauses with exactly one incompatible variable (resolvent):

$$\frac{(\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{l}^{resolvent} \vee \overbrace{l'_{k+1} \vee \ldots \vee l'_m}^{C'}) \qquad (\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{\neg l}^{resolvent} \vee \overbrace{l''_{k+1} \vee \ldots \vee l''_n}^{C''})}{(\underbrace{l_1 \vee \ldots \vee l_k}_{common} \vee \underbrace{l'_{k+1} \vee \ldots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \ldots \vee l''_n}_{C''})}$$

- Ex: $\dfrac{(A \vee B \vee C \vee D \vee E) \qquad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$

- Note: many standard inference rules subcases of resolution:
(recall that $\alpha \rightarrow \beta \Longleftrightarrow \neg\alpha \vee \beta$)

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}\ (trans.) \qquad \frac{A \quad A \rightarrow B}{B}\ (m.\ ponens) \qquad \frac{\neg B \quad A \rightarrow B}{\neg A}\ (m.\ tollens)$$

Improvements: Subsumption & Unit Propagation

Alternative "set" notation (Γ clause set):

$$\frac{\Gamma, \phi_1, ..\phi_n}{\Gamma, \phi'_1, ..\phi'_{n'}} \quad \left(e.g., \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2,} \right)$$

- Removal of valid clauses: $$\frac{\Gamma \wedge (p \vee \neg p \vee C)}{\Gamma}$$
- Clause Subsumption (C clause): $$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$
- Unit Resolution: $$\frac{\Gamma \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma \wedge (l) \wedge (\bigvee_i l_i)}$$
- Unit Subsumption: $$\frac{\Gamma \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma \wedge (l)}$$
- Unit Propagation = Unit Resolution + Unit Subsumption

"Deterministic" rule: applied before other "non-deterministic" rules!

Remark

What happens with more than 1 resolvent?

- Common mistake: the following is not a correct application of the resolution rule:

$$\frac{\Gamma,\ (C_1 \vee l_1 \vee l_2),\ (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma,\ (C_1 \vee l_1 \vee l_2),\ (C_2 \vee \neg l_1 \vee \neg l_2),\ (C_1 \vee C_2)}$$

- Rather, a correct application would be:

$$\frac{\Gamma,\ (C_1 \vee l_1 \vee l_2),\ (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma,\ (C_1 \vee l_1 \vee l_2),\ (C_2 \vee \neg l_1 \vee \neg l_2),\ (C_1 \vee l_2 \vee C_2 \vee \neg l_2)}$$

... but $(C_1 \vee l_2 \vee C_2 \vee \neg l_2)$ is valid and should be removed

$\Longrightarrow$ no clause is produced

Basic Propositional Inference: Resolution [33, 10]

- Assume input formula in CNF
  - if not, apply Tseitin CNF-ization first

$\Longrightarrow$ φ is represented as a set of clauses

- Search for a refutation of φ (is φ unsatisfiable?)
  - recall: $\alpha \models \beta$ iff $\alpha \land \neg\beta$ unsatisfiable
- Basic idea: apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either
  - a false clause is generated, or
  - the resolution rule is no more applicable
- Correct: if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- Complete: if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- Time-inefficient
- Very Memory-inefficient (exponential in memory)
- Many different strategies

Resolution: basic strategy [10]

```
function $DP(\Gamma)$
    if $\bot \in \Gamma$                                   /* unsat */
        then return $False$;
    if (Resolve() is no more applicable to $\Gamma$)       /* sat      */
        then return $True$;
    if {a unit clause $(l)$ occurs in $\Gamma$}            /* unit      */
        then $\Gamma := Unit\_Propagate(l, \Gamma)$);
        return    $DP(\Gamma)$
    $A$ := $select\text{-}variable(\Gamma)$;               /* resolve      */
    $\Gamma = \Gamma \cup \bigcup_{A \in C', \neg A \in C''} \{Resolve(C', C'')\} \setminus \bigcup_{A \in C', \neg A \in C''} \{C', C''\}$;
    return    $DP(\Gamma)$
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$(A_1 \vee A_2)\ \ (A_1 \vee \neg A_2)\ \ (\neg A_1 \vee A_2)\ \ (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2)\ \ (A_2 \vee \neg A_2)\ \ (\neg A_2 \vee A_2)\ \ (\neg A_2)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples (cont.)

$$(A \lor B \lor C) \;\; (B \lor \neg C \lor \neg F) \;\; (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \;\; (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
$$(A \lor E \lor \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples

$$(A \vee B)\ \ (A \vee \neg B)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(A)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(C)\ \ (\neg C)$$

$$\Downarrow$$

$$\perp$$

$\Longrightarrow$ UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
  $\Longrightarrow$ May require exponential space
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

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Semantic tableaux [39]

- Search for an assignment satisfying φ
- applies recursively elimination rules to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i, the branch is closed, otherwise it is open.
- if no rule can be applied to an open branch μ, then $\mu \models \varphi$;
- if all branches are closed, the formula is not satisfiable;

Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \to \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2} \qquad (\wedge\text{-elimination})$$

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi} \qquad (\neg\neg\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \qquad \Gamma, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \qquad \Gamma, \neg\varphi_2} \qquad \frac{\Gamma, (\varphi_1 \to \varphi_2)}{\Gamma, \neg\varphi_1 \qquad \Gamma, \varphi_2} \qquad (\vee\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \qquad \Gamma, \neg\varphi_1\neg\varphi_2} \quad \frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \qquad \Gamma, \neg\varphi_1\varphi_2} \qquad (\leftrightarrow\text{ -elimination}).$$

Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

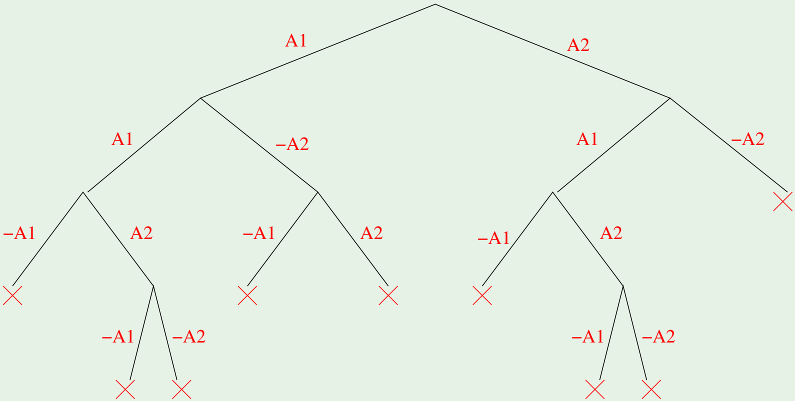
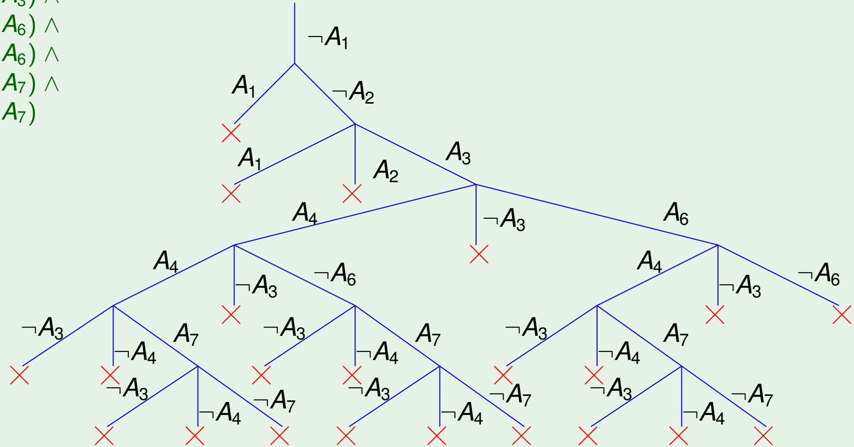


Tableau algorithm

```
function Tableau($\Gamma$)
    if $A_i \in \Gamma$ and $\neg A_i \in \Gamma$                          /* branch closed */
        then return False;
    if $(\varphi_1 \wedge \varphi_2) \in \Gamma$                              /* $\wedge$-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1, \varphi_2\} \backslash \{(\varphi_1 \wedge \varphi_2)\}$);
    if $(\neg\neg\varphi_1) \in \Gamma$                                  /* $\neg\neg$-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1\} \backslash \{(\neg\neg\varphi_1)\}$);
    if $(\varphi_1 \vee \varphi_2) \in \Gamma$                              /* $\vee$-elimination */
        then return     Tableau($\Gamma \cup \{\varphi_1\} \backslash \{(\varphi_1 \vee \varphi_2)\}$)  or
                        Tableau($\Gamma \cup \{\varphi_2\} \backslash \{(\varphi_1 \vee \varphi_2)\}$);
    ...
    return True;                                          /* branch expanded */
```

Semantic Tableaux: Example

$$\begin{array}{lllllll}
 & & & & & (\neg A_1) & \wedge \\
 & & (A_1 & \vee & & \neg A_2) & \wedge \\
(A_1 & \vee & A_2 & \vee & & A_3) & \wedge \\
(A_4 & \vee & \neg A_3 & \vee & & A_6) & \wedge \\
(A_4 & \vee & \neg A_3 & \vee & & \neg A_6) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & & A_7) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & & \neg A_7) &
\end{array}$$



$\Longrightarrow$ unsat

Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend
  $\Longrightarrow$ loved by logicians.
- Rather inefficient
  $\Longrightarrow$ avoided by computer scientists.
- Requires polynomial space

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DPLL [10, 9]

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ;
- At each step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL rules

$$\frac{\varphi_1 \wedge (l)}{\varphi_1[l|\top]}\ (\mathit{Unit})$$

$$\frac{\varphi}{\varphi[l|\top]}\ (l\ \mathit{Pure})$$

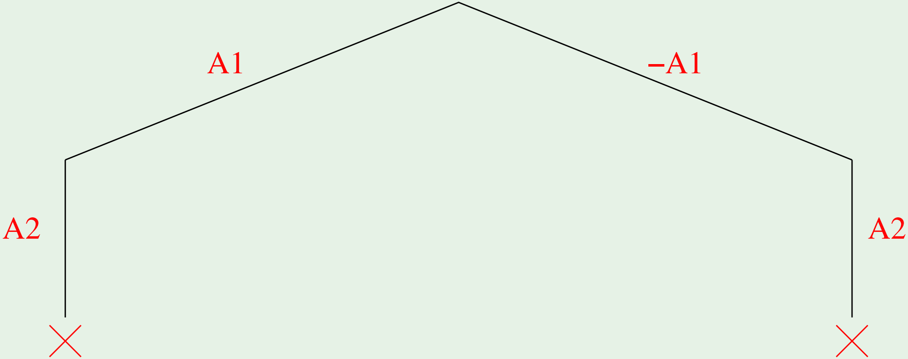
$$\frac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]}\ (\mathit{split})$$

(l is a pure literal in φ iff it occurs only positively).

- Split applied if and only if the others cannot be applied.
- Richer formalisms described in [40, 29, 30]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



The DPLL Algorithm

