

Automated Reasoning and Formal Verification

Module II: Formal Verification

Ch. 07: **SAT-Based Model Checking**

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Outline

- 1 SAT-based Model Checking: Generalities
- 2 Bounded Model Checking
 - Intuitions
 - General Encoding
 - Relevant Subcases
 - An Example
 - Computing Upper Bounds
 - Discussion
- 3 Inductive reasoning on invariants (aka “K-Induction”)
 - K-Induction
 - An Example
- 4 Exercises

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SAT-based Model Checking

- Key problems with BDD's:
 - they can explode in space
- A possible alternative:
 - Propositional Satisfiability Checking (SAT)
 - SAT technology is very advanced
- Advantages:
 - reduced memory requirements
 - limited sensitivity: one good setting, does not require expert users
 - much higher capacity (more variables) than BDD based techniques
- Various techniques:
 - **Bounded Model Checking (BMC)** \implies this chapter
 - **K-induction** \implies this chapter
 - **Counter-example guided abstraction refinement (CEGAR)** \implies next chapter
 - Interpolant-based \implies not presented in this course
 - IC3/PDR \implies not presented in this course
 - ...

SAT-based Bounded Model Checking & K-Induction

Key Ideas:

- **BMC**: look for counter-example paths of increasing length k
⇒ oriented to finding bugs
- **K-Induction**: look for an induction proofs of increasing length k
⇒ oriented to prove correctness
- **BMC [resp. K-induction]**: for each k , build a Boolean formula that is satisfiable [resp. unsatisfiable] iff there is a counter-example [resp. proof] of length k
 - can be expressed using $k \cdot |s|$ variables
 - formula construction is not subject to state explosion
- Satisfiability of the Boolean formulas is checked by a **SAT solver**
 - can manage complex formulae on up to 10^7 Boolean variables (!)
 - returns satisfying assignment (i.e., a counter-example)
 - exploit incrementality

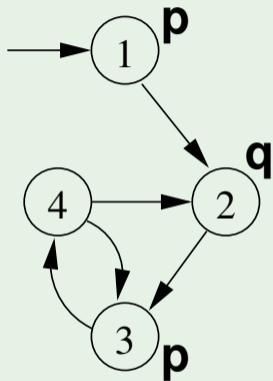
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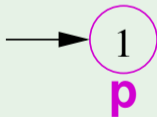
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Bounded Model Checking: Example

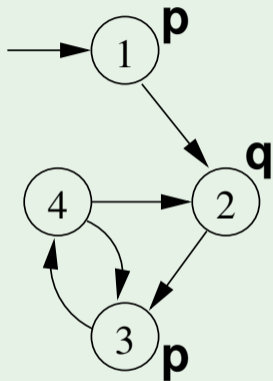


- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 0$:

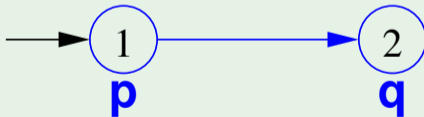


- No counter-example found.

Bounded Model Checking: Example

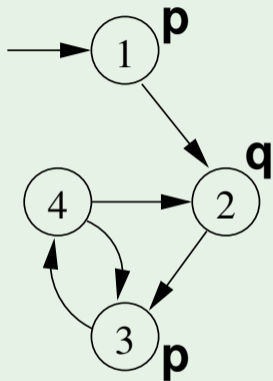


- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 1$:

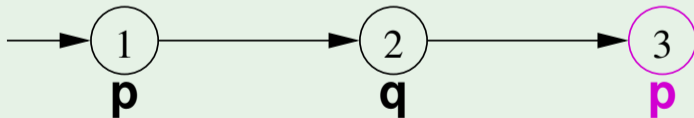


- No counter-example found.

Bounded Model Checking: Example

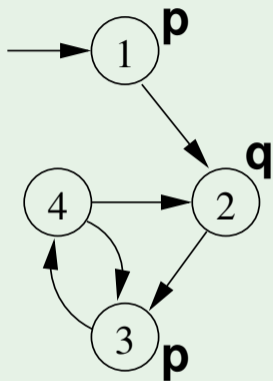


- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 2$:

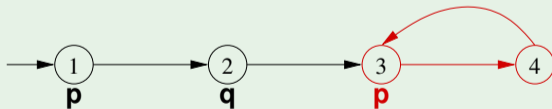
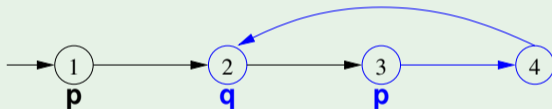


- No counter-example found.

Bounded Model Checking: Example



- LTL Formula: $\mathbf{G}(p \rightarrow \mathbf{F}q)$
- Negated Formula (violation): $\mathbf{F}(p \wedge \mathbf{G}\neg q)$
- $k = 3$:



- The 2nd trace is a counter-example!

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The problem [Biere et al, 1999]

Ingredients:

Assume states represented by an array s of n Boolean variables

- a **system** written as a Kripke structure $M := \langle I(s), R(s, s') \rangle$
- a **property** f written as a **LTL formula**
- an integer $k \geq 0$ (**bound**)

Problem

Is there an execution path π of M of length k satisfying the temporal property f ?

$$M \models_k \mathbf{E}f$$

Note: f is the negation of the property in the LTL model checking problem $M \models \neg f$, and π is a counter-example of length k (bug).

