Automated Reasoning and Formal Verification
Module I: Automated Reasoning
Ch. 04: **Automata-Theoretic LTL Reasoning**

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Outline

1 Büchi Automata

2 The Automata-Theoretic Approach to LTL Reasoning
- General Ideas
- Language-Emptiness Checking of Büchi Automata
- From Kripke Models to Büchi Automata
- From LTL Formulas to Büchi Automata
- Complexity

3 Exercises

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3. Exercises

Infinite Word Languages

Modeling infinite computations of reactive systems

Given an Alphabet Σ (e.g. $\Sigma \overset{\text{def}}{=} \{a, b\}$)

- An ω-word α over Σ is an infinite sequence

  $$a_0,\ a_1,\ a_2 \ldots.$$

  Formally, $\alpha : \mathbb{N} \rightarrow \Sigma$.
- The set of all infinite words is denoted by Σ^ω.
- A ω-language L is collection of ω-words, i.e. $L \subseteq \Sigma^\omega$.
- Example: All words over $\{a, b\}$ with infinitely many a's.

Notation:

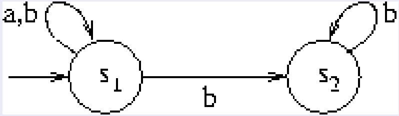
omega words $\alpha, \beta, \gamma \in \Sigma^\omega$.

omega-languages $L, L_1 \subseteq \Sigma^\omega$

For $u \in \Sigma^+$, let $u^\omega = u.u.u\ldots$

Omega-Automata

- We consider automaton running over infinite words.



- Let $\alpha = aabbbb\ldots$.
  There are several (infinite) possible runs.
  Run $\rho_1 = s_1, s_1, s_1, s_1, s_2, s_2 \ldots$
  Run $\rho_2 = s_1, s_1, s_1, s_1, s_1, s_1 \ldots$
- Acceptance Conditions: Büchi (Muller, Rabin, Street):
  Acceptance is based on states occurring infinitely often
- Notation: Let Q be the set of states. Let $\rho \in Q^\omega$. Then,
  $$Inf(\rho) = \{s \in Q \mid \exists^\infty i \in \mathbb{N}.\ \rho(i) = s\}.$$
  (The set of states occurring infinitely many times in ρ.)

Büchi Automata

Nondeterministic Büchi Automaton

- A Nondeterministic Büchi Automaton (NBA) is $(Q, \Sigma, \delta, I, F)$ s.t.
  - Q Finite set of states.
  - Σ is a finite alphabet
  - $I \subseteq Q$ set of initial states.
  - $F \subseteq Q$ set of accepting states.
  - $\delta \subseteq Q \times \Sigma \times Q$ transition relation (edges).
- A Deterministic Büchi Automaton (DBA) is an NBA s.t. the transition relation is functional: $\delta : Q \times \Sigma \longmapsto Q$

Runs and Language of NBAs

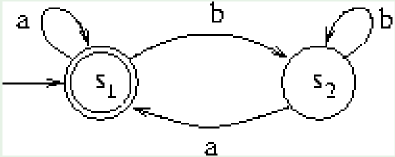
- A run ρ of A on ω-word $\alpha = a_0, a_1, a_2, \ldots$ is an infinite sequence $\rho = q_o, q_1, q_2, \ldots$ s.t. $q_0 \in I$ and $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i$.
- The run ρ is accepting if

$$Inf(\rho) \cap F \neq \emptyset.$$

- The language accepted by A
  $\mathcal{L}(A) = \{\alpha \in \Sigma^\omega \mid A \text{ has an accepting run on } \alpha\}$

Büchi Automaton: Example

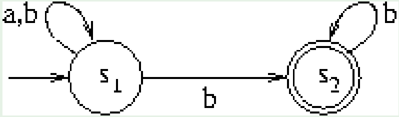
Let $\Sigma = \{a, b\}$.
Let a Deterministic Büchi Automaton (DBA) A_1 be



- With $F = \{s_1\}$ the automaton recognizes words with infinitely many *a*'s.
- With $F = \{s_2\}$ the automaton recognizes words with infinitely many *b*'s.

Büchi Automaton: Example (2)

Let a Nondeterministic Büchi Automaton (NBA) A_2 be



With $F = \{s_2\}$, the automaton A_2 recognizes words with finitely many a. Thus, $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.

Deterministic vs. Nondeterministic Büchi Automata

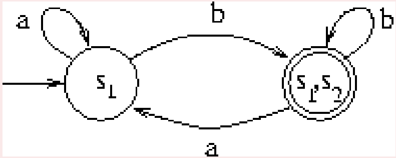
Theorem

*DBA*s are strictly less powerful than *NBA*s.

Remark:

The subset construction of standard Final-State automata does not work!

Let DA_2 be



- DA_2 is not equivalent to A_2 (e.g., it recognizes $(b.a)^\omega$)
- There is no DBA equivalent to A_2

Closure Properties

Theorem (union, intersection)

For the NBAs A_1, A_2 we can construct

- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. $|A| = |A_1| + |A_2|$
- the NBA A s.t. $\mathcal{L}(A) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$. $|A| \leq |A_1| \cdot |A_2| \cdot 2$.

Union of two NBAs

Definition: union of NBAs

Let $A_1 = (Q_1, \Sigma_1, \delta_1, I_1, F_1)$, $A_2 = (Q_2, \Sigma_2, \delta_2, I_2, F_2)$.
Then $A = A_1 \cup A_2 = (Q, \Sigma, \delta, I, F)$ is defined as follows

- $Q := Q_1 \cup Q_2$, $I := I_1 \cup I_2$, $F := F_1 \cup F_2$
- $R(s, s') := \begin{cases} R_1(s, s') \text{ if } s \in Q_1 \\ R_2(s, s') \text{ if } s \in Q_2 \end{cases}$

Theorem

- $\mathcal{L}(A) \;=\; \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$
- $|A| = |A_1| + |A_2|$

Note

A is an automaton which just runs nondeterministically either A_1 or A_2
(same construction as with ordinary automata)

Synchronous Product of NBAs

Definition: synchronous product of NBAs

Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$.
Then, $A_1 \times A_2 = (Q, \Sigma, \delta, I, F)$, where

$$Q = Q_1 \times Q_2 \times \{1, 2\}.$$
$$I = I_1 \times I_2 \times \{1\}.$$
$$F = F_1 \times Q_2 \times \{1\}.$$

$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \notin F_1$.
$\langle p, q, 1\rangle \xrightarrow{a} \langle p', q', 2\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $p \in F_1$.
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$\langle p, q, 2\rangle \xrightarrow{a} \langle p', q', 1\rangle$ iff $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$ and $q \in F_2$.

Theorem

- $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.
- $|A_1 \times A_2| \leq 2 \cdot |A_1| \cdot |A_2|$.

Synchronous Product of NBAs: Intuition

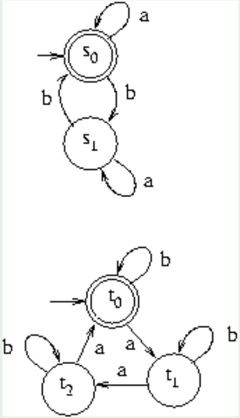
- The automaton remembers two tracks, one for each source NBA, and it points to one of the two tracks
- As soon as it goes through an accepting state of the current track, it switches to the other track

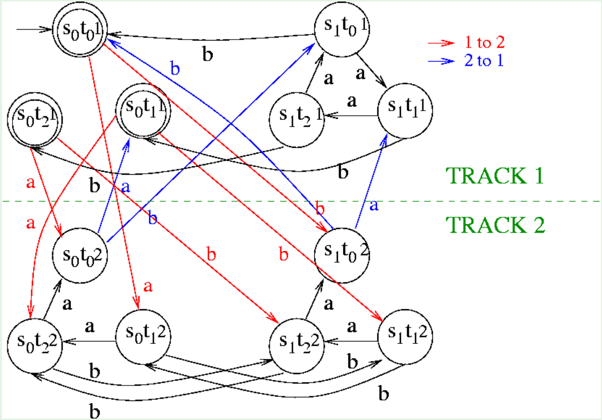
$\Longrightarrow$ to visit infinitely often a state in F (i.e., F_1), it must visit infinitely often some state also in F_2

- Important subcase: If $F_2 = Q_2$, then

$$Q = Q_1 \times Q_2.$$
$$I = I_1 \times I_2.$$
$$F = F_1 \times Q_2.$$

Synchronous Product of NBAs: Example





Closure Properties (2)

Theorem (complementation) [Safra, MacNaughten]

For the NBA A_1 we can construct an NBA A_2 such that $\mathcal{L}(A_2) = \overline{\mathcal{L}(A_1)}$.
$|A_2| = O(2^{|A_1| \cdot \log(|A_1|)})$.

Method: (hint)

(i) convert a Büchi automaton into a Non-Deterministic Rabin automaton
(ii) determinize and Complement the Rabin automaton
(iii) convert the Rabin automaton into a Büchi automaton.

Generalized Büchi Automaton

Definition

- A Generalized Büchi Automaton is a tuple $A := (Q, \Sigma, \delta, I, FT)$ where $FT = \langle F_1, F_2, \ldots, F_k \rangle$ with $F_i \subseteq Q$.
- A run ρ of A is accepting if $Inf(\rho) \cap F_i \neq \emptyset$ for each $1 \leq i \leq k$.

Theorem

For every Generalized Büchi Automaton we can construct a language equivalent plain Büchi Automaton.

Intuition

Let $Q' = Q \times \{1, \ldots, K\}$.
The automaton remains in phase i till it visits a state in F_i. Then, it moves to $(i \; mod \; K) + 1$ mode.

De-generalization of a generalized NBA

Definition: De-generalization of a generalized NBA

Let $A \overset{\text{def}}{=} (Q, \Sigma, \delta, I, FT)$ a generalized BA s.f. $FT \overset{\text{def}}{=} \{F_1, ..., F_K\}$.
Then a language-equivalent BA $A' \overset{\text{def}}{=} (Q', \Sigma, \delta', I', F')$ is built as follows

$$
\begin{aligned}
Q' &= Q_1 \times \{1, ..., K\}. \\
I' &= I \times \{1\}. \\
F' &= F_1 \times \{1\}.
\end{aligned}
$$

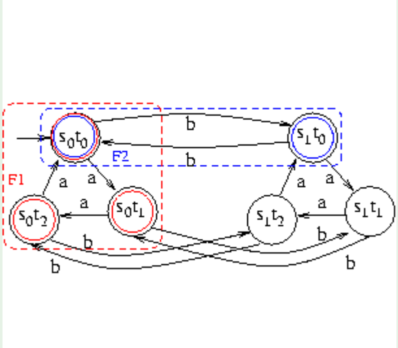
δ' is s.t., for every $i \in [1, ..., K]$:

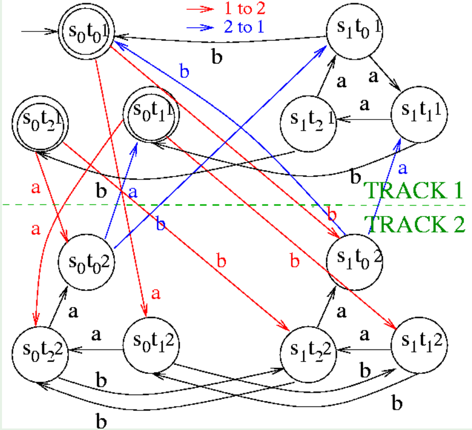
$$
\begin{array}{llllll}
\langle p, i\rangle \xrightarrow{a} \langle q, i\rangle & \text{iff} & p \xrightarrow{a} q & \in \delta & \text{and} & p \notin F_i. \\
\langle p, i\rangle \xrightarrow{a} \langle q, (i \ mod \ K) + 1\rangle & \text{iff} & p \xrightarrow{a} q & \in \delta & \text{and} & p \in F_i.
\end{array}
$$

Theorem

- $\mathcal{L}(A') = \mathcal{L}(A)$.
- $|A'| \leq K \cdot |A|$.

Degeneralizing a Büchi automaton: Example





Omega-regular Expressions

Recall:

A finite-word language is called regular if it it is recognizable by some Finite-State-Automaton (FSA).

Definition

An infinite-word language is called ω-regular if it has the form $\cup_{i=1}^{n} \; U_i.(V_i)^\omega$ where U_i, V_i are regular languages.

Theorem

A language L is ω-regular iff it is NBA-recognizable.

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Automata-Theoretic LTL Satisfiability and Entailment

LTL Validity/Satisfiability

- Let ψ be an LTL formula

$$\begin{array}{ll} & \models \psi \quad \text{(LTL)} \\ \Longleftrightarrow & \neg\psi \text{ unsat} \\ \Longleftrightarrow & \mathcal{L}(A_{\neg\psi}) = \emptyset \end{array}$$

- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

LTL Entailment

- Let φ, ψ be an LTL formula

$$\begin{array}{ll} & \varphi \models \psi \quad \text{(LTL)} \\ & \models \varphi \rightarrow \psi \quad \text{(LTL)} \\ \Longleftrightarrow & \varphi \wedge \neg\psi \text{ unsat} \\ \Longleftrightarrow & \mathcal{L}(A_{\varphi\wedge\neg\psi}) = \emptyset \end{array}$$

- $A_{\varphi\wedge\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\varphi \wedge \neg\psi$ (satisfy φ and do not satisfy ψ)

Automata-Theoretic LTL Satisfiability and Entailment

Two steps for checking $\models \psi$ [resp. $\varphi \models \psi$]

(i) Compute $A_{\neg\psi}$ [resp. $A_{\varphi\wedge\neg\psi}$]

(ii) Check the emptiness of $\mathcal{L}(A_{\neg\psi})$ [resp. $\mathcal{L}(A_{\varphi\wedge\neg\psi})$]

Automata-Theoretic LTL Model Checking

LTL Model Checking

- Let M be a Kripke model and ψ be an LTL formula

$$\begin{aligned}
& M \models \psi \quad \text{(LTL)} \\
\Longleftrightarrow\ & \mathcal{L}(M) \subseteq \mathcal{L}(\psi) \\
\Longleftrightarrow\ & \mathcal{L}(M) \cap \overline{\mathcal{L}(\psi)} = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(M) \cap \mathcal{L}(\neg\psi) = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(A_M) \cap \mathcal{L}(A_{\neg\psi}) = \emptyset \\
\Longleftrightarrow\ & \mathcal{L}(A_M \times A_{\neg\psi}) = \emptyset
\end{aligned}$$

- A_M is a Büchi Automaton equivalent to M (which represents all and only the executions of M)
- $A_{\neg\psi}$ is a Büchi Automaton which represents all and only the paths that satisfy $\neg\psi$ (do not satisfy ψ)

$\Longrightarrow$ $A_M \times A_{\neg\psi}$ represents all and only the paths appearing in M and not in ψ.

Automata-Theoretic LTL Model Checking

Four steps

Let $\varphi \overset{\text{def}}{=} \neg\psi$:

(i) Compute A_M

(ii) Compute A_φ

(iii) Compute the product $A_M \times A_\varphi$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$

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NBA emptiness checking

- Idea: find an accepting cycle reachable from an initial state
  - accepting cycle: a cycle containing some accepting state f
- A naive algorithm (Naive Double Nested DFS algorithm):
  (i) a DFS finds the accepting states *f* reachable from an initial state;
  (ii) for each *f*, a second DFS finds if it can reach f
  (i.e., if there exists a loop)

  Complexity: $O(n^2)$
- SCC-based algorithm:
  (i) Tarjan's algorithm uses a DFS to find the SCCs in linear time;
  (ii) drop all SCCs which do not have at least one arc, and which do not contain at least one accepting state f
  (iii) another DFS finds if the union of non-trivial SCCs is reachable from an initial state.

  Complexity: $O(n)$
- Drawbacks: it stores too much information and does not find directly a counterexample.

(Smart) Double Nested DFS algorithm

(Smart) Double Nested DFS

- Two nested DFSs
  - `DFS1` finds the accepting states f reachable from an initial state
  - for each f, `DFS2` finds if it can reach f (i.e., if there exists a loop)
- Two Hash tables:
  - `T1`: reachable states
  - `T2`: states reachable from a reachable accepting state
- Two stacks:
  - `S1`: current branch of states reachable
  - `S2`: current branch of states reachable from accepting state f
- It stops as soon as it finds a counterexample.
- The counterexample is given by
  - the stack of `DFS2` (an accepting, preceded by cycle)
  - the stack of `DFS1` (a path from an initial state to the cycle)

- DFS1 invokes DFS2 on each f_i only after popping it (postorder)
- T2 passed by reference (or static) $\Longrightarrow$ is not reset at each call of DFS2 !

(Smart) Double Nested DFS - First DFS

```
// returns True if empty language, false otherwise
Bool DFS1(NBA A) {
   stack S1=I; stack S2=∅;
   Hashtable T1=I; Hashtable T2=∅;
   while S1!=∅ {
      v=top(S1);
      if ∃w s.t. w∈δ(v) && T1(w)==0 {
         hash(w,T1);
         push(w,S1);
      } else {
         pop(S1);
         if (v∈F && !DFS2(v,S2,T2,A))     //test after popping!
            return False;
   }    }
   return True;
}
```

(Smart) Double Nested DFS - Second DFS

```
Bool DFS2(state f, stack & S, Hashtable & T, NBA A) {
    hash(f,T);
    S = {f}
    while S!=∅ {
        v=top(S);
        if f∈δ(v) return False;
        if ∃w s.t. w∈δ(v) && T(w)==0 {
            hash(w);
            push(w);
        } else pop(S);
    }
    return True;
}
```

Remark: `T` passed by reference (or static) $\Longrightarrow$ is not reset at each call of `DFS2` !

Double nested DFS: Intuition

DFS1 invokes DFS2 on each $f_1, ..., f_n$ only after popping it (postorder):

- suppose *DFS*2 is invoked on f_j earlier than on f_i

$\Longrightarrow$ f_i not reachable from (any state s which is reachable from) f_j

- If during *DFS*2(f_i, ...) it is encountered a state S which has already been explored by *DFS*2(f_j, ...) for some f_j,
  - can we reach f_i from S?
  - No, because f_i is not reachable from f_j!

$\Longrightarrow$ It is safe to backtrack!

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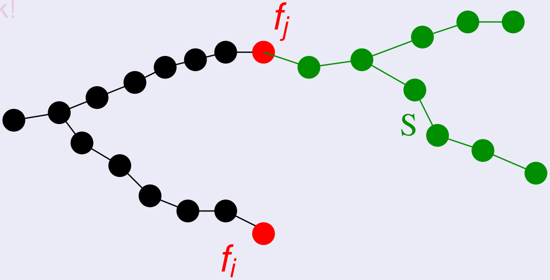

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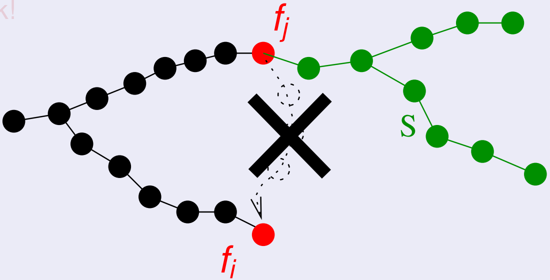

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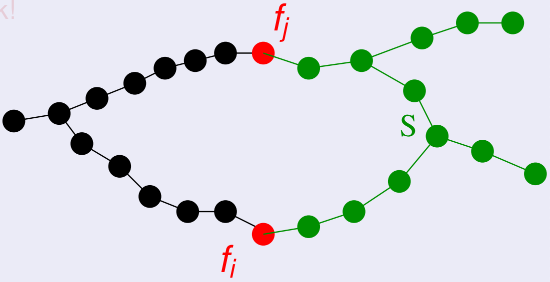

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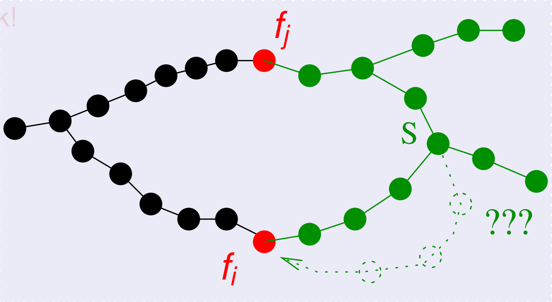

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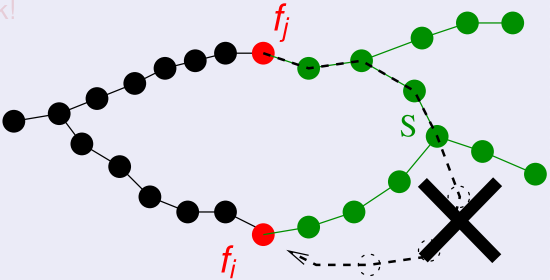

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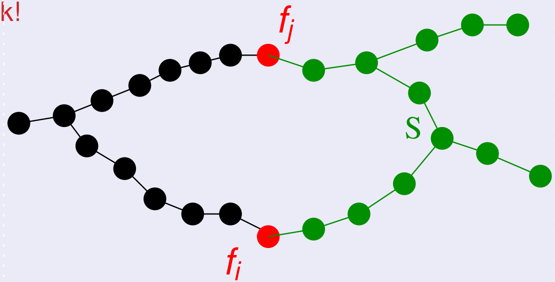
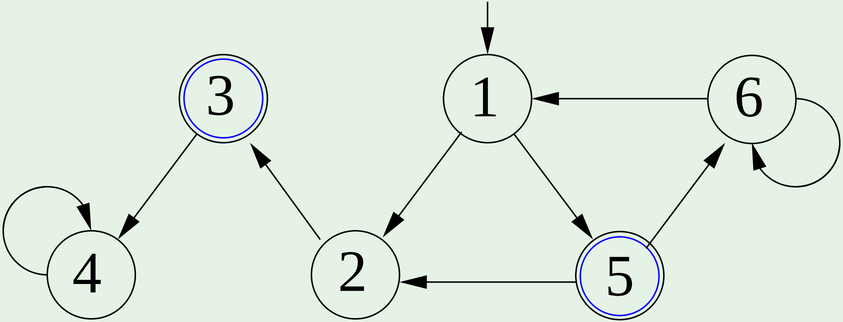


(Smart) Double Nested DFS: example



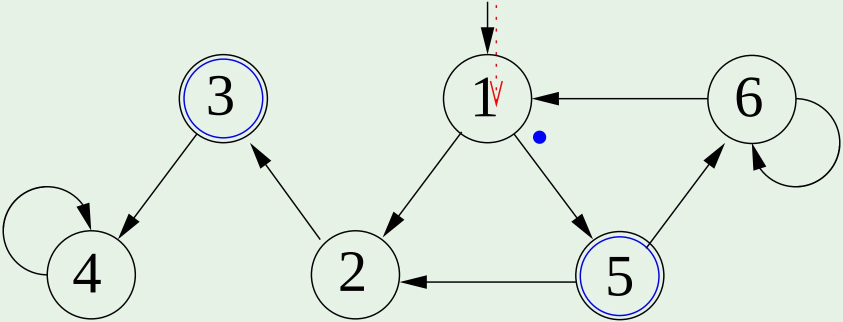
T1

S1

T2

S2

(Smart) Double Nested DFS: example



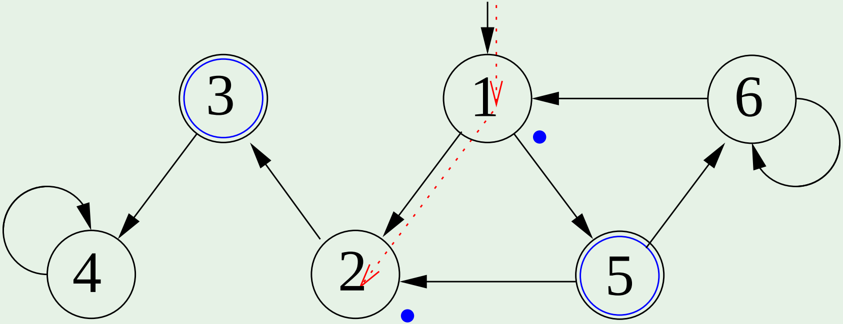
T1 1

S1 1

T2

S2

(Smart) Double Nested DFS: example



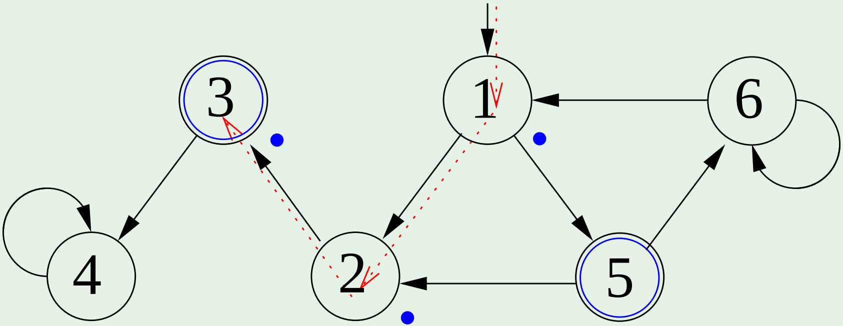
T1 1 2

S1 1 2

T2

S2

(Smart) Double Nested DFS: example



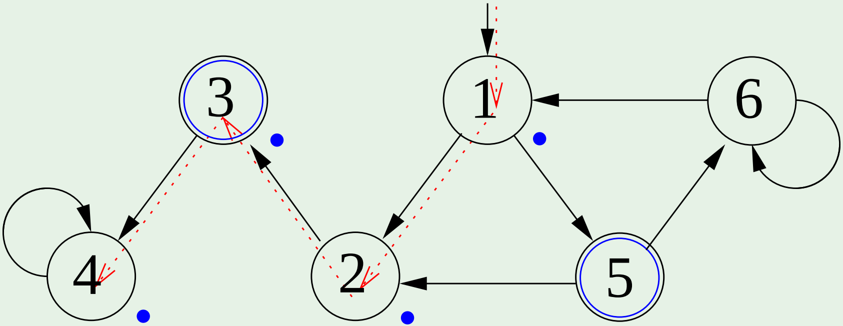
T1 1 2 3

S1 1 2 3

T2

S2

(Smart) Double Nested DFS: example



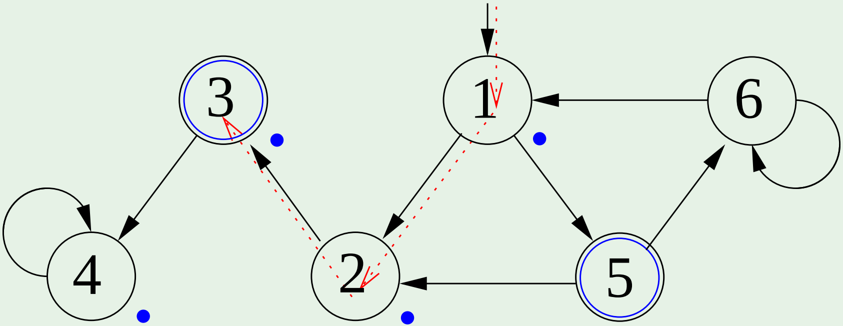
T1 1 2 3 4

S1 1 2 3 4

T2

S2

(Smart) Double Nested DFS: example



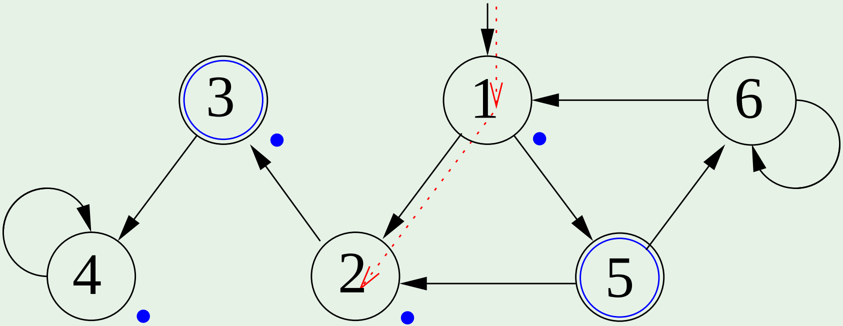
T1 1 2 3 4

S1 1 2 3

T2

S2

(Smart) Double Nested DFS: example



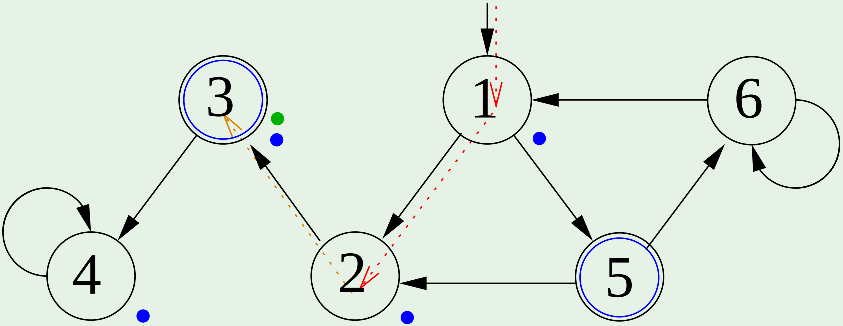
T1 1 2 3 4

S1 1 2

T2

S2

(Smart) Double Nested DFS: example



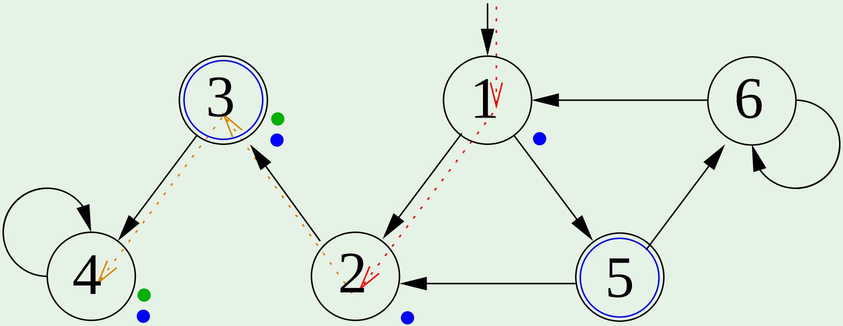
T1 1 2 3 4

S1 1 2

T2 3

S2 3

(Smart) Double Nested DFS: example



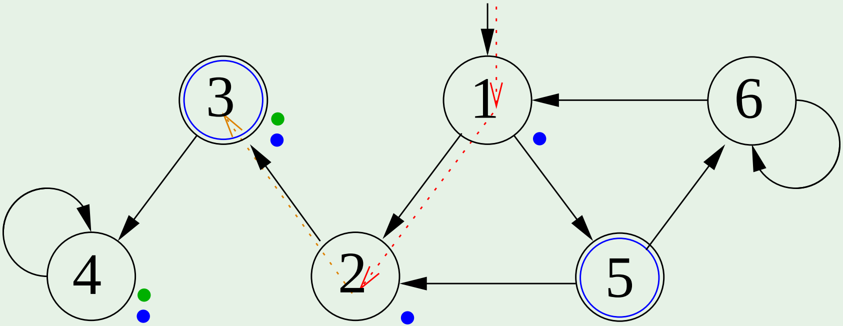
T1 1 2 3 4

T2 3 4

S1 1 2

S2 3 4

(Smart) Double Nested DFS: example



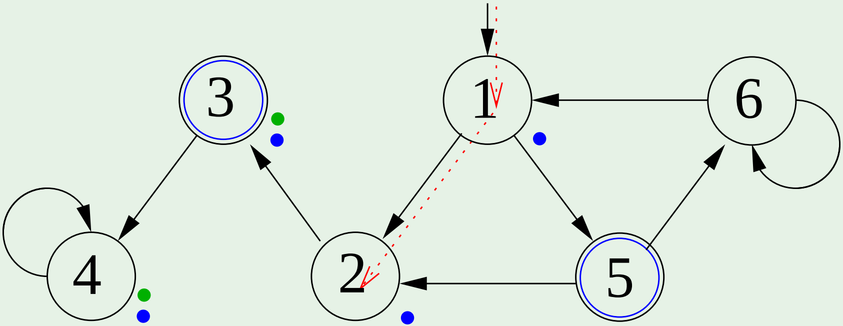
T1 1 2 3 4

T2 3 4

S1 1 2

S2 3

(Smart) Double Nested DFS: example



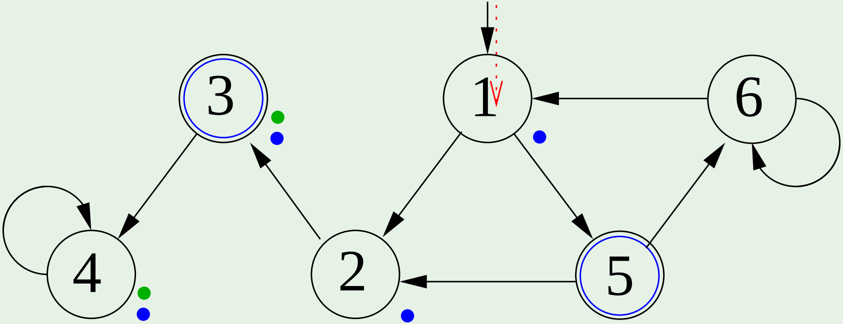
T1 1 2 3 4

S1 1 2

T2 3 4

S2

(Smart) Double Nested DFS: example



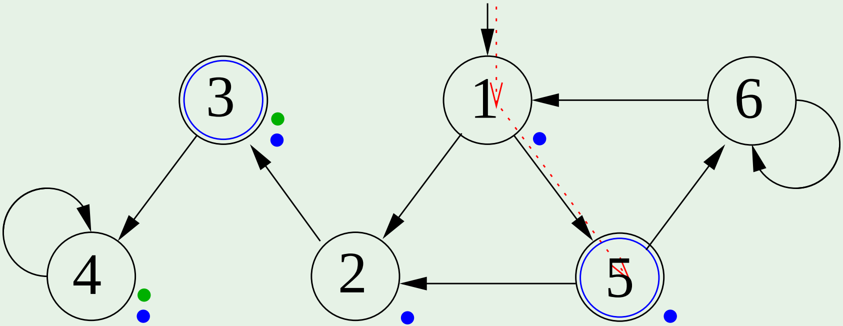
T1 1 2 3 4

S1 1

T2 3 4

S2

(Smart) Double Nested DFS: example



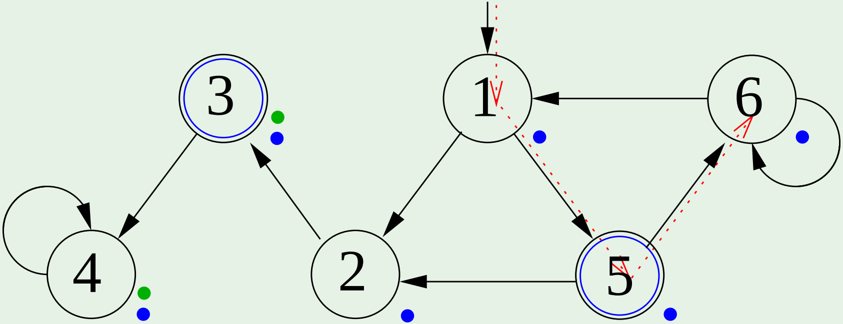
T1 1 2 3 4 5

T2 3 4

S1 1 5

S2

(Smart) Double Nested DFS: example



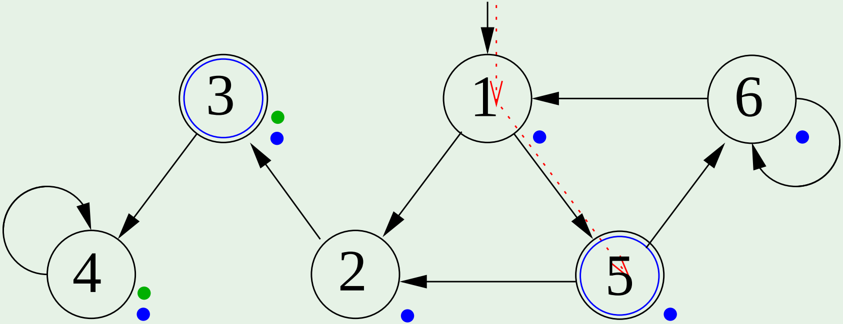
T1 1 2 3 4 5 6

T2 3 4

S1 1 5 6

S2

(Smart) Double Nested DFS: example



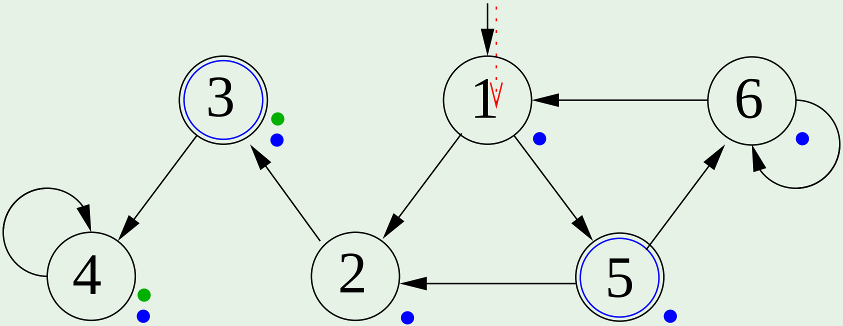
T1 1 2 3 4 5 6

T2 3 4

S1 1 5

S2

(Smart) Double Nested DFS: example



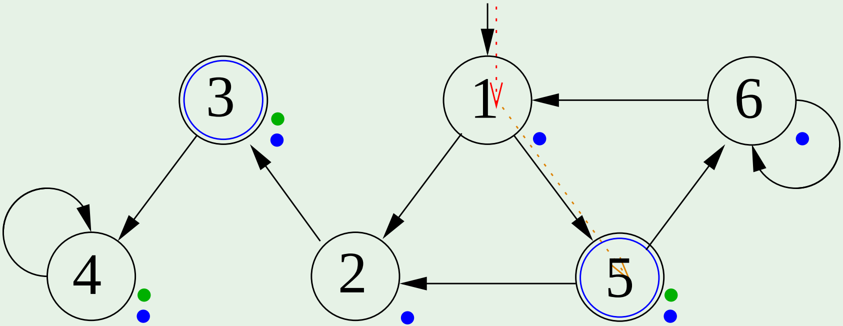
T1 1 2 3 4 5 6

T2 3 4

S1 1

S2

(Smart) Double Nested DFS: example



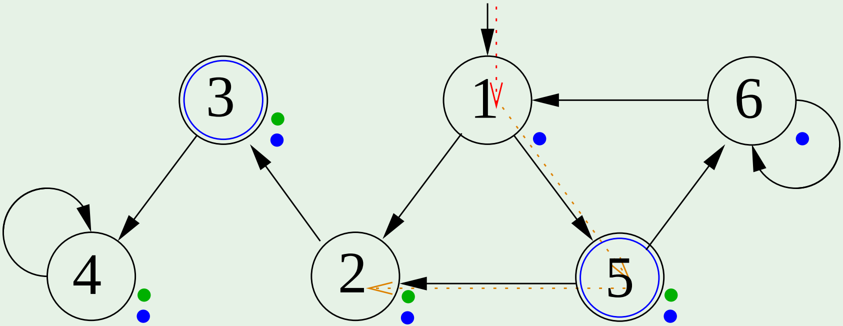
T1 1 2 3 4 5 6

T2 3 4 5

S1 1

S2 5

(Smart) Double Nested DFS: example



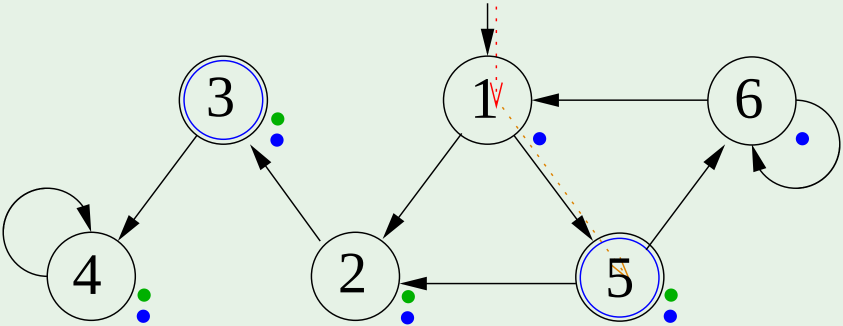
T1 1 2 3 4 5 6

S1 1

T2 3 4 5 2

S2 5 2

(Smart) Double Nested DFS: example



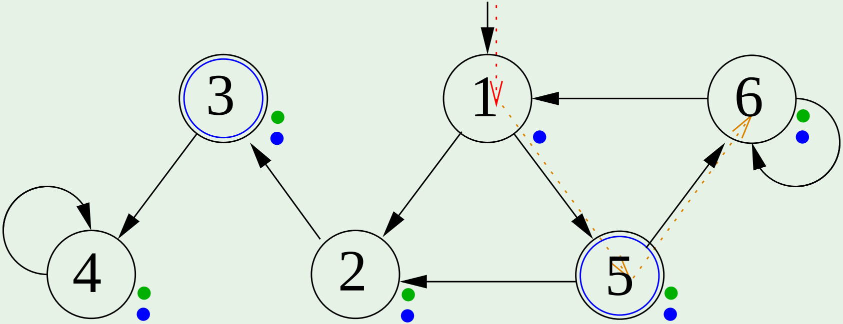
T1 1 2 3 4 5 6

T2 3 4 5 2

S1 1

S2 5

(Smart) Double Nested DFS: example



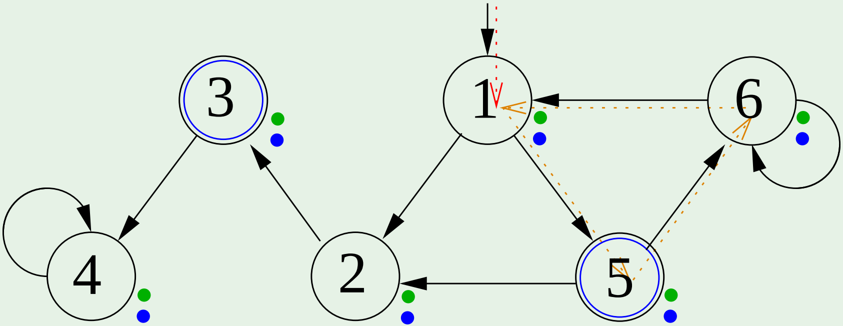
T1 1 2 3 4 5 6

T2 3 4 5 2 6

S1 1

S2 5 6

(Smart) Double Nested DFS: example



T1 1 2 3 4 5 6

T2 3 4 5 2 6 1

S1 1

S2 5 6 1

Outline

1 Büchi Automata

2 The Automata-Theoretic Approach to LTL Reasoning
- General Ideas
- Language-Emptiness Checking of Büchi Automata
- From Kripke Models to Büchi Automata
- From LTL Formulas to Büchi Automata
- Complexity

3 Exercises

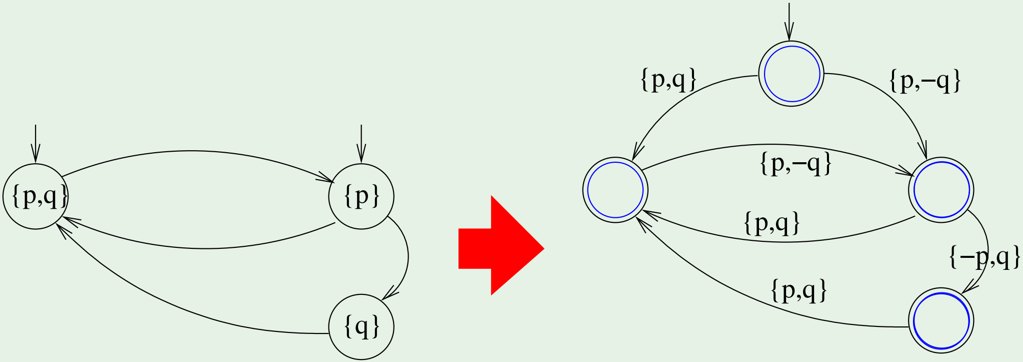
Computing an NBA A_M from a Kripke Structure M

- Transform a Kripke model $M = \langle S, S_0, R, L, AP \rangle$ into an NBA $A_M = \langle Q, \Sigma, \delta, I, F \rangle$ s.t.:
  - States: $Q := S \cup \{init\}$, *init* being a new initial state
  - Alphabet: $\Sigma := 2^{AP}$ (total truth-assignments as alphabet symbols!)
  - Initial State: $I := \{init\}$
  - Accepting States: $F := Q = S \cup \{init\}$
  - Transitions:

$$\begin{aligned} \delta: \quad & q \xrightarrow{a} q' \text{ iff } (q, q') \in R \textit{ and } L(q') = a \\ & init \xrightarrow{a} q \text{ iff } q \in S_0 \textit{ and } L(q) = a \end{aligned}$$

- $\mathcal{L}(A_M) = \mathcal{L}(M)$
- $|A_M| = |M| + 1$

Computing a NBA A_M from a Kripke Structure M: Example



Kripke Structure

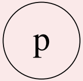
Buechi Automaton

$\Longrightarrow$ Substantially:

1. add one initial state,
2. move labels from states to incoming edges,
3. set all states as accepting states

Labels on Kripke Structures and BA's - Remark

Note that the labels of a Büchi Automaton are different from the labels of a Kripke Structure. Also graphically, they are interpreted differently:



- in a Kripke Structure, it means that *p* is true and all other propositions are false;
- in a Büchi Automaton, it means that *p* is true and all other propositions are irrelevant ("don't care"), i.e. they can be either true or false.

Outline

1. Büchi Automata
2. The Automata-Theoretic Approach to LTL Reasoning
    - General Ideas
    - Language-Emptiness Checking of Büchi Automata
    - From Kripke Models to Büchi Automata
    - From LTL Formulas to Büchi Automata
    - Complexity
3. Exercises

Translation problem

Problem

Given an LTL formula ϕ, find a Büchi Automaton that accepts the same language of ϕ.

- It is a fundamental problem in LTL validity/satisfiability/entailment and model checking
- We translate an LTL formula into a Generalized Büchi Automata (GBA), then into an NBA

LTL Negative Normal Form (NNF)

- Every LTL formula φ can be written into an equivalent formula φ' using only the operators $\wedge$, $\vee$, **X**, **U**, **R** on propositional literals.
- Done by pushing negations down to literal level:

$$\begin{array}{lcl} \neg\neg\varphi_1 & \Longrightarrow & \varphi_1 \\ \neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & (\neg\varphi_1 \wedge \neg\varphi_2) \\ \neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & (\neg\varphi_1 \vee \neg\varphi_2) \\ \neg\mathbf{X}\varphi_1 & \Longrightarrow & \mathbf{X}\neg\varphi_1 \\ \neg(\varphi_1\mathbf{U}\varphi_2) & \Longrightarrow & (\neg\varphi_1\mathbf{R}\neg\varphi_2) \\ \neg(\varphi_1\mathbf{R}\varphi_2) & \Longrightarrow & (\neg\varphi_1\mathbf{U}\neg\varphi_2) \end{array}$$

$\Longrightarrow$ The resulting formula is expressed in terms of $\vee$, $\wedge$, X, **U**, **R** and literals (Negative Normal Form, NNF).

  - the encoding is linear if a DAG representation is used

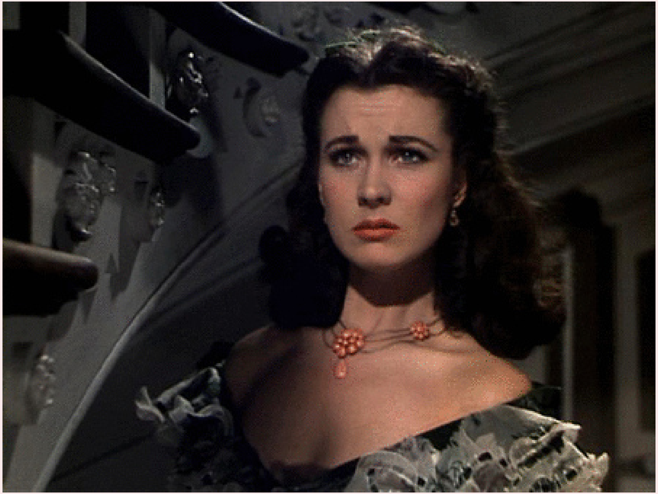
- In the construction of A_φ we now assume that φ is in NNF.
  $\Longrightarrow$ every non-atomic subformula occurs positively in φ
- For convenience, we still use **F**'s and **G**'s as shortcuts: $\mathbf{F}\varphi$ for $\top\mathbf{U}\varphi$ and $\mathbf{G}\varphi$ for $\bot\mathbf{R}\varphi$

On-the-fly Construction of A_φ (Intuition)

(Implicitly) Apply recursively the following steps:

Step 1: Apply the tableau expansion rules to φ:
$\psi_1 \mathbf{U} \psi_2 \Longrightarrow \psi_2 \vee (\psi_1 \wedge \mathbf{X}(\psi_1 \mathbf{U} \psi_2))$ [and $\mathbf{F}\psi \Longrightarrow \psi \vee \mathbf{XF}\psi$]
$\psi_1 \mathbf{R} \psi_2 \Longrightarrow \psi_2 \wedge (\psi_1 \vee \mathbf{X}(\psi_1 \mathbf{R} \psi_2))$ [and $\mathbf{G}\psi \Longrightarrow \psi \wedge \mathbf{XG}\psi$]
until we get a Boolean combination of elementary subformulas of φ
(An elementary formula is a proposition or a **X**-formula.)

Tableaux Rules: a Quote



“After all... tomorrow is another day."
[Scarlett O’Hara, “Gone with the Wind”]

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 2: Convert all formulas into Disjunctive Normal Form, by:
(i) applying recursively the DeMorgan rule: $\varphi_1 \wedge (\varphi_2 \vee \varphi_3) \implies (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$, and then
(ii) pushing the conjunctions inside the next operator:

$$\varphi \xRightarrow{(i)} \bigvee_i (\bigwedge_j l_{ij} \wedge \bigwedge_k \mathbf{X}\psi_{ik}) \xRightarrow{(ii)} \bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik}).$$

- Each disjunct $(\overbrace{\bigwedge_j l_{ij}}^{labels} \wedge \overbrace{\mathbf{X} \bigwedge_k \psi_{ik}}^{next\ part})$ represents a state:
  - the conjunction of literals $\bigwedge_j l_{ij}$ represents a set of labels in Σ
    (e.g., if $Vars(\varphi) = \{p, q, r\}$, $p \wedge \neg q$ represents the two labels $\{p, \neg q, r\}$ and $\{p, \neg q, \neg r\}$)
  - $\mathbf{X} \bigwedge_k \psi_{ik}$ represents the next part of the state
    (obbligations for the successors)
- N.B., if no next part occurs, $\mathbf{X}\top$ is implicitly assumed

On-the-fly Construction of A_φ (Intuition) [cont.]

Step 3: For every state S_i represented by $(\bigwedge_j l_{ij} \wedge \mathbf{X} \overbrace{\bigwedge_k \psi_{ik}}^{\varphi_i})$

- label the incoming edges of S_i with $\bigwedge_j l_{ij}$
- mark that the state S_i satisfies φ
- apply recursively steps 1-2-3 to $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$,
  - rewrite φ_i into $\bigvee_{i'}(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$
  - from each disjunct $(\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$ generate a new state $S_{ii'}$ (if not already present) and label it as satisfying $\varphi_i \stackrel{\text{def}}{=} \bigwedge_k \psi_{ik}$
- draw an edge from S_i to all states $S_{ii'}$ which satisfy $\bigwedge_k \psi_{ik}$
- (if no next part occurs, $\mathbf{X}\top$ is implicitly assumed, so that an edge to a “true” node is drawn)

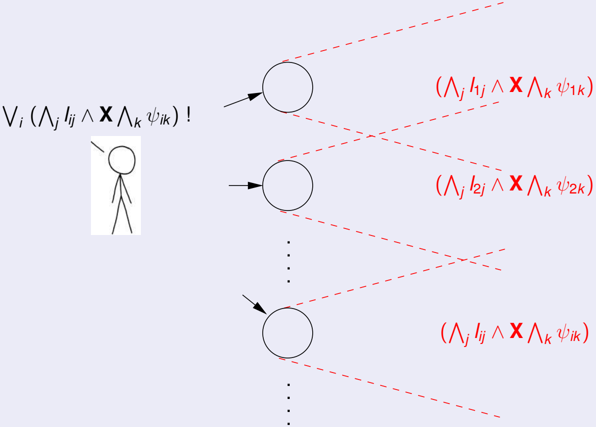
On-the-fly Construction of A_φ (Intuition) [cont.]

φ ??

On-the-fly Construction of A_φ (Intuition) [cont.]

$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

On-the-fly Construction of A_φ (Intuition) [cont.]



$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

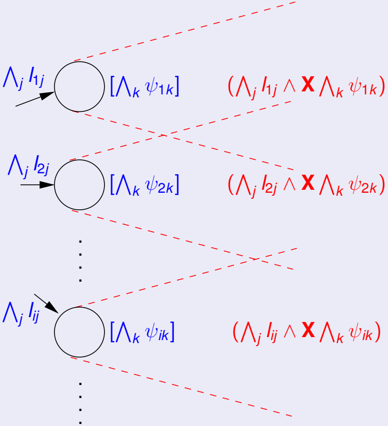
$(\bigwedge_j l_{1j} \wedge \mathbf{X} \bigwedge_k \psi_{1k})$

$(\bigwedge_j l_{2j} \wedge \mathbf{X} \bigwedge_k \psi_{2k})$

$(\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$

On-the-fly Construction of A_φ (Intuition) [cont.]

$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

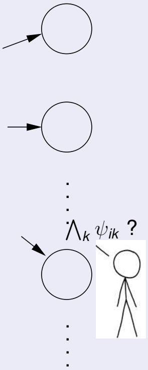


$\bigwedge_j l_{1j}$ $[\bigwedge_k \psi_{1k}]$ $(\bigwedge_j l_{1j} \wedge \mathbf{X} \bigwedge_k \psi_{1k})$

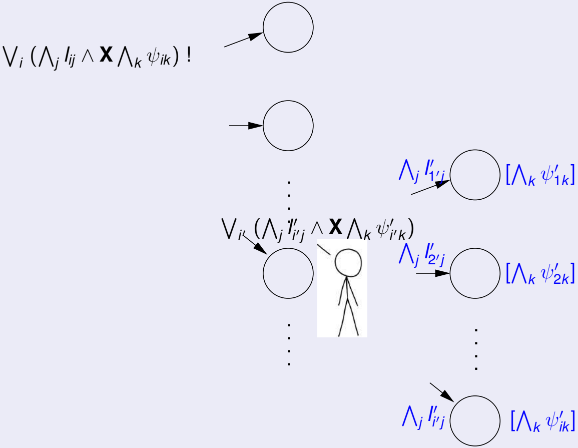
$\bigwedge_j l_{2j}$ $[\bigwedge_k \psi_{2k}]$ $(\bigwedge_j l_{2j} \wedge \mathbf{X} \bigwedge_k \psi_{2k})$

$\bigwedge_j l_{ij}$ $[\bigwedge_k \psi_{ik}]$ $(\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$

On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]



$\bigvee_i (\bigwedge_j l_{ij} \wedge \mathbf{X} \bigwedge_k \psi_{ik})$!

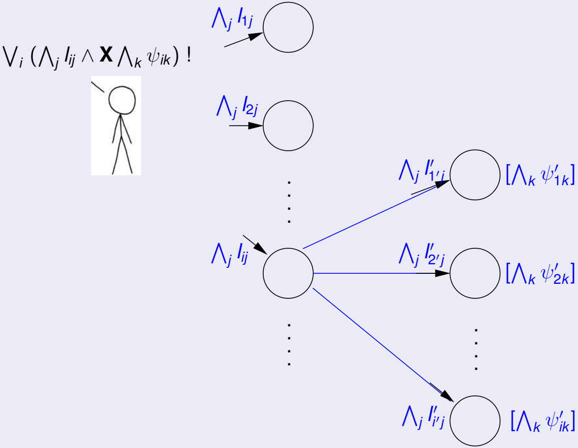
$\bigvee_{i'} (\bigwedge_j l'_{i'j} \wedge \mathbf{X} \bigwedge_k \psi'_{i'k})$

$\bigwedge_j l'_{1'j}$ $[\bigwedge_k \psi'_{1k}]$

$\bigwedge_j l'_{2'j}$ $[\bigwedge_k \psi'_{2k}]$

$\bigwedge_j l'_{i'j}$ $[\bigwedge_k \psi'_{ik}]$

On-the-fly Construction of A_φ (Intuition) [cont.]



On-the-fly Construction of A_φ (Intuition) [cont.]

When the recursive applications of steps 1-3 has terminated and the automata graph has been built, then apply the following:

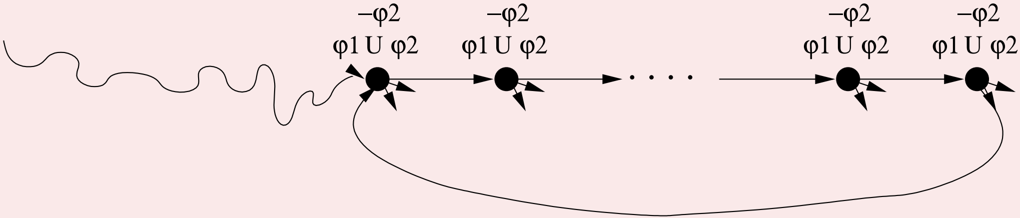
Step 4: For every $\psi_i \mathbf{U} \varphi_i$, for every state q_j, mark q_j with F_i iff $(\psi_i \mathbf{U} \varphi_i) \notin q_j$ *or* $\varphi_i \in q_j$
(If there is no **U**-subformulas, then mark all states with F_1 —i.e., $FT \overset{\text{def}}{=} \{Q\}$).

Remark

The fact that we initially converted the formula into NNF guarantees that only original positive **U**/**F**-subformulas and negative **R**-/**G**-subformulas are considered in step 4

Dealing with **U**-subformulas: Intuition

- Tableaux rules: $\varphi_1 \mathbf{U} \varphi_2 \iff (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X}\varphi_1 \mathbf{U} \varphi_2))$
  are a property, not a definition of **U**:
  $\Longrightarrow$ they implicitly admit a “weaker” semantics of $\varphi_1 \mathbf{U} \varphi_2$, in which $\varphi_1 \mathbf{U} \varphi_2$ always holds and φ_2 never holds
- It cannot happen that we get into a state s' from which we can enter a path π' in which $\varphi_1 \mathbf{U} \varphi_2$ holds forever and φ_2 never holds.



$\Longrightarrow$ every legal path must touch infinitely often a state where $\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ holds

- In LTL: $\neg \mathbf{FG}((\varphi_1 \mathbf{U} \varphi_2) \wedge \neg\varphi_2)$, i.e., $\mathbf{GF}(\neg(\varphi_1 \mathbf{U} \varphi_2) \vee \varphi_2)$ (“avoid bad loops”)

On-the-fly Construction of A_φ - State

- Henceforth, a state is represented by a tuple $s := \langle \lambda, \chi, \sigma \rangle$ where:
  - λ is the set of labels
  - χ is the next part, i.e. the set of X-formulas satisfied by s
  - σ is the set of the subformulas of φ satisfied by s (necessary for the fairness definition)
- Given a set of LTL formulas $\Psi \overset{\text{def}}{=} \{\psi_1, ..., \psi_k\}$, we define $Cover(\Psi) \overset{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$ to be the set of initial states of the Buchi automaton representing $\bigwedge_j \psi_j$.
  - $Expand(\Psi, s)$ takes as input:
    - a set of LTL formulas $\Psi \overset{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ to be expanded
    - a state $s \overset{\text{def}}{=} \langle \lambda, \chi, \sigma \rangle$ under construction

    and returns a set of states $\{\langle \lambda_i, \chi_i, \sigma_i \rangle\}_i$ representing te expansion of Ψ
  - Combines steps 1. and 2. of previous slides

On-the-fly Construction of A_φ - Expand

Given $\Psi \overset{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \overset{\text{def}}{=} \langle\lambda, \chi, \sigma\rangle$, we define *Expand*$(\Psi, s)$ recursively as follows:

- if $\Psi = \emptyset$, *Expand*$(\Psi, s) = \{s\}$
- if $\bot \in \Psi$, *Expand*$(\Psi, s) = \emptyset$
- if $\top \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  Expand$(\Psi, s) =$ *Expand*$(\Psi\backslash\{\top\}, \langle\lambda, \chi, \sigma \cup \{\top\}\rangle)$
- if $l \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$, l propositional literal
  Expand$(\Psi, s) =$ *Expand*$(\Psi\backslash\{l\}, \langle\lambda \cup \{l\}, \chi, \sigma \cup \{l\}\rangle)$
  (add l to the labels of s and to set of satisfied formulas)
- if $\mathbf{X}\psi \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  Expand$(\Psi, s) =$ *Expand*$(\Psi\backslash\{X\psi\}, \langle\lambda, \chi \cup \{\psi\}, \sigma \cup \{\mathbf{X}\psi\}\rangle)$
  (add ψ to the next part of s and $\mathbf{X}\psi$ to set of satisfied formulas)
- if $\psi_1 \wedge \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  Expand$(\Psi, s) =$ *Expand*$(\Psi \cup \{\psi_1, \psi_2\}\backslash\{\psi_1 \wedge \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \wedge \psi_2\}\rangle)$
  (process both ψ_1 and ψ_2 and add $\psi_1 \wedge \psi_2$ to σ)
- ...

On-the-fly Construction of A_φ - Expand

Given $\Psi \overset{\text{def}}{=} \{\psi_1, ..., \psi_k\}$ and $s \overset{\text{def}}{=} \langle\lambda, \chi, \sigma\rangle$, we define *Expand*$(\Psi, s)$ recursively as follows:

- ...
- if $\psi_1 \vee \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\begin{aligned} Expand(\Psi, s) = {} & Expand(\Psi \cup \{\psi_1\} \backslash \{\psi_1 \vee \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\}\rangle) \\ & \cup\ Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \vee \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \vee \psi_2\}\rangle) \end{aligned}$$
  (split *s* into two copies, process ψ_2 on the first, ψ_1 on the second, add $\psi_1 \vee \psi_2$ to σ)
- if $\psi_1 \mathbf{U} \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\begin{aligned} Expand(\Psi, s) = {} & Expand(\Psi \cup \{\psi_1\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle\lambda, \chi \cup \{\psi_1 \mathbf{U} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\}\rangle) \\ & \cup\ Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{U} \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \mathbf{U} \psi_2\}\rangle) \end{aligned}$$
  (split *s* into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{U} \psi_2$ to σ)
- if $\psi_1 \mathbf{R} \psi_2 \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\begin{aligned} Expand(\Psi, s) = {} & Expand(\Psi \cup \{\psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle\lambda, \chi \cup \{\psi_1 \mathbf{R} \psi_2\}, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\}\rangle) \\ & \cup\ Expand(\Psi \cup \{\psi_1, \psi_2\} \backslash \{\psi_1 \mathbf{R} \psi_2\}, \langle\lambda, \chi, \sigma \cup \{\psi_1 \mathbf{R} \psi_2\}\rangle) \end{aligned}$$
  (split *s* into two copies and process ψ_1 on the first, ψ_2 on the second, add $\psi_1 \mathbf{R} \psi_2$ to σ)

On-the-fly Construction of A_φ - Expand

Two relevant subcases: $\mathbf{F}\psi \overset{\text{def}}{=} \top\mathbf{U}\psi$ and $\mathbf{G}\psi \overset{\text{def}}{=} \bot\mathbf{R}\psi$

- if $\mathbf{F}\psi \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\begin{aligned} \mathit{Expand}(\Psi, s) = {} & \mathit{Expand}(\Psi \backslash \{\mathbf{F}\psi\}, \langle\lambda, \chi \cup \{\mathbf{F}\psi\}, \sigma \cup \{\mathbf{F}\psi\}\rangle) \\ & \cup\ \mathit{Expand}(\Psi \cup \{\psi\} \backslash \{\mathbf{F}\psi\}, \langle\lambda, \chi, \sigma \cup \{\mathbf{F}\psi\}\rangle) \end{aligned}$$
- if $\mathbf{G}\psi \in \Psi$ and $s = \langle\lambda, \chi, \sigma\rangle$,
  $$\mathit{Expand}(\Psi, s) = \mathit{Expand}(\Psi \cup \{\psi\} \backslash \{\mathbf{G}\psi\}, \langle\lambda, \chi \cup \{\mathbf{G}\psi\}, \sigma \cup \{\mathbf{G}\psi\}\rangle)$$
  (Note: $\mathit{Expand}(\Psi \cup \{\bot, \psi\} \backslash \{\mathbf{G}\psi\}, \ldots) = \emptyset$.)

Definition of A_φ

Given a set of LTL formulas Ψ, we define $Cover(\Psi) \overset{\text{def}}{=} Expand(\Psi, \langle \emptyset, \emptyset, \emptyset \rangle)$.
For an LTL formula φ, we construct a Generalized NBA $A_\varphi = (Q, \Sigma, \delta, I, FT)$ as follows:

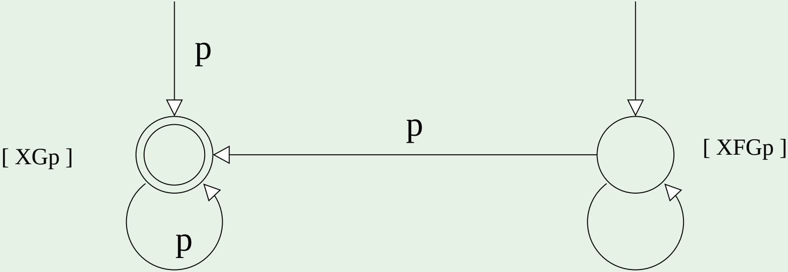
- $\Sigma \ = \ 3^{vars(\varphi)}$ ($v \in \{\top, \bot, *\}$, "$*$" is "don't care")
- Q is the smallest set such that
  - $Cover(\{\varphi\}) \subseteq Q$
  - if $\langle \lambda, \chi, \sigma \rangle \in Q$, then $Cover(\chi) \in Q$
- $Q_0 = Cover(\{\varphi\})$.
- $s \xrightarrow{\lambda'} s' \in \delta$ iff, $s = \langle \lambda, \chi, \sigma \rangle$, $s' = \langle \lambda', \chi', \sigma' \rangle$ and $s' \in Cover(\chi)$
- $FT = \langle F_1, F_2, ..., F_k \rangle$ where, for all $(\psi_i \mathbf{U} \varphi_i)$ occurring positively in φ,
  $F_i \ = \ \{\langle \lambda, \chi, \sigma \rangle \in Q \ | \ (\psi_i \mathbf{U} \varphi_i) \notin \sigma \ \ or \ \ \varphi_i \in \sigma\}$.
  (If there is no **U**-subformulas, then $FT \overset{\text{def}}{=} \{Q\}$).

Example: $\varphi = \mathbf{FG}p$

- $$\begin{aligned}&\mathit{Cover}(\{\mathbf{FG}p\})\\ &= \mathit{Expand}(\{\mathbf{FG}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\emptyset,\{\mathbf{FG}p\},\{\mathbf{FG}p\}\rangle) \cup \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\{\mathbf{FG}p\}\rangle)\\ &= \{\langle\emptyset,\{\mathbf{FG}p\},\{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\{p\}, \langle\emptyset,\{\mathbf{G}p\},\{\mathbf{FG}p,\mathbf{G}p\}\rangle)\\ &= \{\langle\emptyset,\{\mathbf{FG}p\},\{\mathbf{FG}p\}\rangle\} \cup \mathit{Expand}(\emptyset, \langle\{p\},\{\mathbf{G}p\},\{\mathbf{FG}p,\mathbf{G}p,p\}\rangle)\\ &= \{\langle\emptyset,\{\mathbf{FG}p\},\{\mathbf{FG}p\}\rangle, \langle\{p\},\{\mathbf{G}p\},\{\mathbf{FG}p,\mathbf{G}p,p\}\rangle\}\end{aligned}$$
- $$\begin{aligned}\mathit{Cover}(\{\mathbf{G}p\}) &= \mathit{Expand}(\{\mathbf{G}p\}, \langle\emptyset,\emptyset,\emptyset\rangle)\\ &= \mathit{Expand}(\{p\}, \langle\emptyset,\{\mathbf{G}p\},\{\mathbf{G}p\}\rangle)\\ &= \mathit{Expand}(\emptyset, \langle\{p\},\{\mathbf{G}p\},\{\mathbf{G}p,p\}\rangle)\\ &= \{\langle\{p\},\{\mathbf{G}p\},\{\mathbf{G}p,p\}\rangle\}\end{aligned}$$
- Optimization:
  merge $\langle\{p\},\{\mathbf{G}p\},\{\mathbf{FG}p,\mathbf{G}p,p\}\rangle$ and $\langle\{p\},\{\mathbf{G}p\},\{\mathbf{G}p,p\}\rangle$

Example: $\varphi = \mathbf{FG}p$

- Call $s_1 = \langle \emptyset, \{\mathbf{FG}p\}, \{\mathbf{FG}p\} \rangle$, $s_2 = \langle \{p\}, \{\mathbf{G}p\}, \{\mathbf{FG}p, \mathbf{G}p, p\} \rangle$
- $Q = \{s_1, s_2\}$
- $Q_0 = \{s_1, s_2\}$.
- $T: \begin{array}{l} s_1 \to \{s_1, s_2\}, \\ s_2 \to \{s_2\} \end{array}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2\}$.

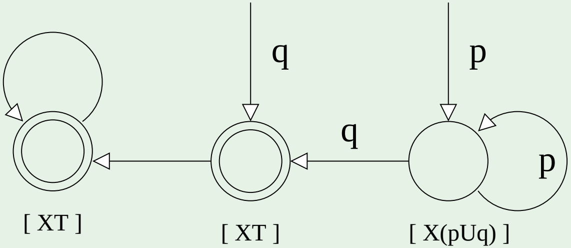


Example: $\varphi = p\mathbf{U}q$

- $$\begin{aligned} & \mathit{Cover}(\{p\mathbf{U}q\}) \\ &= \mathit{Expand}(\{p\mathbf{U}q\}, \langle\emptyset,\emptyset,\emptyset\rangle) \\ &= \mathit{Expand}(\{p\}, \langle\emptyset, \{p\mathbf{U}q\}, \{p\mathbf{U}q\}\rangle) \cup \mathit{Expand}(\{q\}, \langle\emptyset, \emptyset, \{p\mathbf{U}q\}\rangle) \\ &= \mathit{Expand}(\emptyset, \langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle) \cup \mathit{Expand}(\emptyset, \langle\{q\}, \emptyset, \{p\mathbf{U}q, q\}\rangle) \\ &= \{\langle\{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\}\rangle\} \cup \{\langle\{q\}, \{\top\}, \{p\mathbf{U}q, q\}\rangle\} \end{aligned}$$
- $\mathit{Cover}(\{\top\}) = \quad \{\langle\emptyset, \{\top\}, \{\top\}\rangle\}$

Example: $\varphi = p\mathbf{U}q$

- Let $s_1 =_{def} \langle \{p\}, \{p\mathbf{U}q\}, \{p\mathbf{U}q, p\} \rangle$, $s_2 =_{def} \langle \{q\}, \{\top\}, \{p\mathbf{U}q, q\} \rangle$, $s_3 =_{def} \langle \emptyset, \{\top\}, \{\top\} \rangle$.
- $Q = \{s_1, s_2, s_3\}$,
- $Q_0 = \{s_1, s_2\}$,
- $$T: \begin{array}{l} s_1 \to \{s_1, s_2\}, \\ s_2 \to \{s_3\} \\ s_3 \to \{s_3\} \end{array}$$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_2, s_3\}$.



