Automated Reasoning and Formal Verification
Module II: Formal Verification
Ch. 05: **Explicit-State CTL Model Checking**

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Outline

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CTL Model Checking

CTL Model Checking is a formal verification technique where...

- ...the system is represented as a Finite State Machine M:
- ...the property is expressed a CTL formula φ:

$$\mathbf{AG}(p \rightarrow \mathbf{AF}q)$$

- ...the model checking algorithm checks whether in all initial states of M all the executions of the model satisfy the formula ($M \models \varphi$).

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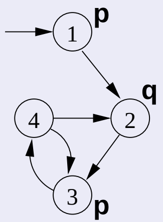

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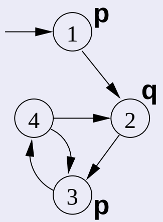

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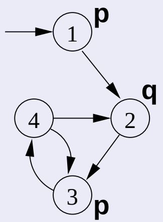

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CTL Model Checking: General Idea

Two macro-steps:

1. construct the set of states where the formula holds:
   $[\varphi] := \{s \in S : M, s \models \varphi\}$
   ($[\varphi]$ is called the denotation of φ)
2. then compare with the set of initial states:
   $I \subseteq [\varphi]$?

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CTL Model Checking: General Idea [cont.]

In order to compute $[\varphi]$:

- proceed “bottom-up” on the structure of the formula, computing $[\varphi_i]$ for each subformula φ_i of $\mathbf{AG}(p \rightarrow \mathbf{AF}q)$:
  - $[q]$,
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In order to compute each $[\varphi_i]$:

- assign Propositional atoms by labeling function
- handle Boolean operators by standard set operations
- handle temporal operators **AX**, **EX** by computing pre-images
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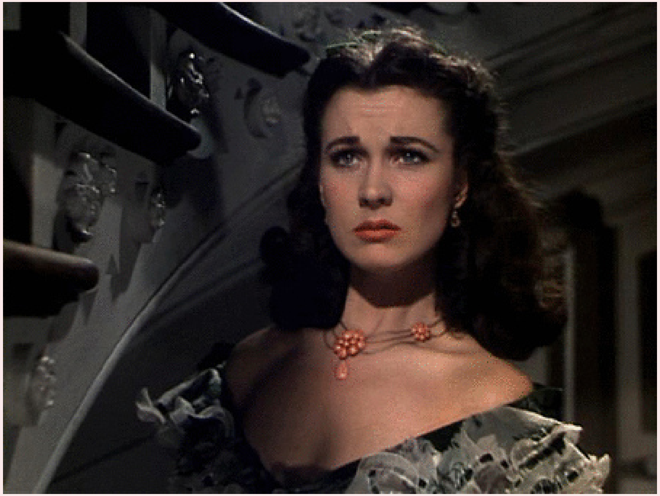
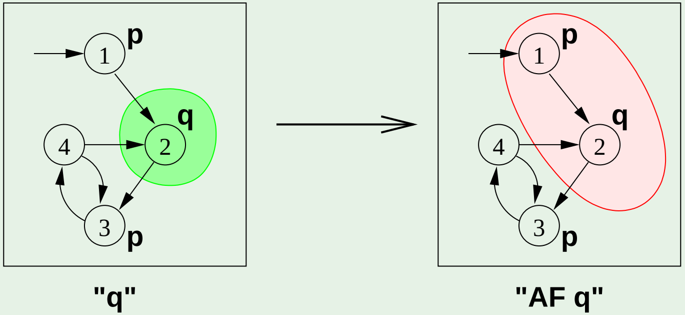
Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

CTL Model Checking: Example: $\mathbf{AG}(p \rightarrow \mathbf{AF}q)$



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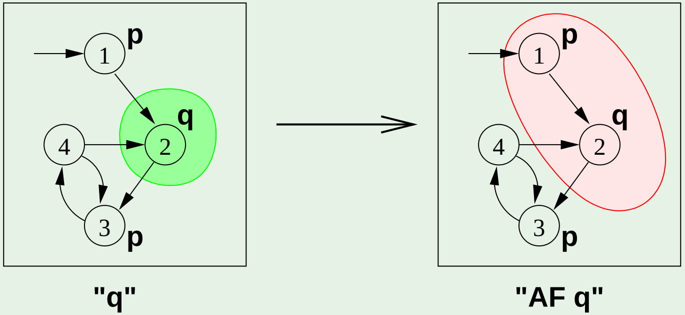

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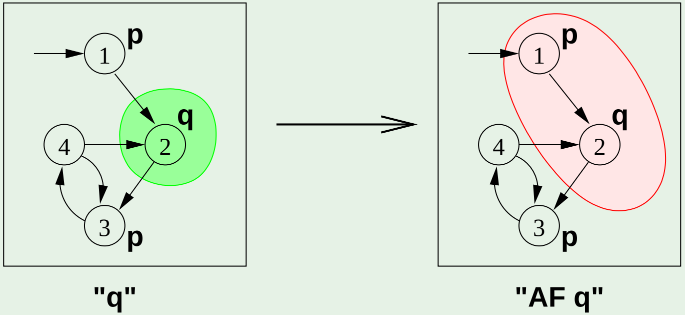

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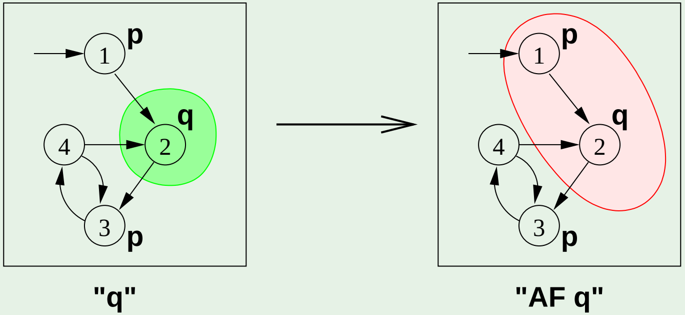

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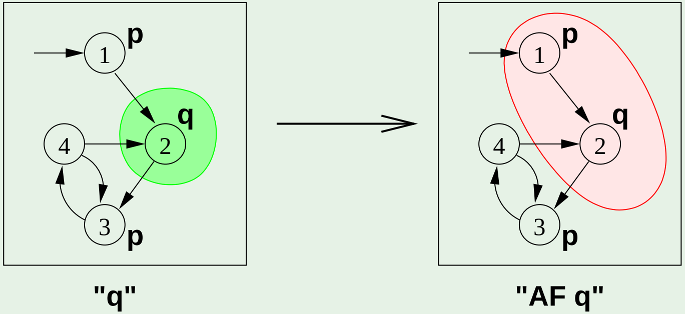

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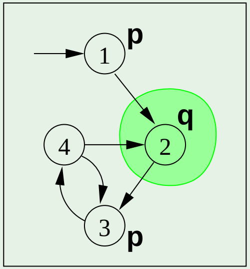

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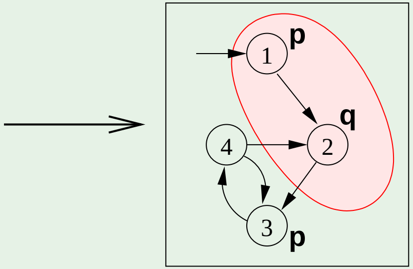

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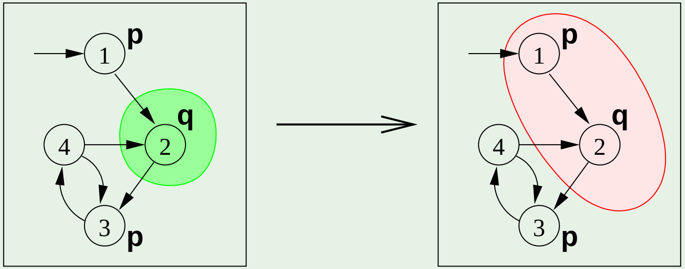