- The check is repeated for increasing values of $k = 0, 1, 2, 3, \dots$

The general encoding

Equivalent to the satisfiability problem of a Boolean formula $[[M, f]]_k$ defined as follows:

$$[[M, f]]_k := [[M]]_k \wedge [[f]]_k$$

$$[[M]]_k := I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}),$$

$$[[f]]_k := (\neg \bigvee_{l=0}^k R(s^k, s^l) \wedge [[f]]_k^0) \vee \bigvee_{l=0}^k (R(s^k, s^l) \wedge I[[f]]_k^l),$$

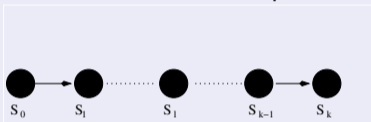
- The vector s of propositional variables is replicated $k+1$ times
 s^0, s^1, \dots, s^k
- $[[M]]_k$ encodes the fact that the k -path is an execution of M
- $[[f]]_k$ encodes the fact that the k -path satisfies f

The general encoding [cont.]

The encoding for a formula f with k steps, $[[f]]_k$ is the disjunction of:

- The constraints needed to express a model without loopback:

$$(\neg(\bigvee_{i=0}^k R(s^k, s^i)) \wedge [[f]]_k^0)$$

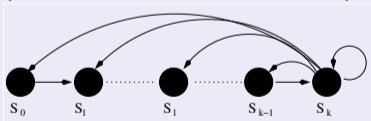


- $[[f]]_k^i, i \in [0, k]$:

“ f holds in s^i under the assumption that s^0, \dots, s^k is a no-loopback path”

- The constraints needed to express a model with some loopback:

$$\bigvee_{i=0}^k (R(s^k, s^i) \wedge {}_i[[f]]_k^0)$$



- ${}_i[[f]]_k^i, i \in [0, k]$:

“ f holds in s^i under the assumption that s^0, \dots, s^k is a path with a loopback from s^k to s^i ”

The Encoding of $[[f]]_k^i$ and ${}_i[[f]]_k^i$

f	$[[f]]_k^i$	${}_i[[f]]_k^i$
p	p_i	p_i
$\neg p$	$\neg p_i$	$\neg p_i$
$h \wedge g$	$[[h]]_k^i \wedge [[g]]_k^i$	${}_i[[h]]_k^i \wedge {}_i[[g]]_k^i$
$h \vee g$	$[[h]]_k^i \vee [[g]]_k^i$	${}_i[[h]]_k^i \vee {}_i[[g]]_k^i$
Xg	$[[g]]_k^{i+1}$ if $i < k$ \perp otherwise.	${}_i[[g]]_k^{i+1}$ if $i < k$ ${}_i[[g]]_k^i$ otherwise.
Gg	\perp	$\bigwedge_{j=\min(i,l)}^k {}_i[[g]]_k^j$
Fg	$\bigvee_{j=i}^k [[g]]_k^j$	$\bigvee_{j=\min(i,l)}^k {}_i[[g]]_k^j$
hUg	$\bigvee_{j=i}^k \left([[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} [[h]]_k^n \right)$	$\bigvee_{j=i}^k \left({}_i[[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_i[[h]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_i[[g]]_k^j \wedge \bigwedge_{n=i}^k {}_i[[h]]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_i[[h]]_k^n \right)$
hRg	$\bigvee_{j=i}^k \left([[h]]_k^j \wedge \bigwedge_{n=i}^j [[g]]_k^n \right)$	$\bigwedge_{j=\min(i,l)}^k {}_i[[g]]_k^j \vee$ $\bigvee_{j=i}^k \left({}_i[[h]]_k^j \wedge \bigwedge_{n=i}^j {}_i[[g]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_i[[h]]_k^j \wedge \bigwedge_{n=i}^k {}_i[[g]]_k^n \wedge \bigwedge_{n=l}^j {}_i[[g]]_k^n \right)$

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Relevant Subcase: $\mathbf{F}p$ (reachability)

- $f := \mathbf{F}p$, s.t. p Boolean:
is there a reachable state in which p holds?
- a finite path can show that the property holds
- $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{j=0}^k p^j$$



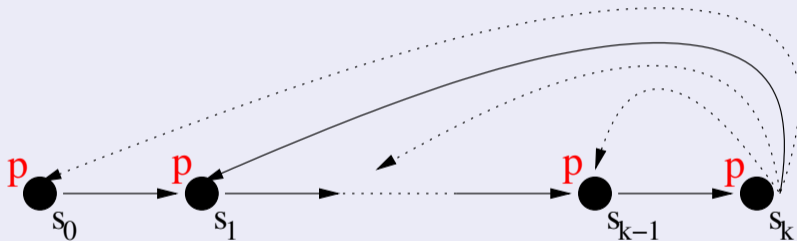
Important: incremental encoding

if done for increasing value of k , then it suffices that $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} (R(s^i, s^{i+1}) \wedge \neg p^i) \wedge p^k$$

Relevant Subcase: Gp

- $f := Gp$, s.t. p Boolean: is there a path where p holds forever?
- We need to produce an infinite behaviour, with a finite number of transitions
- We can do it by imposing that the path loops back

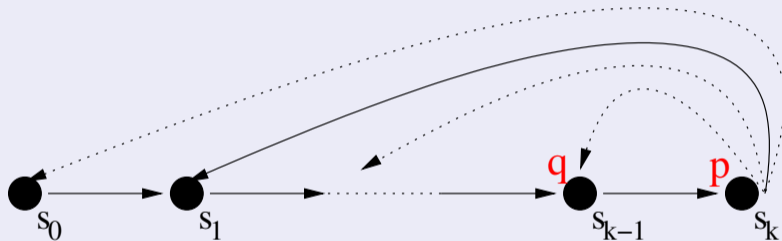


- $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{l=0}^k R(s^k, s^l) \wedge \bigwedge_{j=0}^k p^j$$

