

Automated Reasoning and Formal Verification

Module II: Formal Verification

Ch. 06: **Symbolic Model Checking**

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Outline

- 1 CTL Model Checking with Fair Kripke Models
 - Fairness & Fair Kripke Models
 - Fair CTL Model Checking
 - SCC-Based Approach
 - Emerson-Lei Algorithm
- 2 CTL Symbolic Model Checking
 - Symbolic Representation of Systems
 - Symbolic CTL MC
 - Symbolic Fair CTL MC
 - A simple example
- 3 The Symbolic Approach to LTL Model Checking
 - General Ideas
 - Compute the Tableau T_ψ
 - Compute the Product $M \times T_\psi$
 - Check the Emptiness of $\mathcal{L}(M \times T_\psi)$
- 4 A Complete Example
- 5 Exercises

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The Need for Fairness Conditions: Intuition

Consider a public restroom. A standard access policy is “first come first served” (e.g., a queue-based protocol).

- Does this policy guarantee that everybody entering the queue will eventually access the restroom?
 - **No**: in principle, somebody might remain in the restroom forever, hindering the access to everybody else
 - In practice, it is considered reasonable to assume that everybody exits the restroom after a finite amount of time

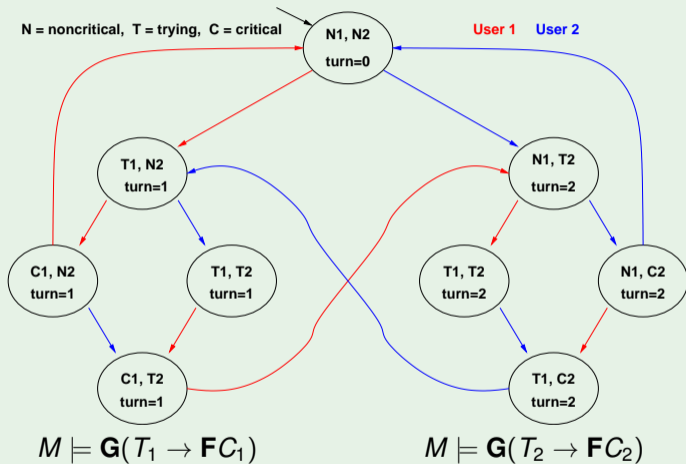
⇒ It is reasonable enough to assume the protocol suitable under the condition that each user is **infinitely often** outside the restroom

- Such a condition is called **fairness condition**

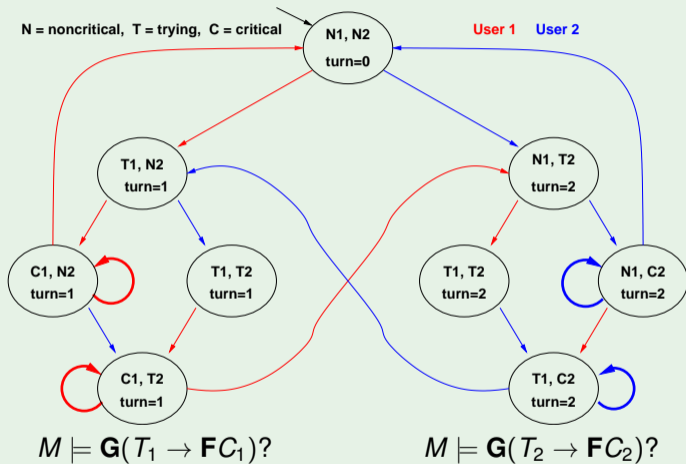
The Need for Fairness Conditions: An Example

- Consider a variant of the mutual exclusion in which one process can stay permanently in the critical zone
- Do $M \models \mathbf{G}(T_1 \rightarrow \mathbf{FC}_1)$, $M \models \mathbf{G}(T_2 \rightarrow \mathbf{FC}_2)$ still hold?

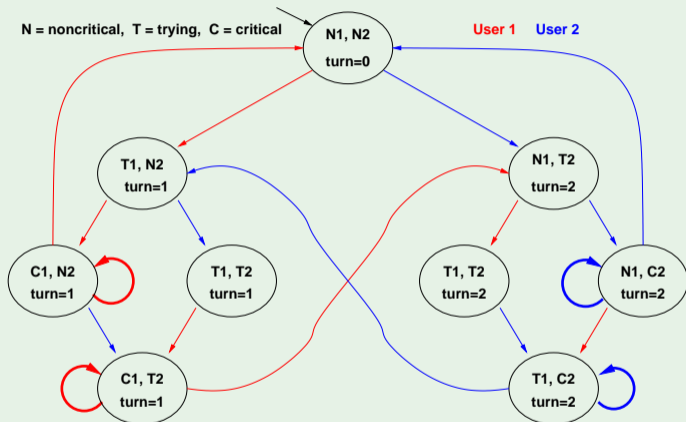
The Need for Fairness Conditions: An Example [cont.]



The need for fairness conditions: an example [cont.]



The need for fairness conditions: an example [cont.]



$G(T_1 \rightarrow FC_1)?$

NO: E.g., it can cycle forever in $\{C_1, T_2, \text{turn} = 1\}$

\Rightarrow **Unfair** protocol: one process might never be served

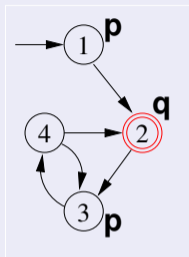
Fairness Conditions

- It is desirable that certain (typically Boolean) conditions φ 's hold infinitely often: **GF** φ
- **GF** φ is called **fairness condition**
- Intuitively, fairness conditions are used to eliminate behaviours in which a certain condition φ never holds:
GF φ : "it is never reached a state from which φ is forever false"
- Example: it is not desirable that, once a process is in the critical section, it never exits:
GF $\neg C_1$
- A fair condition φ_i can be represented also by the set f_i of states where φ_i holds
($f_i := \{s : \pi, s \models \varphi_i, \text{ for each } \pi \in M\}$)

Fair Kripke models

- A **Fair Kripke model** $M_F := \langle S, R, I, AP, L, F \rangle$ consists of:

- a set of states S ;
- a set of initial states $I \subseteq S$;
- a set of transitions $R \subseteq S \times S$;
- a set of atomic propositions AP ;
- a labeling function $L : S \mapsto 2^{AP}$;
- a set of fairness conditions $F = \{f_1, \dots, f_n\}$, with $f_i \subseteq S$.



- E.g., $\{\{2\}\} := \{\{s : L(s) = \{q\}\}\} = \{\mathbf{GF}q\}$ is the set of fairness conditions of the Kripke model above
- **Fair path** π : at least one state for each f_i occurs infinitely often in π
(φ_i holds infinitely often in π : $\pi \models \mathbf{GF}\varphi_i$)
 - E.g., every path visiting infinitely often state 2 is a fair path.
- **Fair state**: a state through which at least one fair path passes
 - E.g., all states 1,2,3,4 are fair states
- Note: fair state \neq state belonging to a fairness condition

Computing an NBA A_M from a Fair Kripke Model M

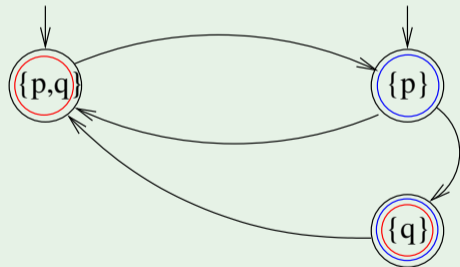
- Transforming a fair K.S. $M = \langle S, S_0, R, L, AP, FT \rangle$, $FT = \{F_1, \dots, F_n\}$, into a generalized NBA $A_M = \langle Q, \Sigma, \delta, I, FT' \rangle$ s.t.:

- States: $Q := S \cup \{init\}$, $init$ being a new initial state
- Alphabet: $\Sigma := 2^{AP}$
- Initial State: $I := \{init\}$
- Accepting States: $FT' := FT$
- Transitions:

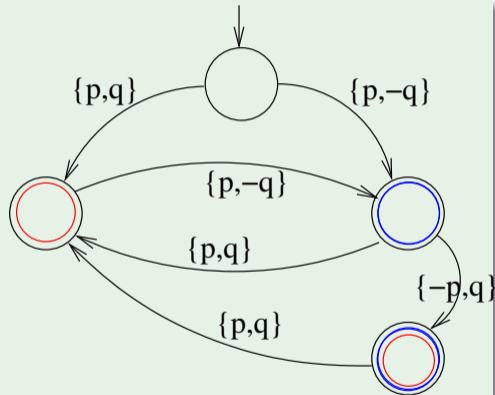
$$\delta : \quad q \xrightarrow{a} q' \text{ iff } (q, q') \in R \text{ and } L(q') = a$$
$$init \xrightarrow{a} q \text{ iff } q \in S_0 \text{ and } L(q) = a$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a (Generalized) BA A_M from a Fair Kripke Structure M : Example



Fair Kripke Structure



Generalized Buchi Automaton

\Rightarrow Substantially, add one initial state, move labels from states to incoming edges, set fair states as accepting states

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CTL M.C. with Fair Kripke Models

Fair Kripke Models restrict the M.C. process to fair paths:

- $M_f \models \varphi$ iff $\pi \models \varphi$ for every **fair** path π
- **Path quantifiers** (from CTL) apply only to fair paths:
 - $M_F, s \models \mathbf{A}\varphi$ iff $\pi, s \models \varphi$ for every **fair** path π s.t. $s \in \pi$
 - $M_F, s \models \mathbf{E}\varphi$ iff $\pi, s \models \varphi$ for some **fair** path π s.t. $s \in \pi$

\Rightarrow a fair state s is a state in M_F iff $M_F, s \models \mathbf{EG}true$.

- We need a procedure to compute the set of fair states: `Check_FairEG(true)`

Example

- $M_f \models \mathbf{EG}true?$ **yes**
- $M_f \models \mathbf{G}(p \rightarrow \mathbf{F}q)?$ **no**
- $M \models \mathbf{G}(p \rightarrow \mathbf{F}q)?$ **no**

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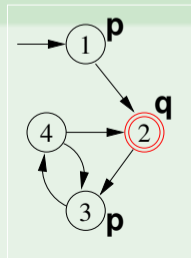
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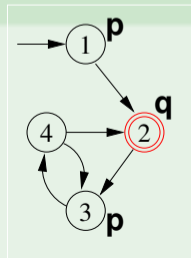
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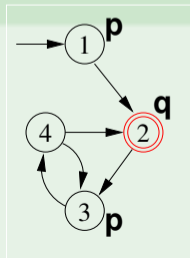
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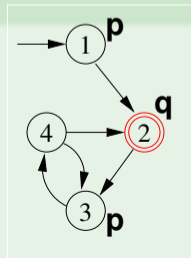
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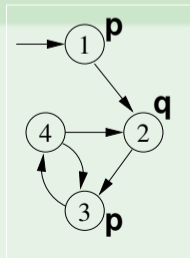
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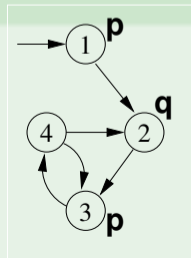
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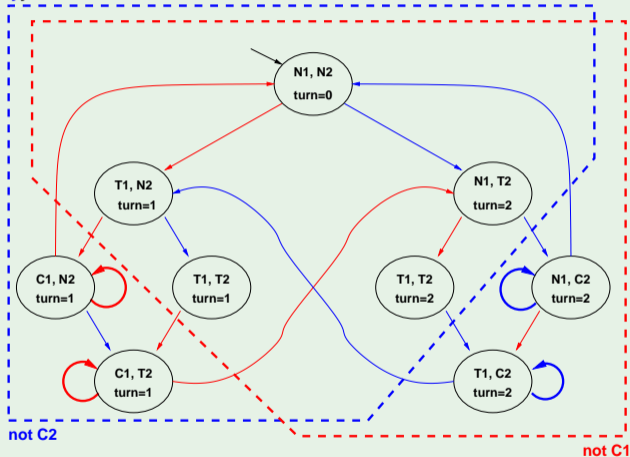
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Fair CTL Model Checking: Example

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$



$M_F \models \mathbf{G}(T_1 \rightarrow \mathbf{FC}_1)?$

$M_F \models \mathbf{G}(T_2 \rightarrow \mathbf{FC}_2)?$ **YES:** every fair path satisfies the conditions

CTL M.C. vs. LTL M.C. with Fair Kripke Models

Remark: fair CTL M.C.

When model checking a **CTL** formula ψ , fairness conditions **cannot** be encoded into the formula:

$$M_{\{f_1, \dots, f_n\}} \models \psi \not\iff M \models \left(\bigwedge_{i=1}^n \mathbf{AGAF} f_i \right) \rightarrow \psi.$$

$$M_{\{f_1, \dots, f_n\}} \models \psi \not\iff M \models \left(\bigwedge_{i=1}^n \mathbf{EGEF} f_i \right) \rightarrow \psi.$$

\implies We need specific procedures for Fair CTL Model Checking.

Remark: fair LTL M.C.

When model checking an **LTL** formula ψ , fairness conditions can be encoded into the formula:

$$M_{\{f_1, \dots, f_n\}} \models \psi \iff M \models \left(\bigwedge_{i=1}^n \mathbf{GF} f_i \right) \rightarrow \psi.$$

\implies There is no need for Fair LTL Model Checking procedures.

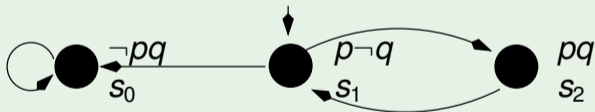
Ex. CTL: $M_{\{f_1, \dots, f_n\}} \models \psi \not\iff M \models (\bigwedge_{i=1}^n \mathbf{AGAF} f_i) \rightarrow \psi$.

[Example provided by the student Davide Kirchner, 2014]

M_p



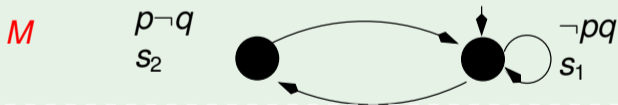
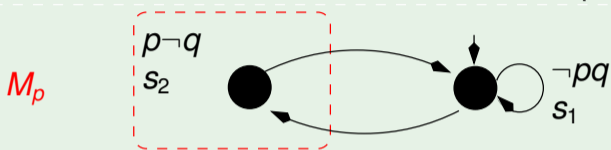
M



- $M_p \not\models \mathbf{AG}q$
- $M \models (\mathbf{AGAF}p) \rightarrow \mathbf{AG}q$

Ex. CTL: $M_{\{f_1, \dots, f_n\}} \models \psi \not\iff M \models (\bigwedge_{i=1}^n \mathbf{EGEF} f_i) \rightarrow \psi$.

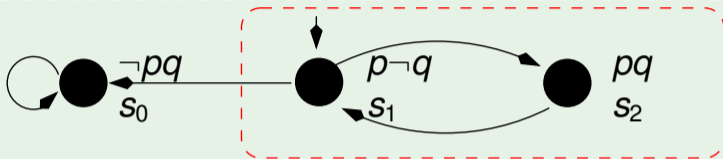
[Example provided by the student Daniele Giuliani, 2019]



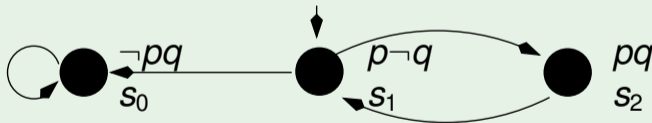
- $M_p \not\models \mathbf{EFEG} q$
- $M \models (\mathbf{EGEF} p) \rightarrow \mathbf{EFEG} q$

Ex. LTL (1): $M_{\{f_1, \dots, f_n\}} \models \psi \iff M \models (\bigwedge_{i=1}^n \mathbf{GF}f_i) \rightarrow \psi$.