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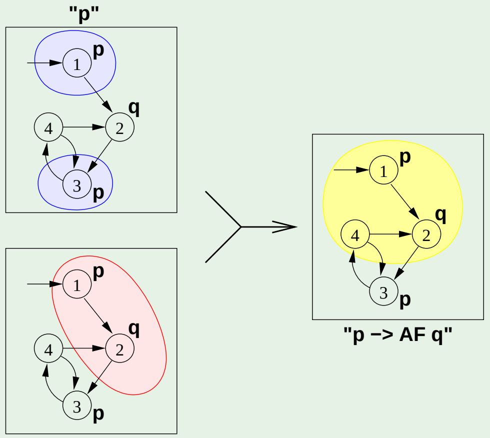

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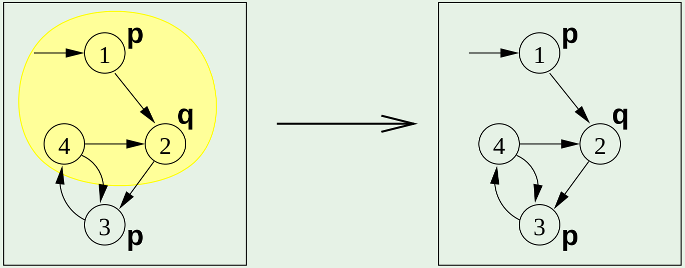

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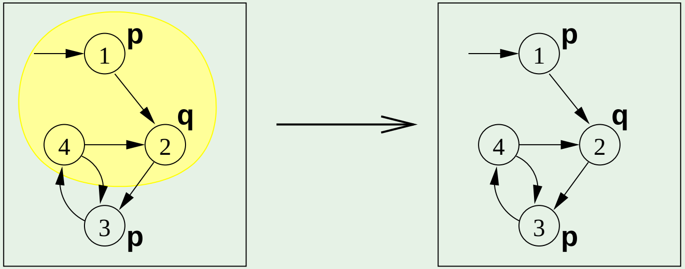

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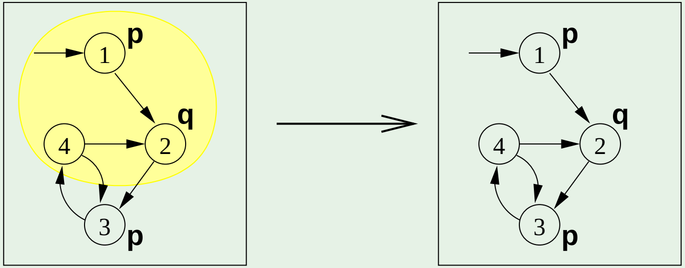

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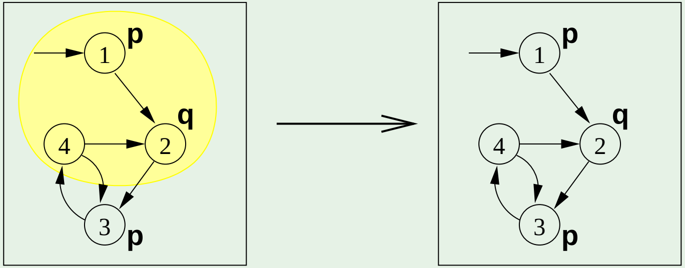

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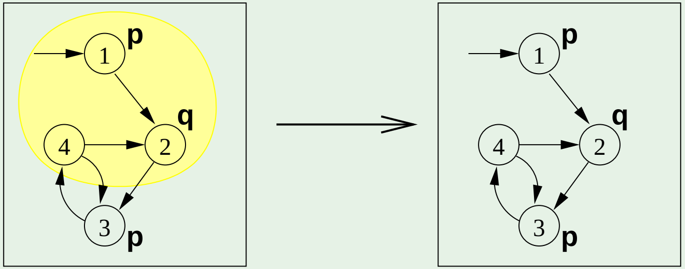

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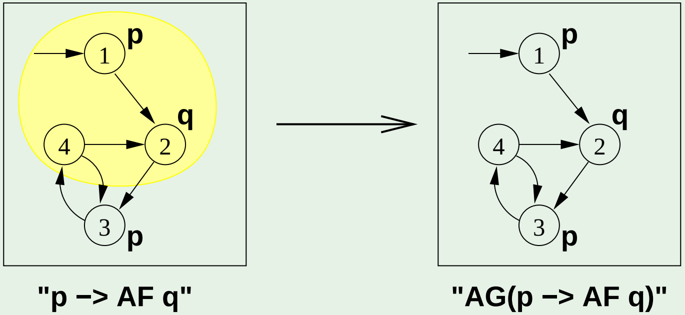

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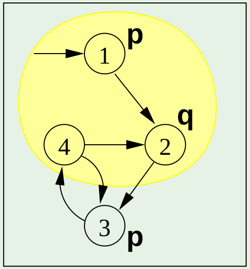

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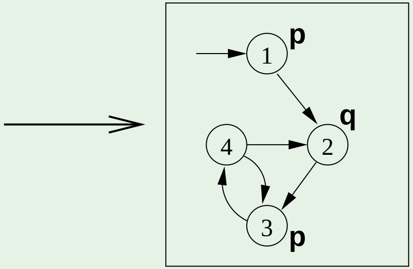

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     $\Longrightarrow$ (fix point reached)

CTL Model Checking: Example: $\mathbf{AG}(p \rightarrow \mathbf{AF}q)$ [cont.]

- The set of states where the formula holds is empty
  $\Longrightarrow$ the initial state does not satisfy the property
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- Counterexample: a lazo-shaped path: $1, 2, \{3, 4\}^{\omega}$ (satisfying $\mathbf{EF}(p \wedge \mathbf{EG}\neg q)$)

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Outline

The fixed-point theory of lattice of sets

Definition

Let 2^S denote the power set of S, i.e., the set of all subsets of S.

- For any finite set S, the structure $\langle 2^S, \subseteq \rangle$ forms a complete lattice with $\cup$ as join and $\cap$ as meet operations.
- A function $F : 2^S \longmapsto 2^S$ is monotonic provided $S_1 \subseteq S_2 \Rightarrow F(S_1) \subseteq F(S_2)$.

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Fixed Points

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Let $\langle 2^S, \subseteq \rangle$ be a complete lattice, S finite.

- Given a function $F : 2^S \longmapsto 2^S$, $a \subseteq S$ is a fixed point of F iff

$$F(a) = a$$

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Iteratively computing fixed points

Tarski's Theorem

A monotonic function over a complete finite lattice has a least and a greatest fixed point.

(A corollary of) Kleene's Theorem

A monotonic function F over a complete finite lattice has a least and a greatest fixed point, which can be computed as follows:

- the least fixed point of F is the limit of the chain $\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \ldots$,
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CTL Model Checking and Lattices

- If $M = \langle S, I, R, L, AP \rangle$ is a Kripke structure, then $\langle 2^S, \subseteq \rangle$ is a complete lattice
- We identify φ with its denotation $[\varphi]$

$\Longrightarrow$ we can see logical operators as functions $F : 2^S \longmapsto 2^S$ on the complete lattice $\langle 2^S, \subseteq \rangle$

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Denotation of a CTL formula φ: $[\varphi]$