Relevant Subcase: $\mathbf{GF}q$ (fair states)

- $f := \mathbf{GF}q$, s.t. q Boolean: **does q hold infinitely often?**
- Again, we need to produce an infinite behaviour, with a finite number of transitions

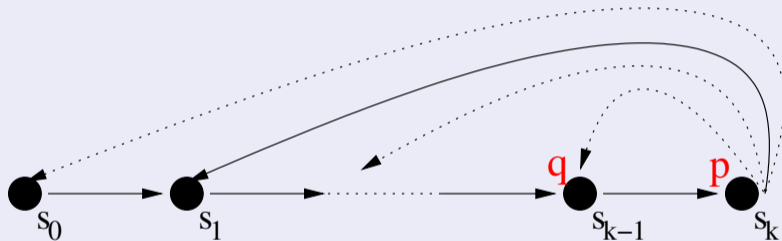


- $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{l=0}^k \left(R(s^k, s^l) \wedge \bigvee_{j=l}^k q^j \right)$$

Subcase Combination: $\mathbf{GF}q \wedge \mathbf{F}p$ (fair reachability)

- $f := \mathbf{GF}q \wedge \mathbf{F}p$, s.t. p, q Boolean: provided that q holds infinitely often, is there a reachable state in which p holds?
- Again, we need to produce an infinite behaviour, with a finite number of transitions



- $[[M, f]]_k$ is:

$$I(s^0) \wedge \bigwedge_{i=0}^{k-1} R(s^i, s^{i+1}) \wedge \bigvee_{j=0}^k p_j \wedge \bigvee_{l=0}^k \left(R(s^k, s^l) \wedge \bigvee_{j=l}^k q^j \right)$$

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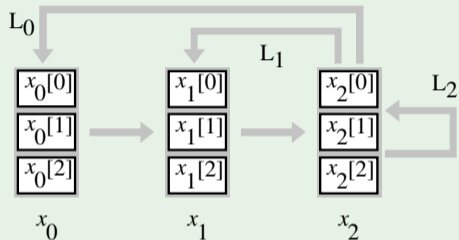
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Example: a bugged 3-bit shift register

- System M :
 - $I(x) := \neg x[0] \wedge \neg x[1] \wedge x[2]$
 - Correct R : $R(x, x') := (x'[0] \leftrightarrow x[1]) \wedge (x'[1] \leftrightarrow x[2]) \wedge (x'[2] \leftrightarrow 0)$
 - Bugged R : $R(x, x') := (x'[0] \leftrightarrow x[1]) \wedge (x'[1] \leftrightarrow x[2]) \wedge (x'[2] \leftrightarrow 1)$
- Property: $\mathbf{F}(\neg x[0] \wedge \neg x[1] \wedge \neg x[2])$
- BMC Problem: is there an execution π of \mathcal{M} of length k s.t. $\pi \models \mathbf{G}((x[0] \vee x[1] \vee x[2]))?$

Example: a bugged 3-bit shift register [cont.]

$k = 0$:



$$\begin{aligned} I : & \quad (\neg x_0[0] \wedge \neg x_0[1] \wedge x_0[2]) \wedge \\ \bigvee_{l=0}^0 L_l : & \quad (((x_0[0] \leftrightarrow x_0[1]) \wedge (x_0[1] \leftrightarrow x_0[2]) \wedge (x_0[2] \leftrightarrow 1))) \wedge \\ \bigwedge_{i=0}^0 (x \neq 0) : & \quad ((x_0[0] \vee x_0[1] \vee x_0[2])) \end{aligned}$$

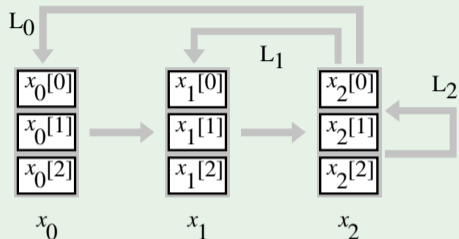
\Rightarrow UNSAT: unit propagation:

$\neg x_0[0], \neg x_0[1], x_0[2]$

\Rightarrow loop violated

Example: a bugged 3-bit shift register [cont.]

$k = 1$:



$$\begin{aligned}
 I: & \quad (\neg x_0[0] \wedge \neg x_0[1] \wedge x_0[2]) \wedge \\
 [[M]]_1: & \quad \left((x_1[0] \leftrightarrow x_0[1]) \wedge (x_1[1] \leftrightarrow x_0[2]) \wedge (x_1[2] \leftrightarrow 1) \right) \wedge \\
 \bigvee_{l=0}^1 L_l: & \quad \left(\begin{aligned} & ((x_0[0] \leftrightarrow x_1[1]) \wedge (x_0[1] \leftrightarrow x_1[2]) \wedge (x_0[2] \leftrightarrow 1)) \vee \\ & ((x_1[0] \leftrightarrow x_1[1]) \wedge (x_1[1] \leftrightarrow x_1[2]) \wedge (x_1[2] \leftrightarrow 1)) \end{aligned} \right) \wedge \\
 \bigwedge_{i=0}^1 (x \neq 0): & \quad \left(\begin{aligned} & (x_0[0] \vee x_0[1] \vee x_0[2]) \wedge \\ & (x_1[0] \vee x_1[1] \vee x_1[2]) \end{aligned} \right)
 \end{aligned}$$

\Rightarrow UNSAT: unit propagation:

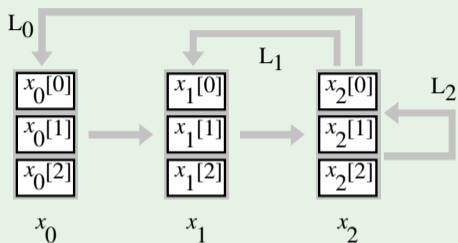
$\neg x_0[0], \neg x_0[1], x_0[2]$

$\neg x_1[0], x_1[1], x_1[2]$

\Rightarrow both loop disjuncts violated

Example: a bugged 3-bit shift register [cont.]