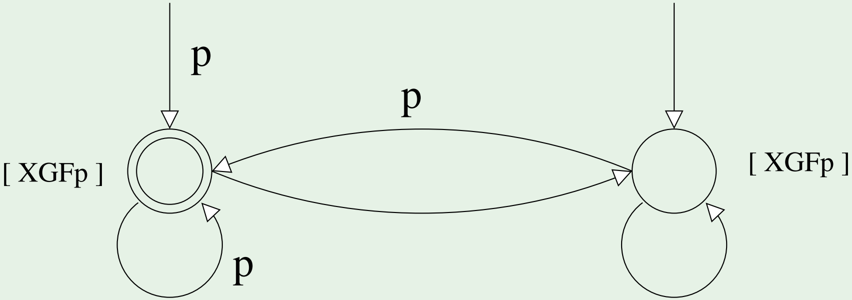
Example: $\varphi = \mathbf{GF}p$

$$
\begin{aligned}
&\mathit{Cover}(\{\mathbf{GF}p\})\\
&\quad = \mathit{Expand}(\{\mathbf{GF}p\}, \langle\emptyset, \emptyset, \emptyset\rangle)\\
&\quad = \mathit{Expand}(\{\mathbf{F}p\}, \langle\emptyset, \{\mathbf{GF}p\}, \{\mathbf{GF}p\}\rangle)\\
&\quad = \mathit{Expand}(\{\}, \langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle) \cup \mathit{Expand}(\{p\}, \langle\{\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle)\\
&\quad = \mathit{Expand}(\{\}, \langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle) \cup \mathit{Expand}(\{\}, \langle\{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\}\rangle)\\
&\quad = \{\langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle\} \cup \{\langle\{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\}\rangle\}
\end{aligned}
$$

Note: $\mathbf{GF}p \wedge \mathbf{F}p \Longleftrightarrow \mathbf{GF}p$, s.t. $\mathit{Cover}(\mathbf{GF}p \wedge \mathbf{F}p) = \mathit{Cover}(\mathbf{GF}p)$

Example: **GF**p

- Let $s_1 =_{def} \langle\{p\}, \{\mathbf{GF}p\}, \{\mathbf{GF}p, \mathbf{F}p, p\}\rangle$, $s_2 =_{def} \langle\emptyset, \{\mathbf{GF}p, \mathbf{F}p\}, \{\mathbf{GF}p, \mathbf{F}p\}\rangle$,
- $Q = \{s_1, s_2\}$,
- $Q_0 = \{s_1, s_2\}$,
- $T: \begin{array}{l} s_1 \to \{s_1, s_2\}, \\ s_2 \to \{s_1, s_2\} \end{array}$
- $FT = \langle F_1 \rangle$ where $F_1 = \{s_1\}$.



NBAs of disjunctions of formulas

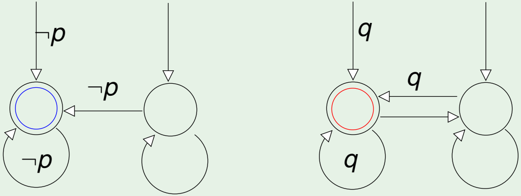
Remark

If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \vee \varphi_2)$ and $A_{\varphi_1}, A_{\varphi_2}$ are NBAs encoding φ_1 and φ_2 resp., then $\mathcal{L}(\varphi) = \mathcal{L}(\varphi_1) \cup \mathcal{L}(\varphi_2)$, so that $A_\varphi \stackrel{\text{def}}{=} A_{\varphi_1} \cup A_{\varphi_2}$ is an NBA encoding φ

- A_φ non necessarily the smallest/best NBA encoding φ

Example

Let $\varphi \stackrel{\text{def}}{=} (\mathbf{GF}p \rightarrow \mathbf{GF}q)$, i.e., $\varphi \equiv (\mathbf{FG}\neg p \vee \mathbf{GF}q)$.
Then $A_{\mathbf{FG}\neg p} \cup A_{\mathbf{GF}q}$ encodes φ:



Suggested Exercises:

- Find an NBA encoding:
  - p
  - $(p \wedge q) \vee (\neg p \wedge \neg q)$
  - $\mathbf{F}p$
  - $\mathbf{G}p$
  - $p\mathbf{R}q$
  - $(\mathbf{GF}p \wedge \mathbf{GF}q) \rightarrow \mathbf{G}r$

Outline

1. Büchi Automata

2. The Automata-Theoretic Approach to LTL Reasoning
   - General Ideas
   - Language-Emptiness Checking of Büchi Automata
   - From Kripke Models to Büchi Automata
   - From LTL Formulas to Büchi Automata
   - Complexity

3. Exercises

Automata-Theoretic LTL Model Checking: Complexity

Four steps:

(i) Compute A_M:
$|A_M| = O(|M|)$

(ii) Compute A_φ:
$|A_\varphi| = O(2^{|\varphi|})$

(iii) Compute the product $A_M \times A_\varphi$:
$|A_M \times A_\varphi| = |A_M| \cdot |A_\varphi| = O(|M| \cdot 2^{|\varphi|})$

(iv) Check the emptiness of $\mathcal{L}(A_M \times A_\varphi)$:
$O(|A_M \times A_\varphi|) = O(|M| \cdot 2^{|\varphi|})$

$\Longrightarrow$ The complexity of LTL M.C. grows linearly wrt. the size of the model M and exponentially wrt. the size of the property φ

Final Remarks

- Büchi automata are in general more expressive than LTL!

$\Longrightarrow$ some tools (e.g., Spin) allow specifications to be expressed directly as NBAs

$\Longrightarrow$ complementation of NBA relevanant in general

- For every LTL formula, there are many possible equivalent NBAs

$\Longrightarrow$ lots of research for finding “the best” conversion algorithm

- Performing the product and checking emptiness very relevant

$\Longrightarrow$ lots of techniques developed (e.g., partial order reduction)

$\Longrightarrow$ lots on ongoing research

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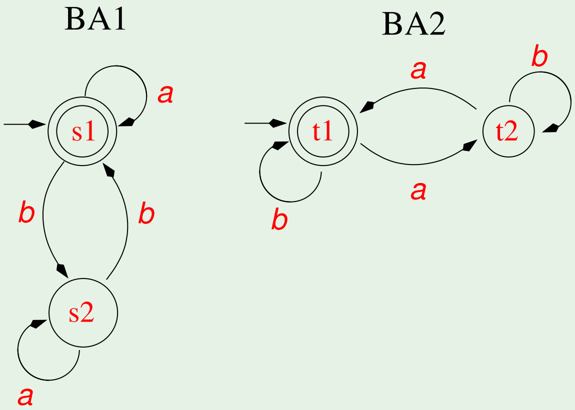
3. Exercises

Ex: Product of Büchi automata

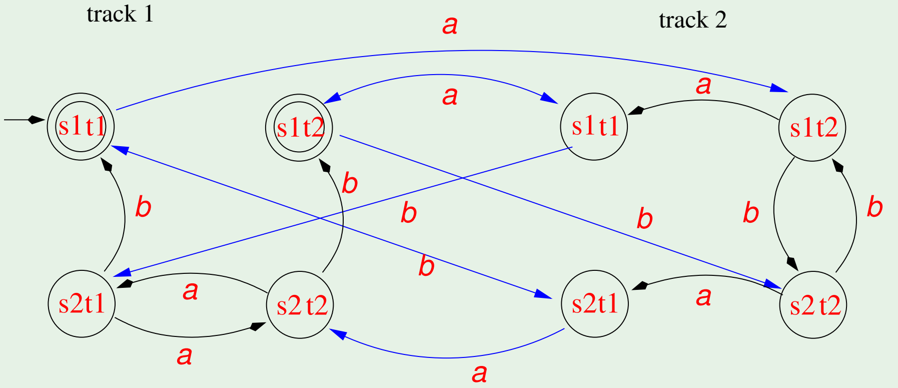
Given the following two Büchi automata (doubly-circled states represent accepting states, *a*, *b* are labels):



Write the product Büchi automaton *BA*1 × *BA*2.

Ex: Product of Büchi automata

[Solution: The product is:



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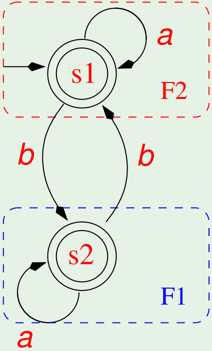
Ex: De-generalization of Büchi Automata

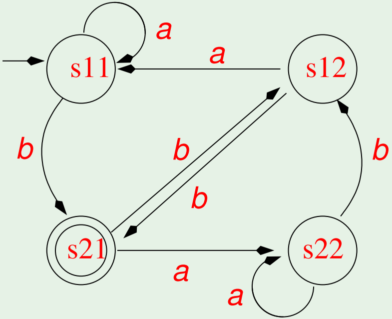
Given the following generalized Büchi automaton $A \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, FT\rangle$, with two sets of accepting states $FT \stackrel{\text{def}}{=} \{F1, F2\}$ s.t. $F1 \stackrel{\text{def}}{=} \{s2\}, F2 \stackrel{\text{def}}{=} \{s1\}$:



convert it into an equivalent plain Büchi automaton.

Ex: De-generalization of Büchi Automata

[Solution: The result is:



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Ex: Construction of Büchi Automata

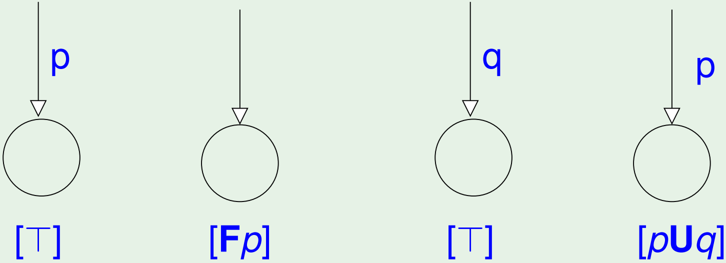
Consider the LTL formula $\varphi \overset{\text{def}}{=} (\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q)$.

(a) rewrite φ into Negative Normal Form

[Solution: $(\mathbf{G}\neg p) \rightarrow (p\mathbf{U}q) \Longrightarrow (\neg\mathbf{G}\neg p) \vee (p\mathbf{U}q) \Longrightarrow (\mathbf{F}p) \vee (p\mathbf{U}q)$]

(b) find the initial states of a corresponding Buchi automaton (for each state, define the labels of the incoming arcs and the “next” section.)

[Solution: Applying tableaux rules we obtain: $p \vee \mathbf{XF}p \vee q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$, which is already in disjunctive normal form. This correspond to the following four initial states:



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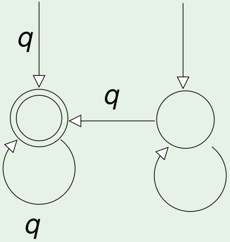
Ex: Büchi automaton

Given the following Büchi automaton BA (doubly-circled states represent accepting states):



Say which of the following sentences are true and which are false.

(*a*) BA accepts all and only the paths verifying **GF**q. [Solution: false]

(*b*) BA accepts all and only the paths verifying **FG**q. [Solution: true]

(*c*) BA accepts only paths verifying **F**q, but not all of them. [Solution: true]

(*d*) BA accepts all the paths verifying **F**q, but not only them. [Solution: false]