M_p



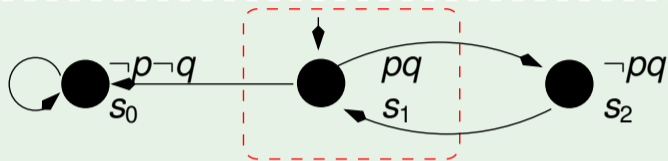
M



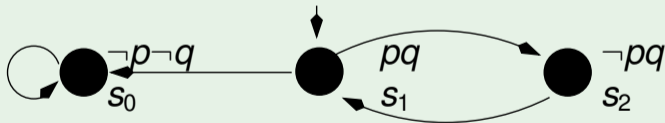
- $M_p \not\models \mathbf{G}q$
- $M \not\models (\mathbf{GF}p) \rightarrow \mathbf{G}q$

Ex. LTL (2): $M_{\{f_1, \dots, f_n\}} \models \psi \iff M \models (\bigwedge_{i=1}^n \mathbf{GF}f_i) \rightarrow \psi$.

M_p



M



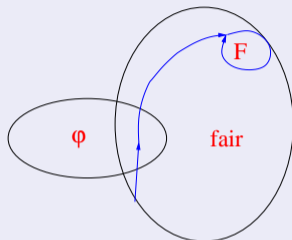
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Fair CTL Model Checking

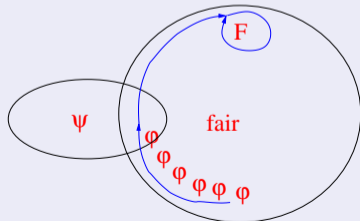
- In order to solve the fair CTL model checking problem, we must be able to compute:
 - $[\varphi_f]$ s.t. φ Boolean (i.e. $[\varphi]$ under fairness conditions f)
 - $[\mathbf{E}_f\mathbf{X}(\varphi)]$ (i.e. $[\mathbf{EX}\varphi]$ under fairness conditions f)
 - $[\mathbf{E}_f(\varphi\mathbf{U}\psi)]$ (i.e. $[\mathbf{E}(\varphi\mathbf{U}\psi)]$ under fairness conditions f)
 - $[\mathbf{E}_f\mathbf{G}\varphi]$ (i.e. $[\mathbf{EG}\varphi]$ under fairness conditions f).
- Suppose we have a procedure `Check_FairEG` to compute $[\mathbf{E}_f\mathbf{G}\varphi]$.
- Let $\text{fair} \stackrel{\text{def}}{=} \mathbf{E}_f\mathbf{G}\text{true}$. ($M, s \models \mathbf{E}_f\mathbf{G}\text{true}$ if s is a fair state.)
- if φ is Boolean, then $M_f, s \models \varphi$ iff $M, s \models (\varphi \wedge \text{fair})$
- We can rewrite all the other fair operators:
 - $\mathbf{E}_f\mathbf{X}(\varphi) \equiv \mathbf{EX}(\varphi \wedge \text{fair})$
 - $\mathbf{E}_f(\varphi\mathbf{U}\psi) \equiv \mathbf{E}(\varphi\mathbf{U}(\psi \wedge \text{fair}))$

Fair CTL Model Checking

- $E_f X(\varphi) \equiv EX(\varphi \wedge \text{fair})$:



- $E_f(\varphi U \psi) \equiv E(\varphi U(\psi \wedge \text{fair}))$:



Language-Emptiness Checking for Fair Kripke Models

Fair_CheckEG

Given: a fair Kripke model $M_F := \langle S, R, I, AP, L, F \rangle$ and a CTL formula φ s.t. $[\varphi] \subseteq S$,
 $\text{Fair_CheckEG}(\varphi)$ returns the subset of the states s in $[\varphi]$ from which at least one fair path π entirely included in $[\varphi]$ passes through

$\text{Fair_CheckEG}(\text{true})$ computes the set of fair states of M_f

$\implies I \subseteq \text{Fair_CheckEG}(\text{true})$ iff $\mathcal{L}(M_f) \neq \emptyset$

Ingredients (from CTL Model Checking)

Some primitive functions from CTL Model Checking:

- $\text{Check_EX}(\phi)$: returns the set of states from which a path verifying $\mathbf{X}\phi$ holds (i.e., the preimage of the set of states where ϕ holds)
- $\text{Check_EG}(\phi)$: returns the set of states from which a path verifying $\mathbf{G}\phi$ holds
- $\text{Check_EU}(\phi_1, \phi_2)$: returns the set of states from which a path verifying $\phi_1 \mathbf{U} \phi_2$ holds

Outline

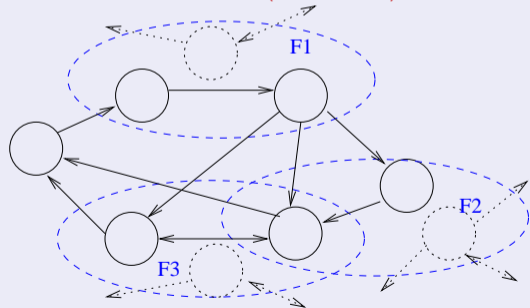
- 1 CTL Model Checking with Fair Kripke Models
 - Fairness & Fair Kripke Models
 - Fair CTL Model Checking
 - **SCC-Based Approach**
 - Emerson-Lei Algorithm
- 2 CTL Symbolic Model Checking
 - Symbolic Representation of Systems
 - Symbolic CTL MC
 - Symbolic Fair CTL MC
 - A simple example
- 3 The Symbolic Approach to LTL Model Checking
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SCC-based Check_FairEG

A **Strongly Connected Component (SCC)** of a directed graph is a maximal subgraph s.t. all its nodes are reachable from each other.

Given a fair Kripke model M , a **fair non-trivial SCC** is an SCC with at least one edge that contains at least one state for every fair condition

\implies all states in a fair (non-trivial) SCC are fair states



SCC-based Check_FairEG (cont.)

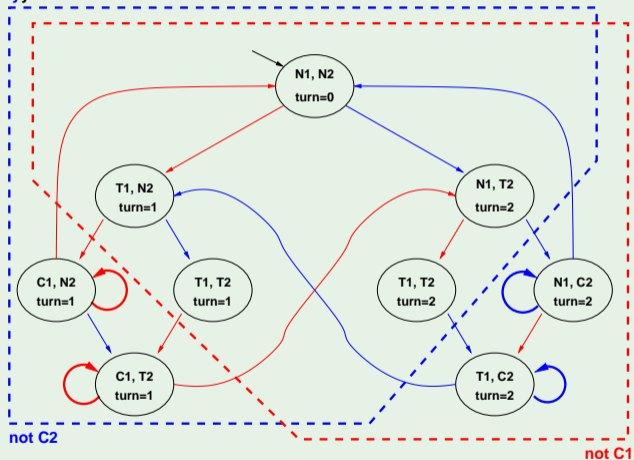
`Check_FairEG($[\phi]$):`

- (i) restrict the graph of M to $[\phi]$;
- (ii) find all fair non-trivial SCCs C_i
- (iii) build $C := \cup_i C_i$;
- (iv) compute the states that can reach C (`Check_EU($[\phi]$, C)`).

$[\phi]$: set of states where ϕ holds (aka **denotation** of ϕ)

Example: Check_FairEG

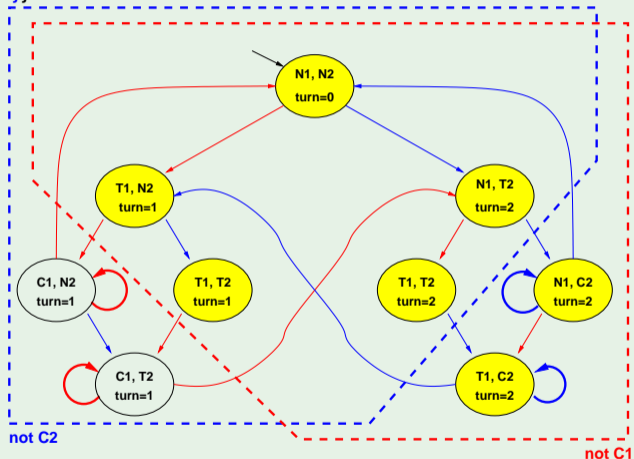
$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$



$EG \neg C_1$

Example: Check_FairEG

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$

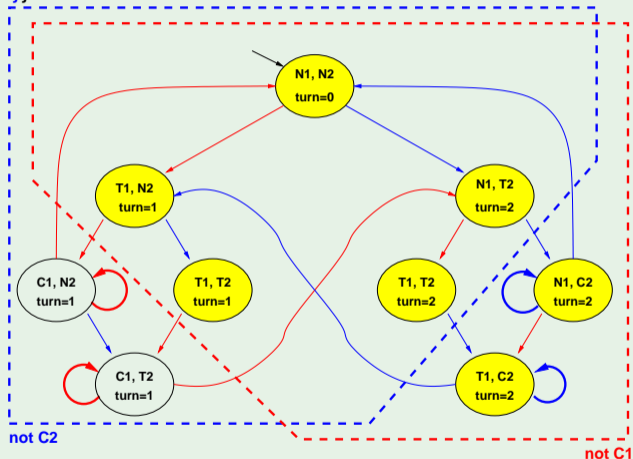


$EG \neg C_1$

Check_FairEG($\neg C_1$): 1. compute $[\neg C_1]$

Example: Check_FairEG

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$

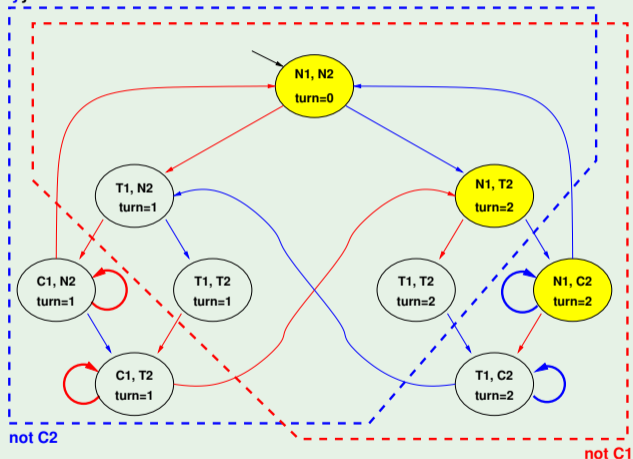


$EG \neg C_1$

Check_FairEG($\neg C_1$): 2. restrict the graph to $[\neg C_1]$

Example: Check_FairEG

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$

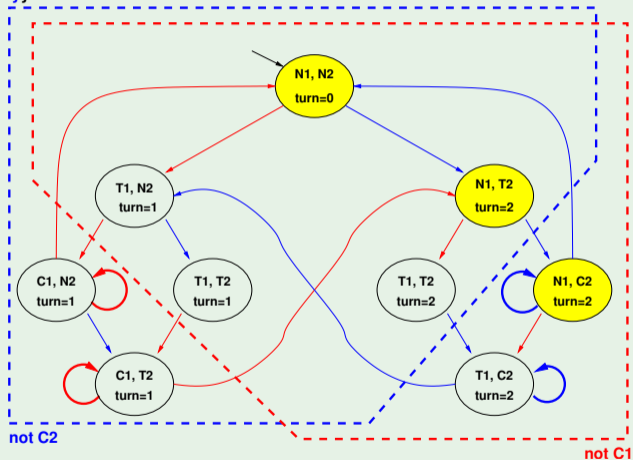


$EG \neg C_1$

Check_FairEG($\neg C_1$): 3. find all fair non-trivial SCC's

Example: Check_FairEG

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$

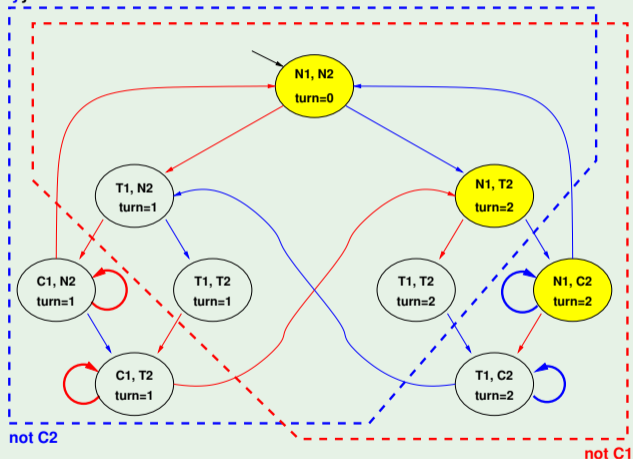


$EG \neg C_1$

Check_FairEG($\neg C_1$): 4. build the union C of all SCC's

Example: Check_FairEG

$F := \{\{\text{not } C1\}, \{\text{not } C2\}\}$



$EG \neg C_1$

Check_FairEG($\neg C_1$): 5. compute the states which can reach it

SCC-based Check_FairEG - Drawbacks

- SCCs computation requires a linear ($O(\#nodes + \#edges)$) DFS (Tarjan).
- The DFS manipulates the states explicitly, storing information for every state.
- A DFS is not suitable for symbolic model checking where we manipulate sets of states.

⇒ We want an algorithm based on (symbolic) preimage computation.

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Emerson-Lei Algorithm

Fixpoint characterization of **EG** and fair **EG**

" $[\phi]$ " denotes the set of states where ϕ holds

- Theorem (Emerson & Clarke): $[EG\phi] = \nu Z.([\phi] \cap [EXZ])$

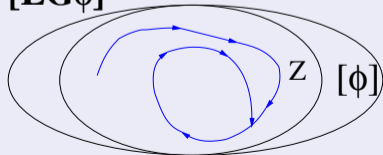
The greatest set Z s.t. every state z in Z satisfies ϕ and reaches another state in Z in one step.

We can characterize fair **EG** (aka "**E_fG**") similarly:

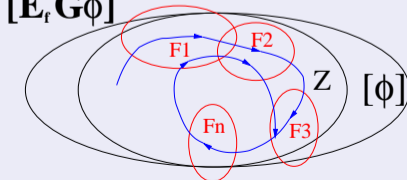
- Theorem (Emerson & Lei): $[E_fG\phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT} [EX E(ZU(Z \cap F_i))])$

The greatest set Z s.t. every state z in Z satisfies ϕ and, for every set $F_i \in FT$, z reaches a state in $F_i \cap Z$ by means of a non-trivial path that lies in Z .