$k = 2$:



$$\begin{aligned}
 I : & \quad (\neg x_0[0] \wedge \neg x_0[1] \wedge x_0[2]) \wedge \\
 [[M]]_2 : & \quad \left(\begin{array}{l} (x_1[0] \leftrightarrow x_0[1]) \wedge (x_1[1] \leftrightarrow x_0[2]) \wedge (x_1[2] \leftrightarrow 1) \wedge \\ (x_2[0] \leftrightarrow x_1[1]) \wedge (x_2[1] \leftrightarrow x_1[2]) \wedge (x_2[2] \leftrightarrow 1) \end{array} \right) \wedge \\
 \bigvee_{i=0}^2 L_i : & \quad \left(\begin{array}{l} ((x_0[0] \leftrightarrow x_2[1]) \wedge (x_0[1] \leftrightarrow x_2[2]) \wedge (x_0[2] \leftrightarrow 1)) \vee \\ ((x_1[0] \leftrightarrow x_2[1]) \wedge (x_1[1] \leftrightarrow x_2[2]) \wedge (x_1[2] \leftrightarrow 1)) \vee \\ ((x_2[0] \leftrightarrow x_2[1]) \wedge (x_2[1] \leftrightarrow x_2[2]) \wedge (x_2[2] \leftrightarrow 1)) \end{array} \right) \wedge \\
 \bigwedge_{i=0}^2 (x \neq 0) : & \quad \left(\begin{array}{l} (x_0[0] \vee x_0[1] \vee x_0[2]) \wedge \\ (x_1[0] \vee x_1[1] \vee x_1[2]) \wedge \\ (x_2[0] \vee x_2[1] \vee x_2[2]) \end{array} \right)
 \end{aligned}$$

\implies SAT: $x_0[0] = x_0[1] = x_1[0] = 0$; $x_i[j] := 1 \forall i, j$

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Basic bounds for k

Theorem [Biere et al. TACAS 1999]

Let f be a LTL formula.

Then $M \models \mathbf{E}f \iff M \models_k \mathbf{E}f$ for some $k \leq |M| \cdot 2^{|f|}$.

- $|M| \cdot 2^{|f|}$ is always a bound of k .
 - $|M|$ huge!
 - \implies not so easy to compute in a symbolic setting.

\implies need to find better bounds!

Note: [Biere et al. TACAS 1999] use " $M \models \mathbf{E}f$ " as "there exists a path of M verifying f ", so that $M \not\models \neg f \iff M \models \mathbf{E}f$

Other bounds for k

ACTL & ECTL

- **ACTL** is a subset of CTL in which “**A...**” (resp. “**E...**”) sub-formulas occur only positively (resp. negatively) in each formula. (e.g. $\mathbf{AG}(p \rightarrow \mathbf{AGAF}q)$)
- Many frequently-used LTL properties $\neg f$ have equivalent ACTL representations $\mathbf{A}\neg f'$
 - e.g. $\mathbf{X}q \iff \mathbf{AX}q$, $\mathbf{G}q \iff \mathbf{AG}q$, $\mathbf{F}q \iff \mathbf{AF}q$, $p\mathbf{U}q \iff \mathbf{A}(p\mathbf{U}q)$,
 $\mathbf{GF}q \iff \mathbf{AGAF}q$, $\mathbf{G}(p \rightarrow \mathbf{GF}q) \iff \mathbf{AG}(p \rightarrow \mathbf{AGAF}q)$
 - ... but not all of them (e.g., $\mathbf{FG} \not\iff \mathbf{AFAG}p$)
- **ECTL** is a subset of CTL in which “**E...**” (resp. “**A...**”) sub-formulas occur only positively (resp. negatively) in each formula. (e.g. $\mathbf{EF}(p \wedge \mathbf{EFEG}\neg q)$)
- ECTL is the dual subset of ACTL: $\phi \in \mathbf{ECTL} \iff \neg\phi \in \mathbf{ACTL}$.

Theorem [Biere et al. TACAS 1999]

Let f be an ECTL formula.

Then $M \models \mathbf{E}f \iff M \models_k \mathbf{E}f$ for some $k \leq |M|$.

Other bounds for k (cont)

Theorem [Biere et al. TACAS 1999]

Let p be a Boolean formula and d be the **diameter** of M .

Then $M \models \mathbf{EF}p \iff M \models_k \mathbf{EF}p$ for some $k \leq d$.

Theorem [Biere et al. TACAS 1999]

Let f be an ECTL formula and d be the **recurrence diameter** of M .

Then $M \models \mathbf{Ef} \iff M \models_k \mathbf{Ef}$ for some $k \leq d$.

The Diameter: Computation

Definition: diameter

- d is the smallest integer d which makes the following formula true:

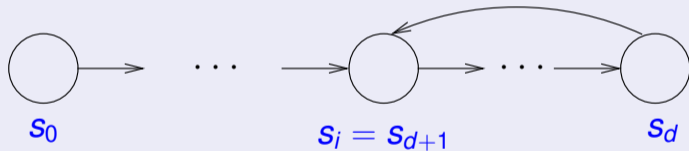
$$\forall s_0, \dots, s_{d+1}. \exists t_0, \dots, t_d. \underbrace{\bigwedge_{i=0}^d T(s_i, s_{i+1})}_{s_0, \dots, s_{d+1} \text{ is a path}} \rightarrow \left(\underbrace{t_0 = s_0 \wedge \bigwedge_{i=0}^{d-1} T(t_i, t_{i+1}) \wedge \bigvee_{i=0}^d t_i = s_{d+1}}_{t_0, \dots, t_i \text{ is another path from } s_0 \text{ to } s_{d+1} \text{ for some } i} \right)$$

- Quantified Boolean formula (QBF): much harder than NP-complete!

The recurrence diameter

Definition: recurrence diameter

Given M , the **recurrence diameter** of M is the smallest integer d s.t. for every path s_0, \dots, s_{d+1} there exist $j \leq d$ s.t. $s_{d+1} = s_j$.



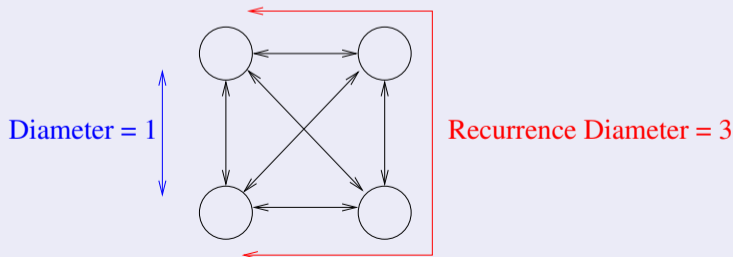
- Intuition: **the maximum length of a non-loop path**

The recurrence diameter: computation

- d is the smallest integer d which makes the following formula true:

$$\forall s_0, \dots, s_{d+1}. \underbrace{\bigwedge_{i=0}^d T(s_i, s_{i+1})}_{s_0, \dots, s_{d+1} \text{ is a path}} \rightarrow \underbrace{\bigvee_{i=0}^d s_i = s_{d+1}}_{s_0, \dots, s_{d+1} \text{ contains a cycle}}$$

- Validity problem: coNP-complete (solvable by SAT).
- Possibly much longer than the diameter!



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 - General Encoding
 - Relevant Subcases
 - An Example
 - Computing Upper Bounds
 - **Discussion**
- 3 Inductive reasoning on invariants (aka “K-Induction”)
 - K-Induction
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- 4 Exercises

Bounded Model Checking: summary

- **Incomplete technique:**
 - if you find all formulas unsatisfiable, it tells you nothing
 - computing the maximum k (diameter) possible but extremely hard
- **Very efficient** for some problems (typically debugging)
- Lots of enhancements
- Current symbolic model checkers embed a SAT based BMC tool

Efficiency Issues in Bounded Model Checking

- Incrementality:
 - exploit the similarities between problems at k and $k + 1$
- Simplification of encodings
 - Reduced Boolean Circuits (RBC)
 - Boolean Expression Diagrams (BED)
 - And-Inverter Graphs (AIG)
 - Simplification based on Binary-Clauses Reasoning
- Computing bounds not very effective
 - ⇒ feasible only on very particular subcases

Other Successful SAT-based MC Techniques

- Inductive reasoning on invariants (aka “K-Induction”)
- Counter-example guided abstraction refinement (CEGAR)
[Clarke et al. CAV 2002]
- Interpolant-based MC
[Mc Millan, TACAS 2005]
- IC3/PDR
[Bradley, VMCAI 2011]
- ...

For a survey see e.g.

[Amla et al., CHARME 2005, Prasad et al. STTT 2005].

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Inductive Reasoning on Invariants

Invariant: “**G***Good*”, *Good* being a Boolean formula

- (i) If all the initial states are good,
 - (ii) and if from good states we only go to good states
- then the system is correct for all reachable states

SAT-based Inductive Reasoning on Invariants

- (i) If all the initial states are good
 - $I(s^0) \rightarrow Good(s^0)$ is valid (i.e. its negation is unsatisfiable)
- (ii) if from good states we only go to good states
 - $(Good(s^{k-1}) \wedge R(s^{k-1}, s^k)) \rightarrow Good(s^k)$ is valid (i.e. its negation is unsatisfiable)

then the **system is correct for all reachable states**

⇒ Check for the (un)satisfiability of the Boolean formulas:

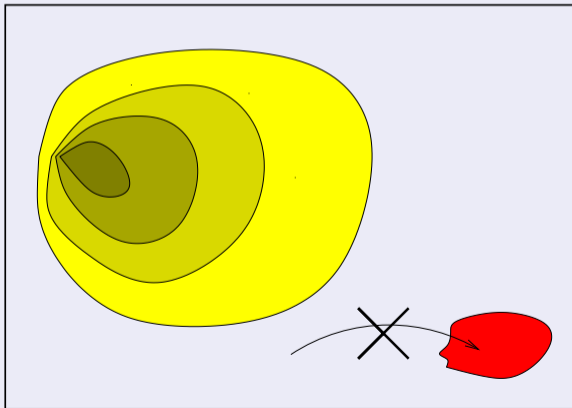
$$\begin{aligned} & (I(s^0) \wedge \neg Good(s^0)); \\ & (Good(s^{k-1}) \wedge R(s^{k-1}, s^k)) \wedge \neg Good(s^k) \end{aligned}$$

Note

“($I(s^0) \wedge \neg Good(s^0)$)” is step-0 incremental BMC encoding for $\mathbf{F}\neg Good$.