```
function DPLL($\varphi, \mu$)
    if $\varphi = \top$                                   /* base      */
        then return True;
    if $\varphi = \bot$                                   /* backtrack */
        then return False;
    if {a unit clause ($l$) occurs in $\varphi$}          /* unit      */
        then return DPLL(assign($l, \varphi$), $\mu \wedge l$);
    if {a literal $l$ occurs pure in $\varphi$}           /* pure      */
        then return DPLL(assign($l, \varphi$), $\mu \wedge l$);
    $l$ := choose-literal($\varphi$);                     /* split     */
    return   DPLL(assign($l, \varphi$), $\mu \wedge l$)  or
             DPLL(assign($\neg l, \varphi$), $\mu \wedge \neg l$);
```

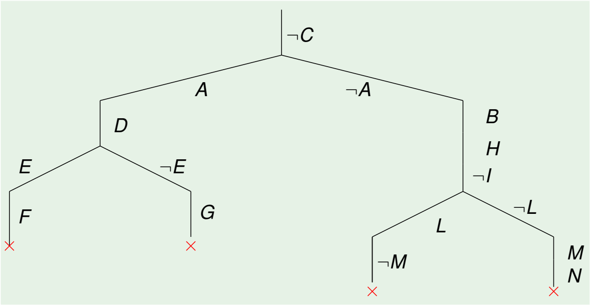
- The pure-literal rule is nowadays obsolete.
- *choose-literal*(φ) picks only variables still occurring in the formula

DPLL – example

DPLL (without pure-literal rule)

Here “choose-literal” selects variables in alphabetic order, selecting true first.

$$
\begin{aligned}
&(\neg C) \land \\
&(B \lor A \lor C) \land \\
&(\neg A \lor D) \land \\
&(\neg E \lor \neg A \lor F) \land \\
&(\neg E \lor \neg F \lor \neg A) \land \\
&(G \lor \neg A \lor E) \land \\
&(E \lor \neg G \lor \neg A) \land \\
&(A \lor H \lor C) \land \\
&(\neg H \lor \neg I \lor A) \land \\
&(I \lor L \lor M) \land \\
&(\neg L \lor C \lor \neg M) \land \\
&(A \lor \neg L \lor M) \land \\
&(L \lor N \lor \neg H) \land \\
&(I \lor L \lor \neg N)
\end{aligned}
$$



$\Longrightarrow$ UNSAT

Remark: “choose-literal” selects only variables which still occur in the formula, after simplification. E.g., in the leftmost branch, after assigning $\neg C$, A, D, it does not select B because the clause $(B \lor A \lor C)$ has been simplified into true, and as such is no more part of the formula, so that B does not occur in the formula anymore.

DPLL – summary

- Handles CNF formulas (non-CNF variant known [1, 15]).
- Branches on truth values
  $\Longrightarrow$ all instances of an atom assigned simultaneously
- Postpones branching as much as possible.
- Mostly ignored by logicians.
- (The grandfather of) the most efficient SAT algorithms
  $\Longrightarrow$ loved by computer scientists.
- Requires polynomial space
- Choose_literal() critical!
- Many very efficient implementations [42, 38, 2, 28].

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Ordered Binary Decision Diagrams (OBDDs) [8]]

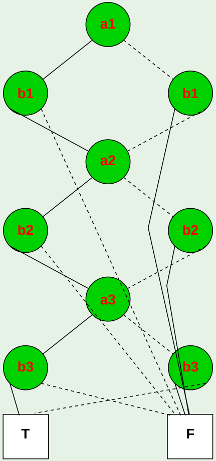
Canonical representation of Boolean formulas

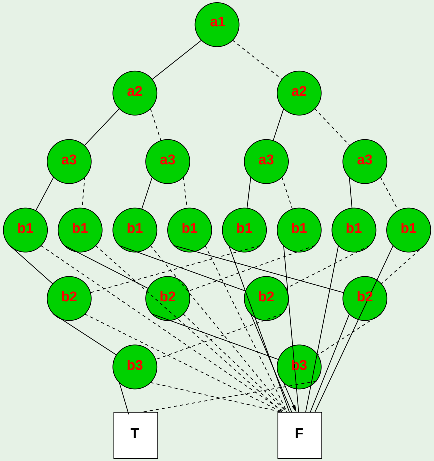
- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top$,$\bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

OBDD - Examples

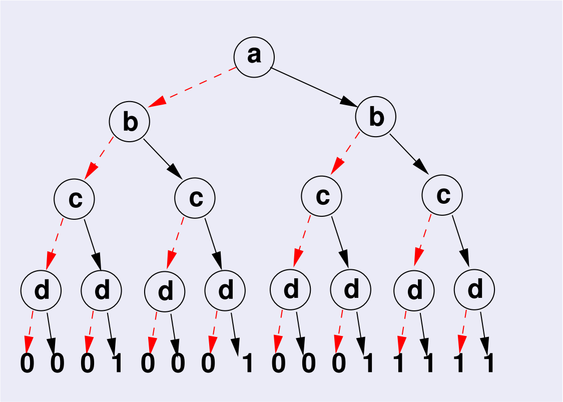




OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- Ordered Decision Tree:
  from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

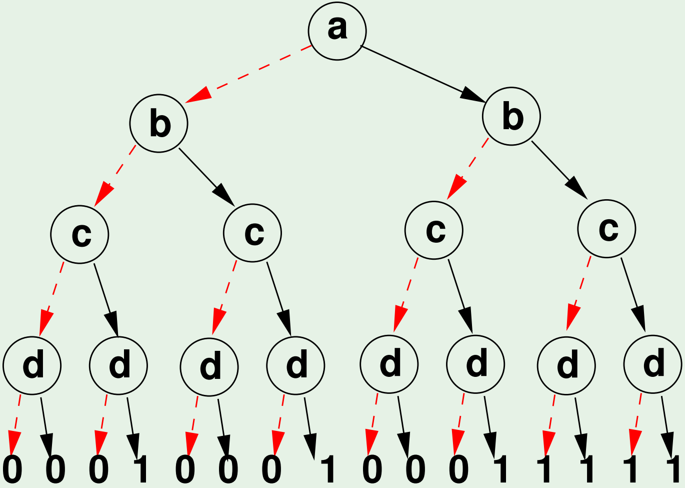


From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following reductions:
  - share subnodes: point to the same occurrence of a subtree (via hash consing)
  - remove redundancies: nodes with same left and right children can be eliminated: "if A then B else B" $\Longrightarrow$ "B"

Reduction: example

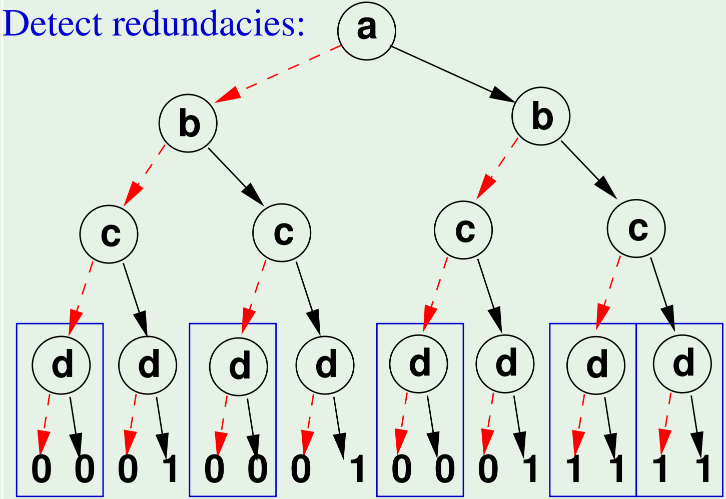
$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

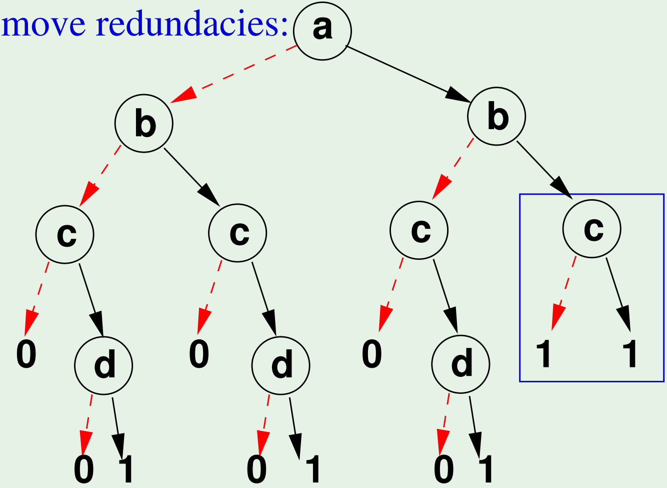
Detect redundacies:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

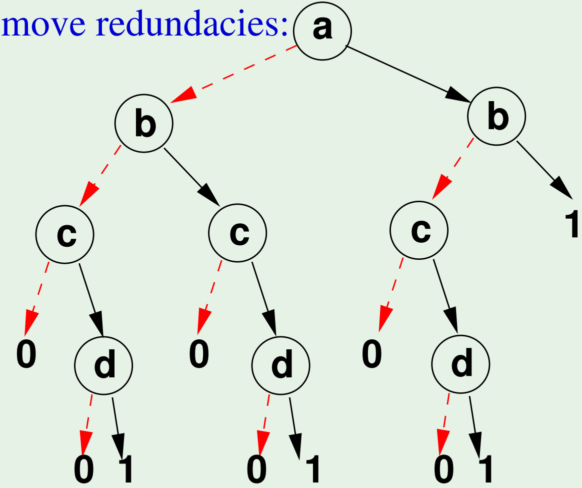
Remove redundacies:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

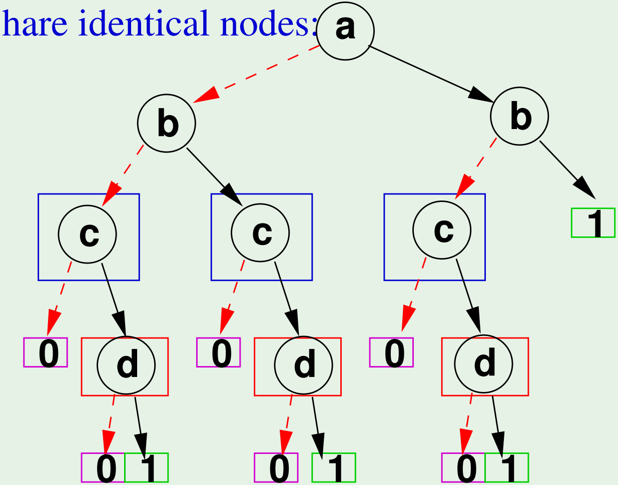
Remove redundacies:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

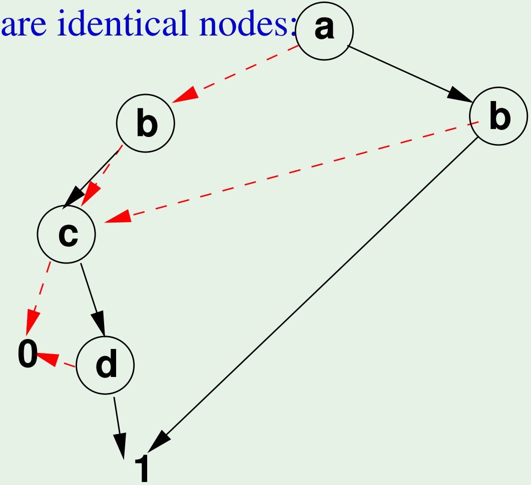
Share identical nodes:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

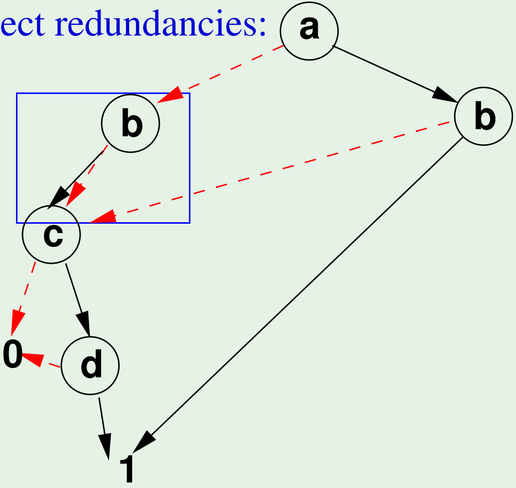
Share identical nodes:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \land b) \lor (c \land d)$

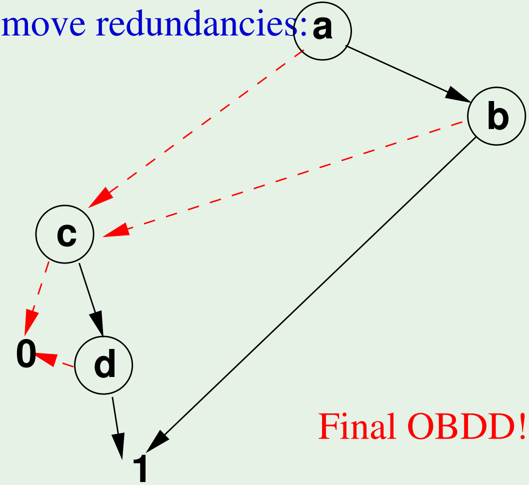
Detect redundancies:



Reduction: example

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

Remove redundancies:



Final OBDD!

If-Then-Else Operators: "*ite*(...)"

If-Then-Else Operators: "*ite*(...)"

- $ite(\phi, \varphi^\top, \varphi^\bot)$: "If ϕ Then $\varphi^\top$ Else $\varphi^\bot$"
- $ite(\phi, \varphi^\top, \varphi^\bot) \overset{\text{def}}{=} ((\neg\phi \vee \varphi^\top) \wedge (\phi \vee \varphi^\bot)) \iff ((\phi \wedge \varphi^\top) \vee (\neg\phi \wedge \varphi^\bot))$
- properties:

$$
\begin{aligned}
ite(\neg\phi, \varphi^\top, \varphi^\bot) &= ite(\phi, \varphi^\bot, \varphi^\top) \\
\neg ite(\phi, \varphi^\top, \varphi^\bot) &= ite(\phi, \neg\varphi^\top, \neg\varphi^\bot) \\
ite(\phi, \varphi_1^\top, \varphi_1^\bot)\ op\ ite(\phi, \varphi_2^\top, \varphi_2^\bot) &= ite(\phi, (\varphi_1^\top\ op\ \varphi_2^\top), (\varphi_1^\bot\ op\ \varphi_2^\bot)) \\
ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot) &= ite(\phi_1, (\varphi_1^\top\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)), \qquad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\} \\
&\qquad\qquad (\varphi_1^\bot\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot))) \\
&= ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) op\ \varphi_2^\top), \\
&\qquad\qquad (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) op\ \varphi_2^\bot))
\end{aligned}
$$

Recursive structure of an OBDD

Assume the variable ordering $A_1, A_2, ..., A_n$:

$$\begin{array}{lcl}
OBDD(\top, \{A_1, A_2, ..., A_n\}) & = & 1 \\
OBDD(\bot, \{A_1, A_2, ..., A_n\}) & = & 0 \\
OBDD(\varphi, \{A_1, A_2, ..., A_n\}) & = & \mathit{if}\ A_1 \\
 & & \mathit{then}\ OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\}) \\
 & & \mathit{else}\ OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\})
\end{array}$$

Incrementally building an OBDD

- $\mathit{obdd\_build}(\top, \{...\}) := \top$,
- $\mathit{obdd\_build}(\bot, \{...\}) := \bot$,
- $\mathit{obdd\_build}(A_i, \{...\}) := \mathit{ite}(A_i, \top, \bot)$,
- $\mathit{obdd\_build}((\neg\varphi), \{A_1, ..., A_n\}) := \mathit{apply}(\neg, \mathit{obdd\_build}(\varphi, \{A_1, ..., A_n\}))$
- $\mathit{obdd\_build}((\varphi_1 \; op \; \varphi_2), \{A_1, ..., A_n\}) :=$

$$
\begin{array}{lll}
\mathit{reduce}(& & \\
\quad \mathit{apply}(& op, & \\
 & \mathit{obdd\_build}(\varphi_1, \{A_1, ..., A_n\}), & op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\} \\
 & \mathit{obdd\_build}(\varphi_2, \{A_1, ..., A_n\}) & \\
)\,) & &
\end{array}
$$

Incrementally building an OBDD (cont.)

- $\mathit{apply}\ (\mathit{op},\ O_i, O_j) := (O_i\ \mathit{op}\ O_j)$ **if** $(O_i \in \{\top, \bot\}$ or $O_j \in \{\top, \bot\})$
- $\mathit{apply}\ (\neg,\ \mathit{ite}(A_i, \varphi_i^\top, \varphi_i^\bot)) :=$
  $\mathit{ite}(A_i, \mathit{apply}(\neg, \varphi_i^\top), \mathit{apply}(\neg, \varphi_i^\bot))$
- $\mathit{apply}\ (\mathit{op},\ \mathit{ite}(A_i, \varphi_i^\top, \varphi_i^\bot),\ \mathit{ite}(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$

$$\begin{array}{ll}
\textbf{if } (A_i = A_j) \textbf{ then } \mathit{ite}(A_i, & \mathit{apply}\ (\mathit{op}, \varphi_i^\top, \varphi_j^\top), \\
 & \mathit{apply}\ (\mathit{op}, \varphi_i^\bot, \varphi_j^\bot)) \\