Definition of $[\varphi]$

$[\varphi] := \{s \in S : M, s \models \varphi\}$

Recursive definition of $[\varphi]$

$$\begin{array}{lcl}
[\top] & = & S \\
[\bot] & = & \{\} \\
[p] & = & \{s | p \in L(s)\} \\
[\neg\varphi_1] & = & S/[\varphi_1] \\
[\varphi_1 \wedge \varphi_2] & = & [\varphi_1] \cap [\varphi_2] \\
[\mathbf{EX}\varphi] & = & \{s \mid \exists s' \in [\varphi] \; s.t. \; \langle s, s' \rangle \in R\} \\
[\mathbf{EG}\beta] & = & \nu Z.(\; [\beta] \cap [\mathbf{EX}Z] \;) \\
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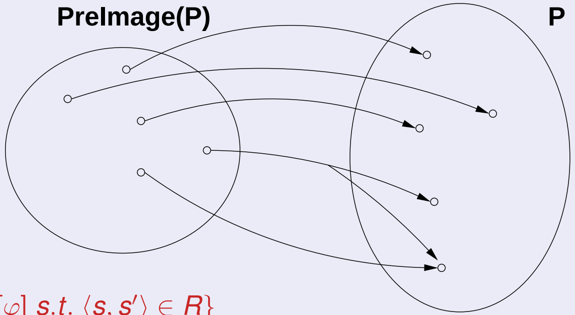
Consider $\mathbf{EX}\varphi$:



- $[\mathbf{EX}\varphi] = \{s \mid \exists s' \in [\varphi]\ s.t.\ \langle s, s'\rangle \in R\}$
- $[\mathbf{EX}\varphi]$ is said to be the Pre-image of $[\varphi]$ ($Preimage([\varphi])$)
- Key step of every CTL M.C. operation

Note

Preimage() is monotonic: $X \subseteq X' \Longrightarrow Preimage(X) \subseteq Preimage(X')$

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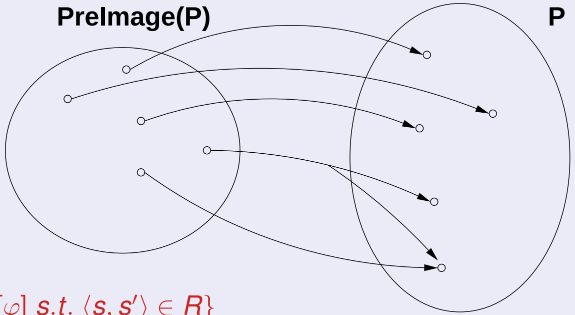

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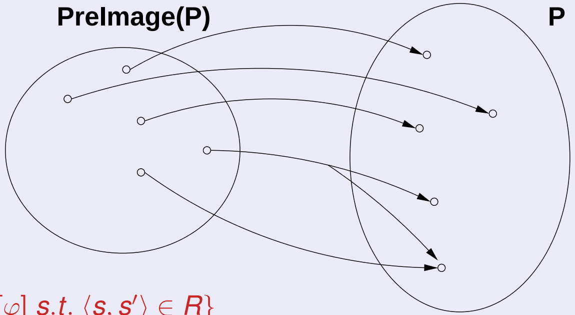

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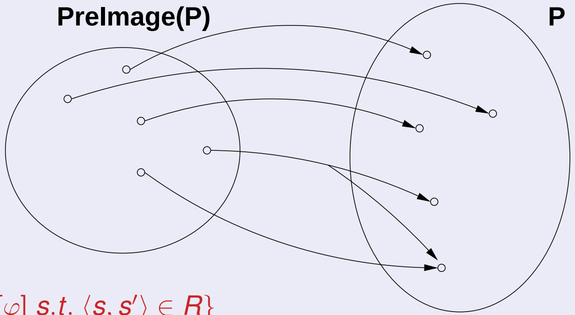

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Consider **EG**β:

- $\nu Z.(\ [\beta] \cap [\mathbf{EX}Z]\ )$: greatest fixed point of the function $F_\beta : 2^S \longmapsto 2^S$, s.t.

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- F_β Monotonic: $a \subseteq a' \Longrightarrow F_\beta(a) \subseteq F_\beta(a')$
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Case **EG** [cont.]

- We can compute $X := [\mathbf{EG}\beta]$ inductively as follows:
$$\begin{array}{lcll} X_0 & := & S & \\ X_1 & := & F_\beta(S) & = [\beta] \\ X_2 & := & F_\beta(F_\beta(S)) & = [\beta] \cap Preimage(X_1) \\ \ldots & & & \\ X_{j+1} & := & F_\beta^{j+1}(S) & = [\beta] \cap Preimage(X_j) \end{array}$$
- Noticing that $X_1 = [\beta]$ and $X_{j+1} \subseteq X_j$ for every $j \geq 0$, and that $([\beta] \cap Y) \subseteq X_j \subseteq [\beta] \Longrightarrow ([\beta] \cap Y) = (X_j \cap Y)$, we can use instead the following inductive schema:
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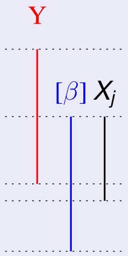

Case **EU**

Consider $\mathbf{E}(\beta_1 \mathbf{U} \beta_2)$:

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$[\mathbf{E}(\beta_1 \mathbf{U} \beta_2)] = \mu Z.(\ [\beta_2] \cup ([\beta_1] \cap [\mathbf{EX}Z])\ )$

Case **EU**

Consider $\mathbf{E}(\beta_1 \mathbf{U} \beta_2)$:

- $\mu Z.(\ [\beta_2] \cup ([\beta_1] \cap [\mathbf{EX}Z])\ )$: least fixed point of the function $F_{\beta_1,\beta_2} : 2^S \longmapsto 2^S$, s.t.
$$\begin{aligned} F_{\beta_1,\beta_2}([\varphi]) &= [\beta_2] \cup ([\beta_1] \cap Preimage([\varphi])) \\ &= [\beta_2] \cup ([\beta_1] \cap \{s \mid \exists s' \in [\varphi]\ s.t.\ \langle s, s' \rangle \in R\}) \end{aligned}$$
- F_{β_1,β_2} Monotonic: $a \subseteq a' \Longrightarrow F_{\beta_1,\beta_2}(a) \subseteq F_{\beta_1,\beta_2}(a')$
  - (Tarski's theorem): $\mu x.F_{\beta_1,\beta_2}(x)$ always exists
  - (Kleene's theorem): $\mu x.F_{\beta_1,\beta_2}(x)$ can be computed as the limit $\emptyset \subseteq F_{\beta_1,\beta_2}(\emptyset) \subseteq F_{\beta_1,\beta_2}(F_{\beta_1,\beta_2}(\emptyset)) \subseteq \ldots$, in a finite number of steps.

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$[\mathbf{E}(\beta_1 \mathbf{U} \beta_2)] = \mu Z.(\ [\beta_2] \cup ([\beta_1] \cap [\mathbf{EX}Z])\ )$

Case **EU** [cont.]

- We can compute $X := [\mathbf{E}(\beta_1 \mathbf{U} \beta_2)]$ inductively as follows:

$$\begin{array}{lcllcl}
X_0 & := & \emptyset & & & \\
X_1 & := & F_{\beta_1,\beta_2}(\emptyset) & = & [\beta_2] \\
X_2 & := & F_{\beta_1,\beta_2}(F_{\beta_1,\beta_2}(\emptyset)) & = & [\beta_2] \cup ([\beta_1] \cap \mathit{Preimage}(X_1)) \\
\ldots & & & & \\
X_{j+1} & := & F^{j+1}_{\beta_1,\beta_2}(\emptyset)) & = & [\beta_2] \cup ([\beta_1] \cap \mathit{Preimage}(X_j))
\end{array}$$

- Noticing that $X_1 = [\beta_2]$ and $X_{j+1} \supseteq X_j$ for every $j \geq 0$, and that $([\beta_2] \cup Y) \supseteq X_j \supseteq [\beta_2] \Longrightarrow ([\beta_2] \cup Y) = (X_j \cup Y)$, we can use instead the following inductive schema:
  - $X_1 \quad := [\beta_2]$
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Case **EU** [cont.]