Strengthening of Invariants

- Problem: Induction may fail because of unreachable states:
 - if $(Good(s^{k-1}) \wedge R(s^{k-1}, s^k)) \rightarrow Good(s^k)$ is not valid, then this does not mean that the property does not hold
 - both s^{k-1} and s^k might be unreachable

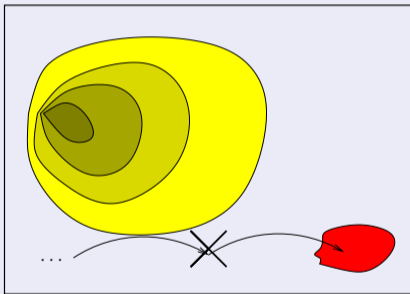


Strengthening of Invariants [cont.]

Solution (once you know you cannot reach $\neg\text{Good}$ in up to 1 step):

- increase the depth of induction

$$(\text{Good}(s^{k-2}) \wedge R(s^{k-2}, s^{k-1}) \wedge \text{Good}(s^{k-1}) \wedge R(s^{k-1}, s^k) \wedge \neg(s^{k-2} = s^{k-1})) \rightarrow \text{Good}(s^k)$$



- force loop freedom with $\neg(s^i = s^j)$ for every $i \neq j$ s.t. $i, j \leq k$
- performed after step-1 BMC step returns “unsat”:
 $I(s^0) \wedge (R(s^0, s^1) \wedge \text{Good}(s^0)) \wedge \neg\text{Good}(s^1)$

Strengthening of Invariants [cont.]

⇒ Check for the [un]satisfiability of the Boolean formulas:

$$I(s^0) \wedge \neg \text{Good}(s^0); \text{ [BMC}_0\text{]}$$

$$(\text{Good}(s^{k-1}) \wedge R(s^{k-1}, s^k)) \wedge \neg \text{Good}(s^k); \text{ [Kind}_0\text{]}$$

$$I(s^0) \wedge (R(s^0, s^1) \wedge \text{Good}(s^0)) \wedge \neg \text{Good}(s^1); \text{ [BMC}_1\text{]}$$

$$(\text{Good}(s^{k-2}) \wedge R(s^{k-2}, s^{k-1}) \wedge \text{Good}(s^{k-1}) \wedge R(s^{k-1}, s^k)) \wedge \neg \text{Good}(s^k)$$

$$\wedge \neg (s^{k-2} = s^{k-1}); \text{ [Kind}_1\text{]}$$

$$I(s^0) \wedge (R(s^0, s^1) \wedge \text{Good}(s^0) \wedge (R(s^1, s^2) \wedge \text{Good}(s^1))) \wedge \neg \text{Good}(s^2); \text{ [BMC}_2\text{]}$$

...

- Repeat for increasing values of the gap 1, 2, 3, 4,
- **Intuition:** increasingly tighten the constraint for “spurious” counterexamples: a spurious counterexample must be a chain s_{k-n}, \dots, s_k of **unreachable** and **different** states s.t. $\neg \text{Good}(s_k)$ and $R(s_i, s_{i+1}), \forall i$.
- Dual to –and interleaved with– **bounded model checking steps**
- K-Induction steps can be shifted ($k \stackrel{\text{def}}{=} 0$) to share the subformulas:
$$\bigwedge_{i=0}^{k-1} (R(s^i, s^{i+1}) \wedge \text{Good}(s^i)) \wedge \neg \text{Good}(s^{k-2})$$

K-Induction Algorithm [Sheeran et al. 2000]

Algorithm

Given:

$$\begin{aligned} \text{Base}_n &:= I(\mathbf{s}_0) \wedge \bigwedge_{i=0}^{n-1} (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_n) \\ \text{Step}_n &:= \bigwedge_{i=0}^n (R(\mathbf{s}_i, \mathbf{s}_{i+1}) \wedge \varphi(\mathbf{s}_i)) \wedge \neg\varphi(\mathbf{s}_{n+1}) \\ \text{Unique}_n &:= \bigwedge_{0 \leq i < j \leq n} \neg(\mathbf{s}_i = \mathbf{s}_{j+1}) \end{aligned}$$

1. **function** CHECK_PROPERTY (I, R, φ)
2. **for** $n := 0, 1, 2, 3, \dots$ **do**
3. **if** (DPLL(Base_n) == SAT)
4. **then return** PROPERTY_VIOLATED;
5. **else if** (DPLL($\text{Step}_n \wedge \text{Unique}_n$) == UNSAT)
6. **then return** PROPERTY_VERIFIED;
7. **end for**;

⇒ Reuses previous search if DPLL is incremental!!

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Example: a correct 3-bit shift register

- System M :
 - $I(x) := (\neg x[0] \wedge \neg x[1] \wedge \neg x[2])$
 - $R(x, x') := ((x'[0] \leftrightarrow x[1]) \wedge (x'[1] \leftrightarrow x[2]) \wedge (x'[2] \leftrightarrow 0))$
- Property: $\mathbf{G}\neg x[0]$

Example: a correct 3-bit shift register [cont.]

- Init (BMC Step 0): $((\neg x^0[0] \wedge \neg x^0[1] \wedge \neg x^0[2]) \wedge x^0[0]) \implies \text{unsat}$
- K-Induction Step 1:

$$\left(\begin{array}{l} (\neg x^0[0] \wedge ((x^1[0] \leftrightarrow x^0[1]) \wedge (x^1[1] \leftrightarrow x^0[2]) \wedge (x^1[2] \leftrightarrow 0))) \\ \wedge x^1[0] \end{array} \right)$$

\implies (partly by unit-propagation)

$$\text{sat: } \left\{ \begin{array}{lll} \neg x^0[0], & x^0[1], & x^0[2], \\ x^1[0], & x^1[1], & \neg x^1[2] \end{array} \right\}$$

\implies not proved

Remark

Both $\{\neg x^0[0], x^0[1], x^0[2]\}$ and $\{x^1[0], x^1[1], \neg x^1[2]\}$ are non-reachable.

Example: a correct 3-bit shift register [cont.]

- BMC Step 1: (...) \implies unsat
- K-Induction Step 2:

$$\left(\begin{array}{l} (\neg x^0[0] \wedge ((x^1[0] \leftrightarrow x^0[1]) \wedge (x^1[1] \leftrightarrow x^0[2]) \wedge (x^1[2] \leftrightarrow 0)) \wedge \\ \neg x^1[0] \wedge ((x^2[0] \leftrightarrow x^1[1]) \wedge (x^2[1] \leftrightarrow x^1[2]) \wedge (x^2[2] \leftrightarrow 0)) \\) \wedge x^2[0] \end{array} \right) \wedge \neg((x^1[0] \leftrightarrow x^0[0]) \wedge (x^1[1] \leftrightarrow x^0[1]) \wedge (x^1[2] \leftrightarrow x^0[2]))$$

$$\implies \text{sat: } \left\{ \begin{array}{lll} \neg x^0[0], & \neg x^0[1], & x^0[2] \\ \neg x^1[0], & x^1[1], & \neg x^1[2] \\ x^2[0], & \neg x^2[1], & \neg x^2[2] \end{array} \right\} \implies \text{not proved}$$

Remark

$\{\neg x^0[0], \neg x^0[1], x^0[2]\}$, $\{\neg x^1[0], x^1[1], \neg x^1[2]\}$, and $\{x^2[0], \neg x^2[1], \neg x^2[2]\}$ are non-reachable.

Example: a correct 3-bit shift register [cont.]

- BMC Step 2: (...) \implies unsat
- K-Induction Step 3:

$$\left(\begin{array}{l} (\neg x^0[0] \wedge ((x^1[0] \leftrightarrow x^0[1]) \wedge (x^1[1] \leftrightarrow x^0[2]) \wedge (x^1[2] \leftrightarrow 0))) \wedge \\ \neg x^1[0] \wedge ((x^2[0] \leftrightarrow x^1[1]) \wedge (x^2[1] \leftrightarrow x^1[2]) \wedge (x^2[2] \leftrightarrow 0)) \wedge \\ \neg x^2[0] \wedge ((x^3[0] \leftrightarrow x^2[1]) \wedge (x^3[1] \leftrightarrow x^2[2]) \wedge (x^3[2] \leftrightarrow 0)) \\) \wedge x^3[0] \end{array} \right)$$
$$\wedge \neg((x^1[0] \leftrightarrow x^0[0]) \wedge (x^1[1] \leftrightarrow x^0[1]) \wedge (x^1[2] \leftrightarrow x^0[2]))$$
$$\wedge \neg((x^2[0] \leftrightarrow x^0[0]) \wedge (x^2[1] \leftrightarrow x^0[1]) \wedge (x^2[2] \leftrightarrow x^0[2]))$$
$$\wedge \neg((x^2[0] \leftrightarrow x^1[0]) \wedge (x^2[1] \leftrightarrow x^1[1]) \wedge (x^2[2] \leftrightarrow x^1[2]))$$

\implies (unit-propagation) $\{x^3[0], x^2[1], x^1[2]\}$

\implies unsat

\implies **proved!**

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Ex: Bounded Model Checking

Given the symbolic representation of a FSM M , expressed in terms of the two Boolean formulas: $I(x, y) \stackrel{\text{def}}{=} \neg x \wedge y$, $T(x, y, x', y') \stackrel{\text{def}}{=} (x' \leftrightarrow (x \leftrightarrow \neg y)) \wedge (y' \leftrightarrow \neg y)$, and the LTL property: $\varphi \stackrel{\text{def}}{=} \neg \mathbf{F}(x \wedge y)$,

1. Write a Boolean formula whose solutions (if any) represent executions of M of length 2 which violate φ .

[Solution: The question corresponds to the Bounded Model Checking problem $M \models_2 \mathbf{E F}f$, s.t. $f(x, y) \stackrel{\text{def}}{=} (x \wedge y)$. Thus we have:

$$\begin{array}{llll} \neg x_0 \wedge y_0 & \wedge & // & I(x_0, y_0) \wedge \\ (x_1 \leftrightarrow (x_0 \leftrightarrow \neg y_0)) \wedge (y_1 \leftrightarrow \neg y_0) & \wedge & // & T(x_0, y_0, x_1, y_1) \wedge \\ (x_2 \leftrightarrow (x_1 \leftrightarrow \neg y_1)) \wedge (y_2 \leftrightarrow \neg y_1) & \wedge & // & T(x_1, y_1, x_2, y_2) \wedge \\ ((x_0 \wedge y_0) & \vee & // & (f(x_0, y_0) \vee \\ (x_1 \wedge y_1) & \vee & // & f(x_1, y_1) \vee \\ (x_2 \wedge y_2)) & & // & f(x_2, y_2)) \end{array}$$

]

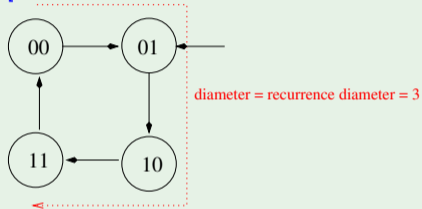
2. Is there a solution? If yes, find the corresponding execution; if no, show why.