\textbf{if } (A_i < A_j) \textbf{ then } \mathit{ite}(A_i, & \mathit{apply}\ (\mathit{op}, \varphi_i^\top, \mathit{ite}(A_j, \varphi_j^\top, \varphi_j^\bot)), \\
 & \mathit{apply}\ (\mathit{op}, \varphi_i^\bot, \mathit{ite}(A_j, \varphi_j^\top, \varphi_j^\bot))) \\
\textbf{if } (A_i > A_j) \textbf{ then } \mathit{ite}(A_j, & \mathit{apply}\ (\mathit{op}, \mathit{ite}(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top), \\
 & \mathit{apply}\ (\mathit{op}, \mathit{ite}(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))
\end{array}$$

$\mathit{op} \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

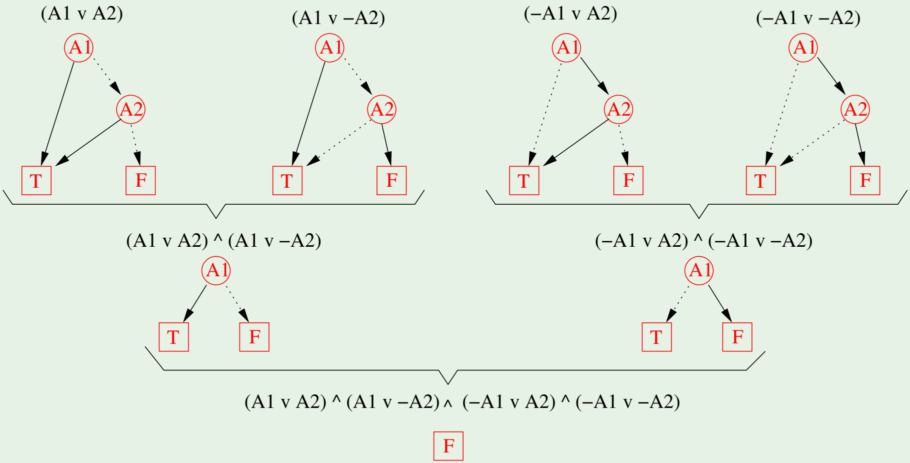
$$
\begin{array}{rl}
 & apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\
= & ite(A_1,\ apply(\vee, \top, ite(A_2, \top, \bot)),\ apply(\vee, \bot, ite(A_2, \top, \bot))\ ) \\
= & ite(A_1,\ \top,\ ite(A_2, \top, \bot)\ )
\end{array}
$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$
\begin{array}{rl}
 & apply(\wedge,\ \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)},\ \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)}, \\
= & ite(A_1,\ apply(\wedge, \top, \top),\ apply(\wedge,\ ite(A_2, \top, \bot),\ ite(A_2, \bot, \top)) \\
= & ite(A_1,\ \top,\ ite(A_2,\ apply(\wedge, \top, \bot),\ apply(\wedge, \bot, \top))) \\
= & ite(A_1,\ \top,\ ite(A_2,\ \bot,\ \bot)) \\
= & ite(A_1,\ \top,\ \bot)
\end{array}
$$

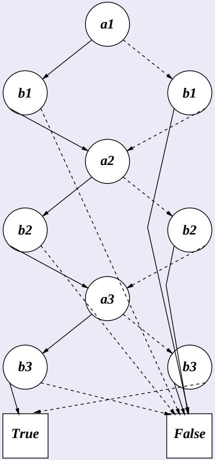
OBDD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

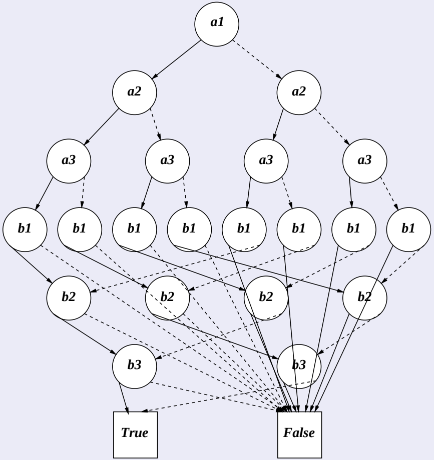


Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$



Linear size



Exponential size

OBDD's as canonical representation of Boolean formulas

- An OBDD is a canonical representation of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff \mathit{OBDD}(\varphi_1) = \mathit{OBDD}(\varphi_2)$$

- equivalence check requires constant time!
  $\Longrightarrow$validity check requires constant time! ($\varphi \leftrightarrow \top$)
  $\Longrightarrow$(un)satisfiability check requires constant time! ($\varphi \leftrightarrow \bot$)
- the set of the paths from the root to 1 represent all the models of the formula
- the set of the paths from the root to 0 represent all the counter-models of the formula

Exponentiality of OBDD's

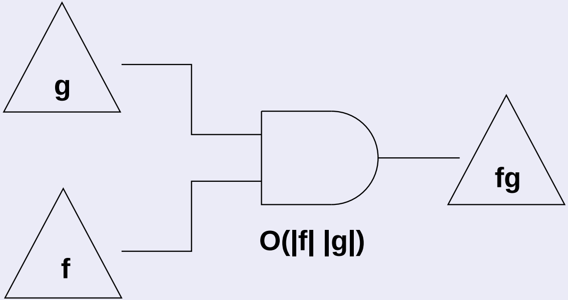
- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless P = co-NP)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

Note

The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

Useful Operations over OBDDs

- the equivalence check between two OBDDs is simple
  - are they the same OBDD? ($\Longrightarrow$ constant time)
- the size of a Boolean composition is up to the product of the size of the operands: $|f\ op\ g| = O(|f| \cdot |g|)$



(but typically much smaller on average).

[Recall] Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then
  $$\begin{array}{lcl} \exists v.\varphi & := & \varphi|_{v=\bot} \vee \varphi|_{v=\top} \\ \forall v.\varphi & := & \varphi|_{v=\bot} \wedge \varphi|_{v=\top} \end{array}$$
- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1,\ldots,w_n).\varphi \ := \ \exists w_1 \ldots \exists w_n.\varphi$

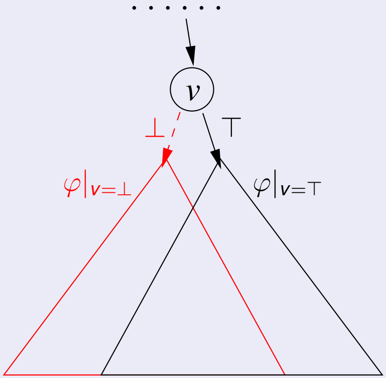
- Intuition:
  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top,\bot\}$ s.t. $\mu \cup \{v := \textit{truthvalue}\} \models \varphi$
  - $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top,\bot\}$, $\mu \cup \{v := \textit{truthvalue}\} \models \varphi$
- Example: $\exists(b,c).((a \wedge b) \vee (c \wedge d)) \ = \ a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if f is a sub-OBDD labeled by variable v, then $\varphi|_{v=\top}$ and $\varphi|_{v=\perp}$ are the "then" and "else" branches of f

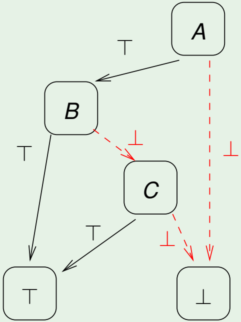


$\Longrightarrow$ lots of sharing of subformulae!

Example

Let $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.\varphi$. Using the variable ordering “A, B, C”, draw the OBDD corresponding to the formulas φ and φ'.

$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$

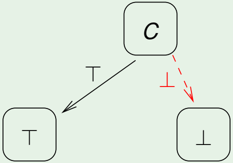


Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

$$
\begin{array}{rlcllcl}
\varphi' \stackrel{\text{def}}{=} & \exists A.\forall B.\varphi & & & & & \\
= & \forall B.(A \wedge (B \vee C)))[A := \top] & & & & \vee & (\forall B.(A \wedge (B \vee C)))[A := \bot] \\
= & \forall B.(B \vee C)) & & & & \vee & \forall B.\bot \\
= & ((B \vee C)[B := \top] & \wedge & (B \vee C)[B := \bot]) & & \vee & \bot \\
= & (\top & \wedge & C) & & & \\
= & C & & & & &
\end{array}
$$

which corresponds to the following OBDD:



OBDD – summary

- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
- Canonical representation of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents all models and counter-models of the formula.
- Require exponential space in worst-case
- Very efficient for some practical problems (circuits, symbolic model checking).

Exercises

1. Let

$$\varphi \overset{\text{def}}{=} \neg \left(\begin{array}{rrrrl} & (& A_1) & \wedge \\ (& A_1 \rightarrow & A_2) & \wedge \\ (& A_2 \rightarrow & A_3) & \wedge \\ (& A_3 \rightarrow & A_4) & \wedge \\ (& A_4 \rightarrow & A_5) & \wedge \end{array} \right)$$

   Using the variable ordering: " A_1 A_2, A_3, A_4, A_5", draw the OBDD corresponding to φ

2. Let

$$\varphi \overset{\text{def}}{=} A_1 \rightarrow (A_2 \leftrightarrow \left(\begin{array}{lllll} (A_3 \vee & A_6 \vee & A_8) & \wedge \\ (A_5 \vee & A_7 \vee & A_8) & \wedge \\ (\neg A_4 \vee & \neg A_6 \vee & \neg A_8) & \wedge \\ (\neg A_6 \vee & A_7 \vee & \neg A_8) & \wedge \\ (\neg A_3 \vee & A_6 \vee & A_9) & \wedge \\ (\neg A_6 \vee & \neg A_8 \vee & \neg A_9) & \wedge \\ (A_3 \vee & A_4 \vee & \neg A_5) & \wedge \\ (\neg A_4 \vee & \neg A_7 \vee & \neg A_9) & \end{array} \right)).$$

   Using the variable ordering: " A_1, A_3, A_4, A_5, A_6, A_7, A_8, A_9", draw the OBDD corresponding to the formula φ' defined as: $\varphi' \overset{\text{def}}{=} \forall A_2.\varphi$.

Outline

1. Boolean Logics and SAT
2. Basic SAT-Solving Techniques
   - Generalities
   - Resolution
   - Tableaux
   - DPLL
3. Ordered Binary Decision Diagrams – OBDDs
4. **Modern CDCL SAT Solvers**
   - Limitations of Chronological Backtracking
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
   - SAT Under Assumptions & Incremental SAT
5. SAT Functionalities: proofs, unsat cores, optimization

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DPLL: “Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment l
- l is toggled, is labeled as “closed”, and the search proceeds.

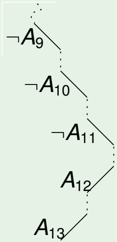
DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible
$\Longrightarrow$ lots of useless search!

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$\ldots$

DPLL Chronological Backtracking: Example



$c_1 : \neg A_1 \lor A_2$
$c_2 : \neg A_1 \lor A_3 \lor A_9$
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$
$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$
$c_8 : A_1 \lor A_8$
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
...

$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ...\}$
(initial assignment)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

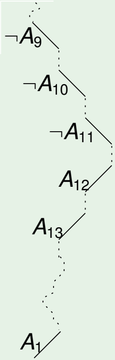
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., A_1\}$