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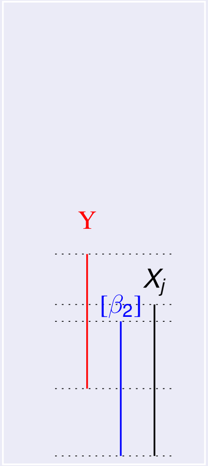

A relevant subcase: **EF**

- $\mathbf{EF}\beta = \mathbf{E}(\top \mathbf{U} \beta)$
- $[\top] = S \implies [\top] \cap \mathit{Preimage}(X_j) = \mathit{Preimage}(X_j)$
- We can compute $X := [\mathbf{EF}\beta]$ inductively as follows:
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Outline

General Schema

- Assume φ written in terms of $\neg$, $\wedge$, **EX**, **EU**, **EG**
- A general M.C. algorithm (fix-point):
  1. for every $\varphi_i \in Sub(\varphi)$, find $[\varphi_i]$
  2. Check if $I \subseteq [\varphi]$
- Subformulas $Sub(\varphi)$ of φ are checked bottom-up
- To compute each $[\varphi_i]$: if the main operator of φ_i is a
  - Propositional atoms: apply labeling function
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General M.C. Procedure

```
state_set Check(CTL_formula $\beta$) {
    case $\beta$ of
    $\top$:                    return $S$;
    $\bot$:                    return $\{\}$;
    $p$:                       return $\{s \mid p \in L(s)\}$;
    $\neg\beta_1$:             return S / Check($\beta_1$);
    $\beta_1 \wedge \beta_2$:  return Check($\beta_1$) $\cap$ Check($\beta_2$);
    EX$\beta_1$:               return PreImage(Check($\beta_1$));
    EG$\beta_1$:               return Check_EG(Check($\beta_1$));
    E($\beta_1$U$\beta_2$):    return Check_EU(Check($\beta_1$),Check($\beta_2$));
}
```

PreImage

Compute [**EX**β]

```
state_set PreImage(state_set [$\beta$]) {
    $X := \{\}$;
    for each $s \in S$ do
        for each $s'$ $s.t.$ $s' \in [\beta]$ and $\langle s, s' \rangle \in R$ do
            $X := X \cup \{s\}$;
return $X$;
}
```

Check_EG

Compute [**EG**β]

```
**state_set** Check_EG(**state_set** [$\beta$]) {
    $X' := [\beta]$;
    **repeat**
        $X := X'$;
        $X' := X \cap PreImage(X)$;
    **until** $(X' = X)$;
**return** $X$;
}
```

Check_EU

Compute $[\mathbf{E}(\beta_1 \mathbf{U} \beta_2)]$

```
**state_set** Check_EU(**state_set** $[\beta_1]$,$[\beta_2]$) {
    $X' := [\beta_2]$;
    **repeat**
        $X := X'$;
        $X' := X \cup ([\beta_1] \cap PreImage(X))$;
    **until** $(X' = X)$;
**return** $X$;
}
```

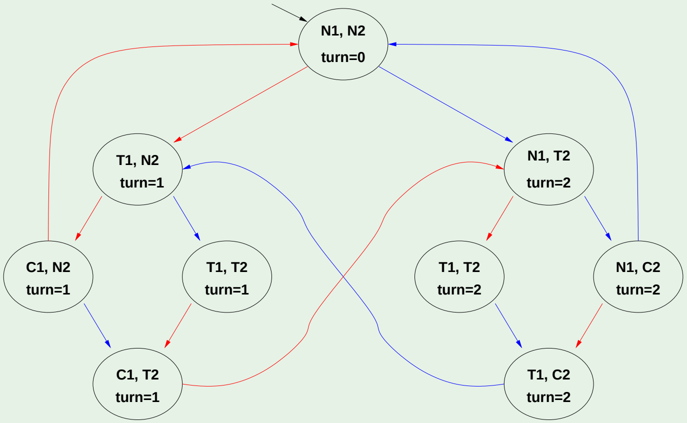
A relevant subcase: Check_EF

Compute [**EF**β]

```
state_set Check_EF(state_set [$\beta$]) {
    $X' := [\beta]$;
    repeat
        $X := X'$;
        $X' := X \cup PreImage(X)$;
    until $(X' = X)$;
return $X$;
}
```

Outline

Example 1: fairness

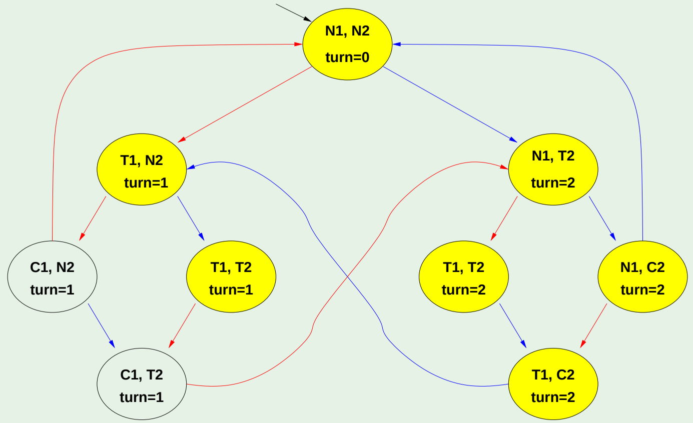


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF}C_1\ ? \Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1\ ?$

Example 1: fairness

$[\neg C_1]$

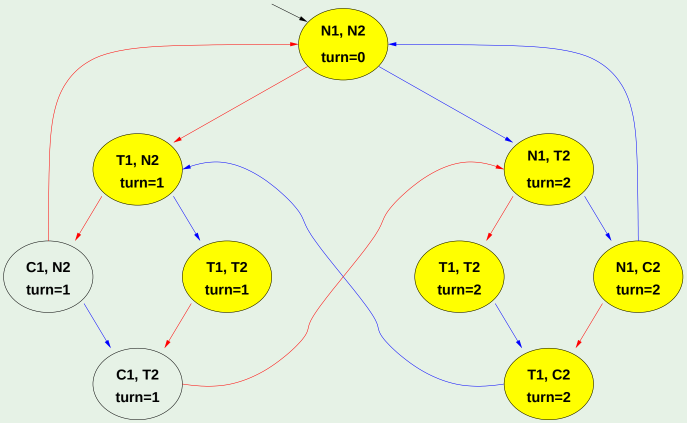


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF} C_1$? $\Longrightarrow M \models \neg \mathbf{EFEG} \neg C_1$?

Example 1: fairness

[$\mathbf{EG}\neg C_1$], step 0:

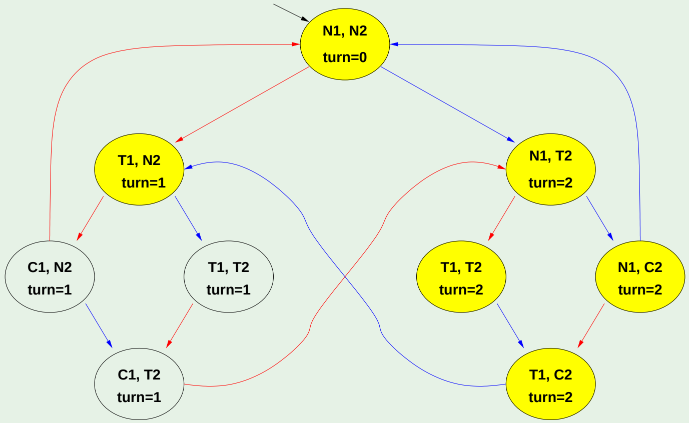


N = noncritical, T = trying, C = critical   User 1   User 2

$M \models \mathbf{AGAF}C_1$? $\Longrightarrow$ $M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

[**EG**$\neg C_1$], step 1:

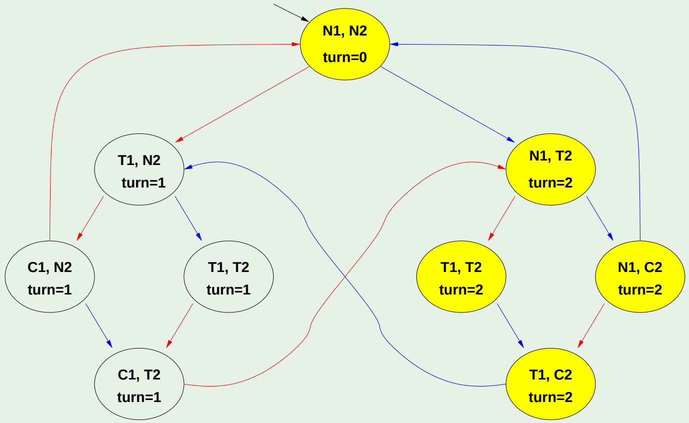


N = noncritical, T = trying, C = critical   User 1   User 2

$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

[**EG**$\neg C_1$], step 2:

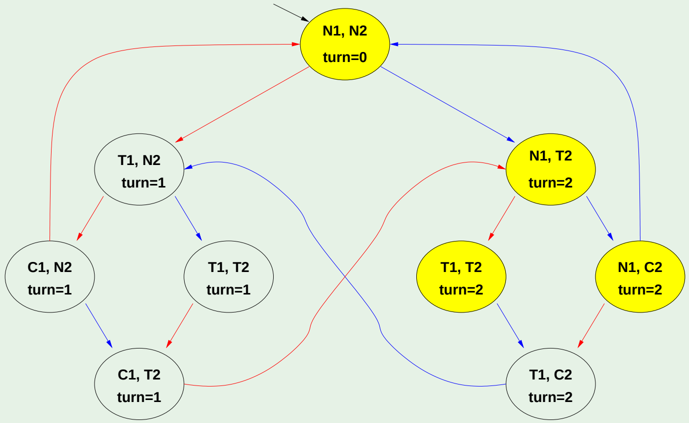


N = noncritical, T = trying, C = critical   User 1   User 2

$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

[$\mathbf{EG}\neg C_1$], step 3:

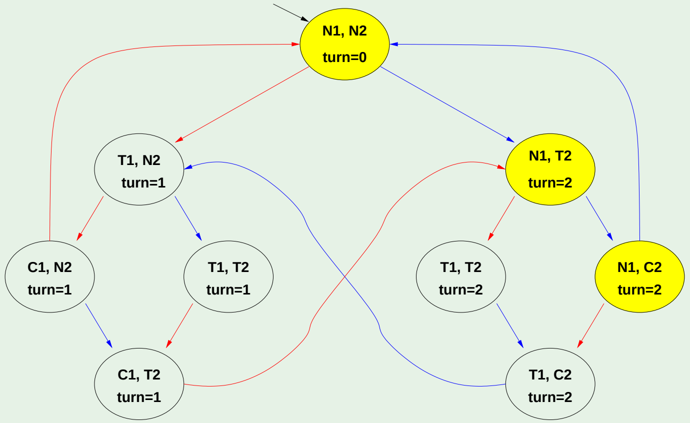


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF} C_1$? $\Longrightarrow$ $M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

[**EG**$\neg C_1$], step 4:

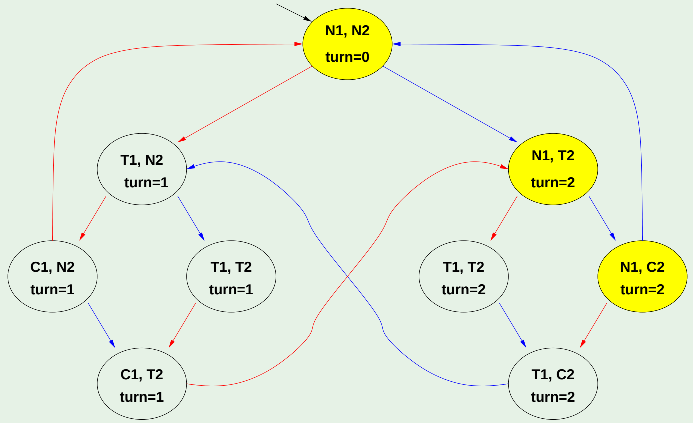


N = noncritical, T = trying, C = critical   User 1   User 2

$M \models$ **AGAF**C_1 ? $\Longrightarrow M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

[**EG**$\neg C_1$], FIXPOINT!

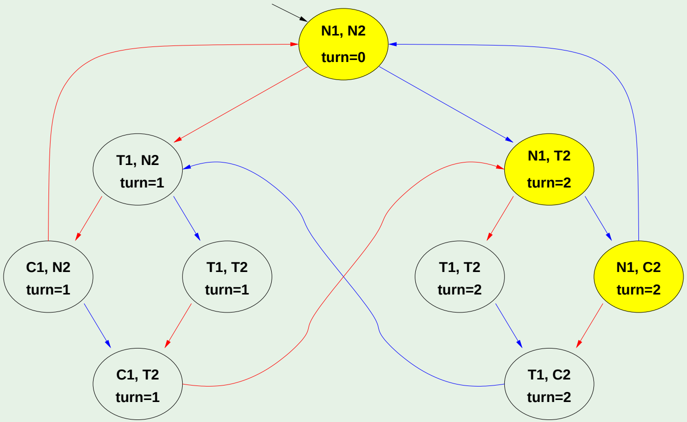


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF}C_1$? $\Longrightarrow$ $M \models \neg\mathbf{EFEG}\neg C_1$?

Example 1: fairness

[**EFEG**$\neg C_1$], STEP 0

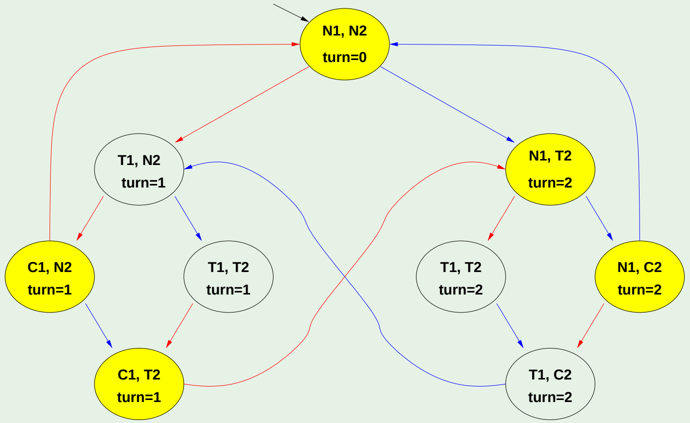


N = noncritical, T = trying, C = critical User 1 User 2

$M \models$ **AGAF**C_1 ? $\Longrightarrow$ $M \models \neg$**EFEG**$\neg C_1$?

Example 1: fairness

$[\mathbf{EFEG}\neg C_1]$, STEP 1

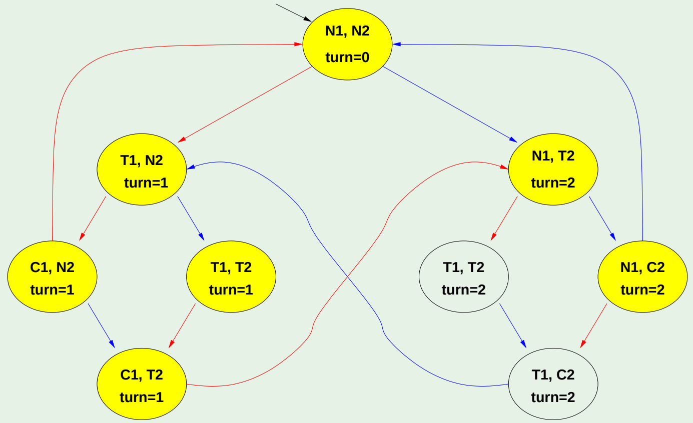


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF}C_1 \; ? \Longrightarrow M \models \neg\mathbf{EFEG}\neg C_1 \; ?$

Example 1: fairness

[**EFEG**$\neg C_1$], STEP 2



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Example 1: fairness

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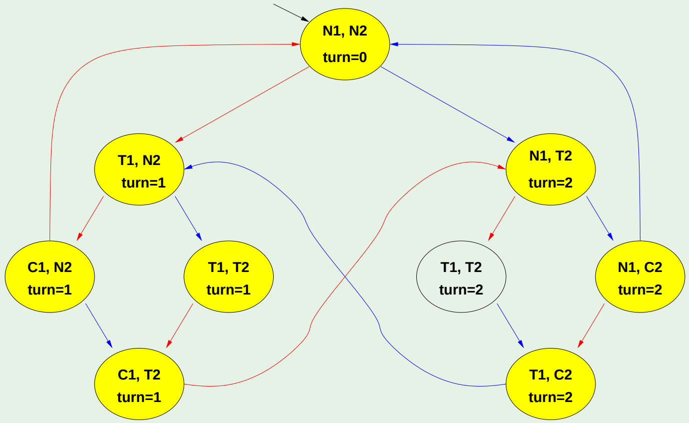

N = noncritical, T = trying, C = critical   User 1   User 2

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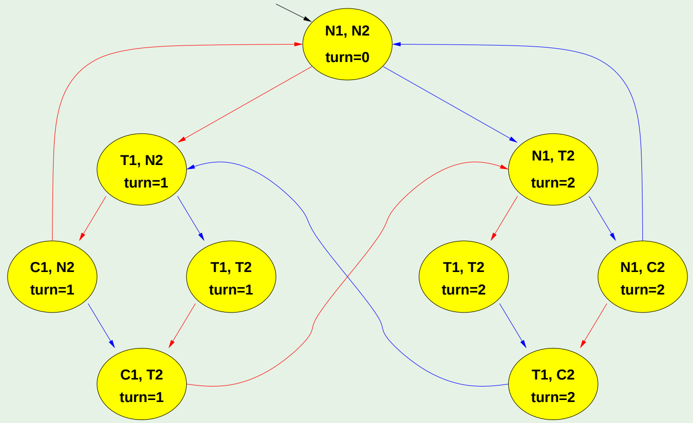
Example 1: fairness

[**EFEG**$\neg C_1$], STEP 4



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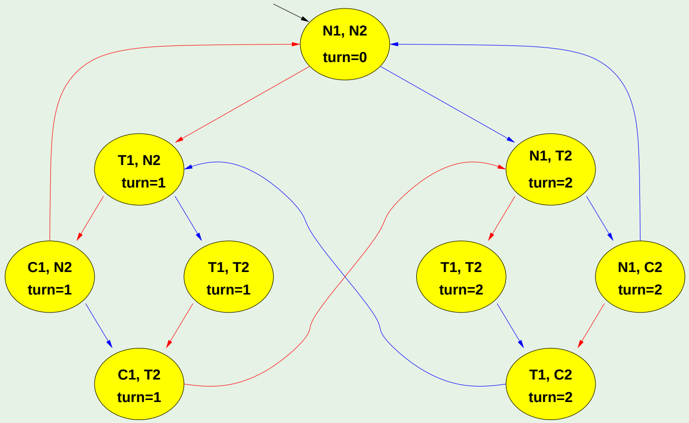
Example 1: fairness

[**EFEG**$\neg C_1$], FIXPOINT!



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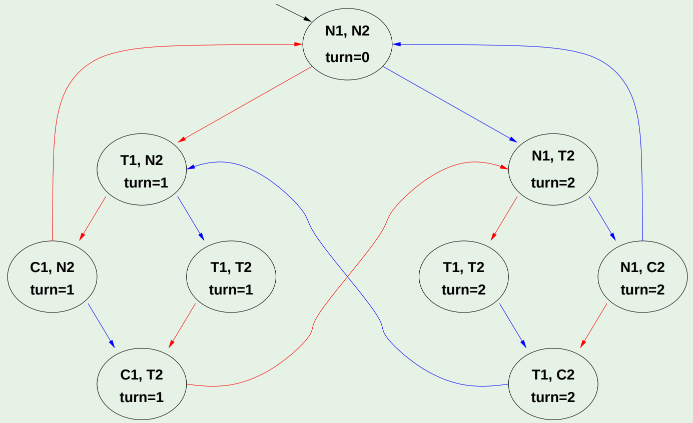
Example 1: fairness

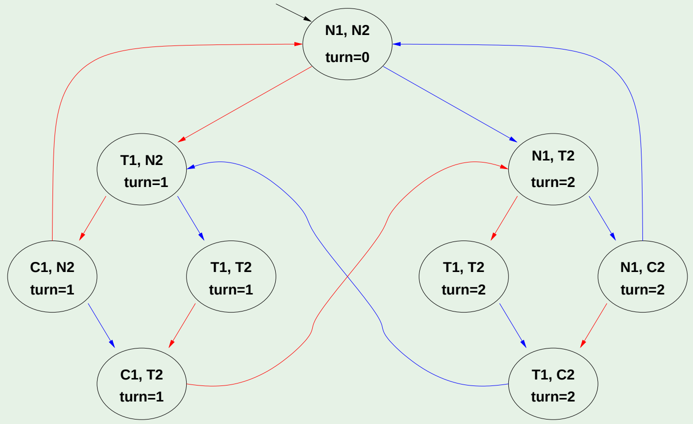
$[\neg \mathbf{EFEG} \neg C_1]$



N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AGAF} C_1$? $\Longrightarrow$ $M \models \neg \mathbf{EFEG} \neg C_1$? $\Longrightarrow$ NO!

Example 2: liveness

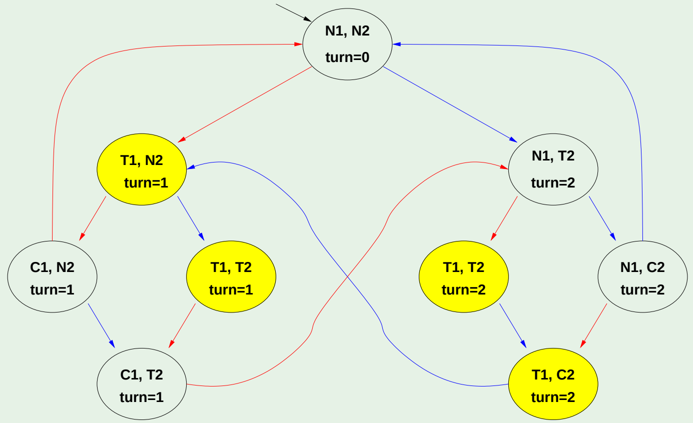


N = noncritical, T = trying, C = critical User 1 User 2

$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF}C_1)$? $\Longrightarrow M \models \neg\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)$?

Example 2: liveness

$[T_1]$:



N = noncritical, T = trying, C = critical User 1 User 2

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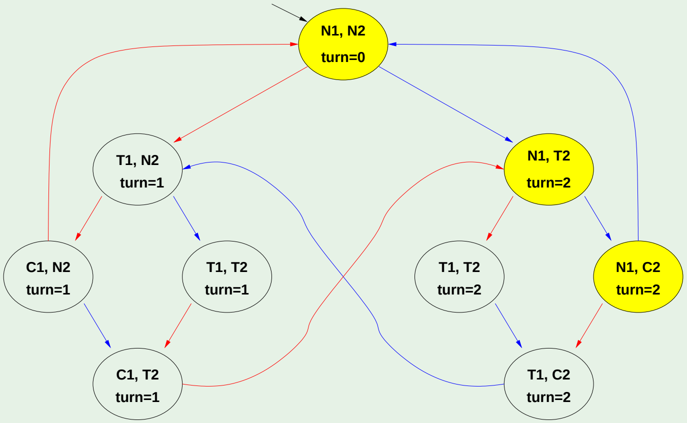
Example 2: liveness

$[\mathbf{EG}\neg C_1]$, STEPS 0-4: (see previous example)



N = noncritical, T = trying, C = critical User 1 User 2

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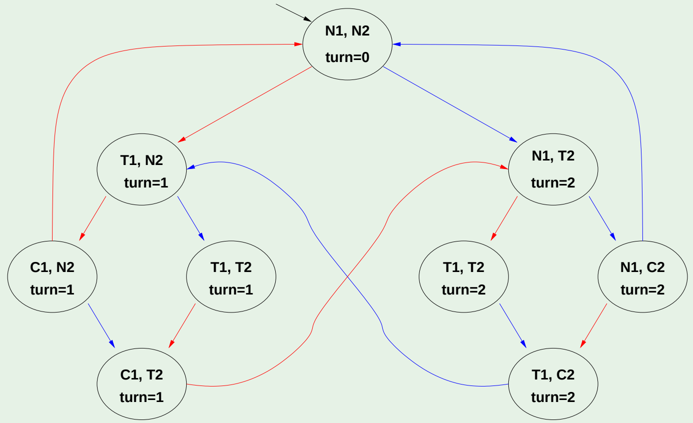
Example 2: liveness

$[T_1 \wedge \mathbf{EG}\neg C_1]$:



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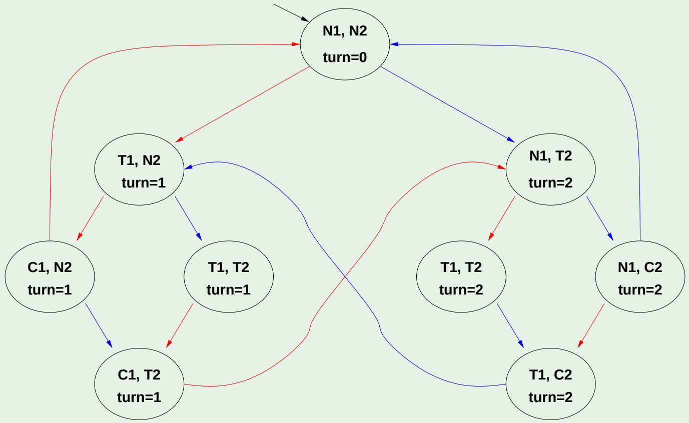
$[\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)]$:



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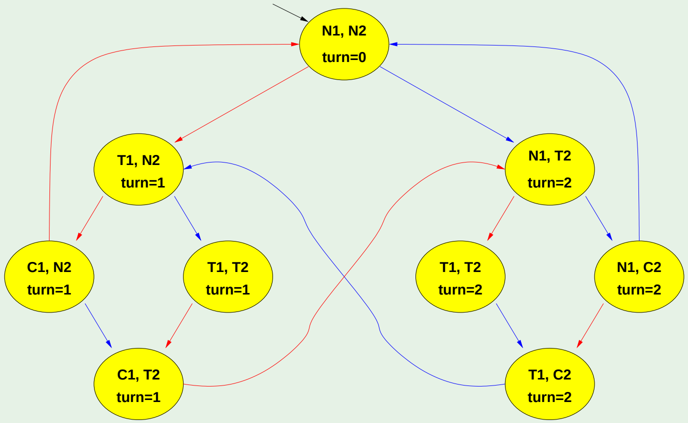
Example 2: liveness

$[\neg \mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)]$:



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$M \models \mathbf{AG}(T_1 \rightarrow \mathbf{AF}C_1)$? $\Longrightarrow M \models \neg\mathbf{EF}(T_1 \wedge \mathbf{EG}\neg C_1)$? YES!

The property verified is...

Homework

Apply the same process to all the CTL examples of Chapter 3.

Complexity of CTL Model Checking: $M \models \varphi$

- Step 1: compute $[\varphi]$
  - Compute $[\varphi]$ bottom-up on the $O(|\varphi|)$ sub-formulas of φ: $O(|\varphi|)$ steps...
  - ... each requiring at most exploring $O(|M|)$ states

  $\Longrightarrow O(|M| \cdot |\varphi|)$ steps
- Step 2: check $I \subseteq [\varphi]$: $O(|M|)$

$\Longrightarrow O(|M| \cdot |\varphi|)$

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Model Checking of Invariants

- Invariant properties have the form **AG p**, where **p** in Boolean (e.g., **AG**$\neg$*bad*)
- Checking invariants is the negation of a reachability problem:
  - is there a reachable state that is also a bad state? (**AG**$\neg$*bad* $=$ $\neg$**EF***bad*)
- Standard M.C. algorithm reasons backward from the *bad* by iteratively applying PreImage:

  $$Y' := Y \cup PreImage(Y)$$

  until a fixed point is reached.
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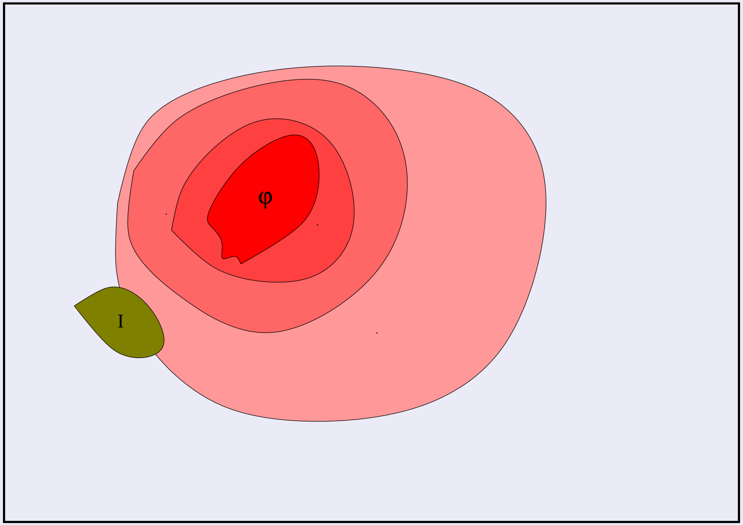
Model Checking of Invariants [cont.]



Forward Model Checking of Invariants

Alternative algorithm (often more efficient): forward checking

- Compute the set of bad states $[bad]$
- Compute the set of initial states I
- Compute incrementally the set of reachable states from I until (i) it intersect $[bad]$ or (ii) a fixed point is reached
- Basic step is the (Forward) Image:

$$Image(Y) \stackrel{\text{def}}{=} \{s' \mid s \in Y \text{ and } R(s, s') \text{ holds}\}$$

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- Simplest form: compute the set of reachable states.

Computing Reachable states: basic

```
**State_Set** Compute_reachable() {
    $Y' := I$; $Y := \emptyset$;
    **while** ($Y' \neq Y$)
        $Y := Y'$;
        $Y' := Y \cup Image(Y)$;
    }
**return** Y;
}
```

Y=reachable

Computing Reachable states: advanced

```
**State_Set** Compute_reachable() {
    $Y := F := I$;
    **while** $(F \neq \emptyset)$
        $F := Image(F) \setminus Y$;
        $Y := Y \cup F$;
    }
**return** Y;
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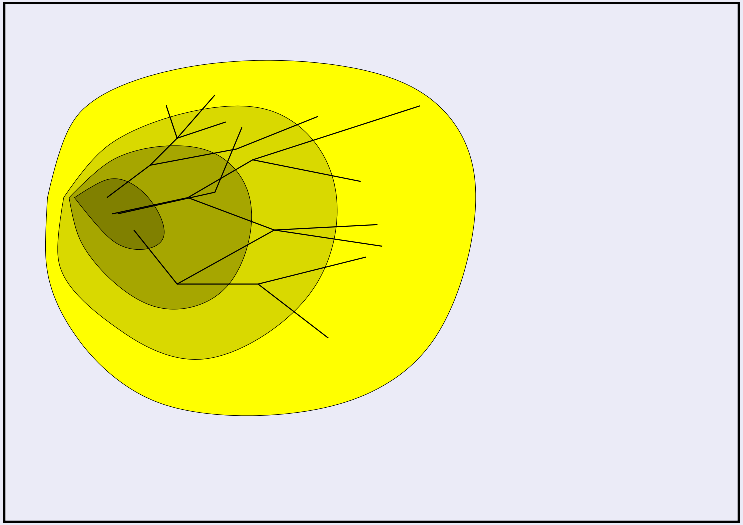
```

Y=reachable;F=frontier (new)

Computing Reachable states [cont.]