$[EG\phi]$



$[E_fG\phi]$



Emerson-Lei Algorithm

Recall: $[E_f G \phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT} [EX E(ZU(Z \cap F_i))])$

```
state_set Check_FairEG(state_set [ $\phi$ ]) {  
   $Z' := [\phi]$ ;  
  repeat  
     $Z := Z'$ ;  
    for each  $F_i$  in FT  
       $Y := \text{Check\_EU}(Z, F_i \cap Z)$ ;  
       $Z' := Z' \cap \text{PreImage}(Y)$ ;  
    end for;  
  until ( $Z' = Z$ );  
  return  $Z$ ;  
}
```

Implementation of the above formula

Emerson-Lei Algorithm

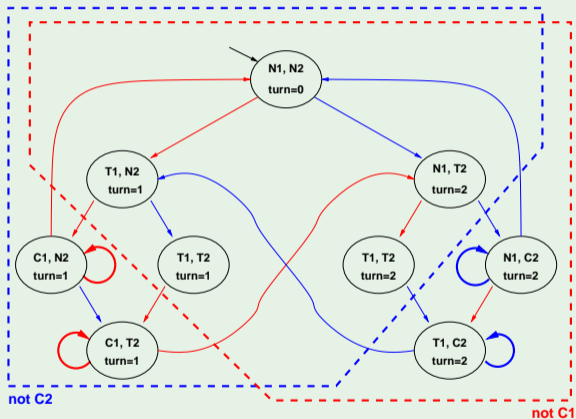
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    end for;  
  until ( $Z' = Z$ );  
  return  $Z$ ;  
}
```

Slight improvement: do not consider states in $Z \setminus Z'$

Example: Check_FairEG

$F := \{ \{ \text{not } C1 \}, \{ \text{not } C2 \} \}$

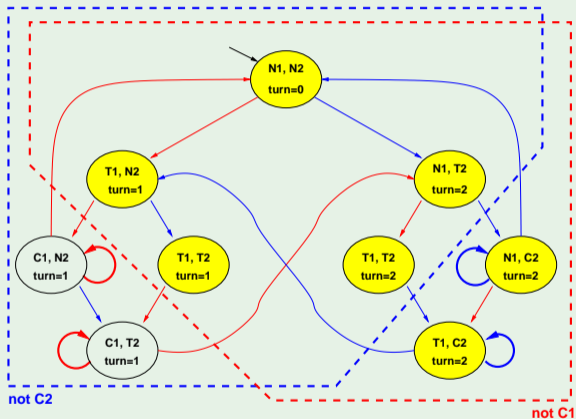


$[E_f G \neg C_1]$

Fixpoint reached

Example: Check_FairEG

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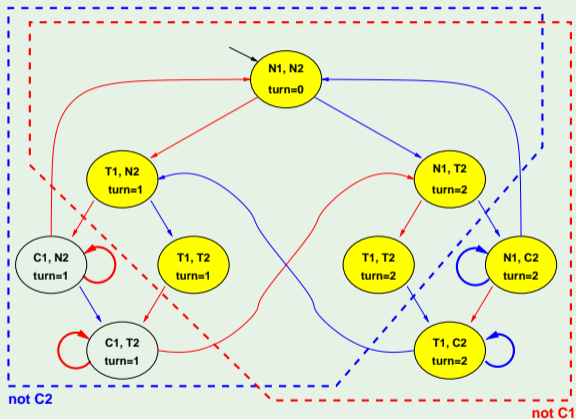


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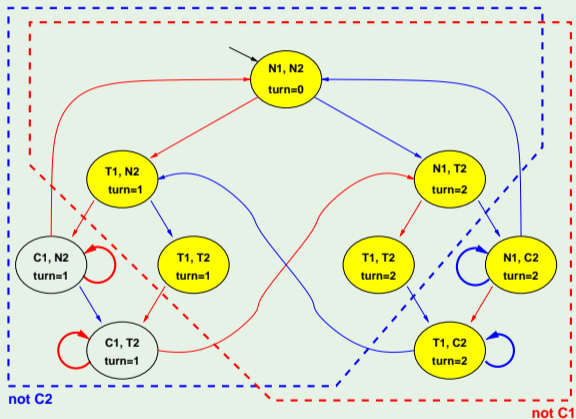
$[E_f G \neg C_1]$

$[E_f G g] = \nu Z. [g] \cap [EXE(ZU(Z \cap F_1))] \cap [EXE(ZU(Z \cap F_2))]$

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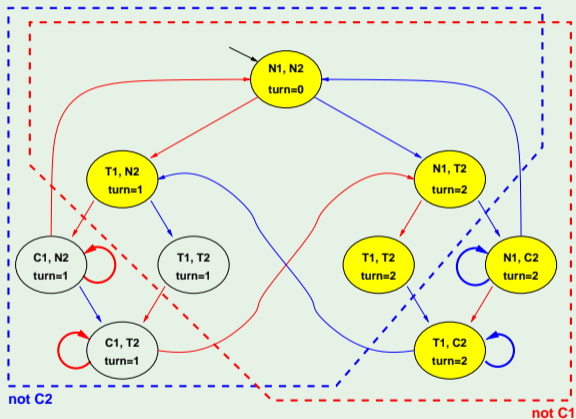
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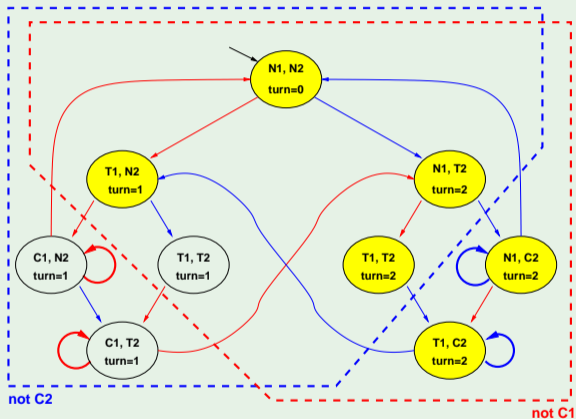
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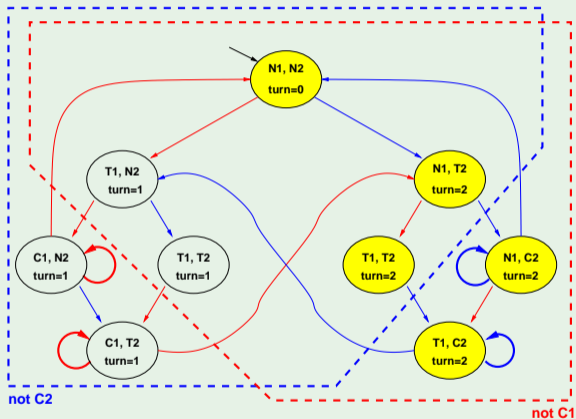
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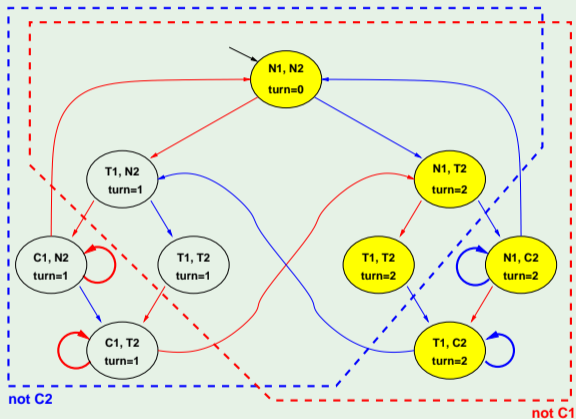
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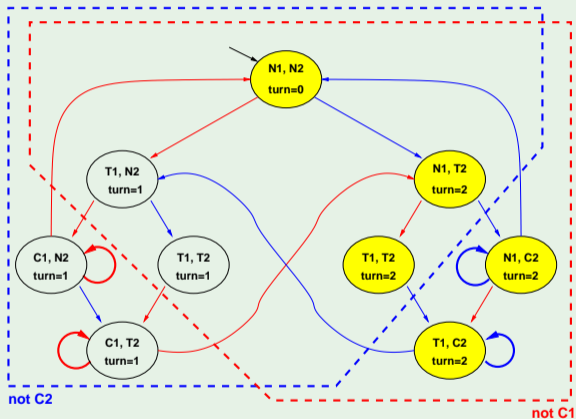
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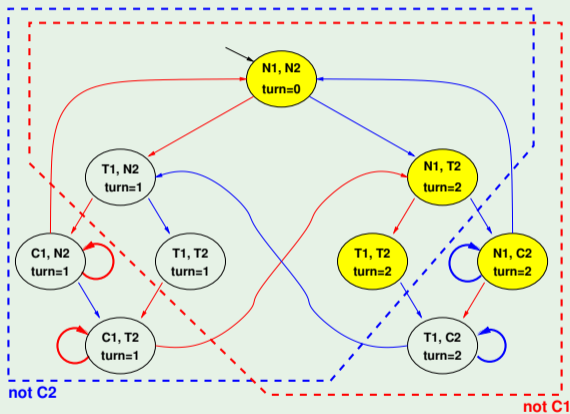
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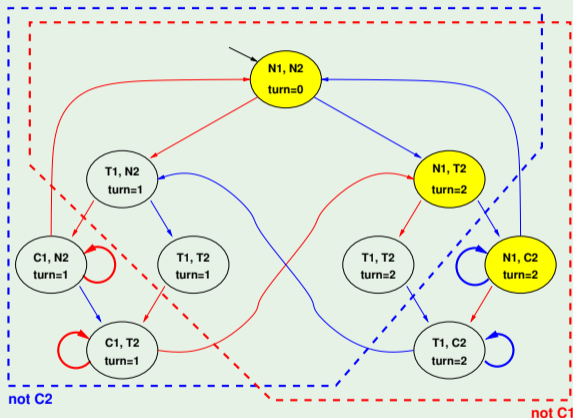
$[E_f G \neg C_1]$

$[E_f G g] = \nu Z. [g] \wedge [EXE(ZU(Z \wedge F_1))] \wedge [EXE(ZU(Z \wedge F_2))]$

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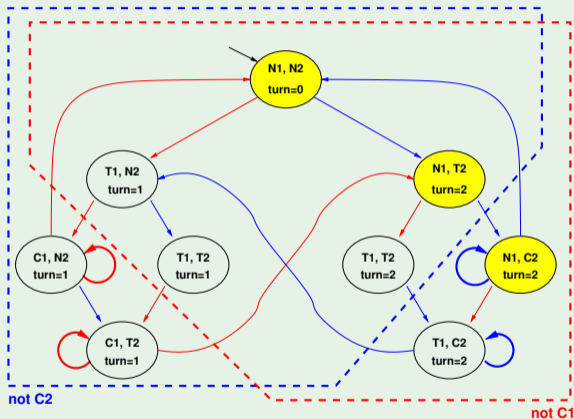
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The Main Problem of M.C.: State Space Explosion

- **The bottleneck:**
 - Exhaustive analysis may require to store all the states of the Kripke structure, and to explore them one-by-one
 - The state space may be exponential in the number of components and variables
 - E.g., 300 Boolean vars \implies up to $2^{300} \approx 10^{100}$ states!
 - State Space Explosion:
 - too much memory required
 - too much CPU time required to explore each state
- **A solution: Symbolic Model Checking**

Symbolic Model Checking

Symbolic representation:

- manipulation of **sets of states** (rather than single states);
- sets of states represented by **formulae in propositional logic**;
 - set cardinality not directly correlated to size
- expansion of **sets of transitions** (rather than single transitions);

Symbolic Model Checking [cont.]

- Two main symbolic techniques:
 - Ordered Binary Decision Diagrams (OBDDs)
 - Propositional Satisfiability Checkers (SAT solvers)
- Different model checking algorithms:
 - Fix-point Model Checking (historically, for CTL)
 - Fix-point Model Checking for LTL (conversion to fair CTL MC)
 - Bounded Model Checking (historically, for LTL)
 - Invariant Checking
 - ...

Symbolic Representation of Kripke Models

- **Symbolic representation:**
 - **sets of states** as their **characteristic function** (Boolean formula)
 - provide logical representation and transformations of characteristic functions
- **Example:**
 - three state variables x_1, x_2, x_3 :
 $\{000, 001, 010, 011\}$ represented as “first bit false”: $\neg x_1$
 - with five state variables x_1, x_2, x_3, x_4, x_5 :
 $\{00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111, \dots, 01111\}$ still represented as “first bit false”: $\neg x_1$

Kripke Models in Propositional Logic

- Let $M = (S, I, R, L, AF)$ be a Kripke model
- States $s \in S$ are described by means of **an array V of Boolean state variables**.
- A **state** is a **truth assignment** to each atomic proposition in V .
 - **0100** is represented by the formula $(\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4)$
 - we call $\xi(s)$ the formula representing the state $s \in S$
(Intuition: $\xi(s)$ holds iff the system is in the state s)
- A set of states $Q \subseteq S$ can be represented by any formula which is logically equivalent to the formula $\xi(Q)$:

$$\bigvee_{s \in Q} \xi(s)$$

(Intuition: $\xi(Q)$ holds iff the system is in one of the states $s \in Q$)

- Bijection between models of $\xi(Q)$ and states in Q

Remark

- Every propositional formula is a (typically very compact) representation of the set of assignments satisfying it
- Any formula equivalent to $\xi(Q)$ is a representation of Q
 \implies Typically Q can be encoded by much smaller formulas than $\bigvee_{s \in Q} \xi(s)$!
- Example: $Q = \{ 00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111, \dots, 01111 \}$
represented as “first bit false”: $\neg x_1$

$$\bigvee_{s \in Q} \xi(s) = \left. \begin{array}{l} (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge x_5) \vee \\ (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5) \vee \\ \dots \\ (\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5) \end{array} \right\} 2^4 \text{ disjuncts}$$

Symbolic Representation of Set Operators

One-to-one correspondence between sets and Boolean operators

- Set of all the states: $\xi(S) := \top$
- Empty set : $\xi(\emptyset) := \perp$
- Union represented by disjunction:
 $\xi(P \cup Q) := \xi(P) \vee \xi(Q)$
- Intersection represented by conjunction:
 $\xi(P \cap Q) := \xi(P) \wedge \xi(Q)$
- Complement represented by negation:
 $\xi(S/P) := \neg \xi(P)$

Symbolic Representation of Transition Relations

- The transition relation R is a set of pairs of states: $R \subseteq S \times S$
- A transition is a pair of states (s, s')
- A new vector of variables V' (the next state vector) represents the value of variables after the transition has occurred
- $\xi(s, s')$ defined as $\xi(s) \wedge \xi(s')$ (Intuition: $\xi(s, s')$ holds iff the system is in the state s and moves to state s' in next step)
- The transition relation R can be represented by any formula equivalent to:

$$\bigvee_{(s,s') \in R} \xi(s, s') = \bigvee_{(s,s') \in R} (\xi(s) \wedge \xi(s'))$$

Each formula equivalent to $\xi(R)$ is a representation of R

\implies Typically R can be encoded by a much smaller formula than $\bigvee_{(s,s') \in R} \xi(s) \wedge \xi(s')$!

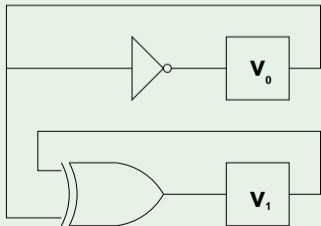
Example: a simple counter

```
MODULE main
VAR
  v0      : boolean;
  v1      : boolean;
  out     : 0..3;