... (branch on A_1)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

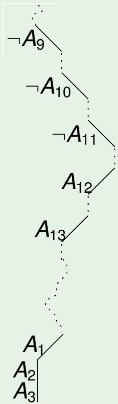
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., A_1, A_2, A_3\}$
(unit A_2, A_3)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

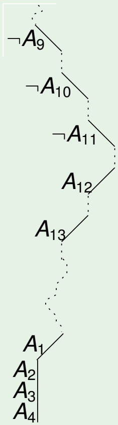
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\dots$



$\{\dots, \neg A_9, \dots, \neg A_{10}, \dots, \neg A_{11}, \dots, A_{12}, \dots, A_{13}, \dots, A_1, A_2, A_3, A_4\}$

(unit A_4)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

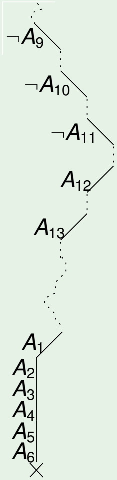
$c_6 : \neg A_5 \lor \neg A_6$ ✗

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...

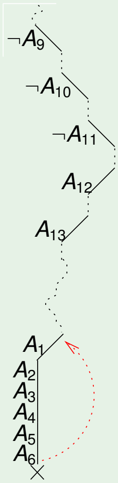


$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., A_1, A_2, A_3, A_4, A_5, A_6\}$

(unit A_5, A_6) $\Longrightarrow$ conflict

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$\ldots$



$\{\ldots, \neg A_9, \ldots, \neg A_{10}, \ldots, \neg A_{11}, \ldots, A_{12}, \ldots, A_{13}, \ldots\}$
$\Longrightarrow$ backtrack up to A_1

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ $\surd$

$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

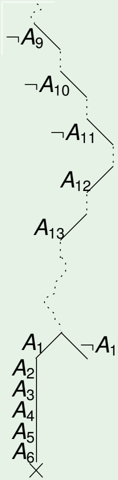
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ $\surd$

$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

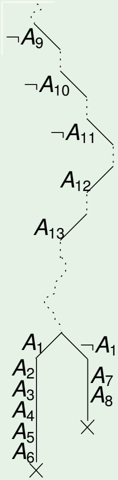
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\surd$

$c_8 : A_1 \vee A_8$ $\surd$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ $\times$

$\dots$

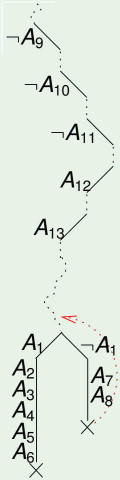


$\{\dots, \neg A_9, \dots, \neg A_{10}, \dots, \neg A_{11}, \dots, A_{12}, \dots, A_{13}, \dots, \neg A_1, A_7, A_8\}$

(unit A_7, A_8) $\Longrightarrow$ conflict

DPLL Chronological Backtracking: Example

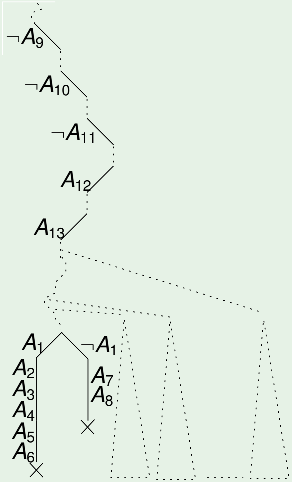
$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ...\}$
$\Longrightarrow$ backtrack to the most recent open branching point

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ...\}$
$\Longrightarrow$ lots of useless search before backtracking up to A_{13}!

Outline

1. Boolean Logics and SAT
2. Basic SAT-Solving Techniques
   - Generalities
   - Resolution
   - Tableaux
   - DPLL
3. Ordered Binary Decision Diagrams – OBDDs
4. Modern CDCL SAT Solvers
   - Limitations of Chronological Backtracking
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
   - SAT Under Assumptions & Incremental SAT
5. SAT Functionalities: proofs, unsat cores, optimization

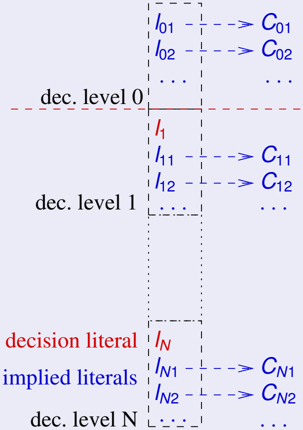
Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
  - inspired to conflict-driven backjumping and learning in CSPs
  - learns implied clauses as nogoods
- Random restarts
  - abandon the current search tree and restart on top level
  - previously-learned clauses maintained
- Smart literal selection heuristics (ex: VSIDS)
  - “static”: scores updated only at the end of a branch
  - “local”: privileges variable in recently learned clauses
- Smart preprocessing/inprocessing technique to simplify formulas
- Smart indexing techniques (e.g. 2-watched literals)
  - efficiently do/undo assignments and reveal unit clauses
- Allow Incremental Calls (stack-based interface)
  - allow for reusing previous search on “similar” problems

Can handle industrial problems with $10^6 - 10^7$ variables and clauses!

Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into decision levels:
  - one decision literal
  - its implied literals
  - each implied literal tagged with the clause causing its unit-propagation (antecedent clause)
- equivalent to an implication graph

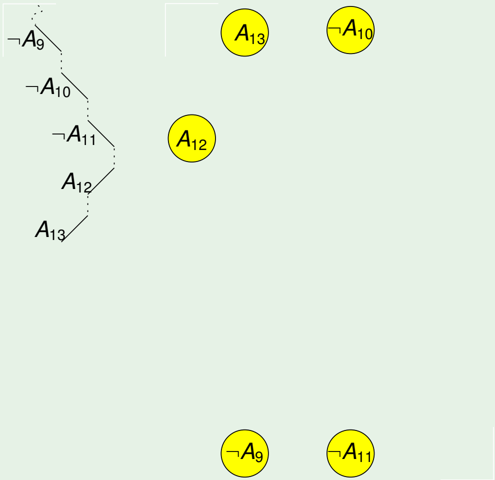


Implication graph

- An implication graph is a DAG s.t.:
  - each node represents a variable assignment (literal)
  - each edge $l_i \overset{c}{\longmapsto} l$ is labeled with a clause
  - the node of a decision literal has no incoming edges
  - all edges incoming into a node l are labeled with the same clause c, s.t. $l_1 \overset{c}{\longmapsto} l, ..., l_n \overset{c}{\longmapsto} l$ iff $c = \neg l_1 \vee ... \vee \neg l_n \vee l$
    (c is said to be the antecedent clause of l)
  - when both l and $\neg l$ occur in the graph, we have a conflict.
- Intuition:
  - representation of the dependencies between literals in μ
  - the graph contains $l_1 \overset{c}{\longmapsto} l, ..., l_n \overset{c}{\longmapsto} l$ iff l has been obtained from l_1, ..., l_n by unit propagation on c
  - a partition of the graph with all decision literals on one side and the conflict on the other represents a conflict set

Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ...\}$
(Initial assignment. Note: $c_1, ..., c_9$ inconsistent.)

Example

$c_1 : \neg A_1 \lor A_2$

$c_2 : \neg A_1 \lor A_3 \lor A_9$

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

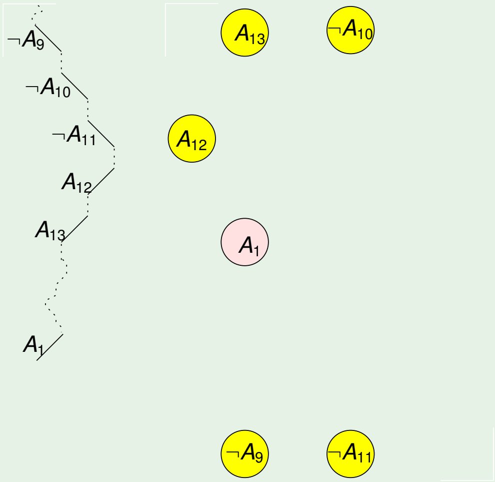
$c_6 : \neg A_5 \lor \neg A_6$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ $\surd$

$c_8 : A_1 \lor A_8$ $\surd$

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., A_1\}$

... (decide A_1)

Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

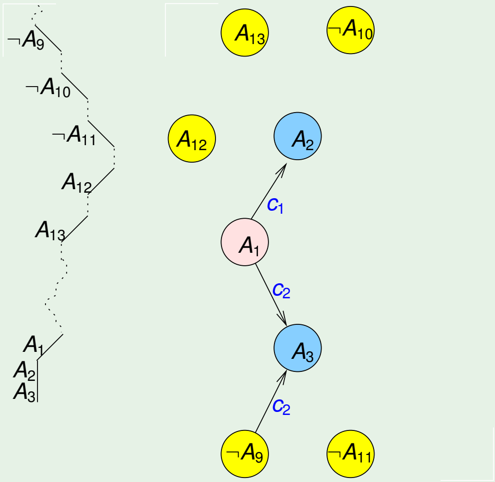
$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\{\ldots, \neg A_9, \ldots, \neg A_{10}, \ldots, \neg A_{11}, \ldots, A_{12}, \ldots, A_{13}, \ldots, A_1, A_2, A_3\}$

(unit A_2, A_3)

Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

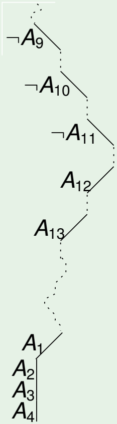
$c_6 : \neg A_5 \lor \neg A_6$

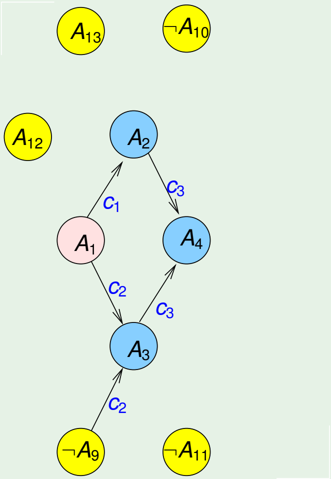
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...





$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., A_1, A_2, A_3, A_4\}$

(unit A_4)

Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ √

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ √

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ √

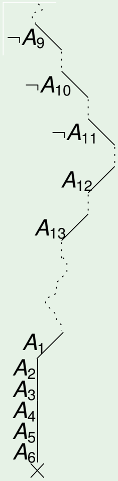
$c_6 : \neg A_5 \vee \neg A_6$ ×

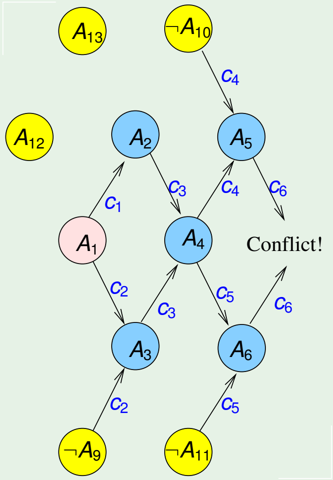
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

…





$\{\ldots, \neg A_9, \ldots, \neg A_{10}, \ldots, \neg A_{11}, \ldots, A_{12}, \ldots, A_{13}, \ldots, A_1, A_2, A_3, A_4, A_5, A_6\}$

(unit A_5, A_6) $\Longrightarrow$ conflict

Unique implication point - UIP [44]

- A node l in an implication graph is an unique implication point (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through l.
  - the most recent decision node is an UIP (last UIP)
  - all other UIP's have been assigned after the most recent decision
- Intuition: if the last decision had beed one of the UIPs instead of the last decision (last UIP), then we would have had the same failure on the same falsified clause.
  $\Longrightarrow$ We can pretend the UIP was the last decision.

Unique implication point - UIP - example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ √

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ √

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ √

$c_6 : \neg A_5 \vee \neg A_6$ ×

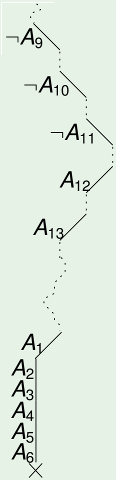
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

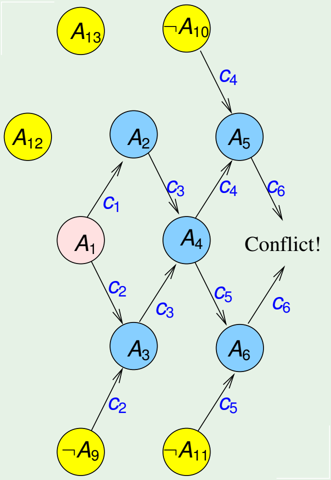
$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

…

- A_1 is the last UIP
- A_4 is the 1^{st} UIP





Schema of a CDCL DPLL solver [38, 45]