[Solution: Yes: $\{\neg x_0, y_0, x_1, \neg y_1, x_2, y_2\}$, corresponding to the execution: $(0, 1) \rightarrow (1, 0) \rightarrow (1, 1)$]

Ex: Bounded Model Checking

3. What are the diameter and the recurrence diameter of this system?

[Solution:



4. From the solutions to question #1 and #2 we can conclude that:

- (a) $M \models \varphi$
- (b) $M \not\models \varphi$
- (c) we can conclude nothing.

[Solution: b)

Ex: Bounded Model Checking

Given the following symbolic representation of a finite state machine M , expressed in terms of the following two formulas:

- $I(x, y) \stackrel{\text{def}}{=} (\neg x \wedge \neg y)$
- $T(x, y, x', y') \stackrel{\text{def}}{=} (x' \leftrightarrow \neg y')$,

and the following LTL property:

- $\varphi \stackrel{\text{def}}{=} \neg \mathbf{F}(x \wedge y)$,
- ① write a Boolean formula whose solutions (if any) represent executions of M of length 2 which violate φ .

[Solution: The question corresponds to the Bounded Model Checking problem $M \models_2 \mathbf{E F}f$, s.t. $f(x, y) \stackrel{\text{def}}{=} (x \wedge y)$. Thus we have:

$$\begin{array}{llll} (\neg x_0 \wedge \neg y_0) & \wedge & // & I(x_0, y_0) \wedge \\ (x_1 \leftrightarrow \neg y_1) & \wedge & // & T(x_0, y_0, x_1, y_1) \wedge \\ (x_2 \leftrightarrow \neg y_2) & \wedge & // & T(x_1, y_1, x_2, y_2) \wedge \\ ((x_0 \wedge y_0) & \vee & // & (f(x_0, y_0) \vee \\ (x_1 \wedge y_1) & \vee & // & f(x_1, y_1) \vee \\ (x_2 \wedge y_2)) & & // & f(x_2, y_2)) \end{array}$$

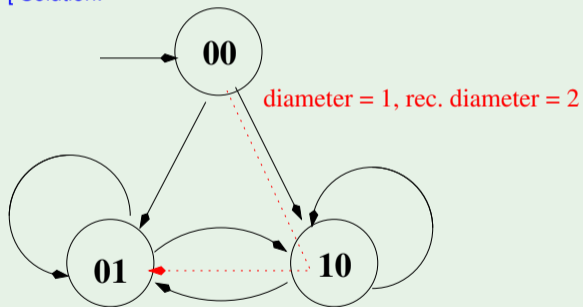
]

- ② is there a solution? If yes, find the corresponding execution.

[Solution: No: it is easy to see that the formula above is inconsistent]

Ex: Bounded Model Checking [cont.]

- 1 ...
- 2 ...
- 3 what are the diameter and the recurrence diameter of this system?
[Solution:



- 4 Can we conclude anything about the model-checking problem $M \models \varphi$? Explain why.
[Solution: yes, we can conclude that $M \models \varphi$, since $M \not\models_2 \mathbf{EF}\neg\varphi$ and rec. diameter=2.]

Ex: K-Induction

Given the following LTL Model Checking problem $M \models \varphi$ expressed in NuSMV input language:

```
MODULE main
VAR x : boolean; y : boolean; z : boolean;
INIT (!x & !y & z)
TRANS ((next(x) <-> (y)) & (next(y) <-> z) & (next(z) <-> x) )
LTLSPEC G (x | y | z) ;
```

- 1 Write the Boolean formulas describing the k-induction encoding of the problem, with $k = 1$.

[Solution: The LTL property is in the form “**G**Good(x, y, z)”, hence, applying k-induction:

$$\begin{aligned} \varphi_{Base} &\stackrel{\text{def}}{=} (\neg x_0 \wedge \neg y_0 \wedge z_0) && \wedge && // I(x_0, y_0, z_0) \wedge \\ &\neg(x_0 \vee y_0 \vee z_0) && && // \neg Good(x_0, y_0, z_0) \\ \varphi_{Ind1} &\stackrel{\text{def}}{=} (x_i \vee y_i \vee z_i) && \wedge && // Good(x_i, y_i, z_i) \wedge \\ &((x_{i+1} \leftrightarrow y_i) \wedge (y_{i+1} \leftrightarrow z_i) \wedge (z_{i+1} \leftrightarrow x_i)) && \wedge && // T(x_i, y_i, z_i, x_{i+1}, y_{i+1}, z_{i+1}) \wedge \\ &\neg(x_{i+1} \vee y_{i+1} \vee z_{i+1}) && && // \neg Good(x_{i+1}, y_{i+1}, z_{i+1}) \end{aligned}$$

]

Ex: K-Induction [cont.]

- 1 ...
- 2 Say if they are satisfiable or not. If yes, show a model. If not, explain why. [Solution:
 - φ_{Base} is not satisfiable. In fact, the second row forces the assignments $\neg x_0, \neg y_0, \neg z_0$, which makes the first row false.
 - φ_{Ind1} is not satisfiable. In fact, the third row forces the assignments $\neg x_{i+1}, \neg y_{i+1}, \neg z_{i+1}$, from which the second row forces the assignments $\neg x_i, \neg y_i, \neg z_i$, which makes the first row false.

]

- 3 From the previous answers we can conclude:

- (a) that $M \models \varphi$;
- (b) that $M \not\models \varphi$;
- (c) we can conclude nothing.

[Solution: a) $M \models \varphi$. In fact, we have proved it in one induction step.

]