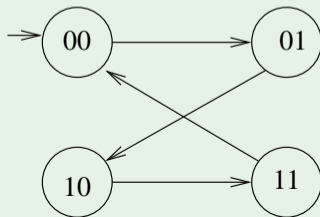
ASSIGN
  init(v0) := 0;
  next(v0) := !v0;

  init(v1) := 0;
  next(v1) := (v0 xor v1);

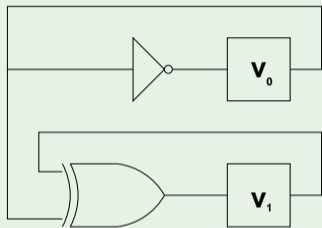
  out := toint(v0) + 2*toint(v1);
```



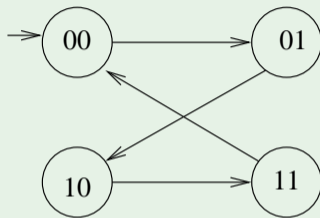
v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



Example: a simple counter [cont.]



v_1	v_0	v_1'	v_0'
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

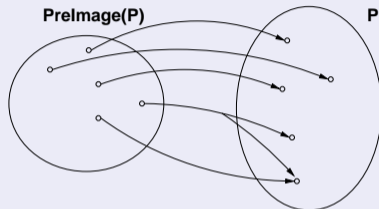


$$\xi(R) = (v_0' \leftrightarrow \neg v_0) \wedge (v_1' \leftrightarrow v_0 \oplus v_1)$$

$$\begin{aligned} \bigvee_{(s,s') \in R} \xi(s) \wedge \xi(s') = & (\neg v_1 \wedge \neg v_0 \wedge \neg v_1' \wedge v_0') \vee \\ & (\neg v_1 \wedge v_0 \wedge v_1' \wedge \neg v_0') \vee \\ & (v_1 \wedge \neg v_0 \wedge v_1' \wedge v_0') \vee \\ & (v_1 \wedge v_0 \wedge \neg v_1' \wedge \neg v_0') \end{aligned}$$

Pre-Image

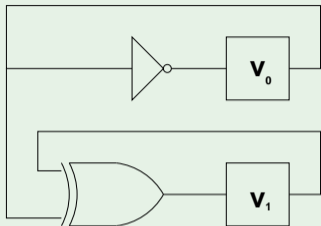
- (Backward) **pre-image** of a set of states:



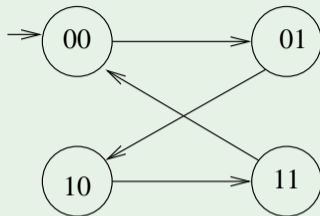
Evaluate one-shot all transitions ending in the states of the set

- Set theoretic view: $PreImage(P, R) := \{s \mid \text{for some } s' \in P, (s, s') \in R\}$
- Logical view: $\xi(PreImage(P, R)) := \exists V'. (\xi(P)[V'] \wedge \xi(R)[V, V'])$
- μ over V is s.t. $\mu \models \exists V'. (\xi(P)[V'] \wedge \xi(R)[V, V'])$ iff,
for some μ' over V' , we have: $\mu \cup \mu' \models (\xi(P)[V'] \wedge \xi(R)[V, V'])$,
i.e., $\mu' \models \xi(P)[V']$ and $\mu \cup \mu' \models \xi(R)[V, V']$
 - Intuition: $\mu \iff s, \mu' \iff s', \mu \cup \mu' \iff \langle s, s' \rangle$

Example: simple counter



v_1	v_0	v'_1	v'_0
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0

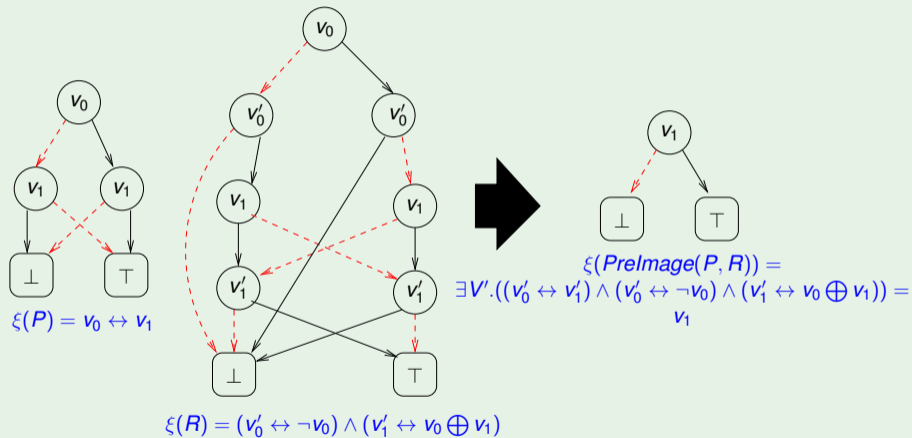


$$\xi(R) = (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \oplus v_1)$$

$$\xi(P) := (v_0 \leftrightarrow v_1) \text{ (i.e., } P = \{00, 11\})$$

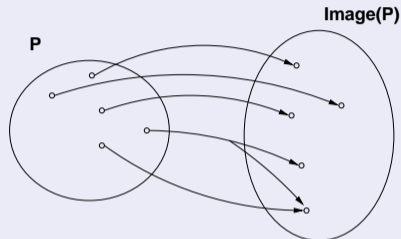
$$\begin{aligned} \xi(\text{PreImage}(P, R)) &= \\ \exists V'. (\xi(P)[V'] \wedge \xi(R)[V, V']) &= \\ \exists v'_0 v'_1. ((v'_0 \leftrightarrow v'_1) \wedge (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \oplus v_1)) &= \\ \underbrace{(\neg v_0 \wedge v_0 \oplus v_1)}_{v'_0=T, v'_1=T} \vee \underbrace{\perp}_{v'_0=T, v'_1=\perp} \vee \underbrace{\perp}_{v'_0=\perp, v'_1=T} \vee \underbrace{(v_0 \wedge \neg(v_0 \oplus v_1))}_{v'_0=\perp, v'_1=\perp} &= \\ v_1 \text{ (i.e., } \{10, 11\}) & \end{aligned}$$

Pre-Image [cont.]



Forward Image

- Forward image of a set:



Evaluate one-shot all transitions from the states of the set

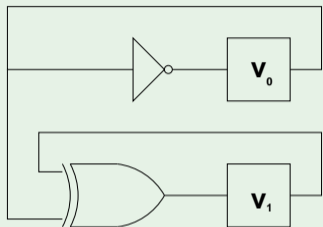
- Set theoretic view:

$$\text{Image}(P, R) := \{s' \mid \text{for some } s \in P, (s, s') \in R\}$$

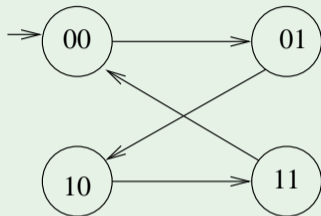
- Logical Characterization:

$$\xi(\text{Image}(P, R)) := \exists V. (\xi(P)[V] \wedge \xi(R)[V, V'])$$

Example: simple counter



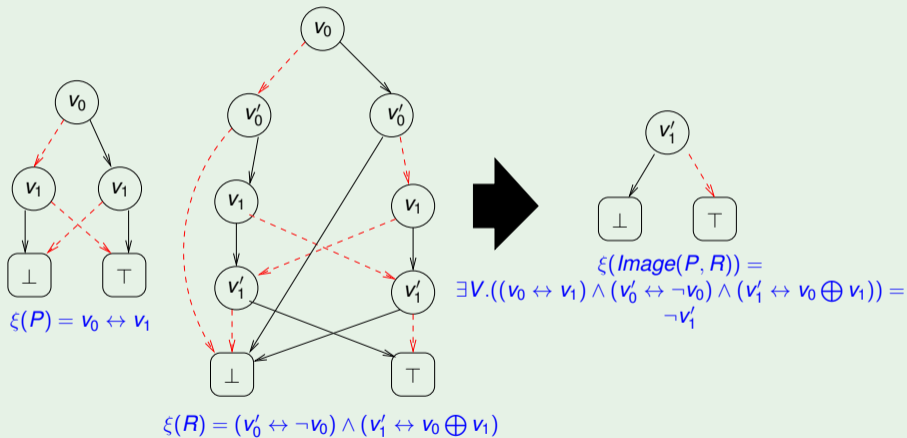
v_1	v_0	v'_1	v'_0
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	0



$$\xi(R) = (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \oplus v_1)$$
$$\xi(P) := (v_0 \leftrightarrow v_1) \text{ (i.e., } P = \{00, 11\})$$

$$\begin{aligned}\xi(\text{Image}(P, R)) &= \exists V. (\xi(P)[V] \wedge \xi(R)[V, V']) \\ &= \exists V. ((v_0 \leftrightarrow v_1) \wedge (v'_0 \leftrightarrow \neg v_0) \wedge (v'_1 \leftrightarrow v_0 \oplus v_1)) \\ &= \dots \\ &= \neg v'_1 \quad (\text{i.e., } \{00, 01\})\end{aligned}$$

Forward Image [cont.]



Application of the Transition Relation

- Image and PreImage of a set of states S computed by means of **quantified Boolean formulae**
- The whole set of transitions can be fired (either forward or backward) in **one logical operation**
- The symbolic computation of PreImage and Image provide the primitives for symbolic search of the state space of FSM's

Notation Remark

Henceforth, for readability sake, we omit the “ $\xi()$ ” notation in symbolic representations of systems.

- Kripke models represented as $\langle I(V), R(V, V') \rangle$
- Fair Kripke models represented as $\langle I(V), R(V, V'), F(V) \rangle$ s.t. $F(V) \stackrel{\text{def}}{=} \{F_1(V), \dots, F_k(V)\}$

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STATE-SET Check(CTL_formula β) {

case β **of**

\top : **return** S ;

\perp : **return** \emptyset ;

$\neg\beta_1$: **return** $S \setminus \text{Check}(\beta_1)$;

$\beta_1 \wedge \beta_2$: **return** $(\text{Check}(\beta_1) \cap \text{Check}(\beta_2))$;

EX β_1 : **return** $\text{PreImage}(\text{Check}(\beta_1))$;

EG β_1 : **return** $\text{Check_EG}(\text{Check}(\beta_1))$;

E(β_1 **U** β_2): **return** $\text{Check_EU}(\text{Check}(\beta_1), \text{Check}(\beta_2))$;

}

General Symbolic CTL MC Procedure

```
OBDD    Check(CTL_formula  $\beta$ ) {  
  if (In_OBDD_Hash( $\beta$ )) return OBDD_Get_From_Hash( $\beta$ );  
  case  $\beta$  of  
     $\top$ :          return obdd_true;  
     $\perp$ :          return obdd_false;  
     $\neg\beta_1$ :      return  $\neg$  Check( $\beta_1$ );  
     $\beta_1 \wedge \beta_2$ : return (Check( $\beta_1$ )  $\wedge$  Check( $\beta_2$ ));  
    EX $\beta_1$ :       return PreImage(Check( $\beta_1$ ));  
    EG $\beta_1$ :       return Check_EG(Check( $\beta_1$ ));  
    E( $\beta_1$  U  $\beta_2$ ): return Check_EU(Check( $\beta_1$ ), Check( $\beta_2$ ));  
  }
```

Some primitive functions from CTL Model Checking:

- **Symbolic Check_EX(ϕ):**
returns **an OBDD representing** the set of states from which a path verifying **X ϕ** begins (i.e., the **symbolic** preimage of the set of states where ϕ holds)
- **Symbolic Check_EG(ϕ):**
returns **an OBDD representing** the set of states from which a path verifying **G ϕ** begins
- **Symbolic Check_EU(ϕ_1, ϕ_2):**
returns **an OBDD representing** the set of states from which a path verifying **ϕ_1 U ϕ_2** begins

Check_EX

Explicit-state

```
State Set Check_EX(State Set X)  
  return {s | for some  $s' \in X, (s, s') \in R$ };
```

Symbolic

```
OBDD Check_EX(OBDD X)  
  return  $\exists V'. (X[V'] \wedge R[V, V'])$ ;
```

Same as Pre-Image computation.

Explicit-State

```
State Set Check_EG(State Set X)  
  Y' := X;  
  repeat  
    Y := Y';  
    Y' := Y  $\cap$  Check_EX(Y);  
  until (Y' = Y);  
return Y;
```

Symbolic

```
OBDD Check_EG(OBDD X)  
  Y' := X;  
  repeat  
    Y := Y';  
    Y' := Y  $\wedge$  Check_EX(Y);  
  until (Y'  $\leftrightarrow$  Y);  
return Y;
```

Hint (tableaux rule): $s \models \mathbf{EG}\phi$ only if $s \models \phi \wedge \mathbf{EXEG}\phi$

Explicit-State

```
State Set Check_EU(State Set  $X_1, X_2$ )  
   $Y' := X_2$ ;  
  repeat  
     $Y := Y'$ ;  
     $Y' := Y \cup (X_1 \cap \text{Check\_EX}(Y))$ ;  
  until ( $Y' = Y$ );  
return  $Y$ ;
```

Symbolic

```
OBDD Check_EU(OBDD  $X_1, X_2$ )  
   $Y' := X_2$ ;  
  repeat  
     $Y := Y'$ ;  
     $Y' := Y \vee (X_1 \wedge \text{Check\_EX}(Y))$ ;  
  until ( $Y' \leftrightarrow Y$ );  
return  $Y$ ;
```

Hint (tableaux rule): $s \models \mathbf{E}(\phi_1 \mathbf{U} \phi_2)$ if $s \models \phi_2 \vee (\phi_1 \wedge \mathbf{EXE}(\phi_1 \mathbf{U} \phi_2))$

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Language-Emptiness Checking for Fair Kripke Models

Fair_CheckEG

Given: a fair Kripke model $M_F := \langle S, R, I, AP, L, F \rangle$ and a CTL formula φ s.t. $[\varphi] \subseteq S$,
 $\text{Fair_CheckEG}(\varphi)$ returns the subset of the states s in $[\varphi]$ from which at least one fair path π entirely included in $[\varphi]$ passes through

Symbolic Fair_CheckEG

Given: the symbolic representation of a fair Kripke model $M_F := \langle I, R, F \rangle$
and a Boolean formula (OBDD) Ψ ,
 $\text{Fair_CheckEG}(\Psi)$ returns a Boolean formula (OBDD) representing the subset of the states s in Ψ from which at least one fair path π entirely included in Ψ passes through

$\text{Fair_CheckEG}(\text{true})$ computes (the symbolic representation of) the set of fair states of M_f
 $\implies I \subseteq \text{Fair_CheckEG}(\text{true})$ iff $\mathcal{L}(M_f) \neq \emptyset$

Ingredients (from Symbolic CTL Model Checking)

Some primitive functions from CTL Model Checking:

- **Symbolic Check_EX(ϕ)**: returns an OBDD representing the set of states from which a path verifying $\mathbf{X}\phi$ begins
(i.e., the symbolic preimage of the set of states where ϕ holds)
- **Symbolic Check_EG(ϕ)**: returns an OBDD representing the set of states from which a path verifying $\mathbf{G}\phi$ begins
- **Symbolic Check_EU(ϕ_1, ϕ_2)**: returns an OBDD representing the set of states from which a path verifying $\phi_1 \mathbf{U} \phi_2$ begins

Emerson-Lei Algorithm

Recall: $[E_f G \phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT} [EX E(Z \cup (Z \cap F_i))])$