```
Function CDCL-SAT (formula: $\varphi$, assignment & $\mu$) {
        status := preprocess($\varphi,\mu$);
        while (1) {
            while (1) {
                status := deduce($\varphi,\mu$);
                if (status == Sat)
                    return Sat;
                if (status == Conflict) {
                    $\langle$blevel,$\eta\rangle$ := analyze_conflict($\varphi,\mu$);
                    //$\eta$ is a conflict set
                    if (blevel == 0)
                        return Unsat;
                    else backtrack(blevel,$\varphi,\mu$);
                }
                else break;
            }
            decide_next_branch($\varphi,\mu$);
}       }
```

Schema of a CDCL DPLL solver [38, 45] (cont.)

- `preprocess`(φ, μ) simplifies φ into an easier equisatisfiable formula, updating μ.
- `decide_next_branch`(φ, μ) chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce`(φ, μ) performs all deterministic assignments (unit-propagations plus others), and updates φ, μ accordingly.
- `analyze_conflict`(φ, μ) Computes the subset η of μ causing the conflict (conflict set), and returns the “wrong-decision” level suggested by η
  - η is s.t. if we had decided all literals in η, we would have obtaned the same failure on the same falsified clause by unit-propagation
  - “`blevel==0`” means that η is entirely assigned at level 0, i.e., a conflict exists even without branching
- `backtrack(blevel,`φ, μ`)` undoes the branches up to blevel, and updates φ, μ accordingly

Backjumping and learning: general ideas [2, 38]

- When a branch μ fails:
  (i) conflict analysis: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (conflict set η)
  (ii) learning: add the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ to the clause set
  (iii) backjumping: use η to decide the point where to backtrack
- Jump back up much more than one decision level in the stack
  $\Longrightarrow$ may avoid lots of redundant search!!.
- We illustrate two main backjumping & learning strategies:
  - the original strategy presented in [38]
  - the state-of-the-art 1^{st}UIP strategy of [44]

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterion: decision

...until C contains only decision literals

$$\dfrac{\neg A_1 \lor A_2 \qquad \dfrac{\neg A_1 \lor A_3 \lor A_9 \qquad \dfrac{\neg A_2 \lor \neg A_3 \lor A_4 \qquad \dfrac{\neg A_4 \lor A_5 \lor A_{10} \qquad \dfrac{\neg A_4 \lor A_6 \lor A_{11} \qquad \overbrace{\neg A_5 \lor \neg A_6}^{\textit{Conflicting cl.}}}{\neg A_4 \lor \neg A_5 \lor A_{11}}\;(A_6)}{\neg A_4 \lor A_{10} \lor A_{11}}\;(A_5)}{\neg A_2 \lor \neg A_3 \lor A_{10} \lor A_{11}}\;(A_4)}{\neg A_2 \lor \neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\;(A_3)}{\neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\;(A_2)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterion: last UIP

... until C contains only one literal assigned at current decision level, and it is the decision literal (last UIP)

$$\dfrac{\neg A_1 \lor A_2 \qquad \dfrac{\neg A_1 \lor A_3 \lor A_9 \qquad \dfrac{\neg A_2 \lor \neg A_3 \lor A_4 \qquad \dfrac{\neg A_4 \lor A_5 \lor A_{10} \qquad \dfrac{\neg A_4 \lor A_6 \lor A_{11} \quad \overbrace{\neg A_5 \lor \neg A_6}^{\textit{Conflicting cl.}}}{\neg A_4 \lor \neg A_5 \lor A_{11}}\;(A_6)}{\neg A_4 \lor A_{10} \lor A_{11}}\;(A_5)}{\neg A_2 \lor \neg A_3 \lor A_{10} \lor A_{11}}\;(A_4)}{\neg A_2 \lor \neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\;(A_3)}{\neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\;(A_2)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat

   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterion: 1st UIP

... until C contains only one literal assigned at current decision level (1st UIP)

$$\dfrac{\neg A_4 \lor A_5 \lor A_{10} \qquad \dfrac{\neg A_4 \lor A_6 \lor A_{11} \qquad \overbrace{\neg A_5 \lor \neg A_6}^{Conflicting\ cl.}}{\neg A_4 \lor \neg A_5 \lor A_{11}}\ (A_6)}{\underbrace{\neg A_4}_{1st\ UIP} \lor A_{10} \lor A_{11}}\ (A_5)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat

   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

Note:

$\varphi \models C$, so that C can be safely added to φ.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis and implication graph - example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

$c_6 : \neg A_5 \vee \neg A_6$ ×

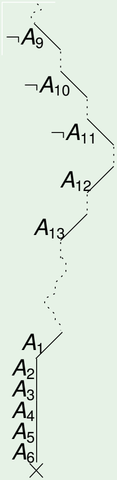
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

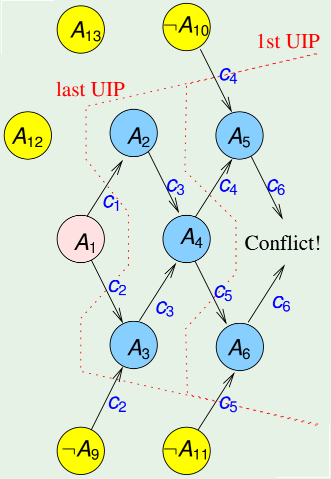
$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...

Note: in this case decision and last-UIP criteria produce the same partition





The original backjumping and learning strategy of [38]

- Idea: when a branch μ fails,
  - (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the “Decision” criterion;
  - (ii) learning: add the conflict clause C to the clause set
  - (iii) backjumping: backtrack to the most recent branching point s.t. the stack does not fully contain η, and then unit-propagate the unassigned literal on C

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ √

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ √

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ √

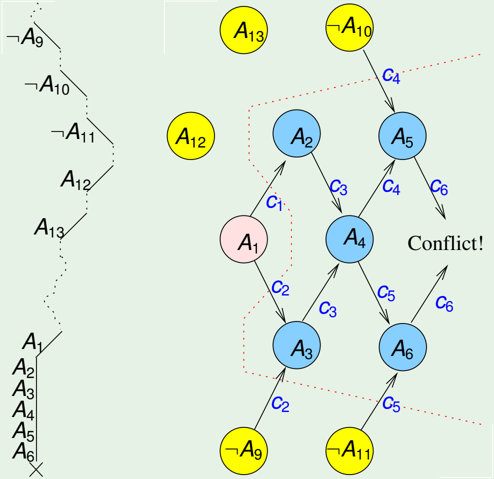
$c_6 : \neg A_5 \vee \neg A_6$ ×

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



$\Longrightarrow$ Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ ("decision" schema)

$\Longrightarrow$ learn the conflict clause $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

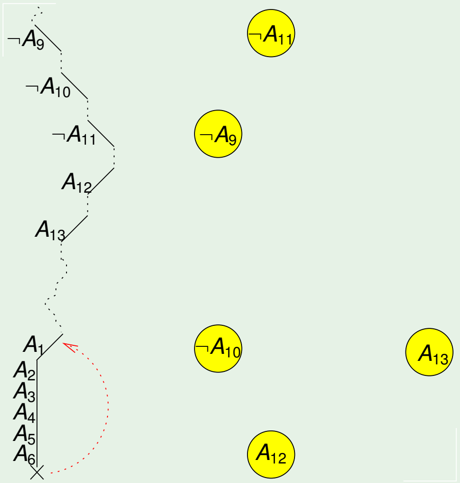
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

...



$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ...\}$

$\Longrightarrow$ backtrack up to A_1

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

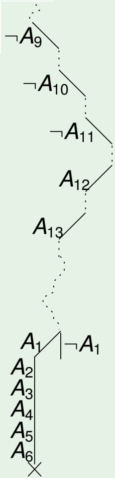
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

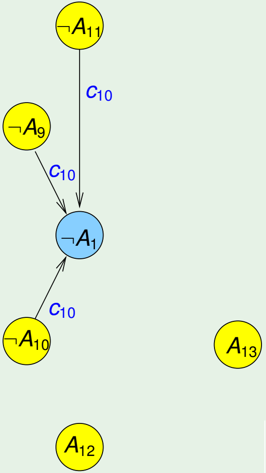
$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √

...





$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., \neg A_1\}$

(unit $\neg A_1$)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

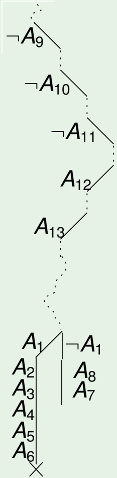
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

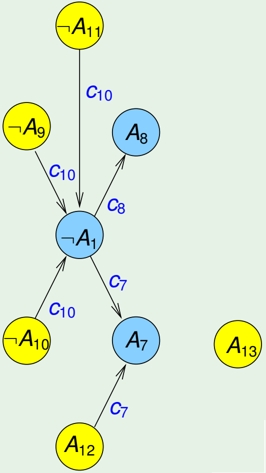
$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √

...





$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., \neg A_1, A_7, A_8\}$
(unit A_7, A_8)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

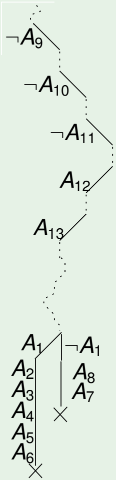
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

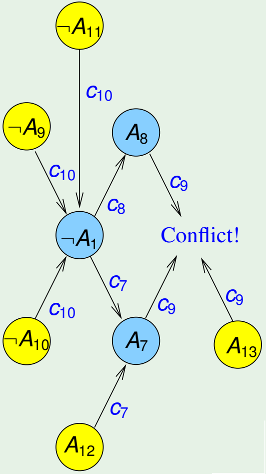
$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √

...





$\{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}, ..., A_{12}, ..., A_{13}, ..., \neg A_1, A_7, A_8\}$

Conflict!

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

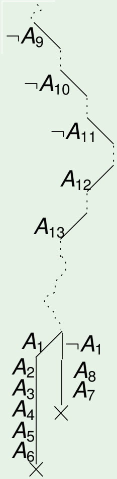
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

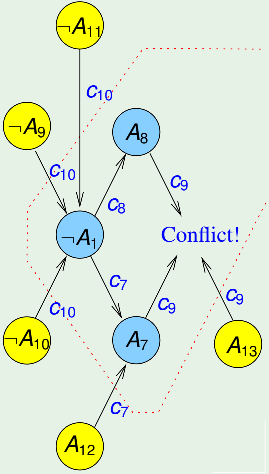
$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓

...





$\Longrightarrow$ conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.

$\Longrightarrow$ learn $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

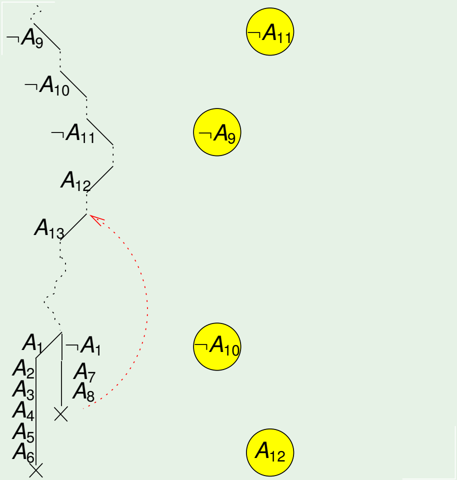
$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

...



$\Longrightarrow$ backtrack to A_{13} $\Longrightarrow$ Lots of search saved!

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ $\surd$

$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\surd$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

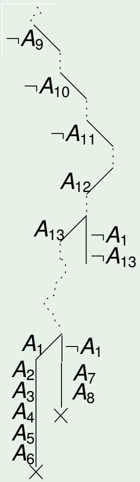
$c_8 : A_1 \vee A_8$

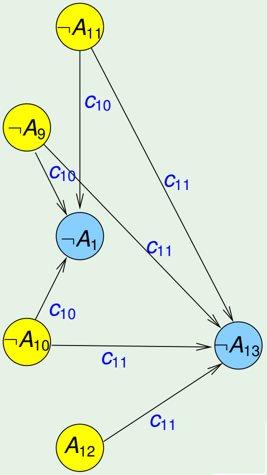
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ $\surd$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ $\surd$

$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ $\surd$

...





$\Longrightarrow$ backtrack to A_{13}, then set A_{13} and A_1 to $\bot$,...

State-of-the-art backjumping and learning [44]

- Idea: when a branch μ fails,

  (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \overset{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the 1^{st}UIP strategy

  (ii) learning: add the conflict clause C to the clause set

  (iii) backjumping: backtrack to the highest branching point s.t. the stack contains all-but-one literals in η, and then unit-propagate the unassigned literal on C

1st UIP strategy – example (7)

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

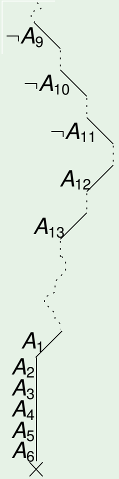
$c_6 : \neg A_5 \vee \neg A_6$ ×

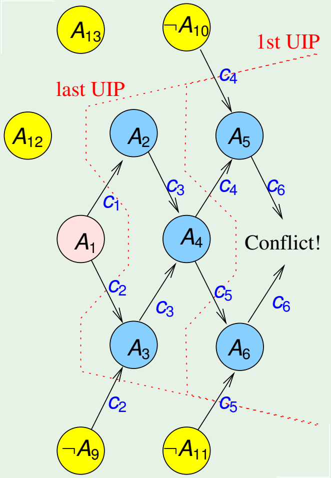
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping [44]

- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$
$\Longrightarrow$ backtrack to A_{11}, then assign $\neg A_4$

1st UIP strategy – example (7)

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

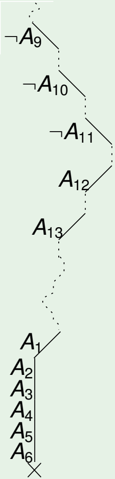
$c_6 : \neg A_5 \vee \neg A_6$ ×

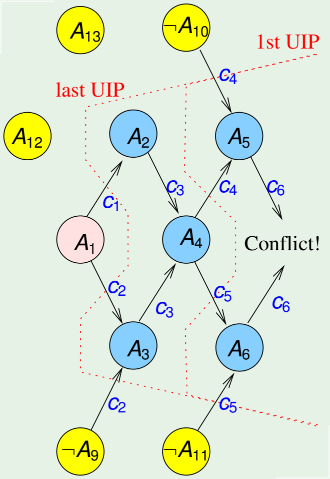
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

…

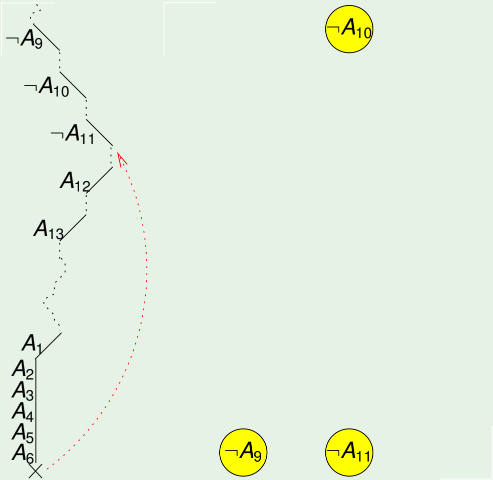




$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy – example (8)

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$
...



$\Longrightarrow$ backtrack up to A_{11} $\Longrightarrow \{..., \neg A_9, ..., \neg A_{10}, ..., \neg A_{11}\}$

1st UIP strategy – example (9)

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ $\surd$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ $\surd$