Checking of Invariant Properties: basic

```
bool Forward_Check_EF(State_Set $BAD$) {
    $Y := I$;  $Y' := \emptyset$;
    while $(Y' \neq Y)$ and $(Y' \cap BAD) = \emptyset$
        $Y := Y'$;
        $Y' := Y \cup Image(Y)$;
    }
    if $(Y' \cap BAD) \neq \emptyset$    // counter-example
        return $true$
    else                    // fixpoint reached
        return $false$
}
```

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Checking of Invariant Properties: advanced

```
bool Forward_Check_EF(State_Set $BAD$) {
    $Y := F := I$;
    while $(F \neq \emptyset)$ and $(F \cap BAD) = \emptyset$
        $F := Image(F) \setminus Y$;
        $Y := Y \cup F$;
    }
    if $(F \cap BAD) \neq \emptyset$      // counter-example
        return $true$
    else                       // fixpoint reached
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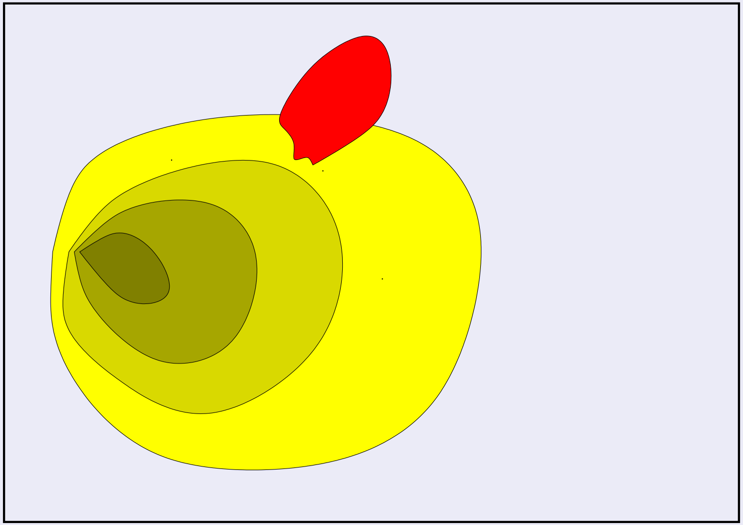
Checking of Invariant Properties [cont.]



Checking of Invariants: Counterexamples

- if layer n intersects with the bad states, then the property is violated
- a counterexample can be reconstructed proceeding backwards
  - (i) select any state of $BAD \cap F[n]$ (we know it is satisfiable), call it $t[n]$
  - (ii) compute $Preimage(t[n])$, i.e. the states that can result in $t[n]$ in one step
  - (iii) compute $Preimage(t[n]) \cap F[n-1]$, and select one state $t[n-1]$
- iterate (i)-(iii) until the initial states are reached
- $t[0], t[1], \ldots, t[n]$ is our counterexample

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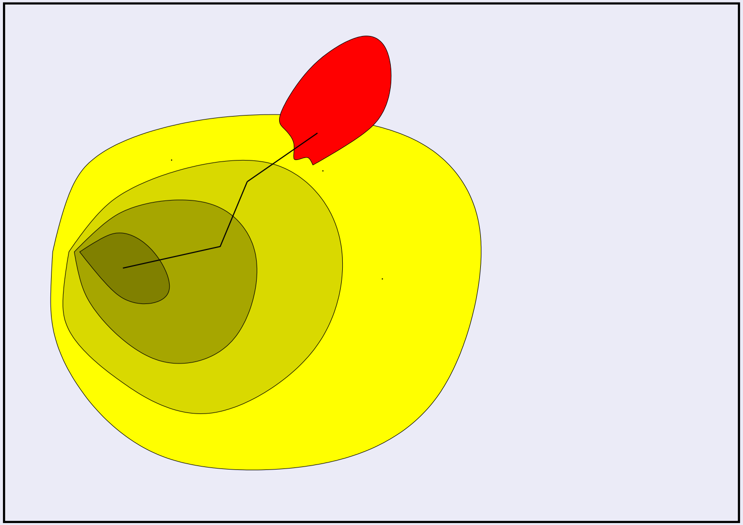
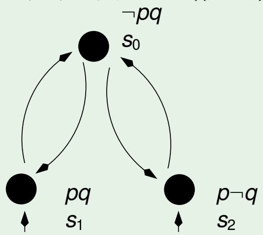


Outline

Ex: CTL Model Checking

Consider the Kripke Model M below, and the CTL property $\varphi \overset{\text{def}}{=} \mathbf{AG}((p \wedge q) \rightarrow \mathbf{EG}q)$.



(a) Rewrite φ into an equivalent formula φ' expressed in terms of **EX**, **EG**, **EU**/**EF** only.

(b) Compute bottom-up the denotations of all subformulas of φ'. (Ex: $[p] = \{s_1, s_2\}$)

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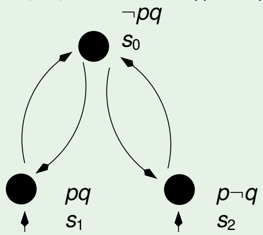

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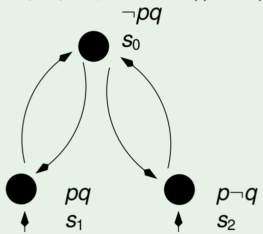

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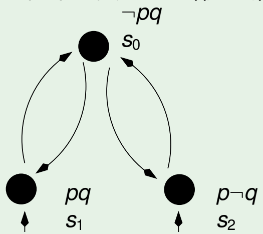

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[Solution: Yes, $\{s_1, s_2\} \subseteq [\varphi']$.]

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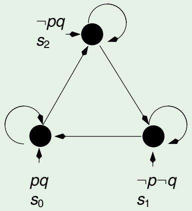

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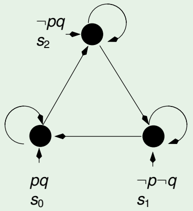

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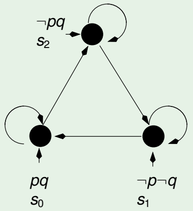

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$[p]$	$=$	$\{s_0\}$	$[\neg q]$	$=$	$\{s_1\}$
$[\neg p]$	$=$	$\{s_1, s_2\}$	$[\mathbf{EG}\neg q]$	$=$	$\{s_1\}$
$[\mathbf{EG}\neg p]$	$=$	$\{s_1, s_2\}$	$[\neg\mathbf{EG}\neg p \wedge \mathbf{EG}\neg q]$	$=$	$\{\}$
$[\neg\mathbf{EG}\neg p]$	$=$	$\{s_0\}$	$[\mathbf{EF}(\neg\mathbf{EG}\neg p \wedge \mathbf{EG}\neg q)]$	$=$	$\{\}$
$[q]$	$=$	$\{s_0, s_2\}$	$[\neg\mathbf{EF}(\neg\mathbf{EG}\neg p \wedge \mathbf{EG}\neg q)]$	$=$	$\{s_0, s_1, s_2\}$

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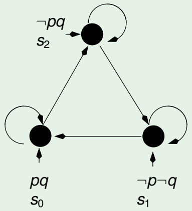

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