```
state_set Check_FairEG(state_set [ $\phi$ ]) {  
   $Z' := [\phi]$ ;  
  repeat  
     $Z := Z'$ ;  
    for each  $F_i$  in FT  
       $Y := \text{Check\_EU}(Z', F_i \cap Z')$ ;  
       $Z' := Z' \cap \text{PreImage}(Y)$ ;  
    end for;  
  until ( $Z' = Z$ );  
  return  $Z$ ;  
}
```

Slight improvement: do not consider states in $Z \setminus Z'$

Emerson-Lei Algorithm (symbolic version)

Recall: $[E_f G \phi] = \nu Z.([\phi] \cap \bigcap_{F_i \in FT} [EX E(ZU(Z \wedge F_i))])$

```
Obdd Check_FairEG( Obdd  $\phi$  ) {  
   $Z' := \phi$ ;  
  repeat  
     $Z := Z'$ ;  
    for each  $F_i$  in FT  
       $Y := \text{Check\_EU}(Z', F_i \wedge Z')$ ;  
       $Z' := Z' \wedge \text{PreImage}(Y)$ ;  
    end for;  
  until ( $Z' \leftrightarrow Z$ );  
  return  $Z$ ;  
}
```

Symbolic version.

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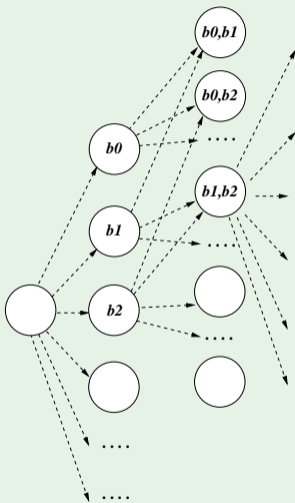
A simple example