$c_6 : \neg A_5 \vee \neg A_6$

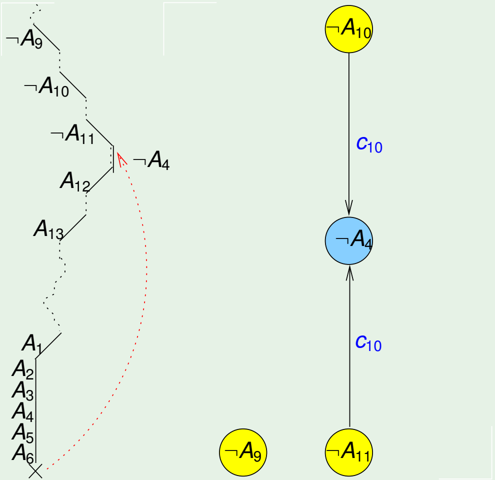
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$ $\surd$

...



$\Longrightarrow$ unit propagate $\neg A_4 \Longrightarrow \{\ldots, \neg A_9, \ldots, \neg A_{10}, \ldots, \neg A_{11}, A_4\}\ldots$

1st UIP strategy and backjumping – intuition

- An UIP is a single reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
  - does not enlarge the conflict
  - requires less resolution steps to compute C
  - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
  - jump higher
  - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [2, 38]

Idea: When a conflict set η is revealed, then $C \overset{\text{def}}{=} \neg\eta$ is added to φ
$\Longrightarrow$ the solver will no more generate an assignment containing η:
when $|\eta| - 1$ literals in η are assigned, the other is set $\perp$ by unit-propagation on C
$\Longrightarrow$ Drastic pruning of the search!

Learning – example

$c_1 : \neg A_1 \lor A_2$

$c_2 : \neg A_1 \lor A_3 \lor A_9$

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

$c_6 : \neg A_5 \lor \neg A_6$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$

$c_8 : A_1 \lor A_8$

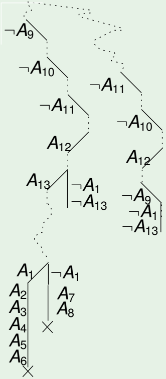
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$ √

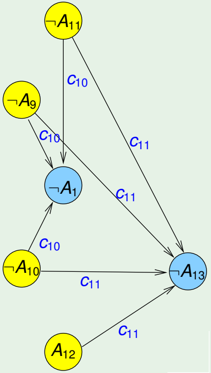
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ √

$c_{11} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_{12} \lor \neg A_{13}$ √

...

$\Longrightarrow$ Unit: $\{\neg A_1, \neg A_{13}\}$





Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their activity.

A clause is currently active if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [30])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.
$\Longrightarrow$ CDCL solvers require polynomial space

"Lazy" Strategy

- when a clause is involved in conflict analisis, increase its activity
- when needed, drop the least-active clauses

State-of-the-art backjumping and learning: intuitions

- Backjumping: allows for climbing up to many decision levels in the stack
  - intuition: “ go back to the oldest decision where you’d have done something different if only you had known C”

  $\Longrightarrow$ may avoid lots of redundant search
- Learning: in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
  - intuition: “when you’re about to repeat the mistake, do the opposite of the last step”

  $\Longrightarrow$ avoid finding the same conflict again

Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
  - Backjumping: if causes the highest backjump (“local” role)
  - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented (see, e.g., [2, 38, 44])

Outline

1. Boolean Logics and SAT
2. Basic SAT-Solving Techniques
   - Generalities
   - Resolution
   - Tableaux
   - DPLL
3. Ordered Binary Decision Diagrams – OBDDs
4. Modern CDCL SAT Solvers
   - Limitations of Chronological Backtracking
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
   - SAT Under Assumptions & Incremental SAT
5. SAT Functionalities: proofs, unsat cores, optimization

Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
  - detect and remove subsumed clauses
  - detect & collapse equivalent literals
  - apply basic resolution steps
  - ...

Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\varphi_1 \land (l_2 \lor l_1) \land \varphi_2 \land (l_2 \lor l_3 \lor l_1) \land \varphi_3$$

$$\Downarrow$$

$$\varphi_1 \land (l_1 \lor l_2) \land \varphi_2 \land \varphi_3$$

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [7]

Repeat:

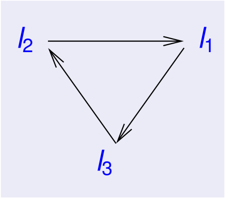
(i) build the implication graph induced by binary clauses

(ii) detect strongly connected cycles $\Longrightarrow$ equivalence classes of literals

(iii) perform substitutions

(iv) perform unit and pure literal.

Until (no more simplification is possible).



- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$
$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$
$$\Downarrow_{resolve}$$
$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$
$$\Downarrow_{unit-propagate}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- Branch is the source of non-determinism for DPLL
  $\Longrightarrow$ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

Some Heuristics

- MOMS heuristics (DPLL): pick the literal occurring most often in the minimal size clauses
  $\Longrightarrow$ fast and simple, many variants
- Jeroslow-Wang (DPLL): choose the literal with maximum

  $$score(l) := \Sigma_{l \in c \ \& \ c \in \varphi} \ 2^{-|c|}$$

  $\Longrightarrow$ estimates l's contribution to the satisfiability of φ
- Satz [21] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
  $\Longrightarrow$ maximizes the effects of unit propagation
- VSIDS [28] (CDCL+): variable state independent decaying sum
  - "static": scores updated only at the end of a branch
  - "local": privileges variable in recently learned clauses

Restarts [16]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- avoid getting stuck in certain areas of the search space

$\Longrightarrow$ may significantly reduce the overall search space

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   - Limitations of Chronological Backtracking
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
   - SAT Under Assumptions & Incremental SAT
5. SAT Functionalities: proofs, unsat cores, optimization

SAT under assumptions: $SAT(\varphi, \{l_1, ..., l_n\})$ [12]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, ..., l_n\}$: $SAT(\varphi, \{l_1, ..., l_n\})$
  - $SAT(\varphi, \{l_1, ..., l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^{n} l_i)$
  - often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2), ...$
- Idea:
  - $l_1, ..., l_n$ "decided" at decision level 0 before starting the search
  - if backjump to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT

Property

If the "decision" strategy for conflict analysis is used,
then η is the subset of assumptions causing the inconsistency

Selection of sub-formulas

Idea: select clauses [12, 23]

Let φ be $\bigwedge_{i=1}^{n} C_i$.

- let $S_1 ... S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, ..., S_{i_k}\} \subseteq \{S_1, ..., S_n\}$

$\Longrightarrow$ SAT($\bigwedge_{i=1}^{n}(\neg S_i \vee C_i), \mathcal{A}$): same as SAT($\bigwedge_{i=i_1}^{i_k}(C_i)$)

- if S_i is not assumed, then $\neg S_i \vee C_i$ does not contribute to search

$\Longrightarrow$ "Select" (activate) only a subset of the clauses in φ at each call.

Generalised Idea: select blocks of clauses

Let φ be $\bigwedge_{i=1}^{n}(\bigwedge_{j=1}^{n_i} C_{ij})$.

- let $S_1 ... S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, ..., S_{i_k}\} \subseteq \{S_1, ..., S_n\}$
- SAT($\bigwedge_{i=1}^{n}(\bigwedge_{j=1}^{n_i}(\neg S_i \vee C_{ij})), \mathcal{A}$): same as SAT($\bigwedge_{i=i_1}^{i_k}(\bigwedge_{j=1}^{n_i} C_{ij})$)

$\Longrightarrow$ Allows for "selecting" block of clauses at each call.

Example

- Initial formula φ:

  $(\ A_1 \lor \neg A_2 \lor \neg A_3\ ) \land$ // *group* 1
  $(\neg A_3 \lor A_2 \lor \neg A_5\ ) \land$ // *group* 1
  $(\neg A_2 \lor A_5 \lor A_7\ ) \land$ // *group* 2
  $(\ A_3 \lor A_5 \lor \neg A_8\ ) \land$ // *group* 2
  $(\neg A_1 \lor \neg A_3 \lor A_8\ ) \land$ // *group* 3

- Augmented formula φ':

  $(\neg S_1 \lor A_1 \lor \neg A_2 \lor \neg A_3\ ) \land$ // *group* 1, *inactive*
  $(\neg S_1 \lor \neg A_3 \lor A_2 \lor \neg A_5\ ) \land$ // *group* 1, *inactive*
  $(\neg S_2 \lor \neg A_2 \lor A_5 \lor A_7\ ) \land$ // *group* 2, *inactive*
  $(\neg S_2 \lor A_2 \lor A_5 \lor \neg A_8\ ) \land$ // *group* 2, *inactive*
  $(\neg S_3 \lor \neg A_1 \lor \neg A_3 \lor A_8\ ) \land$ // *group* 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ:

  $(\ A_1 \ \lor \neg A_2 \ \lor \neg A_3 \ )\land$ *// group 1*
  $(\neg A_3 \ \lor \ A_2 \ \lor \neg A_5 \ )\land$ *// group 1*
  $(\neg A_2 \ \lor \ A_5 \ \lor \ A_7 \ )\land$ *// group 2*
  $(\ A_3 \ \lor \ A_5 \ \lor \neg A_8 \ )\land$ *// group 2*
  $(\neg A_1 \ \lor \neg A_3 \ \lor \ A_8 \ )\land$ *// group 3*

- Augmented formula φ':

  $(\neg S_1 \ \lor \ A_1 \ \lor \neg A_2 \ \lor \neg A_3 \ )\land$ *// group 1, inactive*
  $(\neg S_1 \ \lor \neg A_3 \ \lor \ A_2 \ \lor \neg A_5 \ )\land$ *// group 1, inactive*
  $(\neg S_2 \ \lor \neg A_2 \ \lor \ A_5 \ \lor \ A_7 \ )\land$ *// group 2, inactive*
  $(\neg S_2 \ \lor \ A_2 \ \lor \ A_5 \ \lor \neg A_8 \ )\land$ *// group 2, inactive*
  $(\neg S_3 \ \lor \neg A_1 \ \lor \neg A_3 \ \lor \ A_8 \ )\land$ *// group 3*

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ:

$$\begin{array}{llllll}
(& A_1 & \lor\neg A_2 & \lor\neg A_3 &)\land & //\ \textit{group}\ 1 \\
(\neg & A_3 & \lor\ A_2 & \lor\neg A_5 &)\land & //\ \textit{group}\ 1 \\
(\neg & A_2 & \lor\ A_5 & \lor\ A_7 &)\land & //\ \textit{group}\ 2 \\
(& A_3 & \lor\ A_5 & \lor\neg A_8 &)\land & //\ \textit{group}\ 2 \\
(\neg & A_1 & \lor\neg A_3 & \lor\ A_8 &)\land & //\ \textit{group}\ 3
\end{array}$$

- Augmented formula φ':

$$\begin{array}{llllll}
(\neg S_1 & \lor\ A_1 & \lor\neg A_2 & \lor\neg A_3 &)\land & //\ \textit{group}\ 1, \textit{inactive} \\
(\neg S_1 & \lor\neg A_3 & \lor\ A_2 & \lor\neg A_5 &)\land & //\ \textit{group}\ 1, \textit{inactive} \\
(\neg S_2 & \lor\neg A_2 & \lor\ A_5 & \lor\ A_7 &)\land & //\ \textit{group}\ 2, \textit{inactive} \\
(\neg S_2 & \lor\ A_2 & \lor\ A_5 & \lor\neg A_8 &)\land & //\ \textit{group}\ 2, \textit{inactive} \\
(\neg S_3 & \lor\neg A_1 & \lor\neg A_3 & \lor\ A_8 &)\land & //\ \textit{group}\ 3