```
MODULE main
VAR
  b0 : boolean;
  b1 : boolean;
  ...
ASSIGN
  init(b0) := 0;
  next(b0) := case
    b0 : 1;
    !b0 : {0,1};
  esac;
  init(b1) := 0;
  next(b1) := case
    b1 : 1;
    !b1 : {0,1};
  esac;
  ...
```

A simple example [cont.]

- N Boolean variables b_0, b_1, \dots
- Initially, all variables set to 0
- Each variable can pass from 0 to 1, but not vice-versa
- 2^N states, all reachable
- (Simplified) model of a student career behaviour.

A simple example: FSM

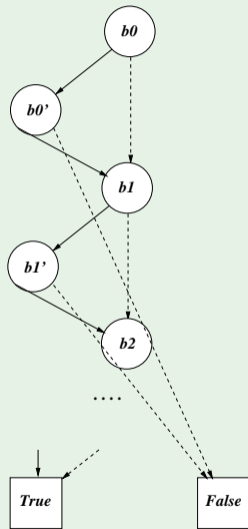


(transitive transitions omitted)

2^N STATES

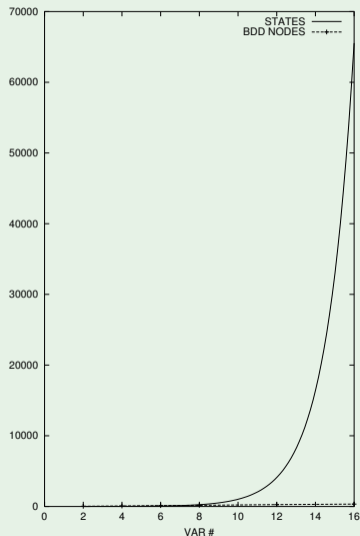
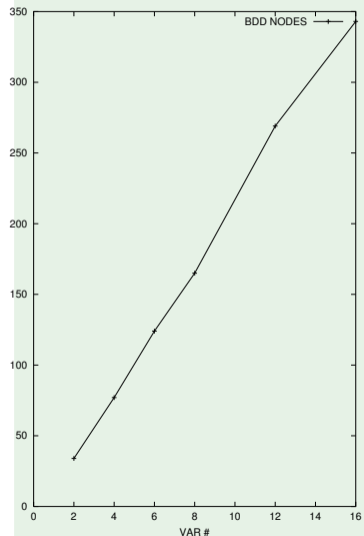
$O(2^N)$ TRANSITIONS

A simple example: $OBDD(\xi(R))$



$2N + 2$ NODES

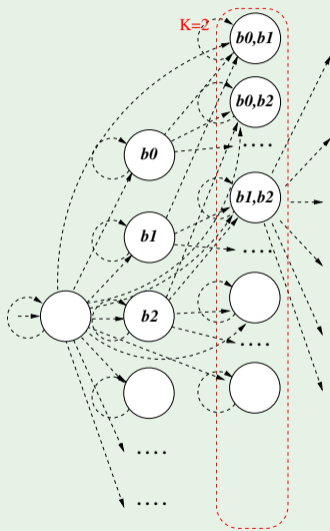
A simple example: states vs. OBDD nodes [NuSMV.2]



A simple example: reaching K bits true

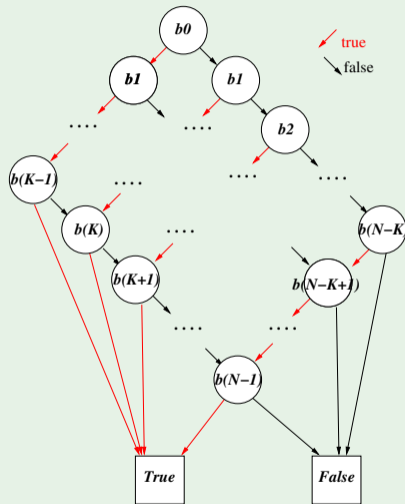
- Property **EF**($b_0 + b_1 + \dots + b_{(N-1)} \geq K$) ($K \leq N$)
(it may be reached a state in which K bits are true)
- E.g.: “it is reachable a state where K exams are passed”

A simple example: FSM



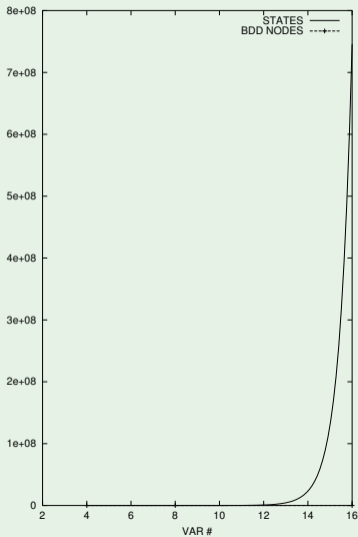
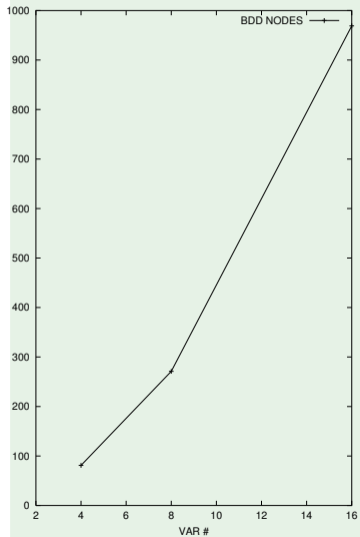
$$\binom{N}{K} + \binom{N}{K+1} + \dots + \binom{N}{N}$$

A simple example: $OBDD(\xi(\varphi))$



$(N - K + 1) \cdot K + 2$ NODES

A simple example: states vs. OBDD nodes [NuSMV.2]



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Symbolic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$\models \psi \quad (\text{LTL})$$

$$\iff \neg\psi \text{ unsat}$$

$$\iff \mathcal{L}(T_{\neg\psi}) = \emptyset$$

- $T_{\neg\psi}$ is a **fair Kripke model** (aka **tableaux**) which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

$$\varphi \models \psi \quad (\text{LTL})$$

$$\models \varphi \rightarrow \psi \quad (\text{LTL})$$

$$\iff \varphi \wedge \neg\psi \text{ unsat}$$

$$\iff \mathcal{L}(T_{\varphi \wedge \neg\psi}) = \emptyset$$

- $T_{\varphi \wedge \neg\psi}$ is a **fair Kripke model** (aka **tableaux**) which represents all and only the paths that satisfy $\varphi \wedge \neg\psi$ (satisfy φ and do not satisfy ψ)

Symbolic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$M \models \psi \quad (\text{LTL})$$

$$\iff \mathcal{L}(M) \subseteq \mathcal{L}(\psi)$$

$$\iff \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset$$

$$\iff \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset$$

$$\iff \mathcal{L}(M) \cap \mathcal{L}(T_{\neg\psi}) = \emptyset$$

$$\iff \mathcal{L}(M \times T_{\neg\psi}) = \emptyset$$

- $T_{\neg\psi}$ is a **fair Kripke model** (aka **tableaux**) which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\implies M \times T_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ .

Symbolic LTL Model Checking

Three steps

Let $\varphi \stackrel{\text{def}}{=} \neg\psi$:

- (i) Compute T_φ
- (ii) Compute the product $M \times T_\varphi$
- (iii) Check the emptiness of $\mathcal{L}(M \times T_\varphi)$

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The Set of States

- Elementary subformulas of ψ : $el(\psi)$
 - $el(p) := \{p\}$
 - $el(\neg\varphi_1) := el(\varphi_1)$
 - $el(\varphi_1 \wedge \varphi_2) := el(\varphi_1) \cup el(\varphi_2)$
 - $el(\mathbf{X}\varphi_1) = \{\mathbf{X}\varphi_1\} \cup el(\varphi_1)$
 - $el(\varphi_1 \mathbf{U}\varphi_2) := \{\mathbf{X}(\varphi_1 \mathbf{U}\varphi_2)\} \cup el(\varphi_1) \cup el(\varphi_2)$
- Intuition: $el(\psi)$ is the set of propositions and \mathbf{X} -formulas occurring in ψ , ψ' being the result of applying recursively the tableau expansion rules to ψ
- The set of states S_{T_ψ} of T_ψ is given by $2^{el(\psi)}$
- The labeling function L_{T_ψ} of T_ψ comes straightforwardly (the label is the Boolean component of each state)

Example: $\psi := p\mathbf{U}q$

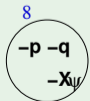
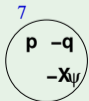
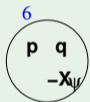
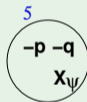
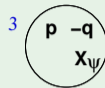
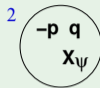
- $el(p\mathbf{U}q) = el((q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))) = \{p, q, \mathbf{X}(p\mathbf{U}q)\}$

$$\implies S_{T_\psi} = \{$$

- | | | |
|-----|--|-----------------------|
| 1 : | $\{p, q, \mathbf{X}(p\mathbf{U}q)\},$ | $[p\mathbf{U}q]$ |
| 2 : | $\{\neg p, q, \mathbf{X}(p\mathbf{U}q)\},$ | $[p\mathbf{U}q]$ |
| 3 : | $\{p, \neg q, \mathbf{X}(p\mathbf{U}q)\},$ | $[p\mathbf{U}q]$ |
| 4 : | $\{\neg p, q, \neg\mathbf{X}(p\mathbf{U}q)\},$ | $[p\mathbf{U}q]$ |
| 5 : | $\{\neg p, \neg q, \mathbf{X}(p\mathbf{U}q)\},$ | $[\neg p\mathbf{U}q]$ |
| 6 : | $\{p, q, \neg\mathbf{X}(p\mathbf{U}q)\},$ | $[p\mathbf{U}q]$ |
| 7 : | $\{p, \neg q, \neg\mathbf{X}(p\mathbf{U}q)\},$ | $[\neg p\mathbf{U}q]$ |
| 8 : | $\{\neg p, \neg q, \neg\mathbf{X}(p\mathbf{U}q)\}$ | $[\neg p\mathbf{U}q]$ |

}

Example: $\psi := p \mathbf{U} q$ [cont.]



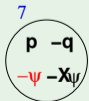
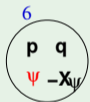
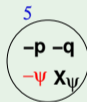
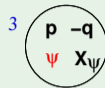
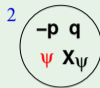
sat()

- Set of states in S_{T_ψ} satisfying φ_i : $sat(\varphi_i)$
 - $sat(\varphi_1) := \{s \mid \varphi_1 \in s\}, \varphi_1 \in el(\psi)$
 - $sat(\neg\varphi_1) := S_{T_\psi} / sat(\varphi_1)$
 - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \cap sat(\varphi_2)$
 - $sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \cup (sat(\varphi_1) \cap sat(\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)))$
- intuition: $sat()$ establishes in which states subformulas are true

Remark

- Semantics of “ $\varphi_1 \mathbf{U} \varphi_2$ ” here induced by tableaux rule: $\varphi_1 \mathbf{U} \varphi_2 \stackrel{\text{def}}{=} \varphi_2 \vee (\varphi_1 \wedge \mathbf{X}(\varphi_1 \mathbf{U} \varphi_2))$
- \implies weaker than standard semantics (aka “**weak until**”, “ $\varphi_1 \mathbf{W} \varphi_2$ ”):
a path where φ_1 is always true and φ_2 is always false satisfies it

Example: $\psi := p \mathbf{U} q$ [cont.]



Initial States and Transition Relation

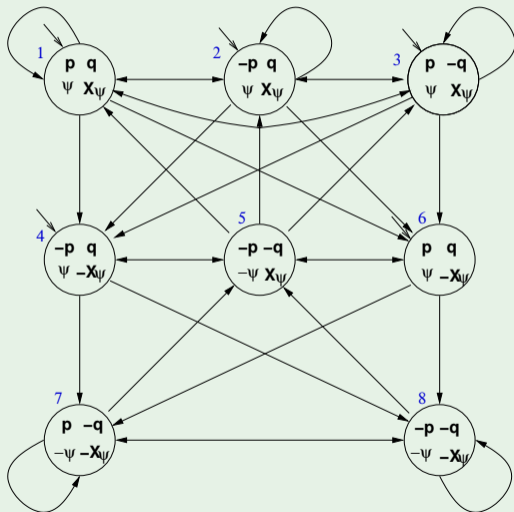
- Set of states in S_{T_ψ} satisfying φ_i : $sat(\varphi_i)$
 - $sat(\varphi_1) := \{s \mid \varphi_1 \in s\}, \varphi_1 \in el(\psi)$
 - $sat(\neg\varphi_1) := S_{T_\psi} / sat(\varphi_1)$
 - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \cap sat(\varphi_2)$
 - $sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \cup (sat(\varphi_1) \cap sat(\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)))$
- Intuition: $sat()$ establishes in which states subformulas are true
- The set of initial states I_{T_ψ} is defined as

$$I_{T_\psi} = sat(\psi)$$

- The transition relation R_{T_ψ} is defined as

$$R_{T_\psi}(s, s') = \bigcap_{\mathbf{X}\varphi_i \in el(\psi)} \{(s, s') \mid s \in sat(\mathbf{X}\varphi_i) \Leftrightarrow s' \in sat(\varphi_i)\}$$

Example: $\psi := p \mathbf{U} q$ [cont.]



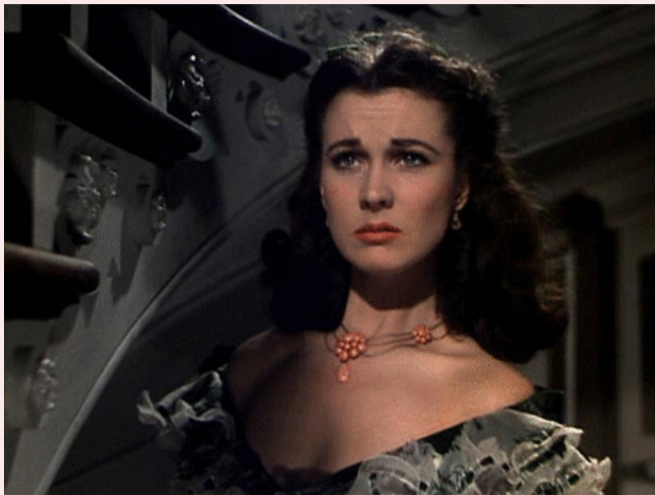
Problems with **U**-subformulas

- R_{T_ψ} does not guarantee that the **U**-subformulas are fulfilled
- Example: state 3 $\{p, \neg q, \mathbf{X}(p\mathbf{U}q)\}$:
although state 3 belongs to

$$sat(p\mathbf{U}q) := sat(q) \cup (sat(p) \cap sat(\mathbf{X}(p\mathbf{U}q))),$$

the path which loops forever in state 3 does not satisfy $p\mathbf{U}q$, as q never holds in that path.

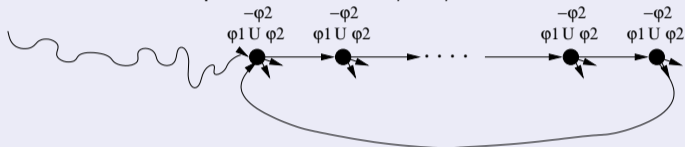
Tableaux Rules: a Quote



*"After all... tomorrow is another day."
[Scarlett O'Hara, "Gone with the Wind"]*

Fairness conditions for every **U**-subformula

- It must never happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.



\Rightarrow For every [positive] **U**-subformula $\varphi_1 \mathbf{U} \varphi_2$ of ψ , we must add a **fairness LTL condition** $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$

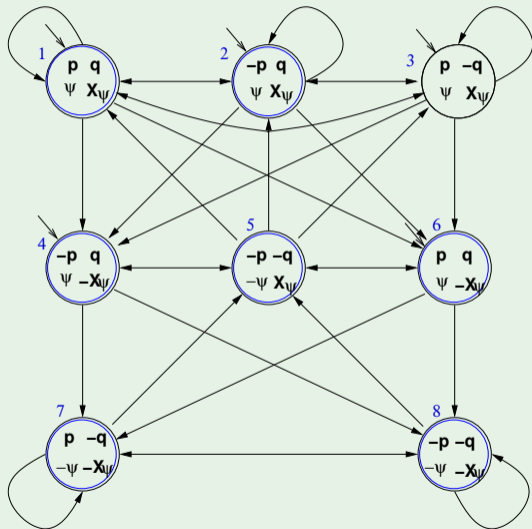
If no [positive] **U**-subformulas, then add one fairness condition $\mathbf{GF}\top$.

\Rightarrow We restrict the admissible paths of T_ψ to those which verify the fairness condition:

$$T_\psi := \langle S_{T_\psi}, I_{T_\psi}, R_{T_\psi}, L_{T_\psi}, F_{T_\psi} \rangle$$

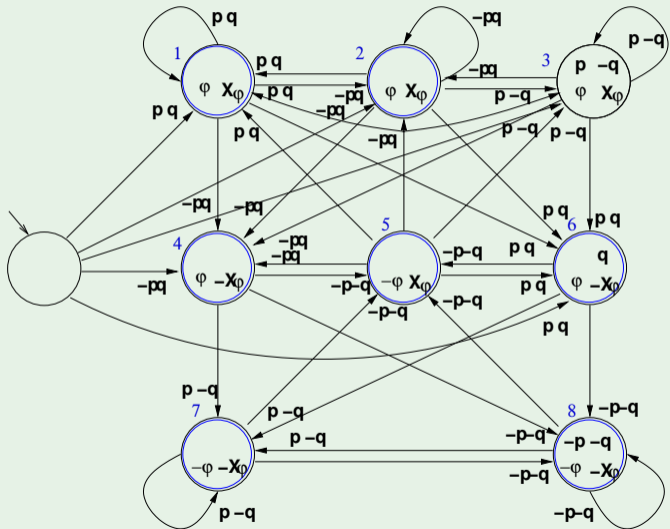
$$F_{T_\psi} := \{ \text{sat}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2) \text{ s.t. } (\varphi_1 \mathbf{U} \varphi_2) \text{ occurs [positively] in } \psi \}$$

Example: $\psi := p \mathbf{U} q$ [cont.]



Note: easily transformed into a generalized Büchi automaton

Example: $\psi := p \mathbf{U} q$ [cont.]



Note: easily transformed into a generalized Büchi automaton

Symbolic Representation of T_ψ

- State variables: one Boolean variable for each formula in $eI(\psi)$
 - EX: p, q and x and primed versions p', q' and x'
[x is a Boolean label for $\mathbf{X}(p\mathbf{U}q)$]
- $sat(\varphi_i)$:
 - $sat(p) := p$, s.t. p Boolean state variable
 - $sat(\neg\varphi_1) := \neg sat(\varphi_1)$
 - $sat(\varphi_1 \wedge \varphi_2) := sat(\varphi_1) \wedge sat(\varphi_2)$
 - $sat(\mathbf{X}\varphi_i) := x_{[x_{\varphi_i}]}$, s.t. $x_{[x_{\varphi_i}]}$ Boolean state variable
 - $sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \vee (sat(\varphi_1) \wedge sat(\mathbf{X}(\varphi_1 \mathbf{U} \varphi_2)))$
 - $\Rightarrow sat(\varphi_1 \mathbf{U} \varphi_2) := sat(\varphi_2) \vee (sat(\varphi_1) \wedge x_{[x_{\varphi_1 \mathbf{U} \varphi_2}]})$
- ...

Symbolic Representation of T_ψ [cont.]

- ...
- Initial states: $I_{T_\psi} = \text{sat}(\psi)$
 - EX: $I(p, q, x) = q \vee (p \wedge x)$
- Transition Relation: $R_{T_\psi}(s, s') = \bigcap_{\mathbf{x}\varphi_i \in \text{el}(\psi)} \{(s, s') \mid s \in \text{sat}(\mathbf{X}\varphi_i) \leftrightarrow s' \in \text{sat}(\varphi_i)\}$
 - $R_{T_\psi} = \bigwedge_{\mathbf{x}\varphi_i \in \text{el}(\psi)} (\text{sat}(\mathbf{X}\varphi_i) \leftrightarrow \text{sat}'(\varphi_i))$
where $\text{sat}'(\varphi_i)$ is $\text{sat}(\varphi_i)$ on primed variables
 - EX: $R_{T_\psi}(p, q, x, p', q', x') = x \leftrightarrow (q' \vee (p' \wedge x'))$
- **Fairness Conditions:** $F_{T_\psi} := \{\text{sat}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)\}$ s.t. $(\varphi_1 \mathbf{U} \varphi_2)$ occurs [positively] in ψ
 - EX: $F_{T_\psi}(p, q, x) = \neg(q \vee (p \wedge x)) \vee q = \dots = \neg p \vee \neg x \vee q$

Symbolic Representation of T_ψ : Examples

- $I_{T_\psi}(p, q, x) = q \vee (p \wedge x)$

- 1 : $\{p, q, x\} \models I_{T_\psi}$

- 3 : $\{p, \neg q, x\} \models I_{T_\psi}$

- 5 : $\{\neg p, \neg q, x\} \not\models I_{T_\psi}$

- $R_{T_\psi}(p, q, x, p', q', x') = x \leftrightarrow (q' \vee (p' \wedge x'))$

- 1 \Rightarrow 1 : $\{p, q, x, p', q', x'\} \models R_{T_\psi}$

- 6 \Rightarrow 7 : $\{p, q, \neg x, p', \neg q', \neg x'\} \models R_{T_\psi}$

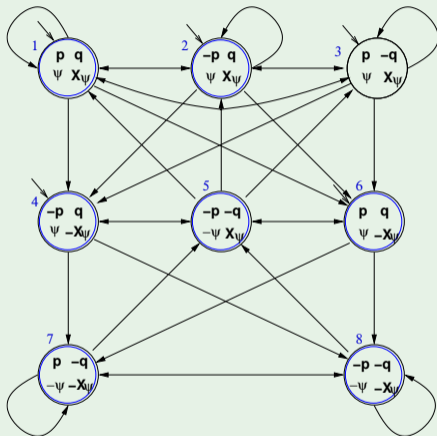
- 6 $\not\Rightarrow$ 1 : $\{p, q, \neg x, p', q', x'\} \not\models R_{T_\psi}$

- $F_{T_\psi}(p, q, x) = \neg p \vee \neg x \vee q$

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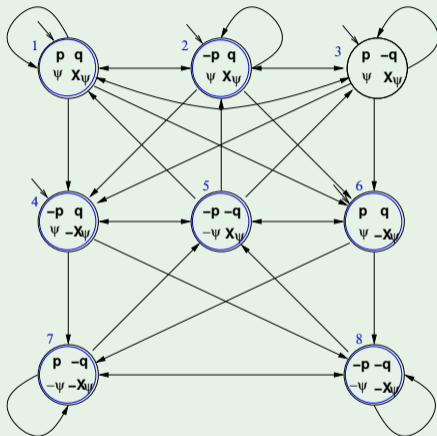
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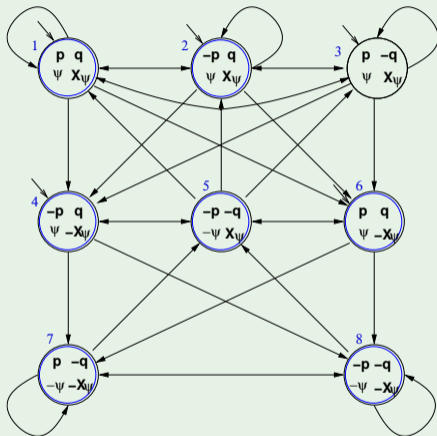
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Symbolic Representation of T_ψ : Examples

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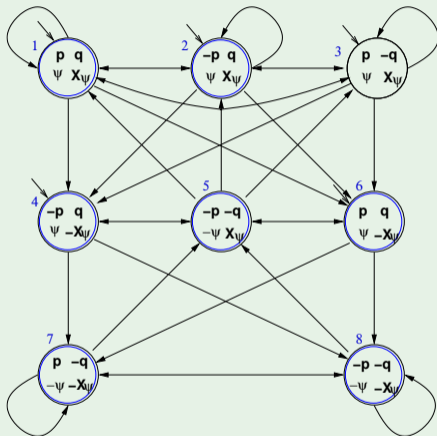
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 - Fairness & Fair Kripke Models
 - Fair CTL Model Checking
 - SCC-Based Approach
 - Emerson-Lei Algorithm
- 2 CTL Symbolic Model Checking
 - Symbolic Representation of Systems
 - Symbolic CTL MC
 - Symbolic Fair CTL MC
 - A simple example