\end{array}$$

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ:

$$\begin{array}{llllll}
(& A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \textit{group}\ 1 \\
(\neg A_3 & \vee & A_2 & \vee\neg A_5 &)\wedge & //\ \textit{group}\ 1 \\
(\neg A_2 & \vee & A_5 & \vee & A_7 &)\wedge\ //\ \textit{group}\ 2 \\
(& A_3 & \vee & A_5 & \vee\neg A_8 &)\wedge\ //\ \textit{group}\ 2 \\
(\neg A_1 & \vee\neg A_3 & \vee & A_8 &)\wedge & //\ \textit{group}\ 3
\end{array}$$

- Augmented formula φ':

$$\begin{array}{lllll}
(\neg S_1 & \vee\ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge\ //\ \textit{group}\ 1, \textit{inactive} \\
(\neg S_1 & \vee\neg A_3 & \vee\ A_2 & \vee\neg A_5 &)\wedge\ //\ \textit{group}\ 1, \textit{inactive} \\
(\neg S_2 & \vee\neg A_2 & \vee\ A_5 & \vee\ A_7 &)\wedge\ //\ \textit{group}\ 2, \textit{inactive} \\
(\neg S_2 & \vee\ A_2 & \vee\ A_5 & \vee\neg A_8 &)\wedge\ //\ \textit{group}\ 2, \textit{inactive} \\
(\neg S_3 & \vee\neg A_1 & \vee\neg A_3 & \vee\ A_8 &)\wedge\ //\ \textit{group}\ 3
\end{array}$$

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Incremental SAT solving [12, 11]

- Many CDCL solvers provide a stack-based incremental interface
  - it is possible to push/pop ϕ_i into a stack of subformulas $\{\phi_1, ..., \phi_k\}$
  - check incrementally the satisfiability of $\varphi \overset{\text{def}}{=} \bigwedge_{i=1}^{k} \phi_i$.
- Maintains the status of the search from one call to the other
  - in particular it records the learned clauses (plus other information)

  $\Longrightarrow$ reuses search from one call to another
- Very useful in many applications (in particular in FV)

- Idea: incremental calls $SAT(\varphi', \mathcal{A}_1)$, $SAT(\varphi', \mathcal{A}_2)$,...
  - $\varphi' \overset{\text{def}}{=} \bigwedge_i (\neg S_i \vee \phi_i)$, $\mathcal{A}_j \subseteq \{S_1, ..., S_k\}$, $(\neg S_i \vee \bigwedge_j C_{ij}) \overset{\text{def}}{=} \bigwedge_j (\neg S_i \vee C_{ij})$
  - push/pop selection variables S_i
  - in practice, also subformulas ϕ_i can be pushed/popped
- Key efficiency issue: learned clauses safely reused from call to call (even if assumptions have been popped)
  - a learned clause $C \overset{\text{def}}{=} \bigvee_j \neg S_j \vee C'$ is s.t. $\bigwedge_j (\neg S_j \vee \phi_j) \models C$

  $\Longrightarrow$ C contains the vars selecting the subformulas it is derived from

  $\Longrightarrow$ in $SAT(\varphi', \mathcal{A})$, if some $S_j \notin \mathcal{A}$, then C is not active

Example

- Initial formula φ:

$$\begin{array}{lllll}
\dots & & & \wedge & \\
(\ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg A_3 & \vee\ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1
\end{array}$$

- Augmented formula φ':

$$\begin{array}{llllll}
\dots & & & & \wedge & \\
(\neg S_1 & \vee\ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee\ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1
\end{array}$$

[push(S_1)]: $SAT(\varphi', \{..., S_1\})$: ϕ_1 active $\Longrightarrow$ learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$\begin{array}{llllll}
\ldots & & & & \wedge & \\
(\ A_1 & \vee\neg A_2 & \vee\neg A_3 &) & \wedge & // \ \phi_1 \\
(\neg A_3 & \vee \ A_2 & \vee\neg A_5 &) & \wedge & // \ \phi_1
\end{array}$$

- Augmented formula φ':

$$\begin{array}{lllllll}
\ldots & & & & & \wedge & \\
(\neg S_1 & \vee \ A_1 & \vee\neg A_2 & \vee\neg A_3 &) & \wedge & // \ \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee \ A_2 & \vee\neg A_5 &) & \wedge & // \ \phi_1 \\
 & & & & & & \\
(\neg S_1 & \vee \ A_1 & \vee\neg A_3 & \vee\neg A_5 &) & \wedge & // \ \textit{learned } C_1
\end{array}$$

[push(S_1)]: $SAT(\varphi', \{..., S_1\})$: ϕ_1 active $\Longrightarrow$ learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$\begin{array}{lllll}
\ldots & & & \wedge & \\
(\ \ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg A_3 & \vee\ \ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1 \\
(\neg A_2 & \vee\ \ A_5 & \vee\ \ A_7 &)\wedge & //\ \phi_2 \\
(\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \phi_2
\end{array}$$

- Augmented formula φ':

$$\begin{array}{llllll}
\ldots & & & & \wedge & \\
(\neg S_1 & \vee\ \ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee\ \ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1 \\
(\neg S_2 & \vee\neg A_2 & \vee\ \ A_5 & \vee\ \ A_7 &)\wedge & //\ \phi_2\text{, } \textit{inactive} \\
(\neg S_2 & \vee\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \phi_2\text{, } \textit{inactive} \\
 & & & & & \\
(\neg S_1 & \vee\ \ A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \textit{learned } C_1
\end{array}$$

[push(S_2)]: $SAT(\varphi', \{..., S_1, S_2\})$: ϕ_1, ϕ_2 active $\Longrightarrow$ learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$\begin{array}{lllll}
\ldots & & & \wedge & \\
(\ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg A_3 & \vee\ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1 \\
(\neg A_2 & \vee\ A_5 & \vee\ A_7 &)\wedge & //\ \phi_2 \\
(\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \phi_2
\end{array}$$

- Augmented formula φ':

$$\begin{array}{llllll}
\ldots & & & & \wedge & \\
(\neg S_1 & \vee\ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & //\ \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee\ A_2 & \vee\neg A_5 &)\wedge & //\ \phi_1 \\
(\neg S_2 & \vee\neg A_2 & \vee\ A_5 & \vee\ A_7 &)\wedge & //\ \phi_2\text{, inactive} \\
(\neg S_2 & \vee\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \phi_2\text{, inactive} \\
 & & & & & \\
(\neg S_1 & \vee\ A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \mathit{learned}\ C_1 \\
(\neg S_1 & \vee\neg S_2 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & //\ \mathit{learned}\ C_2\text{, inactive}
\end{array}$$

[push(S_2)]: $SAT(\varphi', \{\ldots, S_1, S_2\})$: ϕ_1, ϕ_2 active $\Longrightarrow$ learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$\begin{array}{llllll}
\ldots & & & & \wedge & \\
(\; A_1 & \vee\neg A_2 & \vee\neg A_3 &) & \wedge & // \; \phi_1 \\
(\neg A_3 & \vee \; A_2 & \vee\neg A_5 &) & \wedge & // \; \phi_1 \\
 & & & & & \\
(\neg A_1 & \vee\neg A_3 & \vee \; A_8 &) & \wedge & // \; \phi_3
\end{array}$$

- Augmented formula φ':

$$\begin{array}{lllllll}
\ldots & & & & & \wedge & \\
(\neg S_1 & \vee \; A_1 & \vee\neg A_2 & \vee\neg A_3 &) & \wedge & // \; \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee \; A_2 & \vee\neg A_5 &) & \wedge & // \; \phi_1 \\
(\neg S_2 & \vee\neg A_2 & \vee \; A_5 & \vee \; A_7 &) & \wedge & // \; \phi_2, \textit{inactive} \\
(\neg S_2 & \vee\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &) & \wedge & // \; \phi_2, \textit{inactive} \\
(\neg S_3 & \vee\neg A_1 & \vee\neg A_3 & \vee \; A_8 &) & \wedge & // \; \phi_3 \\
(\neg S_1 & \vee \; A_1 & \vee\neg A_3 & \vee\neg A_5 &) & \wedge & // \; \textit{learned } C_1 \\
(\neg S_1 & \vee\neg S_2 & \vee\neg A_3 & \vee\neg A_5 &) & \wedge & // \; \textit{learned } C_2, \textit{inactive}
\end{array}$$

[pop(S_2);push(S_3)]: $SAT(\varphi', \{..., S_1, S_3\})$: ϕ_1, ϕ_3 active $\Longrightarrow$...

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$\begin{array}{lllll}
\ldots & & & \wedge & \\
(\ \ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & // \ \phi_1 \\
(\neg A_3 & \vee \ \ A_2 & \vee\neg A_5 &)\wedge & // \ \phi_1 \\
 & & & & \\
 & & & & \\
(\neg A_1 & \vee\neg A_3 & \vee \ \ A_8 &)\wedge & // \ \phi_3
\end{array}$$

- Augmented formula φ':

$$\begin{array}{llllll}
\ldots & & & & \wedge & \\
(\neg S_1 & \vee \ \ A_1 & \vee\neg A_2 & \vee\neg A_3 &)\wedge & // \ \phi_1 \\
(\neg S_1 & \vee\neg A_3 & \vee \ \ A_2 & \vee\neg A_5 &)\wedge & // \ \phi_1 \\
(\neg S_2 & \vee\neg A_2 & \vee \ \ A_5 & \vee \ \ A_7 &)\wedge & // \ \phi_2, \mathit{inactive} \\
(\neg S_2 & \vee\neg A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & // \ \phi_2, \mathit{inactive} \\
(\neg S_3 & \vee\neg A_1 & \vee\neg A_3 & \vee \ \ A_8 &)\wedge & // \ \phi_3 \\
(\neg S_1 & \vee \ \ A_1 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & // \ \mathit{learned}\ C_1 \\
(\neg S_1 & \vee\neg S_2 & \vee\neg A_3 & \vee\neg A_5 &)\wedge & // \ \mathit{learned}\ C_2, \mathit{inactive}
\end{array}$$

[pop(S_2);push(S_3)]: $SAT(\varphi', \{..., S_1, S_3\})$: ϕ_1, ϕ_3 active $\Longrightarrow$...

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Outline

1. Boolean Logics and SAT
2. Basic SAT-Solving Techniques
   - Generalities
   - Resolution
   - Tableaux
   - DPLL
3. Ordered Binary Decision Diagrams – OBDDs
4. Modern CDCL SAT Solvers
   - Limitations of Chronological Backtracking
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Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

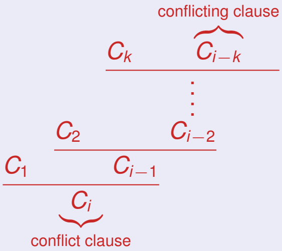
- Building proofs of unsatisfiability
- Extracting unsatisfiable Cores
- Enumeration in SAT: AllSAT (hints)
- Optimization in SAT: MaxSAT (hints)

Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
  - to verify the result of the solver
  - to understand a “reason” for unsatisfiability
  - to build unsatisfiable cores and interpolants
- Can be built by keeping track of the resolution steps performed when constructing the conflict clauses.

Building Proofs of Unsatisfiability

- Recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal



conflicting clause

C_k C_{i-k}

C_2 C_{i-2}

C_1 C_{i-1}

C_i

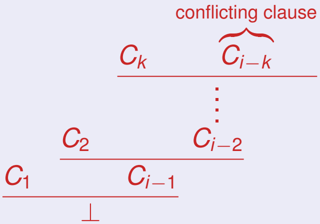
conflict clause

- $C_1, \ldots, C_k$, and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

```
k i-k -> i-k-1
...
2 i-2 -> i-1
1 i-1 -> i
```

Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces “false” as conflict clause:



$$\overbrace{C_{i-k}}^{\text{conflicting clause}}$$

C_k

$\vdots$

C_2 C_{i-2}

C_1 C_{i-1}

$\bot$

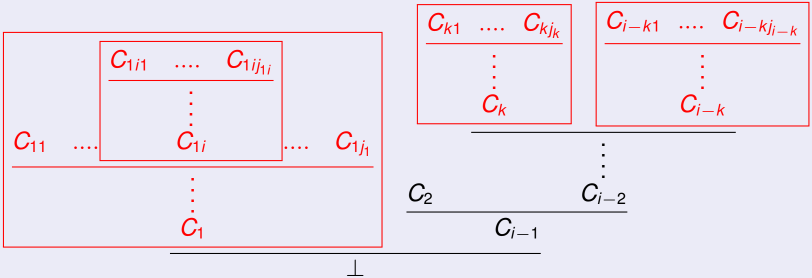
- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- $C_1, ..., C_k$, and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

- for every learned leaf clause C_i, substitute C_i with the resolution proof generating it

until all leaf clauses are original clauses

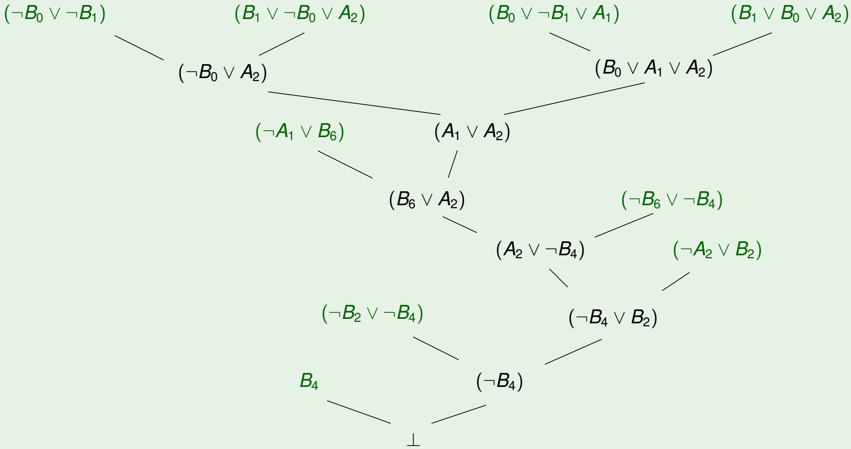


C_{1i1} .... $C_{1ij_{1i}}$
C_{11} .... C_{1i} .... C_{1j_1}
C_1
C_{k1} .... C_{kj_k}
C_k
C_{i-k1} .... $C_{i-kj_{i-k}}$
C_{i-k}
C_2
C_{i-2}
C_{i-1}
$\bot$

$\Longrightarrow$ We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$



$(\neg B_0 \vee \neg B_1)$ $(B_1 \vee \neg B_0 \vee A_2)$ $(B_0 \vee \neg B_1 \vee A_1)$ $(B_1 \vee B_0 \vee A_2)$

$(\neg B_0 \vee A_2)$ $(B_0 \vee A_1 \vee A_2)$

$(\neg A_1 \vee B_6)$ $(A_1 \vee A_2)$

$(B_6 \vee A_2)$ $(\neg B_6 \vee \neg B_4)$

$(A_2 \vee \neg B_4)$ $(\neg A_2 \vee B_2)$

$(\neg B_2 \vee \neg B_4)$ $(\neg B_4 \vee B_2)$

B_4 $(\neg B_4)$

$\perp$

Extraction of unsatisfiable cores

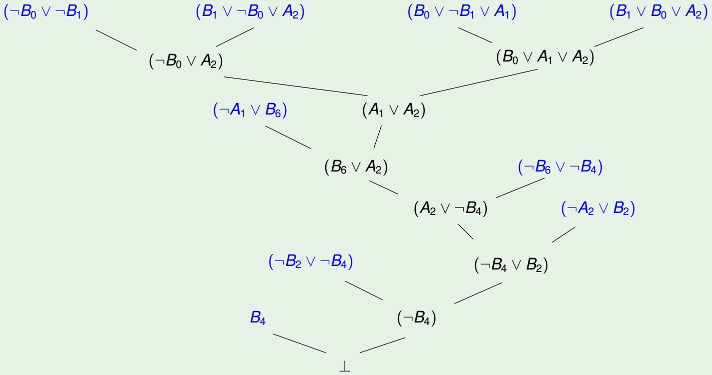
- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset

  $\Longrightarrow$ unsatisfiable cores (aka (Minimal) Unsatisfiable Subsets, (M)US)
- Lots of literature on the topic [46, 24, 26, 31, 43, 19, 13, 6]
- We recognize two main approaches:
  - Proof-based approach [46]: byproduct of finding a resolution proof
  - Assumption-based approach [24]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
  - repeat the process up to fixpoit
  - remove clauses one-by one, until satisfiability is obtained
  - combinations of the two processed above
  - ...

The proof-based approach to core extraction [46]

Unsat core: the set of leaf clauses of a resolution proof

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$



$(\neg B_0 \vee \neg B_1)$ $(B_1 \vee \neg B_0 \vee A_2)$ $(B_0 \vee \neg B_1 \vee A_1)$ $(B_1 \vee B_0 \vee A_2)$

$(\neg B_0 \vee A_2)$ $(B_0 \vee A_1 \vee A_2)$

$(\neg A_1 \vee B_6)$ $(A_1 \vee A_2)$

$(B_6 \vee A_2)$ $(\neg B_6 \vee \neg B_4)$

$(A_2 \vee \neg B_4)$ $(\neg A_2 \vee B_2)$

$(\neg B_2 \vee \neg B_4)$ $(\neg B_4 \vee B_2)$

B_4 $(\neg B_4)$

$\perp$

The assumption-based approach to core extraction [24]

Based on the following process:

(i) each clause C_i is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh “selector” variable

(ii) before starting the search each S_i is forced to true.

(iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$

$\Longrightarrow$ $\{C_j\}_j$ is the unsat core!

The assumption-based approach to core extraction

Example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge$
$B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

(i) add selector variables:
$(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge$
$(\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge$
$(\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7)$

(ii) The conflict analysis returns: $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$,

(iii) corresponding to the unsat core:

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge$
$B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$

All-SAT (hints)

All-SAT & Projected All-SAT

- All-SAT: enumerate all truth assignments satisfying φ
  - Ex: $\varphi \stackrel{\text{def}}{=} (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2)$
    $\Longrightarrow AllSAT(\varphi) = \{\{A_1, A_2\}, \{\neg A_1, \neg A_2\}\}$
- Projected All-SAT: given an “important” subset $\mathbf{A} \stackrel{\text{def}}{=} \{A_i\}_i$ of $Atoms(\varphi) \stackrel{\text{def}}{=} \mathbf{A} \cup \mathbf{B}$, enumerate all assignments over $\mathbf{A}$ which can be extended to truth assignments on $\mathbf{B}$ satisfying φ
  - Equivalent to compute $AllSAT(\exists \mathbf{B}.\varphi)$
  - Ex: $\varphi \stackrel{\text{def}}{=} B_1 \wedge B_2 \wedge (B_1 \leftrightarrow (A_1 \vee \neg A_2)) \wedge (B_2 \leftrightarrow (\neg A_1 \vee A_2))$
    $\Longrightarrow AllSAT(\exists \mathbf{B}.\varphi) = \{\{A_1, A_2\}, \{\neg A_1, \neg A_2\}\}$

Model Counting, aka #SAT (hints)

Model Counting & Projected Model Counting

- #SAT: count all truth assignments on $\mathbf{A} \supseteq \mathit{Atoms}(\varphi)$ satisfying φ
  - Ex: $\varphi \stackrel{\text{def}}{=} (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2)$
    $\Longrightarrow \#SAT(\varphi) = 2$
- Projected #SAT: given an "important" subset $\mathbf{A} \stackrel{\text{def}}{=} \{A_i\}_i$ of $\mathbf{A} \cup \mathbf{B} \supseteq \mathit{Atoms}(\varphi)$, count all assignments over $\mathbf{A}$ which can be extended to truth assignments on $\mathbf{B}$ satisfying φ
  - Equivalent to compute $\#SAT(\exists \mathbf{B}.\varphi)$
  - Ex: $\varphi \stackrel{\text{def}}{=} B_1 \wedge B_2 \wedge (B_1 \leftrightarrow (A_1 \vee \neg A_2)) \wedge (B_2 \leftrightarrow (\neg A_1 \vee A_2))$
    $\Longrightarrow \#SAT(\exists \mathbf{B}.\varphi) = 2$

MaxSAT (hints)

- MaxSAT: given a pair of CNF formulas $\langle\varphi_h,\varphi_s\rangle$ s.t. $\varphi_h \wedge \varphi_s \models \bot$, $\varphi_s \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s.
- Weighted MaxSAT: given also the positive integer penalties $\{w_1, ..., w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to optimization

  $\Longrightarrow$ much harder than SAT
- Many different approaches (see e.g. [22])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \qquad \varphi_s \stackrel{\text{def}}{=} \left(\begin{array}{ccc} (\ \ A_1 \vee \neg A_2) & \wedge & [4] \\ (\neg A_1 \vee \ \ A_2) & \wedge & [3] \\ (\neg A_1 \vee \neg A_2) & \wedge & [2] \end{array}\right)$$

$\Longrightarrow \mu = \{A_1, A_2\}$ (penalty = 2)

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