- 3 The Symbolic Approach to LTL Model Checking
 - General Ideas
 - Compute the Tableau T_ψ
 - **Compute the Product $M \times T_\psi$**
 - Check the Emptiness of $\mathcal{L}(M \times T_\psi)$
- 4 A Complete Example
- 5 Exercises

Computing the product $P := T_\psi \times M$

- Given $M := \langle S_M, I_M, R_M, L_M \rangle$ and $T_\psi := \langle S_{T_\psi}, I_{T_\psi}, R_{T_\psi}, L_{T_\psi}, F_{T_\psi} \rangle$, we compute the product $P := T_\psi \times M = \langle S, I, R, L, F \rangle$ as follows:
 - $S := \{(s, s') \mid s \in S_{T_\psi}, s' \in S_M \text{ and } L_M(s')|_\psi = L_{T_\psi}(s)\}$
 - $I := \{(s, s') \mid s \in I_{T_\psi}, s' \in I_M \text{ and } L_M(s')|_\psi = L_{T_\psi}(s)\}$
 - Given $(s, s'), (t, t') \in S$, $((s, s'), (t, t')) \in R$ iff $(s, t) \in R_{T_\psi}$ and $(s', t') \in R_M$
 - $L((s, s')) = L_{T_\psi}(s) \cup L_M(s')$
- Extension of $\text{sat}()$ and F_{T_ψ} to P :
 - $(s, s') \in \text{sat}(\psi) \iff s \in \text{sat}(\psi)$
 - $F := \{\text{sat}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2) \text{ s.t. } (\varphi_1 \mathbf{U} \varphi_2) \text{ occurs [positively] in } \psi\}$

Computing the product $P := T_\psi \times M$ symbolically

Let V, W be the array of Boolean state variables of T_ψ and M respectively:

- Initial states: $I(V \cup W) = I_{T_\psi}(V) \wedge I_M(W)$
- Transition Relation: $R(V \cup W, V' \cup W') = R_{T_\psi}(V, V') \wedge R_M(W, W')$
- Fairness conditions: $\{F_1(V \cup W), \dots, F_k(V \cup W)\} = \{F_{T_\psi 1}(V), \dots, F_{T_\psi k}(V)\}$

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Main theorem [Clarke, Grumberg & Hamaguchi; 94]

Theorem

THEOREM: $M.s' \models \mathbf{E}\psi$ iff there is a state s in T_ψ s.t. $(s, s') \in \text{sat}(\psi)$ and $T_\psi \times M, (s, s') \models \mathbf{EG}true$ under the fairness conditions:

$\{\text{sat}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)\}$ s.t. $(\varphi_1 \mathbf{U} \varphi_2)$ occurs in ψ .

$\implies M \models \mathbf{E}\psi$ iff $T_\psi \times M \models \mathbf{E}_f \mathbf{G}true$

$\implies M \models \neg\psi$ iff $T_\psi \times M \not\models \mathbf{E}_f \mathbf{G}true$

- LTL M.C. reduced to Fair CTL M.C.!!!
- Symbolic OBDD-based techniques apply.

Note

The transition relation R of $T_\psi \times M$ may not be total.

$\implies \text{Check_FairEG}$ does not consider states without successors, restricting R to the remaining states.

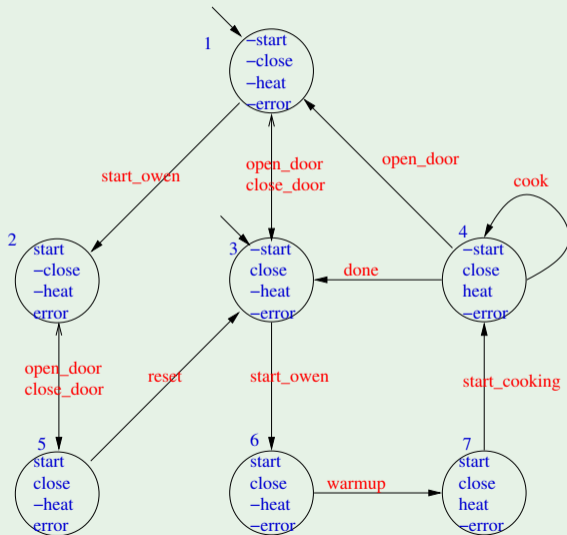
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A microwave oven

- 4 state variables: **start, close, heat, error**
- Actions (implicit): start_oven, open_door, close_door, reset, warmup, start_cooking, cook, done
- Error situation: if oven is started while the door is open
- Represented as a Kripke structure (and hence as a OBDD's)

A microwave oven [cont.]



A microwave oven: symbolic representation

- Initial states: $I_M(s, c, h, e) = \neg s \wedge \neg h \wedge \neg e$
- Transition relation: $R_M(s, c, h, e, s', c', h', e') =$ [a simplification of]
 - ($\neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e'$) \vee (close_door, no error)
 - ($s \wedge \neg c \wedge \neg h \wedge e \wedge s' \wedge c' \wedge \neg h' \wedge e'$) \vee (close_door, error)
 - ($\neg s \wedge c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge \neg c' \wedge \neg h' \wedge \neg e'$) \vee (open_door, no error)
 - ($s \wedge c \wedge \neg h \wedge e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e'$) \vee (open_door, error)
 - ($\neg s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge \neg h' \wedge \neg e'$) \vee (start_oven, no error)
 - ($\neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e'$) \vee (start_oven, error)
 - ($s \wedge c \wedge \neg h \wedge e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e'$) \vee (reset)
 - ($s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge h' \wedge \neg e'$) \vee (warmup)
 - ($s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e'$) \vee (start_cooking)
 - ($\neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e'$) \vee (cook)
 - ($\neg s \wedge c \wedge h \wedge e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e'$) \vee (done)

Note: the third row represents two transitions: $3 \rightarrow 1$ and $4 \rightarrow 1$.

- “necessarily, the oven’s door eventually closes and, till there, the oven does not heat”:

$$M \models \neg \text{heat } \mathbf{U} \text{ close},$$

i.e.,

$$M \models \neg \mathbf{E} \neg (\neg \text{heat } \mathbf{U} \text{ close})$$

Tableau construction for $\psi = \neg(\neg\text{heat } \mathbf{U} \text{ close})$

- $\varphi := \neg\psi = (\neg\text{heat } \mathbf{U} \text{ close})$
- Tableaux expansion: $\psi = \neg(\neg\text{heat } \mathbf{U} \text{ close}) = \neg(\text{close} \vee (\neg\text{heat} \wedge \mathbf{X}(\neg\text{heat } \mathbf{U} \text{ close})))$
- $el(\psi) = el(\varphi) = \{\text{heat}, \text{close}, \mathbf{X}\varphi\}$ ($\{h, c, \mathbf{X}\varphi\}$)
- States:
1 := $\{\neg h, c, \mathbf{X}\varphi\}$, 2 := $\{h, c, \mathbf{X}\varphi\}$, 3 := $\{\neg h, \neg c, \mathbf{X}\varphi\}$,
4 := $\{h, c, \neg\mathbf{X}\varphi\}$, 5 := $\{h, \neg c, \mathbf{X}\varphi\}$, 6 := $\{\neg h, c, \neg\mathbf{X}\varphi\}$,
7 := $\{\neg h, \neg c, \neg\mathbf{X}\varphi\}$, 8 := $\{h, \neg c, \neg\mathbf{X}\varphi\}$

Tableau construction for $\psi = \neg(\neg heat \text{ U } close)$ [cont.]



Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$

- ...

- States:

$$\begin{aligned} 1 &:= \{\neg h, c, \mathbf{X}\varphi\}, & 2 &:= \{h, c, \mathbf{X}\varphi\}, & 3 &:= \{\neg h, \neg c, \mathbf{X}\varphi\}, \\ 4 &:= \{h, c, \neg\mathbf{X}\varphi\}, & 5 &:= \{h, \neg c, \mathbf{X}\varphi\}, & 6 &:= \{\neg h, c, \neg\mathbf{X}\varphi\}, \\ 7 &:= \{\neg h, \neg c, \neg\mathbf{X}\varphi\}, & 8 &:= \{h, \neg c, \neg\mathbf{X}\varphi\} \end{aligned}$$

- $sat()$:

$$\begin{aligned} sat(h) &= \{2, 4, 5, 8\} \implies sat(\neg h) = \{1, 3, 6, 7\}, \\ sat(c) &= \{1, 2, 4, 6\} \implies sat(\neg c) = \{3, 5, 7, 8\}, \\ sat(\mathbf{X}\varphi) &= \{1, 2, 3, 5\} \implies sat(\neg\mathbf{X}\varphi) = \{4, 6, 7, 8\}, \\ sat(\varphi) &= sat(c) \cup (sat(\neg h) \cap sat(\mathbf{X}(\neg h \mathbf{U} c))) = \{1, 2, 3, 4, 6\} \\ \implies sat(\psi) &= sat(\neg\varphi) = \{5, 7, 8\} \end{aligned}$$

Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$ [cont.]

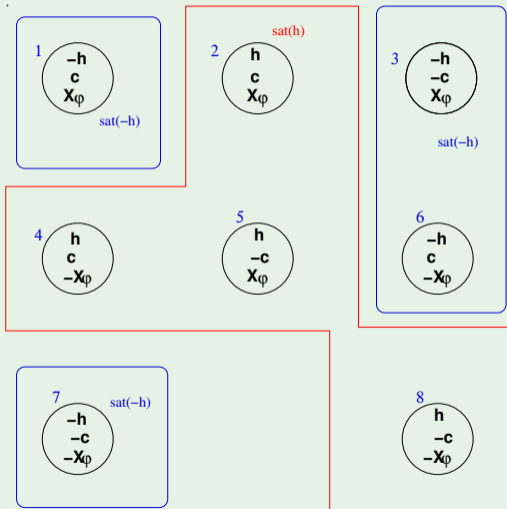


Tableau construction for $\psi = \neg(\neg heat \text{ U } close)$ [cont.]

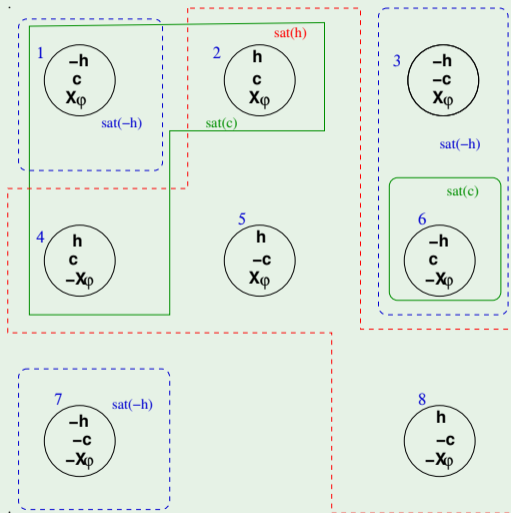


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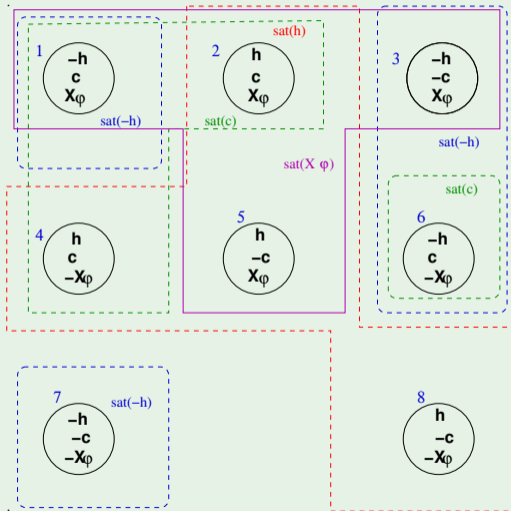


Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$ [cont.]

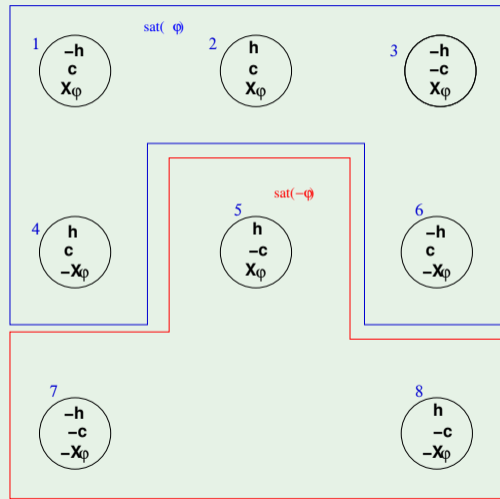


Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$ [cont.]

- ...

- $sat()$:

$$sat(h) = \{2, 4, 5, 8\} \implies sat(\neg h) = \{1, 3, 6, 7\},$$

$$sat(c) = \{1, 2, 4, 6\} \implies sat(\neg c) = \{3, 5, 7, 8\},$$

$$sat(\mathbf{X}\varphi) = \{1, 2, 3, 5\} \implies sat(\neg\mathbf{X}\varphi) = \{4, 6, 7, 8\},$$

$$sat(\varphi) = sat(c) \cup (sat(\neg h) \cap sat(\mathbf{X}(\neg h \mathbf{U} c))) = \{1, 2, 3, 4, 6\}$$

- Initial states I : $sat(\psi) = sat(\neg\varphi) = \{5, 7, 8\}$

- **Transition Relation R :**

- add an edge from every state in $sat(\mathbf{X}\varphi)$ to every state in $sat(\varphi)$
- add an edge from every state in $sat(\neg\mathbf{X}\varphi)$ to every state in $sat(\neg\varphi)$

Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$ [cont.]

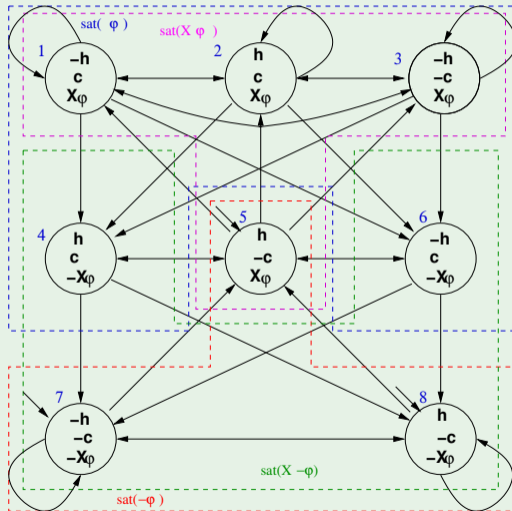
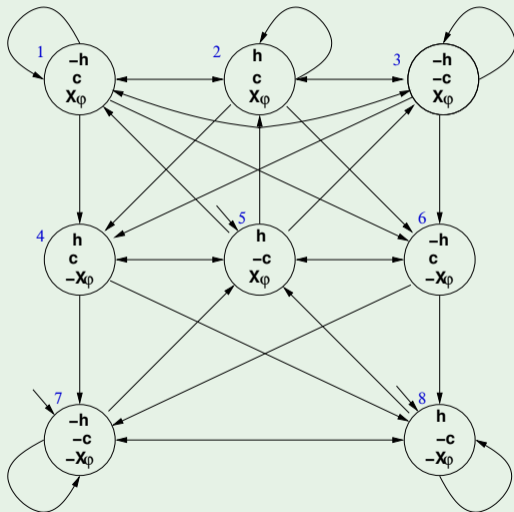


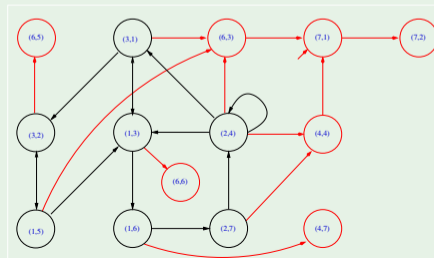
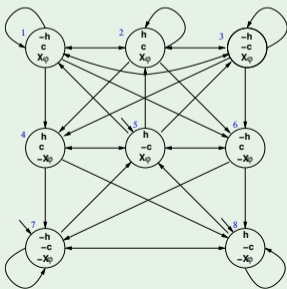
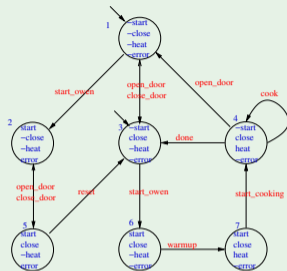
Tableau construction for $\psi = \neg(\neg heat \mathbf{U} close)$ [cont.]



Symbolic representation of T_ψ , s.t. $\psi := \neg(\neg h \mathbf{U} c)$

- State variables: h , c and x and primed versions h' , c' and x'
[x is a Boolean label for $\mathbf{X}(\neg h \mathbf{U} c)$]
- Initial states: $I_{T_\psi} = \text{sat}(\psi)$
 $\implies I(h, c, x) = \neg(c \vee (\neg h \wedge x))$
- Transition Relation: $R_{T_\psi} = \bigwedge_{\mathbf{x}\varphi_i \in \text{el}(\psi)} (\text{sat}(\mathbf{X}\varphi_i) \leftrightarrow \text{sat}'(\varphi_i))$
 $\implies R_{T_\psi}(h, c, x, h', c', x') = x \leftrightarrow (c' \vee (\neg h' \wedge x'))$
- Fairness Property: (due to negative polarity of $(\neg h \mathbf{U} c)$ in ψ):
 $F_{T_\psi}(h, c, x) = \top$

$$\text{Product } P = T_{\psi} \times M$$



Product $P = T_\psi \times M$: symbolic representation

- Initial states: $I(s, c, h, e, x) = (\neg s \wedge \neg h \wedge \neg e) \wedge \neg(c \vee (\neg h \wedge x)) = \neg s \wedge \neg h \wedge \neg e \wedge \neg c \wedge \neg x$
- Transition relation: $R(s, c, h, e, x, s', c', h', e', x') =$ (an OBDD for $(x \leftrightarrow (c' \vee (\neg h' \wedge x')))) \wedge$
 - $(\neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') \vee$ (close_door, no error)
 - $(s \wedge \neg c \wedge \neg h \wedge e \wedge s' \wedge c' \wedge \neg h' \wedge e') \vee$ (close_door, error)
 - $(\neg s \wedge c \wedge \neg h \wedge \neg e \wedge \neg s' \wedge \neg c' \wedge \neg h' \wedge \neg e') \vee$ (open_door, no error)
 - $(s \wedge c \wedge \neg h \wedge e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') \vee$ (open_door, error)
 - $(\neg s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge \neg h' \wedge \neg e') \vee$ (start_oven, no error)
 - $(\neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge s' \wedge \neg c' \wedge \neg h' \wedge e') \vee$ (start_oven, error)
 - $(s \wedge c \wedge \neg h \wedge e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e') \vee$ (reset)
 - $(s \wedge c \wedge \neg h \wedge \neg e \wedge s' \wedge c' \wedge h' \wedge \neg e') \vee$ (warmup)
 - $(s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') \vee$ (start_cooking)
 - $(\neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge h' \wedge \neg e') \vee$ (cook)
 - $(\neg s \wedge c \wedge h \wedge \neg e \wedge \neg s' \wedge c' \wedge \neg h' \wedge \neg e')$ (done)

[**EGtrue**]: symbolic representation

- Emerson-Lei returns (an OBDD equivalent to):

EGtrue =

$$(\neg s \wedge \neg c \wedge \neg h \wedge \neg e \wedge x) \vee \quad (3, 1)$$

$$(s \wedge \neg c \wedge \neg h \wedge e \wedge x) \vee \quad (3, 2)$$

$$(\neg s \wedge c \wedge \neg h \wedge \neg e \wedge x) \vee \quad (1, 3)$$

$$(\neg s \wedge c \wedge h \wedge \neg e \wedge x) \vee \quad (2, 4)$$

$$(s \wedge c \wedge \neg h \wedge e \wedge x) \vee \quad (1, 5)$$

$$(s \wedge c \wedge \neg h \wedge \neg e \wedge x) \vee \quad (1, 5)$$

$$(s \wedge c \wedge h \wedge \neg e \wedge x) \vee \quad (2, 7)$$

...

(other unreachable states)

- Initial states: $I(s, c, h, e, x) = \neg s \wedge \neg h \wedge \neg e \wedge \neg c \wedge \neg x$

$$\implies I(s, c, h, e, x) \not\models \mathbf{EGtrue}$$

$$\implies I \not\subseteq [\mathbf{EGtrue}]$$

$$\implies T_\psi \times M \not\models \mathbf{EGtrue}$$

\implies Property verified!



The property verified is...

Outline

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 - Fairness & Fair Kripke Models
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 - Emerson-Lei Algorithm
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 - Symbolic CTL MC
 - Symbolic Fair CTL MC
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- 3 The Symbolic Approach to LTL Model Checking
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Ex: Symbolic CTL Model Checking

Given the following finite state machine expressed in NuSMV input language:

```
MODULE main
VAR v1 : boolean; v2 : boolean;
INIT (!v1 & !v2)
TRANS (next(v1) <-> !v1) & (next(v2) <-> (v1<->v2))
```

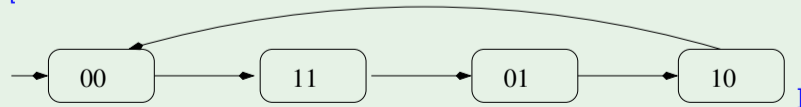
and consider the property $P \stackrel{\text{def}}{=} (v_1 \wedge v_2)$. Write:

- the Boolean formulas $I(v_1, v_2)$ and $T(v_1, v_2, v'_1, v'_2)$ representing respectively the initial states and the transition relation of M .

[Solution: $I(v_1, v_2)$ is $(\neg v_1 \wedge \neg v_2)$, $T(v_1, v_2, v'_1, v'_2)$ is $(v'_1 \leftrightarrow \neg v_1) \wedge (v'_2 \leftrightarrow (v_1 \leftrightarrow v_2))$]

- the graph representing the FSM. (Assume the notation “ $v_1 v_2$ ” for labeling the states: e.g. “10” means “ $v_1 = 1, v_2 = 0$ ”.)

[Solution:



Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula representing symbolically **EXP**. [The formula must be computed symbolically, not simply inferred from the graph of the previous question!]

[Solution:

$$\begin{aligned}\mathbf{EX}(P) &= \exists v'_1, v'_2. (T(v_1, v_2, v'_1, v'_2) \wedge P(v'_1, v'_2)) \\ &= \exists v'_1, v'_2. ((v'_1 \leftrightarrow \neg v_1) \wedge (v'_2 \leftrightarrow (v_1 \leftrightarrow v_2))) \wedge \underbrace{(v'_1 \wedge v'_2)}_{\implies v'_1=T, v'_2=T}) \\ &= \underbrace{v'_1=T, v'_2=T}_{(\neg v_1 \wedge \neg v_2) \vee \perp \vee \perp \vee \perp} \\ &= (\neg v_1 \wedge \neg v_2)\end{aligned}$$

.]

Ex: Symbolic CTL Model Checking

Given the following finite state machine expressed in NuSMV input language:

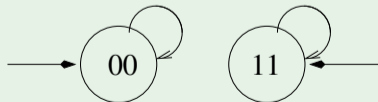
```
VAR    v1 : boolean;  v2 : boolean;
INIT   init(v1) <-> init(v2)
TRANS  (v1 <-> next(v2)) & (v2 <-> next(v1));
```

write:

- the Boolean formulas $I(v_1, v_2)$ and $T(v_1, v_2, v'_1, v'_2)$ representing the initial states and the transition relation of M respectively.

[Solution: $I(v_1, v_2)$ is $(v_1 \leftrightarrow v_2)$, $T(v_1, v_2, v'_1, v'_2)$ is $(v_1 \leftrightarrow v'_1) \wedge (v_2 \leftrightarrow v'_2)$]

- the graph representing the FSM. (Assume the notation " $v_1 v_2$ " for labeling the states. E.g., "10" means " $v_1 = 1, v_2 = 0$ ".)



[Solution:

]]

Ex: Symbolic CTL Model Checking (cont.)

- the Boolean formula $R^1(v'_1, v'_2)$ representing the set of states which can be reached after exactly 1 step.
NOTE: this must be computed symbolically, not simply deduced from the graph of question b).

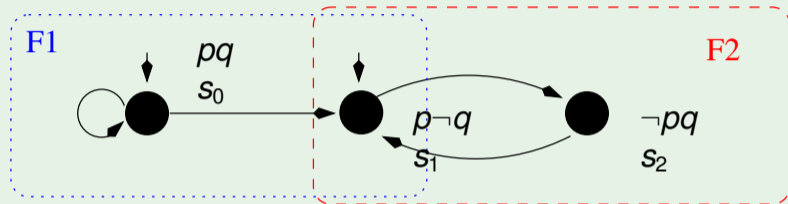
[Solution:

$$\begin{aligned}R^1(v'_1, v'_2) &= \exists v_1, v_2. (I(v_1, v_2) \wedge T(v_1, v_2, v'_1, v'_2)) \\ &= \exists v_1, v_2. ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1)) \\ &= ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \perp, v_2 = \perp] \vee \\ &\quad ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \perp, v_2 = \top] \vee \\ &\quad ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \top, v_2 = \perp] \vee \\ &\quad ((v_1 \leftrightarrow v_2) \wedge (v_1 \leftrightarrow v'_2) \wedge (v_2 \leftrightarrow v'_1))[v_1 = \top, v_2 = \top] \\ &= (\neg v'_1 \wedge \neg v'_2) \vee \perp \vee \perp \vee (v'_1 \wedge v'_2) \\ &= (\neg v'_1 \wedge \neg v'_2) \vee (v'_1 \wedge v'_2) \\ &= (v'_1 \leftrightarrow v'_2)\end{aligned}$$

.]

Ex: Fair CTL Model Checking

Consider the following *fair* Kripke Model M :

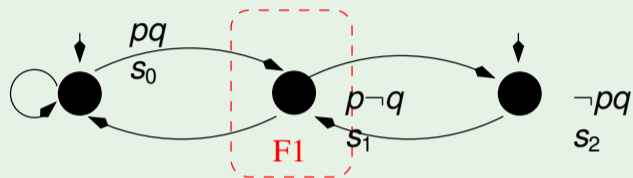


For each of the following facts, say if it is true or false in CTL.

- (a) $M \models \mathbf{AF}\neg p$
[Solution: true]
- (b) $M \models \mathbf{A}(p\mathbf{U}\neg q)$
[Solution: true]
- (c) $M \models \mathbf{AX}\neg q$
[Solution: false]
- (d) $M \models \mathbf{AGAF}\neg p$
[Solution: true]

Ex: Fair CTL Model Checking

Consider the following *fair* Kripke Model M :



For each of the following facts, say if it is true or false in CTL.

- (a) $M \models \mathbf{EF}(p \wedge q)$
[Solution: true]
- (b) $M \models \mathbf{AGAF}p$
[Solution: true]
- (c) $M \models \mathbf{AF}\neg q$
[Solution: true]
- (d) $M \models \mathbf{AG}(\neg p \vee \neg q)$
[Solution: false]

Ex: Symbolic LTL Model Checking

Given the following LTL formula: $\varphi \stackrel{\text{def}}{=} \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{GF}r)$

(a) Compute the Negative Normal Form of φ ($\mathbf{NNF}(\varphi)$).

$$\begin{aligned}
 \varphi &\iff \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{GF}r) \\
 \text{[Solution: } &\iff \neg(\neg(\mathbf{GF}p \wedge \mathbf{GF}q) \vee \mathbf{GF}r) \\
 &\iff (\mathbf{GF}p \wedge \mathbf{GF}q \wedge \neg\mathbf{GF}r) \\
 &\iff (\mathbf{GF}p \wedge \mathbf{GF}q \wedge \mathbf{FG}\neg r) \iff \mathbf{NNF}(\varphi) \\
 \text{]} &
 \end{aligned}$$

(b) Compute the set of elementary subformulas of φ .

[Solution: First write the formula in terms of \mathbf{X} and \mathbf{U} 's (write " $\mathbf{F}\psi$ " for " $\top\mathbf{U}\psi$ "):

$$\begin{aligned}
 \varphi &\iff \neg((\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{GF}r) \\
 &\iff \neg((\neg\mathbf{F}\neg\mathbf{F}p \wedge \neg\mathbf{F}\neg\mathbf{F}q) \rightarrow \neg\mathbf{F}\neg\mathbf{F}r)
 \end{aligned}$$

$$el(\mathbf{F}\neg\mathbf{F}p) = \{\mathbf{X}\mathbf{F}\neg\mathbf{F}p\} \cup el(\neg\mathbf{F}p) = \{\mathbf{X}\mathbf{F}\neg\mathbf{F}p\} \cup \{\mathbf{X}\mathbf{F}p\} \cup el(p) = \{\mathbf{X}\mathbf{F}\neg\mathbf{F}p, \mathbf{X}\mathbf{F}p, p\}.$$

$$\begin{aligned}
 \text{Hence: } el(\varphi) &= el(\neg((\neg\mathbf{F}\neg\mathbf{F}p \wedge \neg\mathbf{F}\neg\mathbf{F}q) \rightarrow \neg\mathbf{F}\neg\mathbf{F}r)) \\
 &= el(\mathbf{F}\neg\mathbf{F}p) \cup el(\mathbf{F}\neg\mathbf{F}q) \cup el(\mathbf{F}\neg\mathbf{F}r) \\
 &= \{\mathbf{X}\mathbf{F}\neg\mathbf{F}p, \mathbf{X}\mathbf{F}p, p, \mathbf{X}\mathbf{F}\neg\mathbf{F}q, \mathbf{X}\mathbf{F}q, q, \mathbf{X}\mathbf{F}\neg\mathbf{F}r, \mathbf{X}\mathbf{F}r, r\}
 \end{aligned}$$

(c) What is the (maximum) number of states of a fair Kripke Model representing φ ?

[Solution: By definition it is $2^{|el(\varphi)|} = 2^9 = 512$.]

Ex: Symbolic LTL Model Checking

Given the following LTL formula $\psi \stackrel{\text{def}}{=} \neg \mathbf{F} \neg p$, compute and draw the tableau \mathcal{T}_ψ of ψ . [Solution:

(i) The set of elementary subformulas of ψ is $el(\psi) \stackrel{\text{def}}{=} \{p, \mathbf{X}\mathbf{F}\neg p\}$. Hence, the set of states is

$$\{s_1 : (p, \neg \mathbf{X}\mathbf{F}\neg p), s_2 : (p, \mathbf{X}\mathbf{F}\neg p), s_3 : (\neg p, \neg \mathbf{X}\mathbf{F}\neg p), s_4 : (\neg p, \mathbf{X}\mathbf{F}\neg p)\}$$

(ii) The set of initial states of \mathcal{T}_ψ is $sat(\psi) \stackrel{\text{def}}{=} S \setminus (sat(\neg p) \cup sat(\mathbf{X}\mathbf{F}\neg p)) = \{s_1\}$.

(iii) Since s_1 is the only state in $sat(\neg \mathbf{F}\neg p)$, then s_1 is the only successor of itself, so that the only relevant transition is a self-loop over s_1 .

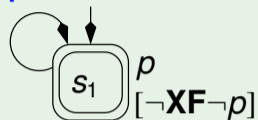
(One can also —un-necessarily— draw all transitions from states where $\neg \mathbf{X}\mathbf{F}\neg p$ holds into $\{s_1\}$ and from from states where $\mathbf{X}\mathbf{F}\neg p$ holds into $\{s_2, s_3, s_4\}$.)

(iv) There is one **U**-subformula, $\mathbf{F}\neg p$, so that there is one fairness condition defined as $sat(\neg \mathbf{F}\neg p \vee \neg p)$. Since $\mathbf{F}\neg p$ is false in s_1 , then s_1 is part of the fairness condition. [Alternatively: there is no **positive U**-subformula, so that we must add a **AGAF^T** fairness condition, which is equivalent to say that all states belong to the fairness condition.]

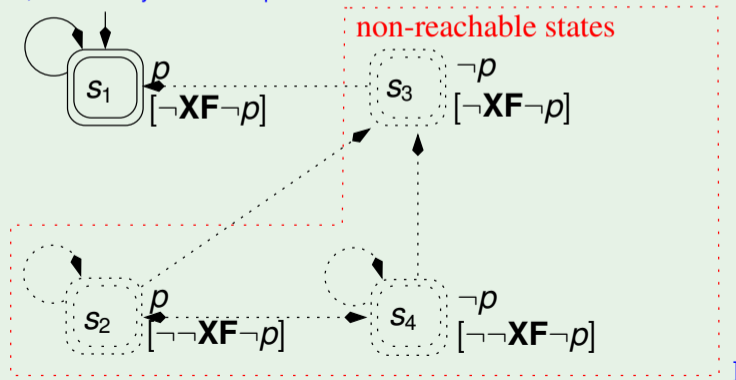
]

Ex: Symbolic LTL Model Checking (cont.)

[Solution:



or, alternatively without simplifications:



Ex: Symbolic LTL Model Checking

Given the following LTL formula $\psi \stackrel{\text{def}}{=} \mathbf{G}p$, compute and draw the tableau \mathcal{T}_ψ of ψ . [Without converting anything into \mathbf{X} , \mathbf{U}].

[Solution:

- (i) The set of elementary subformulas of ψ is $el(\psi) \stackrel{\text{def}}{=} \{p, \mathbf{XG}p\}$. Hence, the set of states is

$$\{s_1 : (p, \mathbf{XG}p), s_2 : (p, \neg\mathbf{XG}p), s_3 : (\neg p, \mathbf{XG}p), s_4 : (\neg p, \neg\mathbf{XG}p)\}$$

- (ii) The set of initial states of \mathcal{T}_ψ is $sat(\psi) \stackrel{\text{def}}{=} sat(p) \cap sat(\mathbf{XG}p) = \{s_1\}$.

- (iii) Since s_1 is the only state in $sat(\mathbf{G}p)$, then s_1 is the only successor of itself, so that the only relevant transition is a self-loop over s_1 .

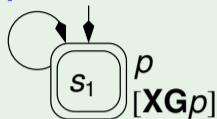
(One can also —un-necessarily— draw all transitions from states where $\mathbf{XG}p$ holds into $\{s_1\}$ and from from states where $\neg\mathbf{XG}p$ holds into $\{s_2, s_3, s_4\}$.)

- (iv) Since there is no “ \mathbf{U} ” subformula, we must add a **AGAF** \top fairness condition, which is equivalent to say that all states belong to the fairness condition.

]

Ex: Symbolic LTL Model Checking (cont.)

[Solution:



or, alternatively without simplifications:

