Automated Reasoning and Formal Verification
Module I: Automated Reasoning
Ch. 01: **Propositional Satisfiability (SAT)**

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Outline

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Propositional Logic (aka Boolean Logic)

Basic Definitions

- Propositional formula (aka Boolean formula)
    - $\top, \bot$ are formulas
    - a propositional atom $A_1, A_2, A_3, ...$ is a formula;
    - if φ_1 and φ_2 are formulas, then
      $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$
      are formulas.
- Ex: $\varphi \stackrel{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4))))$
- $Atoms(\varphi)$: the set $\{A_1, ..., A_N\}$ of atoms occurring in φ.
    - Ex: $Atoms(\varphi) = \{A_1, A_2, A_3, A_4\}$
- Literal: a propositional atom A_i (positive literal) or its negation $\neg A_i$ (negative literal)
    - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- Clause: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee ...)$)
- Cube: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge ...)$)

Semantics of Boolean operators

Truth Table

α	β	$\neg\alpha$	$\alpha\wedge\beta$	$\alpha\vee\beta$	$\alpha\rightarrow\beta$	$\alpha\leftarrow\beta$	$\alpha\leftrightarrow\beta$	$\alpha\oplus\beta$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

English meaning of connectives

English	Logic
A and B	$A \wedge B$
A if B | A when B | A whenever B	$A \leftarrow B$
if A, then B | A implies B | A forces B | A requires B	$A \rightarrow B$
A precisely when B | A if and only if B	$A \leftrightarrow B$
A or B (or both) | A unless B	$A \vee B$ (logical or)
either A or B (but not both)	$A \oplus B$ (exclusive or)

Semantics of Boolean operators (cont.)

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Note

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are commutative:
  $$(\alpha \wedge \beta) \iff (\beta \wedge \alpha)$$
  $$(\alpha \vee \beta) \iff (\beta \vee \alpha)$$
  $$(\alpha \leftrightarrow \beta) \iff (\beta \leftrightarrow \alpha)$$
  $$(\alpha \oplus \beta) \iff (\beta \oplus \alpha)$$

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are associative:
  $$((\alpha \wedge \beta) \wedge \gamma) \iff (\alpha \wedge (\beta \wedge \gamma)) \iff (\alpha \wedge \beta \wedge \gamma)$$
  $$((\alpha \vee \beta) \vee \gamma) \iff (\alpha \vee (\beta \vee \gamma)) \iff (\alpha \vee \beta \vee \gamma)$$
  $$((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) \iff (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) \iff (\alpha \leftrightarrow \beta \leftrightarrow \gamma)$$
  $$((\alpha \oplus \beta) \oplus \gamma) \iff (\alpha \oplus (\beta \oplus \gamma)) \iff (\alpha \oplus \beta \oplus \gamma)$$

- $\rightarrow$, $\leftarrow$ are neither commutative nor associative:
  $$(\alpha \rightarrow \beta) \not\iff (\beta \rightarrow \alpha)$$
  $$((\alpha \rightarrow \beta) \rightarrow \gamma) \not\iff (\alpha \rightarrow (\beta \rightarrow \gamma))$$

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Equivalences with Boolean Operators

$$
\begin{array}{rcl}
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\neg(\alpha \leftrightarrow \beta) & \Longleftrightarrow & (\neg\alpha \leftrightarrow \beta) \\
& \Longleftrightarrow & (\alpha \leftrightarrow \neg\beta) \\
& \Longleftrightarrow & ((\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta)) \\
(\alpha \oplus \beta) & \Longleftrightarrow & \neg(\alpha \leftrightarrow \beta)
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Tree & DAG Representations of Formulas

- Formulas can be represented either as trees or as DAGS (Directed Acyclic Graphs)
- DAG representation can be up to exponentially smaller
  - in particular, when $\leftrightarrow$'s are involved

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

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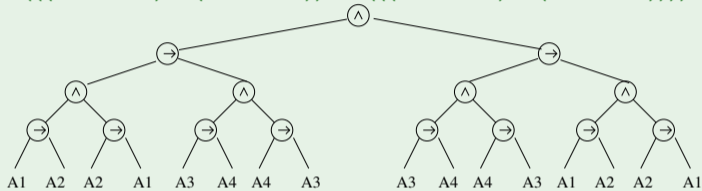$$((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2)))$$

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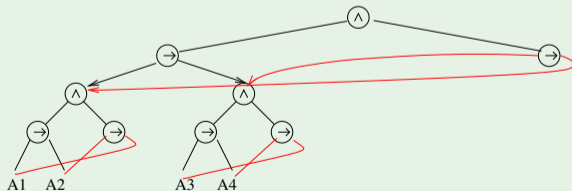 - in particular, when $\leftrightarrow$'s are involved

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Tree Representation

DAG Representation

Semantics: Basic Definitions

- Total truth assignment μ for φ:
  $\mu : \mathit{Atoms}(\varphi) \longmapsto \{\top, \bot\}$.
    - represents a possible world or a possible state of the world
- Partial Truth assignment μ for φ:
  $\mu : \mathcal{A} \longmapsto \{\top, \bot\}$, $\mathcal{A} \subset \mathit{Atoms}(\varphi)$.
    - represents 2^k total assignments, k is # unassigned variables
- Notation: set and formula representations of an assignment
    - μ can be represented as a set of literals:
      EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies \{A_1, \neg A_2\}$
    - μ can be represented as a formula (cube):
      EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies (A_1 \wedge \neg A_2)$

Semantics: Basic Definitions [cont.]

- A **total** truth assignment μ satisfies φ (μ is a model of φ, $\mu \models \varphi$):

  $\mu \models A_i \Longleftrightarrow \mu(A_i) = \top$

  $\mu \models \neg\varphi \Longleftrightarrow \text{not } \mu \models \varphi$

  $\mu \models \alpha \wedge \beta \Longleftrightarrow \mu \models \alpha \text{ and } \mu \models \beta$

  $\mu \models \alpha \vee \beta \Longleftrightarrow \mu \models \alpha \text{ or } \mu \models \beta$

  $\mu \models \alpha \rightarrow \beta \Longleftrightarrow \text{ if } \mu \models \alpha, \text{ then } \mu \models \beta$

  $\mu \models \alpha \leftrightarrow \beta \Longleftrightarrow \mu \models \alpha \text{ iff } \mu \models \beta$

  $\mu \models \alpha \oplus \beta \Longleftrightarrow \mu \models \alpha \text{ iff not } \mu \models \beta$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$ (the set of models of φ)

- A **partial** truth assignment μ satisfies φ iff all total assignments extending μ satisfy φ
  - Ex: $\{A_1\} \models (A_1 \vee A_2)$) because both $\{A_1, A_2\} \models (A_1 \vee A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$

- φ is satisfiable iff $\mu \models \varphi$ for some μ (i.e. $M(\varphi) \neq \emptyset$)

- α entails β ($\alpha \models \beta$): $\alpha \models \beta$ iff $\mu \models \alpha \Longrightarrow \mu \models \beta$ for all μs
  (i.e., $M(\alpha) \subseteq M(\beta)$)

- φ is valid ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for all μs
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Properties & Results

Property

φ is valid iff $\neg\varphi$ is not satisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

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Equivalence and Equi-Satisfiability

- α and β are equivalent iff, for every μ, $\mu \models \alpha$ iff $\mu \models \beta$
  (i.e., if $M(\alpha) = M(\beta)$)
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- α, β equivalent
  $$\Downarrow \quad \not\Uparrow$$
  α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
  $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α: $\beta \stackrel{\text{def}}{=} T(\alpha)$:
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Boolean Quantification

Shannon's expansion:

- If *v* is a Boolean variable and f is a Boolean formula, then
  $$\exists v.\varphi \quad := \quad \varphi|_{v=\bot} \vee \varphi|_{v=\top}$$
  $$\forall v.\varphi \quad := \quad \varphi|_{v=\bot} \wedge \varphi|_{v=\top}$$
- *v* does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
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- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

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  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \perp\}$ s.t. $\mu \cup \{v := truthvalue\} \models \varphi$
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Complexity

NP-Completeness of SAT

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is NP-complete
- $\implies$ The most important logical problems (validity, inference, entailment, equivalence, ...) can be straightforwardly reduced to (un)satisfiability, and are thus (co)NP-complete.

$$\Downarrow$$

No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
    - φ occurs positively in φ;
    - if $\neg\varphi_1$ occurs positively [negatively] in φ,
      then φ_1 occurs negatively [positively] in φ
    - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ,
      then φ_1 and φ_2 occur positively [negatively] in φ;
    - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ,
      then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
    - if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ,
      then φ_1 and φ_2 occur positively and negatively in φ;

Negative Normal Form (NNF)

- φ is in Negative normal form iff it is given only by the recursive applications of $\wedge, \vee$ to literals.
- every φ can be reduced into NNF:
  (i) substituting all $\rightarrow$'s and $\leftrightarrow$'s:

$$\begin{aligned} \alpha \rightarrow \beta &\implies \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta &\implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta) \end{aligned}$$

  (ii) pushing down negations recursively:

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- Every non-atomic subformula in $NNF(\varphi)$ occurs with positive polarity only
  $\implies$ a subformula ψ occurring with both polarities in φ is encoded as both $NNF(\psi)$ and $NNF(\neg\psi)$
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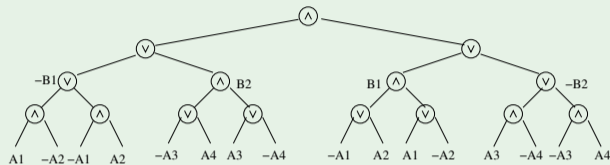
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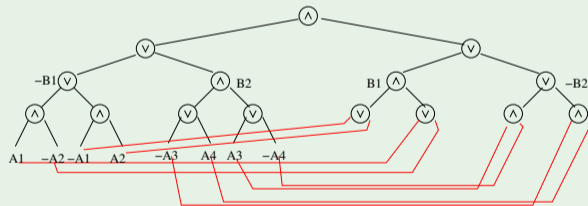
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NNF: Example [cont.]

Note

Tree Representation

DAG Representation

For each non-literal subformula φ, φ and $\neg\varphi$ have different representations $\implies$ they are not shared.

Optimized polynomial representations

- Maximize the sharing in DAG representations:
  $\{\land, \leftrightarrow, \neg\}$-only, negations on arcs, sorting of subformulae, lifting of $\neg$'s over $\leftrightarrow$'s,...
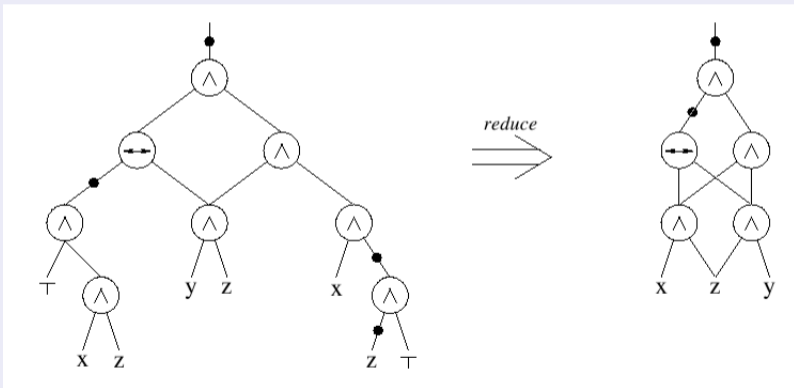
Conjunctive Normal Form (CNF)

- φ is in Conjunctive normal form iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
- Easier to handle: list of lists of literals.
  $\implies$ no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,
  - (i) expanding implications and equivalences:
    - $\alpha \to \beta \implies \neg\alpha \vee \beta$
    - $\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$
  - (ii) pushing down negations recursively:
    - $\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$
    - $\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$
    - $\neg\neg\alpha \implies \alpha$
    - $\implies$ Negation Normal Form, NNF (see previous slides)
  - (iii) applying recursively the DeMorgan's Rule: $(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
- Resulting formula worst-case exponential:
  - ex: $||CNF(\bigvee_{i=1}^{N}(l_{i1} \wedge l_{i2})|| = ||(l_{11} \vee l_{21} \vee ... \vee l_{N1}) \wedge (l_{12} \vee l_{21} \vee ... \vee l_{N1}) \wedge ... \wedge (l_{12} \vee l_{22} \vee ... \vee l_{N2})|| = 2^N$
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Labeling CNF conversion $CNF_{label}(\varphi)$

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- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:
  $$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$
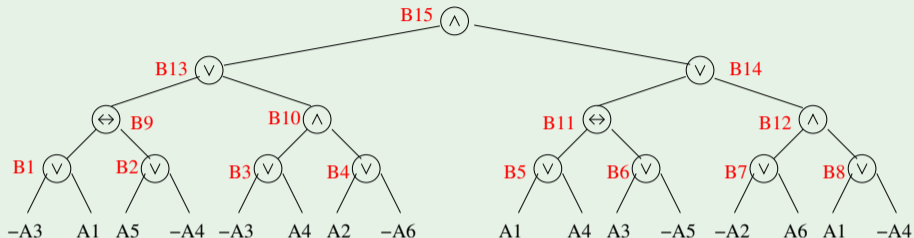 $$\varphi \implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j))$$
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  l_i, l_j being literals and B being a "new" variable.
- Worst-case linear!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$ is equi-satisfiable (but not equivalent) to φ.
  - moreover: $\exists B_1, ..., B_k . CNF_{label}(\varphi)$ equivalent to φ, s.t. $B_1, ..., B_k$ all fresh variables introduced
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$CNF(B \leftrightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j) \wedge$
		$(B \vee \neg l_i) \wedge$
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$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$
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		$(B \vee l_i \vee l_j) \wedge$
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$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \wedge$$
$$... \wedge$$
$$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \wedge$$
$$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \wedge$$

$$... \wedge$$
$$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \wedge$$
$$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \wedge$$
$$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \wedge$$
$$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \wedge$$
$$B_{15}$$

$$=$$

$$(\neg B_1 \vee \neg A_3 \vee A_1) \wedge (B_1 \vee A_3) \wedge (B_1 \vee \neg A_1) \wedge$$
$$... \wedge$$
$$(\neg B_8 \vee A_1 \vee \neg A_4) \wedge (B_8 \vee \neg A_1) \wedge (B_8 \vee A_4) \wedge$$
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$$... \wedge$$
$$(B_{12} \vee \neg B_7 \vee \neg B_8) \wedge (\neg B_{12} \vee B_7) \wedge (\neg B_{12} \vee B_8) \wedge$$
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$$B_{15}$$

Improved Labeling CNF conversion CNF_{label}

Polarity-based Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Plaisted&Greenbaum CNF-ization)

- As in the previous case, applying instead the rules:

$$\begin{array}{rcll}
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \vee l_j)) & \textit{if } (l_i \vee l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF((l_i \vee l_j) \rightarrow B) & \textit{if } (l_i \vee l_j) \textit{ neg.} \\
\varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \wedge l_j)) & \textit{if } (l_i \wedge l_j) \textit{ pos.} \\
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\end{array}$$

- Smaller in size:

$$\begin{array}{rll}
CNF(B \rightarrow (l_i \vee l_j)) & = & (\neg B \vee l_i \vee l_j) \\
CNF(((l_i \vee l_j) \rightarrow B)) & = & (\neg l_i \vee B) \wedge (\neg l_j \vee B)
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- As in the previous case, applying instead the rules:

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\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF(B \to (l_i \vee l_j)) & \textit{if } (l_i \vee l_j)\ \textit{pos.} \\
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Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i) \wedge$
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$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(B \vee l_i \vee l_j) \wedge$
		$(B \vee \neg l_i \vee \neg l_j)$

Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))$ $\wedge$
... $\wedge$
$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4))$ $\wedge$
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Improved

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Labeling CNF conversion CNF_{label} – further improvements

- Do not apply CNF_{label} when not necessary:
  (e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \Longrightarrow CNF_{label}(\varphi_1) \wedge \varphi_2$, if φ_2 already in CNF)
- Apply DeMorgan's rules where it is more effective:
  (e.g., $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \Longrightarrow CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$
- Exploit the associativity of $\wedge$'s and $\vee$'s:
  $... \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} ... \Longrightarrow ... CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3)) ...$
- Before applying CNF_{label}, rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Exercises

1. Consider the following Boolean formula φ:
   $\neg(((\neg A_1 \to A_2) \land (\neg A_3 \to A_4)) \lor ((A_5 \to A_6) \land (A_7 \to \neg A_8)))$
   Compute the Negative Normal Form of φ

2. Consider the following Boolean formula φ:
   $((\neg A_1 \land A_2) \lor (A_7 \land A_4) \lor (\neg A_3 \land \neg A_2) \lor (A_5 \land \neg A_4))$
   1. Produce the CNF formula $CNF(\varphi)$.
   2. Produce the CNF formula $CNF_{label}(\varphi)$.
   3. Produce the CNF formula $CNF_{label}(\varphi)$ (improved version)

Outline

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Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
  - AI, formal verification, circuit synthesis, operational research,....
- Important in AI: $KB \models \alpha$: entail fact α from knowledge base KB (aka Model Checking: $M(KB) \subseteq M(\alpha)$)
  - typically $|KB| \gg |\alpha|$
- All propositional reasoning tasks reduced to satisfiability (SAT)
  - $KB \models \alpha \implies$ SAT($KB \wedge \neg \alpha$)=false
  - input formula CNF-ized and fed to a SAT solver
- Current SAT solvers dramatically efficient:
  - handle industrial problems with $10^6 - 10^7$ variables & clauses!
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Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|Atoms(\varphi)|}$ lines.
- Never used in practice.

Outline

The Resolution Rule

- Resolution: deduction of a new clause from a pair of clauses with exactly one incompatible variable (resolvent):

$$\frac{(\ \overbrace{l_1 \vee ... \vee l_k}^{common} \vee\ \overbrace{l}^{resolvent} \vee\ \overbrace{l'_{k+1} \vee ... \vee l'_m}^{C'}\ )\qquad (\ \overbrace{l_1 \vee ... \vee l_k}^{common} \vee\ \overbrace{\neg l}^{resolvent} \vee\ \overbrace{l''_{k+1} \vee ... \vee l''_n}^{C''}\ )}{(\ \underbrace{l_1 \vee ... \vee l_k}_{common} \vee\ \underbrace{l'_{k+1} \vee ... \vee l'_m}_{C'} \vee\ \underbrace{l''_{k+1} \vee ... \vee l''_n}_{C''}\ )}$$

- Ex: $\dfrac{(\ A \vee B \vee C \vee D \vee E\ )\qquad (\ A \vee B \vee \neg C \vee F\ )}{(\ A \vee B \vee D \vee E \vee F\ )}$

- Note: many standard inference rules subcases of resolution:
  (recall that $\alpha \rightarrow \beta \iff \neg\alpha \vee \beta$)

  $$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}\ (trans.) \qquad \frac{A \quad A \rightarrow B}{B}\ (m.\ ponens) \qquad \frac{\neg B \quad A \rightarrow B}{\neg A}\ (m.\ tollens)$$

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Improvements: Subsumption & Unit Propagation

Alternative "set" notation (Γ clause set):

$$\frac{\Gamma, \phi_1, ..\phi_n}{\Gamma, \phi'_1, ..\phi'_{n'}} \quad \left(e.g., \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2,} \right)$$

- Removal of valid clauses:

$$\frac{\Gamma \wedge (p \vee \neg p \vee C)}{\Gamma}$$

- Clause Subsumption (C clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

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- Unit Propagation = Unit Resolution + Unit Subsumption

"Deterministic" rule: applied before other "non-deterministic" rules!

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$$\frac{\Gamma \wedge (p \vee \neg p \vee C)}{\Gamma}$$

- Clause Subsumption (C clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

$$\frac{\Gamma \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma \wedge (l) \wedge (\bigvee_i l_i)}$$

- Unit Subsumption:

$$\frac{\Gamma \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma \wedge (l)}$$

- Unit Propagation = Unit Resolution + Unit Subsumption

"Deterministic" rule: applied before other "non-deterministic" rules!

Remark

What happens with more than 1 resolvent?

- Common mistake: the following is <u>not</u> a correct application of the resolution rule:

$$\frac{\Gamma, \ (C_1 \vee l_1 \vee l_2), \ (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, \ (C_1 \vee l_1 \vee l_2), \ (C_2 \vee \neg l_1 \vee \neg l_2), \ (C_1 \vee C_2)}$$

- Rather, a correct application would be:

$$\frac{\Gamma, \ (C_1 \vee l_1 \vee l_2), \ (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, \ (C_1 \vee l_1 \vee l_2), \ (C_2 \vee \neg l_1 \vee \neg l_2), \ (C_1 \vee l_2 \vee C_2 \vee \neg l_2)}$$

... but $(C_1 \vee l_2 \vee C_2 \vee \neg l_2)$ is valid and should be removed

$\Longrightarrow$ no clause is produced

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Basic Propositional Inference: Resolution [33, 10]

- Assume input formula in CNF
  - if not, apply Tseitin CNF-ization first
$\implies$ φ is represented as a set of clauses
- Search for a refutation of φ (is φ unsatisfiable?)
  - recall: $\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ unsatisfiable
- Basic idea: apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either
  - a false clause is generated, or
  - the resolution rule is no more applicable
- Correct: if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- Complete: if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- Time-inefficient
- Very Memory-inefficient (exponential in memory)
- Many different strategies

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Resolution: basic strategy [10]

```
function DP(Γ)
    if ⊥ ∈ Γ                                    /* unsat */
        then return False;
    if (Resolve() is no more applicable to Γ)    /* sat      */
        then return True;
    if {a unit clause (l) occurs in Γ}           /* unit     */
        then Γ := Unit_Propagate(l, Γ));
        return   DP(Γ)
    A := select-variable(Γ);                     /* resolve      */
    Γ = Γ ∪ ⋃_{A∈C', ¬A∈C''}{Resolve(C', C'')} \ ⋃_{A∈C', ¬A∈C''}{C', C''}};
    return   DP(Γ)
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$(A_1 \lor A_2) \; (A_1 \lor \neg A_2) \; (\neg A_1 \lor A_2) \; (\neg A_1 \lor \neg A_2)$$
$$\Downarrow$$
$$(A_2) \; (A_2 \lor \neg A_2) \; (\neg A_2 \lor A_2) \; (\neg A_2)$$
$$\Downarrow$$
$$\bot$$

$$\Longrightarrow \text{UNSAT}$$

$$(A_1 \vee A_2) \ (A_1 \vee \neg A_2) \ (\neg A_1 \vee A_2) \ (\neg A_1 \vee \neg A_2)$$
$$\Downarrow$$
$$(A_2) \ (A_2 \vee \neg A_2) \ (\neg A_2 \vee A_2) \ (\neg A_2)$$
$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A_1 \lor A_2) \ (A_1 \lor \neg A_2) \ (\neg A_1 \lor A_2) \ (\neg A_1 \lor \neg A_2)$$
$$\Downarrow$$
$$(A_2) \ (A_2 \lor \neg A_2) \ (\neg A_2 \lor A_2) \ (\neg A_2)$$
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Resolution: Examples

$$(A_1 \vee A_2) \ (A_1 \vee \neg A_2) \ (\neg A_1 \vee A_2) \ (\neg A_1 \vee \neg A_2)$$
$$\Downarrow$$
$$(A_2) \ (A_2 \vee \neg A_2) \ (\neg A_2 \vee A_2) \ (\neg A_2)$$
$$\Downarrow$$
$$\bot$$

$\implies$ UNSAT

Resolution: Examples (cont.)

$$(A \lor B \lor C) \;\; (B \lor \neg C \lor \neg F) \;\; (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \;\; (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
$$(A \lor E \lor \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$(A \lor B \lor C) \ (B \lor \neg C \lor \neg F) \ (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \ (\neg C \lor \neg F \lor E)$$
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Resolution: Examples (cont.)

$$(A \lor B \lor C) \ (B \lor \neg C \lor \neg F) \ (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \ (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
$$(A \lor E \lor \neg F)$$

$\implies$ SAT

Resolution: Examples (cont.)

$$(A \vee B \vee C) \ (B \vee \neg C \vee \neg F) \ (\neg B \vee E)$$
$$\Downarrow$$
$$(A \vee C \vee E) \ (\neg C \vee \neg F \vee E)$$
$$\Downarrow$$
$$(A \vee E \vee \neg F)$$

$\Longrightarrow$ SAT

$$(A \lor B) \ (A \lor \neg B) \ (\neg A \lor C) \ (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(A) \ (\neg A \lor C) \ (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(C) \ (\neg C)$$
$$\Downarrow$$
$$\bot$$

$$\implies \text{UNSAT}$$

Resolution: Examples

$$(A \lor B) \quad (A \lor \neg B) \quad (\neg A \lor C) \quad (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(A) \quad (\neg A \lor C) \quad (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(C) \quad (\neg C)$$
$$\Downarrow$$
$$\bot$$

$$\Longrightarrow \text{UNSAT}$$

Resolution: Examples

$$(A \lor B) \ (A \lor \neg B) \ (\neg A \lor C) \ (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(A) \ (\neg A \lor C) \ (\neg A \lor \neg C)$$
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$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A \vee B) \;\; (A \vee \neg B) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$
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$$(A) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(C) \;\; (\neg C)$$
$$\Downarrow$$
$$\bot$$

$\implies$ UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
  $\implies$ May require exponential space
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

Outline

Semantic tableaux [39]

- Search for an assignment satisfying φ
- applies recursively elimination rules to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i, the branch is closed, otherwise it is open.
- if no rule can be applied to an open branch μ, then $\mu \models \varphi$;
- if all branches are closed, the formula is not satisfiable;

Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \rightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2} \qquad (\wedge\text{-elimination})$$

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi} \qquad (\neg\neg\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \quad \Gamma, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \neg\varphi_2} \qquad \frac{\Gamma, (\varphi_1 \rightarrow \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \varphi_2} \qquad (\vee\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \quad \Gamma, \neg\varphi_1\neg\varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \quad \Gamma, \neg\varphi_1\varphi_2} \qquad (\leftrightarrow\text{-elimination}).$$

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

Semantic Tableaux – Example

$$\varphi = (A_1 \lor A_2) \land (A_1 \lor \neg A_2) \land (\neg A_1 \lor A_2) \land (\neg A_1 \lor \neg A_2)$$
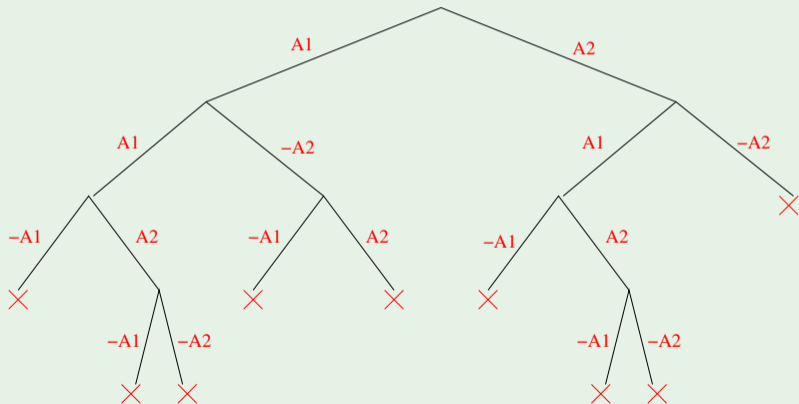
Tableau algorithm

```
function Tableau(Γ)
    if Aᵢ ∈ Γ and ¬Aᵢ ∈ Γ                                    /* branch closed */
        then return False;
    if (φ₁ ∧ φ₂) ∈ Γ                                          /* ∧-elimination */
        then return Tableau(Γ ∪ {φ₁, φ₂}\{(φ₁ ∧ φ₂)});
    if (¬¬φ₁) ∈ Γ                                             /* ¬¬-elimination */
        then return Tableau(Γ ∪ {φ₁}\{(¬¬φ₁)});
    if (φ₁ ∨ φ₂) ∈ Γ                                          /* ∨-elimination */
        then return    Tableau(Γ ∪ {φ₁}\{(φ₁ ∨ φ₂)}) or
                       Tableau(Γ ∪ {φ₂}\{(φ₁ ∨ φ₂)});
    ...
    return True;                                              /* branch expanded */
```

Semantic Tableaux: Example

$$
\begin{array}{lllll}
& & & & (\neg A_1) \wedge \\
& & (A_1 & \vee & \neg A_2) \wedge \\
(A_1 & \vee & A_2 & \vee & A_3) \wedge \\
(A_4 & \vee & \neg A_3 & \vee & A_6) \wedge \\
(A_4 & \vee & \neg A_3 & \vee & \neg A_6) \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & A_7) \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & \neg A_7)
\end{array}
$$

$\Longrightarrow$ unsat

Semantic Tableaux: Example

$$(\neg A_1) \wedge$$
$$(A_1 \vee \neg A_2) \wedge$$
$$(A_1 \vee A_2 \vee A_3) \wedge$$
$$(A_4 \vee \neg A_3 \vee A_6) \wedge$$
$$(A_4 \vee \neg A_3 \vee \neg A_6) \wedge$$
$$(\neg A_3 \vee \neg A_4 \vee A_7) \wedge$$
$$(\neg A_3 \vee \neg A_4 \vee \neg A_7)$$

$\implies$ unsat

Semantic Tableaux: Example

$$(\neg A_1) \land$$
$$(\ A_1 \quad \lor \quad \neg A_2)\ \land$$
$$(\ A_1 \quad \lor \quad A_2 \quad \lor \quad A_3)\ \land$$
$$(\ A_4 \quad \lor \quad \neg A_3 \quad \lor \quad A_6)\ \land$$
$$(\ A_4 \quad \lor \quad \neg A_3 \quad \lor \quad \neg A_6)\ \land$$
$$(\neg A_3 \quad \lor \quad \neg A_4 \quad \lor \quad A_7)\ \land$$
$$(\neg A_3 \quad \lor \quad \neg A_4 \quad \lor \quad \neg A_7)$$

$\Longrightarrow$ unsat

Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend
  $\implies$ loved by logicians.
- Rather inefficient
  $\implies$ avoided by computer scientists.
- Requires polynomial space

Outline

DPLL [10, 9]

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ;
- At each step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL rules

$$\frac{\varphi_1 \wedge (l)}{\varphi_1[l|\top]} \ (Unit)$$

$$\frac{\varphi}{\varphi[l|\top]} \ (l \ Pure)$$

$$\frac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} \ (split)$$

(*l* is a pure literal in φ iff it occurs only positively).

- Split applied if and only if the others cannot be applied.
- Richer formalisms described in [40, 29, 30]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

A1

−A1

A2

A2

×

×

DPLL Algorithm

```
function DPLL(φ, μ)
    if φ = ⊤                                    /* base      */
        then return True;
    if φ = ⊥                                    /* backtrack */
        then return False;
    if {a unit clause (l) occurs in φ}          /* unit      */
        then return DPLL(assign(l, φ), μ ∧ l);
    if {a literal l occurs pure in φ}           /* pure      */
        then return DPLL(assign(l, φ), μ ∧ l);
    l := choose-literal(φ);                     /* split     */
    return    DPLL(assign(l, φ), μ ∧ l)  or
              DPLL(assign(¬l, φ), μ ∧ ¬l);
```

- The pure-literal rule is nowadays obsolete.
- choose-literal(φ) picks only variables still occurring in the formula

DPLL Algorithm

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    if φ = ⊤                              /* base      */
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- The pure-literal rule is nowadays obsolete.
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DPLL – example

DPLL (without pure-literal rule)

Here "choose-literal" selects variable in alphabetic, selecting true first.

$$
\begin{aligned}
(\neg C & & &) \wedge \\
(B & \vee A & \vee C) & \wedge \\
(\neg A & \vee D & &) \wedge \\
(\neg E & \vee \neg A & \vee F) & \wedge \\
(\neg E & \vee \neg F & \vee \neg A) & \wedge \\
(G & \vee \neg A & \vee E) & \wedge \\
(E & \vee \neg G & \vee \neg A) & \wedge \\
(A & \vee H & \vee C) & \wedge \\
(\neg H & \vee \neg I & \vee A) & \wedge \\
(I & \vee L & \vee M) & \wedge \\
(\neg L & \vee C & \vee \neg M) & \wedge \\
(A & \vee \neg L & \vee M) & \wedge \\
(L & \vee N & \vee \neg H) & \wedge \\
(I & \vee L & \vee \neg N) &
\end{aligned}
$$

$\implies$ UNSAT

DPLL – example

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Here "choose-literal" selects variable in alphabetic, selecting true first.

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&(\neg A \ \vee \ D \qquad\ ) \wedge \\
&(\neg E \ \vee \neg A \ \vee \ F) \wedge \\
&(\neg E \ \vee \neg F \ \vee \neg A) \wedge \\
&(\ G \ \vee \neg A \ \vee \ E) \wedge \\
&(\ E \ \vee \neg G \ \vee \neg A) \wedge \\
&(\ A \ \vee \ H \ \vee \ C) \wedge \\
&(\neg H \ \vee \neg I \ \vee \ A) \wedge \\
&(\ I \ \vee \ L \ \vee \ M) \wedge \\
&(\neg L \ \vee \ C \ \vee \neg M) \wedge \\
&(\ A \ \vee \neg L \ \vee \ M) \wedge \\
&(\ L \ \vee \ N \ \vee \neg H) \wedge \\
&(\ I \ \vee \ L \ \vee \neg N)
\end{aligned}
$$

$\implies$ UNSAT

58 / 167

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$$
\begin{aligned}
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&(\neg A &\vee D &&) \wedge \\
&(\neg E &\vee \neg A &\vee F) \wedge \\
&(\neg E &\vee \neg F &\vee \neg A) \wedge \\
&(G &\vee \neg A &\vee E) \wedge \\
&(E &\vee \neg G &\vee \neg A) \wedge \\
&(A &\vee H &\vee C) \wedge \\
&(\neg H &\vee \neg I &\vee A) \wedge \\
&(I &\vee L &\vee M) \wedge \\
&(\neg L &\vee C &\vee \neg M) \wedge \\
&(A &\vee \neg L &\vee M) \wedge \\
&(L &\vee N &\vee \neg H) \wedge \\
&(I &\vee L &\vee \neg N)
\end{aligned}
$$

$\Longrightarrow$ UNSAT

58/167

DPLL – summary

- Handles CNF formulas (non-CNF variant known [1, 15]).
- Branches on truth values
  $\implies$ all instances of an atom assigned simultaneously
- Postpones branching as much as possible.
- Mostly ignored by logicians.
- (The grandfather of) the most efficient SAT algorithms
  $\implies$ loved by computer scientists.
- Requires polynomial space
- Choose_literal() critical!
- Many very efficient implementations [42, 38, 2, 28].

Outline

Ordered Binary Decision Diagrams (OBDDs) [8]]

Canonical representation of Boolean formulas

- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top, \bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

Ordered Binary Decision Diagrams (OBDDs) [8]]

Canonical representation of Boolean formulas

- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top, \bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

OBDD - Examples
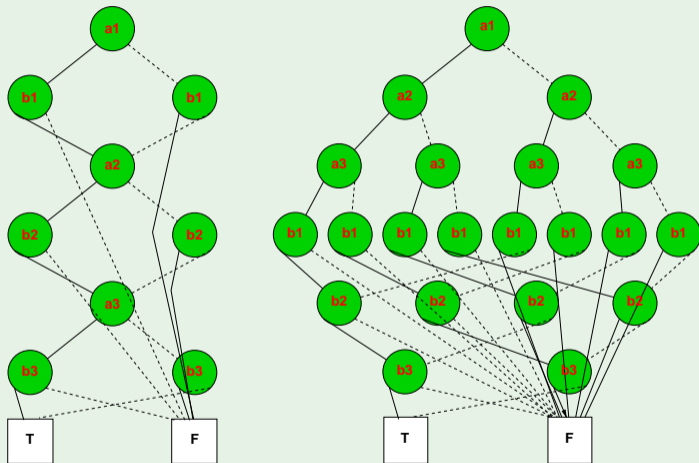
OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- Ordered Decision Tree:
  from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$
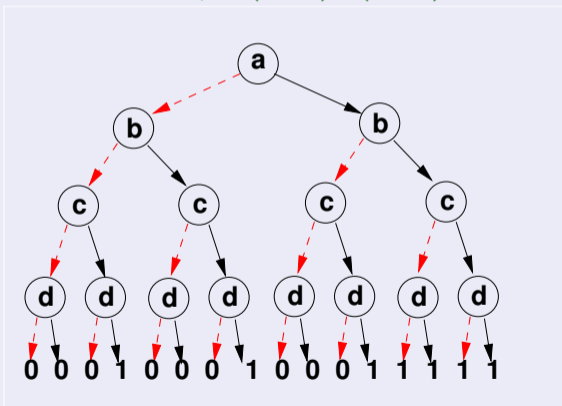
From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following reductions:
  - share subnodes: point to the same occurrence of a subtree
    (via hash consing)
  - remove redundancies: nodes with same left and right children can be eliminated:
    "if A then B else B" $\Longrightarrow$ "B"

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Reduction: example

$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

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Detect redundacies:

$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

Remove redundacies:

$\varphi \overset{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

Remove redundacies:

$\varphi \overset{\text{def}}{=} (a \land b) \lor (c \land d)$

Share identical nodes:
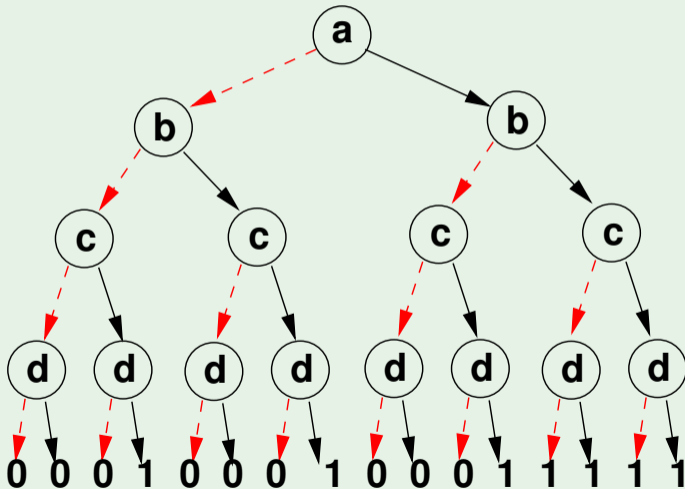
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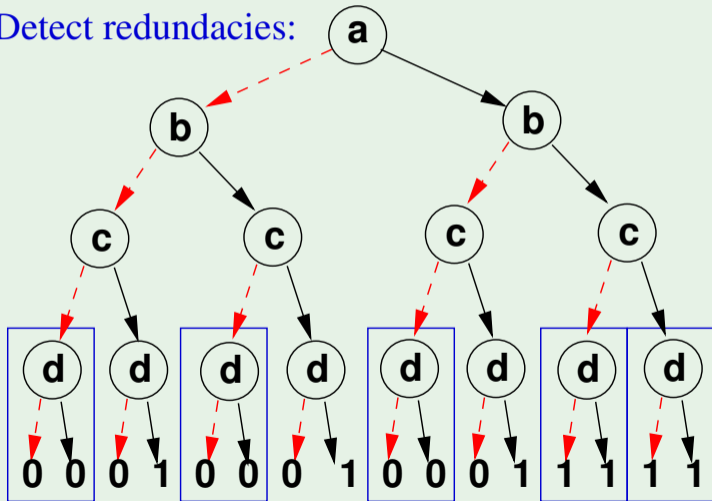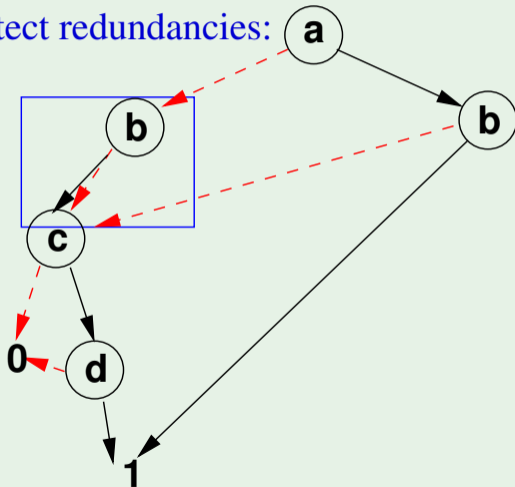
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Remove redundancies:
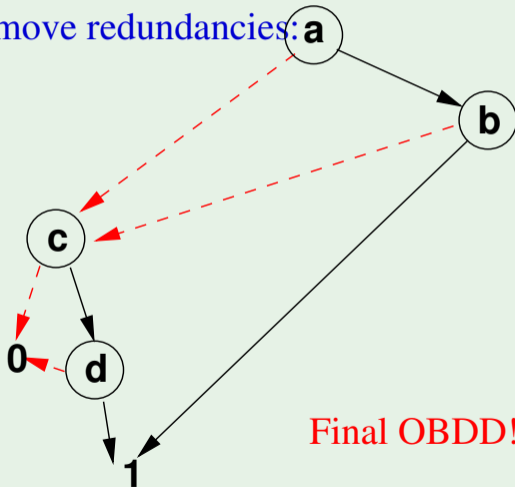
Final OBDD!

If-Then-Else Operators: "$ite(...)$"

If-Then-Else Operators: "$ite(...)$"

- $ite(\phi, \varphi^\top, \varphi^\perp)$: "If ϕ Then $\varphi^\top$ Else $\varphi^\perp$"
- $ite(\phi, \varphi^\top, \varphi^\perp) \stackrel{def}{=} ((\neg\phi \vee \varphi^\top) \wedge (\phi \vee \varphi^\perp)) \iff ((\phi \wedge \varphi^\top) \vee (\neg\phi \wedge \varphi^\perp))$
- properties:

$$ite(\neg\phi, \varphi^\top, \varphi^\perp) = ite(\phi, \varphi^\perp, \varphi^\top)$$
$$\neg ite(\phi, \varphi^\top, \varphi^\perp) = ite(\phi, \neg\varphi^\top, \neg\varphi^\perp)$$
$$ite(\phi, \varphi^\top, \varphi^\perp) \, op \, ite(\phi, \psi^\top, \psi^\perp) = ite(\phi, (\varphi^\top \, op \, \psi^\top), (\varphi^\perp \, op \, \psi^\perp))$$
$$ite(\phi_1, \varphi^\top, \varphi^\perp) \, op \, ite(\phi_2, \psi^\top, \psi^\perp) = ite(\phi_1, (\varphi^\top \, op \, ite(\phi_2, \psi^\top, \psi^\perp)),$$
$$(\varphi^\perp \, op \, ite(\phi_2, \psi^\top, \psi^\perp)))$$
$$= ite(\phi_2, (ite(\phi_1, \varphi^\top, \varphi^\perp) \, op \, \psi^\top),$$
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$$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$$

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$$ite(\phi, \varphi^\top, \varphi^\bot) \; op \; ite(\phi, \psi^\top, \psi^\bot) = ite(\phi, (\varphi^\top \; op \; \psi^\top), (\varphi^\bot \; op \; \psi^\bot))$$
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$$
\begin{aligned}
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\neg ite(\phi, \varphi^\top, \varphi^\bot) &= ite(\phi, \neg\varphi^\top, \neg\varphi^\bot) \\
ite(\phi, \varphi_1^\top, \varphi_1^\bot)\ op\ ite(\phi, \varphi_2^\top, \varphi_2^\bot) &= ite(\phi, (\varphi_1^\top\ op\ \varphi_2^\top), (\varphi_1^\bot\ op\ \varphi_2^\bot)) \\
ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot) &= ite(\phi_1, (\varphi_1^\top\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)), \qquad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\} \\
& \qquad (\varphi_1^\bot\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot))) \\
&= ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ \varphi_2^\top), \\
& \qquad (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ \varphi_2^\bot))
\end{aligned}
$$

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$$ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) \; op \; ite(\phi_2, \varphi_2^\top, \varphi_2^\bot) = ite(\phi_1, (\varphi_1^\top \; op \; ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)), \quad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$$
$$(\varphi_1^\bot \; op \; ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)))$$
$$= ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) op \; \varphi_2^\top),$$
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$$ite(\phi, \varphi_1^\top, \varphi_1^\bot) \; op \; ite(\phi, \varphi_2^\top, \varphi_2^\bot) = ite(\phi, (\varphi_1^\top \, op \, \varphi_2^\top), (\varphi_1^\bot \, op \, \varphi_2^\bot))$$

$$ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) \; op \; ite(\phi_2, \varphi_2^\top, \varphi_2^\bot) = ite(\phi_1, (\varphi_1^\top \, op \, ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)), \quad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$$
$$(\varphi_1^\bot \, op \, ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)))$$

$$= ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot) \, op \, \varphi_2^\top),$$
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$$(\varphi_1^\perp \, op \, ite(\phi_2, \varphi_2^\top, \varphi_2^\perp)))$$
$$= ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\perp) op \, \varphi_2^\top),$$
$$(ite(\phi_1, \varphi_1^\top, \varphi_1^\perp) op \, \varphi_2^\perp))$$

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If-Then-Else Operators: "*ite*(...)"

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- properties:

$$\begin{aligned}
\textit{ite}(\neg\phi, \varphi^\top, \varphi^\perp) &= \textit{ite}(\phi, \varphi^\perp, \varphi^\top) \\
\neg\textit{ite}(\phi, \varphi^\top, \varphi^\perp) &= \textit{ite}(\phi, \neg\varphi^\top, \neg\varphi^\perp) \\
\textit{ite}(\phi, \varphi_1^\top, \varphi_1^\perp) \textit{ op } \textit{ite}(\phi, \varphi_2^\top, \varphi_2^\perp) &= \textit{ite}(\phi, (\varphi_1^\top \textit{ op } \varphi_2^\top), (\varphi_1^\perp \textit{ op } \varphi_2^\perp)) \\
\textit{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \textit{ op } \textit{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp) &= \textit{ite}(\phi_1, (\varphi_1^\top \textit{ op } \textit{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp)), \quad \textit{op} \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\} \\
&\quad\quad\quad (\varphi_1^\perp \textit{ op } \textit{ite}(\phi_2, \varphi_2^\top, \varphi_2^\perp))) \\
&= \textit{ite}(\phi_2, (\textit{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \textit{op } \varphi_2^\top), \\
&\quad\quad\quad (\textit{ite}(\phi_1, \varphi_1^\top, \varphi_1^\perp) \textit{op } \varphi_2^\perp))
\end{aligned}$$

Recursive structure of an OBDD

Assume the variable ordering $A_1, A_2, ..., A_n$:

$$OBDD(\top, \{A_1, A_2, ..., A_n\}) = 1$$
$$OBDD(\bot, \{A_1, A_2, ..., A_n\}) = 0$$
$$OBDD(\varphi, \{A_1, A_2, ..., A_n\}) = \text{ if } A_1$$
$$\text{then } OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\})$$
$$\text{else } OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\})$$

Incrementally building an OBDD

- *obdd_build*($\top, \{...\}) := \top$,
- *obdd_build*($\bot, \{...\}) := \bot$,
- *obdd_build*($A_i, \{...\}) := ite(A_i, \top, \bot)$,
- *obdd_build*($(\neg\varphi), \{A_1, ..., A_n\}) := apply(\neg, obdd\_build(\varphi, \{A_1, ..., A_n\}))$
- *obdd_build*($(\varphi_1 \text{ op } \varphi_2), \{A_1, ..., A_n\}) :=$
  reduce(
   apply(*op*,
            obdd_build($\varphi_1, \{A_1, ..., A_n\}$), *op* $\in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
            obdd_build($\varphi_2, \{A_1, ..., A_n\}$)
))

Incrementally building an OBDD

- $obdd\_build(\top, \{...\}) := \top,$
- $obdd\_build(\bot, \{...\}) := \bot,$
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- $obdd\_build((\neg\varphi), \{A_1, ..., A_n\}) := apply(\neg, obdd\_build(\varphi, \{A_1, ..., A_n\}))$
- $obdd\_build((\varphi_1 \; op \; \varphi_2), \{A_1, ..., A_n\}) :=$
  $reduce($
  $apply(\quad op,$
  $\qquad\qquad obdd\_build(\varphi_1, \{A_1, ..., A_n\}), \quad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
  $\qquad\qquad obdd\_build(\varphi_2, \{A_1, ..., A_n\})$
  $))$

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   $apply(\quad op,$
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  $\qquad\qquad obdd\_build(\varphi_2, \{A_1, ..., A_n\})$
  $\quad))$

Incrementally building an OBDD (cont.)

- *apply* $(op, O_i, O_j) := (O_i \; op \; O_j)$ **if** $(O_i \in \{\top, \bot\}$ or $O_j \in \{\top, \bot\})$
- *apply* $(\neg, \; ite(A_i, \varphi_i^\top, \varphi_i^\bot)) :=$
  $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\bot))$
- *apply* $(op, \; ite(A_i, \varphi_i^\top, \varphi_i^\bot), \; ite(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$
  if $(A_i = A_j)$ **then** $ite(A_i, \quad apply \; (op, \varphi_i^\top, \varphi_j^\top),$
  $\qquad\qquad\qquad\qquad\qquad apply \; (op, \varphi_i^\bot, \varphi_j^\bot))$
  if $(A_i < A_j)$ **then** $ite(A_i, \quad apply \; (op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\bot)),$
  $\qquad\qquad\qquad\qquad\qquad apply \; (op, \varphi_i^\bot, ite(A_j, \varphi_j^\top, \varphi_j^\bot)))$
  if $(A_i > A_j)$ **then** $ite(A_j, \quad apply \; (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top),$
  $\qquad\qquad\qquad\qquad\qquad apply \; (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))$

  $op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD (cont.)

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    if $(A_i = A_j)$ **then** $ite(A_i,\quad apply\ (op, \varphi_i^\top, \varphi_j^\top),$
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- $apply\ (op,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot),\ ite(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$
  if $(A_i = A_j)$ **then** $ite(A_i,\quad apply\ (op, \varphi_i^\top, \varphi_j^\top),$
  $\qquad\qquad\qquad\qquad\qquad apply\ (op, \varphi_i^\bot, \varphi_j^\bot))$
  if $(A_i < A_j)$ **then** $ite(A_i,\quad apply\ (op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\bot)),$
  $\qquad\qquad\qquad\qquad\qquad apply\ (op, \varphi_i^\bot, ite(A_j, \varphi_j^\top, \varphi_j^\bot)))$
  if $(A_i > A_j)$ **then** $ite(A_i,\quad apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top),$
  $\qquad\qquad\qquad\qquad\qquad apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))$

  $op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \lor A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$apply(\lor, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2})$$
$$= ite(A_1, \; apply(\lor, \top, ite(A_2, \top, \bot)), \; apply(\lor, \bot, ite(A_2, \top, \bot)) \;)$$
$$= ite(A_1, \; \top, \; ite(A_2, \top, \bot) \;)$$

- Ex: build the obdd for $(A_1 \lor A_2) \land (A_1 \lor \neg A_2)$ from those of $(A_1 \lor A_2)$, $(A_1 \lor \neg A_2)$ (order: A_1, A_2):
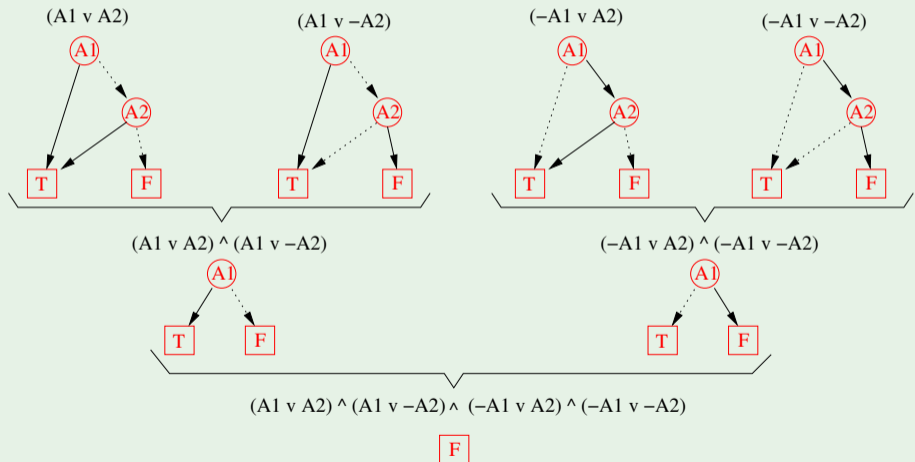
$$apply(\land, \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \lor A_2)}, \; \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \lor \neg A_2)},$$
$$= ite(A_1, \; apply(\land, \top, \top), \; apply(\land, \; ite(A_2, \top, \bot), \; ite(A_2, \bot, \top))$$
$$= ite(A_1, \; \top, \; ite(A_2, \; apply(\land, \top, \bot), \; apply(\land, \bot, \top)))$$
$$= ite(A_1, \; \top, \; ite(A_2, \; \bot, \; \bot))$$
$$= ite(A_1, \; \top, \; \bot)$$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2})$$

$$= ite(A_1, apply(\vee, \top, ite(A_2, \top, \bot)), apply(\vee, \bot, ite(A_2, \top, \bot)))$$

$$= ite(A_1, \top, ite(A_2, \top, \bot))$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$apply(\wedge, \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)}, \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)},$$

$$= ite(A_1, apply(\wedge, \top, \top), apply(\wedge, ite(A_2, \top, \bot), ite(A_2, \bot, \top))$$

$$= ite(A_1, \top, ite(A_2, apply(\wedge, \top, \bot), apply(\wedge, \bot, \top)))$$

$$= ite(A_1, \top, ite(A_2, \bot, \bot))$$

$$= ite(A_1, \top, \bot)$$

OBDD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

OBDD incremental building – example

$\varphi = (A_1 \lor A_2) \land (A_1 \lor \neg A_2) \land (\neg A_1 \lor A_2) \land (\neg A_1 \lor \neg A_2)$

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$
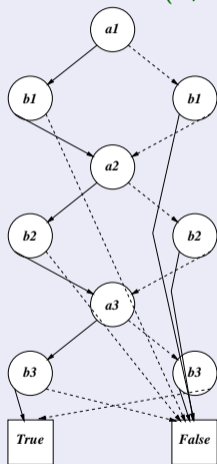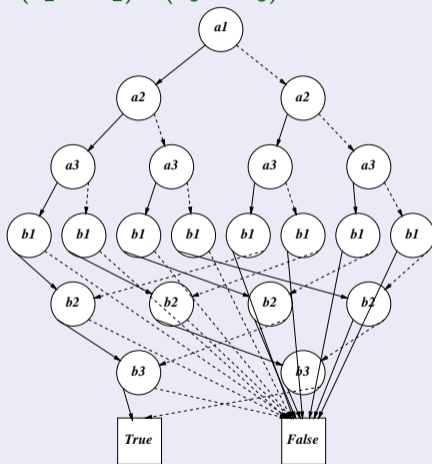
Linear size   Exponential size

Critical choice of variable Orderings in OBDD's
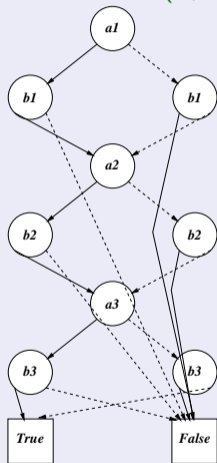
$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$

Linear size   Exponential size

Critical choice of variable Orderings in OBDD's

$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$

Linear size

Exponential size

OBDD's as canonical representation of Boolean formulas

- An OBDD is a canonical representation of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff \textit{OBDD}(\varphi_1) = \textit{OBDD}(\varphi_2)$$

- equivalence check requires constant time!
  $\implies$ validity check requires constant time! ($\varphi \leftrightarrow \top$)
  $\implies$ (un)satisfiability check requires constant time! ($\varphi \leftrightarrow \bot$)
- the set of the paths from the root to 1 represent all the models of the formula
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- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless P = co-NP)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

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Useful Operations over OBDDs

- the equivalence check between two OBDDs is simple
  - are they the same OBDD? ($\implies$ constant time)
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  $|f\ op\ g| = O(|f| \cdot |g|)$

(but typically much smaller on average).

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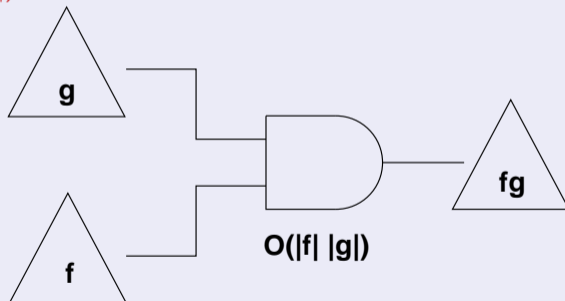
(but typically much smaller on average).

[Recall] Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then
  $$\exists v.\varphi \quad := \quad \varphi|_{v=\bot} \vee \varphi|_{v=\top}$$
  $$\forall v.\varphi \quad := \quad \varphi|_{v=\bot} \wedge \varphi|_{v=\top}$$
- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \ldots, w_n).\varphi := \exists w_1 \ldots \exists w_n.\varphi$

- Intuition:
  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \bot\}$ s.t. $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
  - $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \bot\}$, $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if f is a sub-OBDD labeled by variable v, then $\varphi|_{v=\top}$ and $\varphi|_{v=\bot}$ are the "then" and "else" branches of f
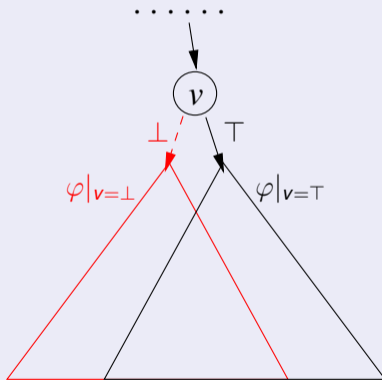
$\Longrightarrow$ lots of sharing of subformulae!

Let $\varphi \overset{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \overset{\text{def}}{=} \exists A.\forall B.\varphi$. Using the variable ordering "A, B, C", draw the OBDD corresponding to the formulas φ and φ'.
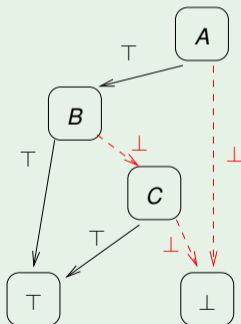
Example

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$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$

Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

which corresponds to the following OBDD:

Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \land (B \lor C))$

$$
\begin{aligned}
\varphi' \stackrel{\text{def}}{=}\ & \exists A.\forall B.\varphi \\
=\ & \forall B.(A \land (B \lor C)))[A := \top] && \lor\quad (\forall B.(A \land (B \lor C)))[A := \bot] \\
=\ & \forall B.(B \lor C)) && \lor\quad \forall B.\bot \\
=\ & ((B \lor C)[B := \top] \quad \land\quad (B \lor C)[B := \bot]) && \lor\quad \bot \\
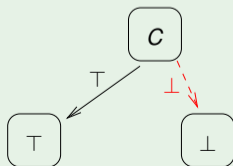=\ & (\top \qquad\qquad\qquad\qquad \land\quad C) \\
=\ & C
\end{aligned}
$$

which corresponds to the following OBDD:

Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

$$
\begin{aligned}
\varphi' &\stackrel{\text{def}}{=} \exists A.\forall B.\varphi \\
&= \forall B.(A \wedge (B \vee C)))[A := \top] & \vee & \quad (\forall B.(A \wedge (B \vee C)))[A := \bot] \\
&= \forall B.(B \vee C)) & \vee & \quad \forall B.\bot \\
&= ((B \vee C)[B := \top] \quad \wedge \quad (B \vee C)[B := \bot]) & \vee & \quad \bot \\
&= (\top \quad\quad\quad\quad\quad \wedge \quad C) \\
&= C
\end{aligned}
$$

which corresponds to the following OBDD:

OBDD – summary

- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
- Canonical representation of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents all models and counter-models of the formula.
- Require exponential space in worst-case
- Very efficient for some practical problems (circuits, symbolic model checking).

Outline

Outline

DPLL: "Classic" chronological backtracking

DPLL implements "classic" chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as "unit", "open", "closed"
- when a conflict is encountered, the stack is popped up to the most recent open assignment *l*
- *l* is toggled, is labeled as "closed", and the search proceeds.

DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible
⟹ lots of useless search!

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
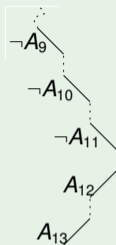$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

DPLL Chronological Backtracking: Example

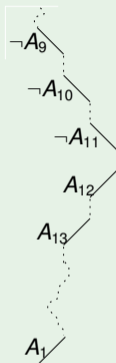$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(initial assignment)

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$c_1 : \neg A_1 \lor A_2$
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$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ $\checkmark$
$c_8 : A_1 \lor A_8$ $\checkmark$
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (branch on A_1)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ $\checkmark$
$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\checkmark$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\checkmark$
$c_8 : A_1 \vee A_8$ $\checkmark$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$
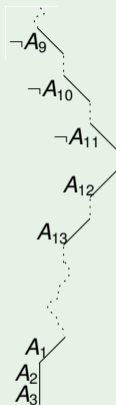(unit A_2, A_3)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ ✓
$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
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$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4\}$
(unit A_4)

DPLL Chronological Backtracking: Example
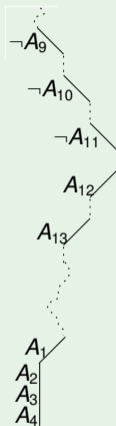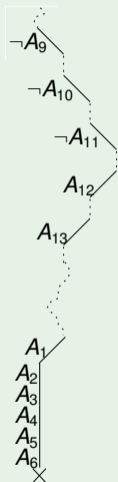

$c_1 : \neg A_1 \vee A_2$ $\checkmark$
$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\checkmark$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ $\checkmark$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ $\checkmark$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ $\checkmark$
$c_6 : \neg A_5 \vee \neg A_6$ $\times$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\checkmark$
$c_8 : A_1 \vee A_8$ $\checkmark$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{1 \neg A_4 1}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) $\Longrightarrow$ conflict

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
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$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\implies$ backtrack up to A_1

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$c_1 : \neg A_1 \vee A_2$ $\checkmark$
$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\checkmark$
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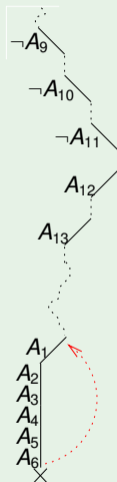$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

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$c_2 : \neg A_1 \vee A_3 \vee A_9$ $\checkmark$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\checkmark$
$c_8 : A_1 \vee A_8$ $\checkmark$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ $\times$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$
(unit A_7, A_8) $\implies$ conflict

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
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$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...
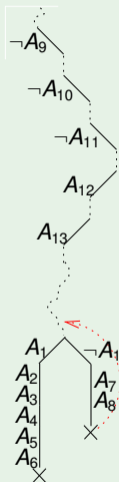
$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\implies$ backtrack to the most recent open branching point

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
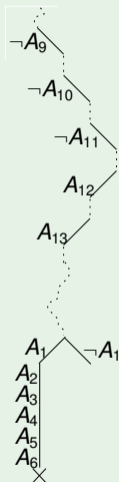$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
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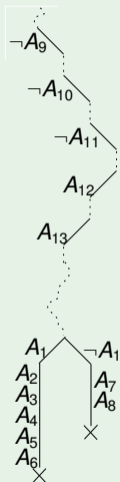$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
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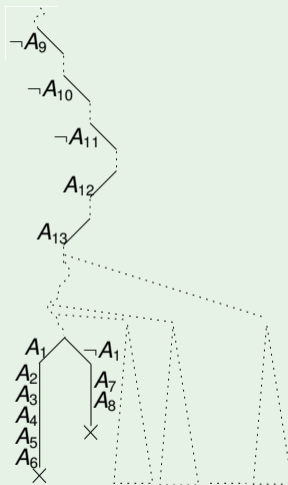
$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\implies$ lots of useless search before backtracking up to A_{13}!

Outline

Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
  - inspired to conflict-driven backjumping and learning in CSPs
  - learns implied clauses as nogoods
- Random restarts
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  - previously-learned clauses maintained
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Can handle industrial problems with $10^6 - 10^7$ variables and clauses!

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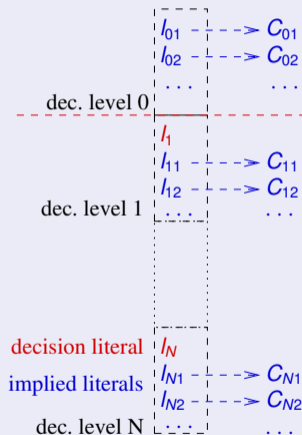Can handle industrial problems with $10^6 - 10^7$ variables and clauses!

Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into decision levels:
  - one decision literal
  - its implied literals
  - each implied literal tagged with the clause causing its unit-propagation (antecedent clause)
- equivalent to an implication graph

$$l_{01} \dashrightarrow C_{01}$$
$$l_{02} \dashrightarrow C_{02}$$
$$\ldots \qquad \ldots$$

dec. level 0

$$l_1$$
$$l_{11} \dashrightarrow C_{11}$$
$$l_{12} \dashrightarrow C_{12}$$

dec. level 1 $\quad \ldots \qquad \ldots$

decision literal $\quad l_N$

$$l_{N1} \dashrightarrow C_{N1}$$

implied literals $\quad l_{N2} \dashrightarrow C_{N2}$

dec. level N $\quad \ldots \qquad \ldots$

Implication graph

- An implication graph is a DAG s.t.:
  - each node represents a variable assignment (literal)
  - each edge $l_i \overset{c}{\longmapsto} l$ is labeled with a clause
  - the node of a decision literal has no incoming edges
  - all edges incoming into a node l are labeled with the same clause c, s.t. $l_1 \overset{c}{\longmapsto} l,...,l_n \overset{c}{\longmapsto} l$ iff
    $c = \neg l_1 \vee ... \vee \neg l_n \vee l$
    (c is said to be the antecedent clause of l)
  - when both l and $\neg l$ occur in the graph, we have a conflict.
- Intuition:
  - representation of the dependencies between literals in μ
  - the graph contains $l_1 \overset{c}{\longmapsto} l,...,l_n \overset{c}{\longmapsto} l$ iff l has been obtained from $l_1, ..., l_n$ by unit propagation on c
  - a partition of the graph with all decision literals on one side and the conflict on the other represents a conflict set

Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

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...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(Initial assignment. Note: $c_1, ..., c_9$ inconsistent.)

Example

$c_1 : \neg A_1 \lor A_2$
$c_2 : \neg A_1 \lor A_3 \lor A_9$
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
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$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓
$c_8 : A_1 \lor A_8$ ✓
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (decide A_1)

Example

$c_1 : \neg A_1 \lor A_2$ ✓
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...
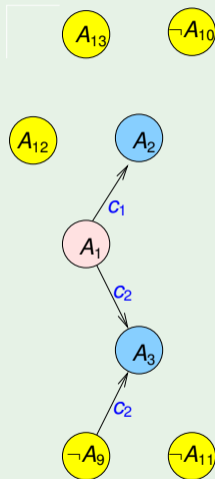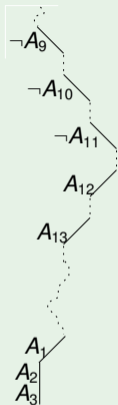
$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$
(unit A_2, A_3)

Example

$c_1 : \neg A_1 \vee A_2$ ✓
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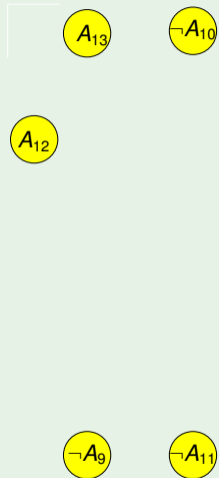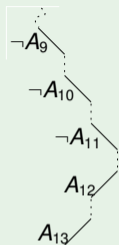$c_6 : \neg A_5 \vee \neg A_6$
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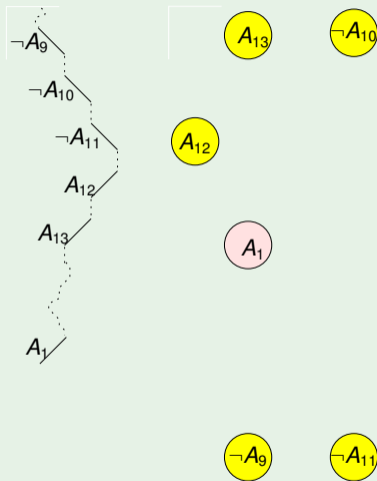$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4\}$
(unit A_4)

Example
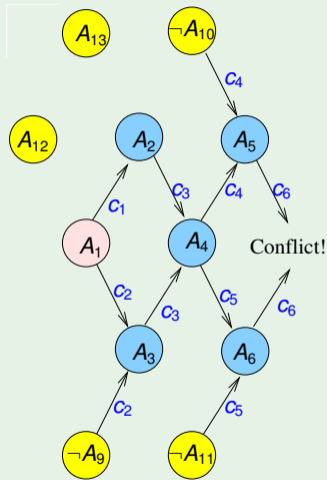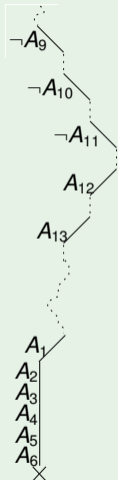
- A node *I* in an implication graph is an unique implication point (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through *I*.
  - the most recent decision node is an UIP (last UIP)
  - all other UIP's have been assigned after the most recent decision

Unique implication point - UIP - example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

$c_6 : \neg A_5 \lor \neg A_6$ ✗

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...

- A_1 is the last UIP
- A_4 is the 1st UIP
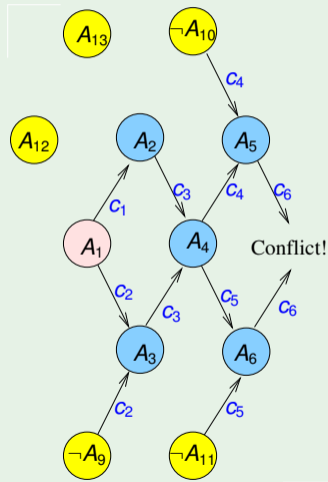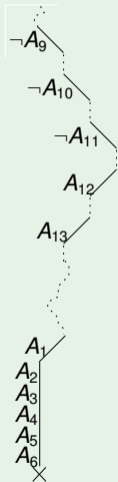
Schema of a CDCL DPLL solver [38, 45]

```
Function CDCL-SAT (formula: φ, assignment & μ) {
        status := preprocess(φ,μ);
        while (1) {
           while (1) {
               status := deduce(φ,μ);
               if (status == Sat)
                  return Sat;
               if (status == Conflict) {
                  ⟨blevel,η⟩ := analyze_conflict(φ,μ);
                  //η is a conflict set
                  if (blevel == 0)
                      return Unsat;
                  else backtrack(blevel,φ,μ);
               }
               else break;
           }
           decide_next_branch(φ,μ);
}       }
```

- `preprocess`(φ, μ) simplifies φ into an easier equisatisfiable formula, updating μ.
- `decide_next_branch`(φ, μ) chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce`(φ, μ) performs all deterministic assignments (unit-propagations plus others), and updates φ, μ accordingly.
- `analyze_conflict`(φ, μ) Computes the subset η of μ causing the conflict (conflict set), and returns the "wrong-decision" level suggested by η ("0" means that η is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack(blevel,`φ, μ`)` undoes the branches up to blevel, and updates φ, μ accordingly

Backjumping and learning: general ideas [2, 38]

- When a branch μ fails:
  - (i) conflict analysis: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (conflict set η)
  - (ii) learning: add the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ to the clause set
  - (iii) backjumping: use η to decide the point where to backtrack
- Jump back up much more than one decision level in the stack
  $\implies$ may avoid lots of redundant search!!.
- We illustrate two main backjumping & learning strategies:
  - the original strategy presented in [38]
  - the state-of-the-art 1^{st}UIP strategy of [44]

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

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criterion: decision

...until C contains only decision literals

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criterion: last UIP

... until C contains only one literal assigned at current decision level, and it is the decision literal (last UIP)

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\neg A_4 \vee A_6 \vee A_{11} \quad \overbrace{\neg A_5 \vee \neg A_6}^{\text{Conflicting cl.}}
}{\neg A_4 \vee A_5 \vee A_{10} \quad \neg A_4 \vee \neg A_5 \vee A_{11}} \ (A_6)
}{\neg A_2 \vee \neg A_3 \vee A_4 \quad \neg A_4 \vee A_{10} \vee A_{11}} \ (A_5)
}{\neg A_1 \vee A_3 \vee A_9 \quad \neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}} \ (A_4)
}{\neg A_1 \vee A_2 \quad \neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \ (A_3)
}{\neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \ (A_2)
$$

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1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C
   until C verifies some given termination criteria

criterion: 1st UIP

... until C contains only one literal assigned at current decision level (1st UIP)

$$\cfrac{\neg A_4 \vee A_5 \vee A_{10} \quad \cfrac{\neg A_4 \vee A_6 \vee A_{11} \quad \overbrace{\neg A_5 \vee \neg A_6}^{Conflicting\ cl.}}{\neg A_4 \vee \neg A_5 \vee A_{11}} \ (A_6)}{\underbrace{\neg A_4}_{1st\ UIP} \vee A_{10} \vee A_{11}} \ (A_5)$$

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Note:

$\varphi \models C$, so that C can be safely added to φ.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

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Conflict analysis and implication graph - example

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$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

$c_6 : \neg A_5 \vee \neg A_6$ ✗   Note: in

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...

this case decision and last-UIP criteria
produce the same partition
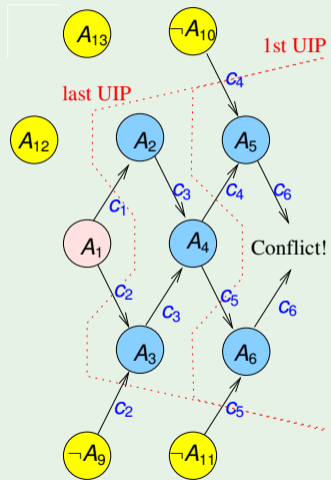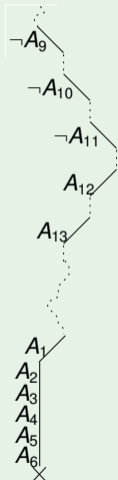
The original backjumping and learning strategy of [38]

- Idea: when a branch μ fails,
  - (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the "Decision" criterion;
  - (ii) learning: add the conflict clause C to the clause set
  - (iii) backjumping: backtrack to the most recent branching point s.t. the stack does not fully contain η, and then unit-propagate the unassigned literal on C

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

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$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

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$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...
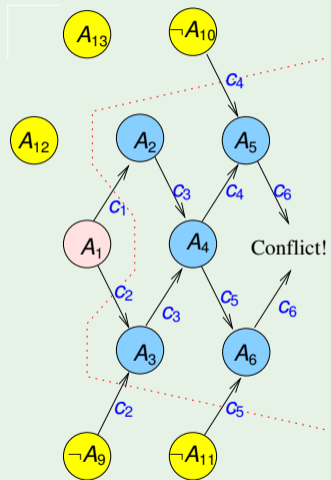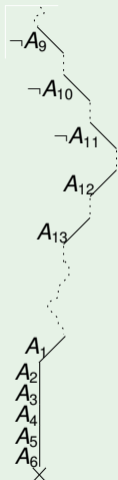
$\implies$ Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ ("decision" schema)

$\implies$ learn the conflict clause $c_{10} := A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
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$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ $\checkmark$

$c_2 : \neg A_1 \lor A_3 \lor A_9$ $\checkmark$

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

$c_6 : \neg A_5 \lor \neg A_6$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$

$c_8 : A_1 \lor A_8$

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ $\checkmark$

...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$

(unit $\neg A_1$)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ ✓
$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
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$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓
$c_8 : A_1 \lor A_8$ ✓
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ ✓
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$
(unit A_7, A_8)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ ✓
$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$
$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓
$c_8 : A_1 \lor A_8$ ✓
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$ ✗
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ ✓
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$
Conflict!
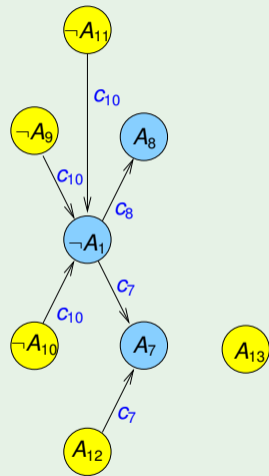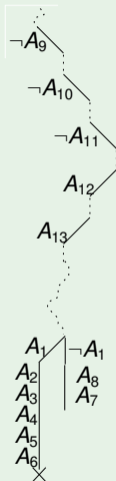
The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

$c_6 : \neg A_5 \lor \neg A_6$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$ ✗

$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ ✓

...
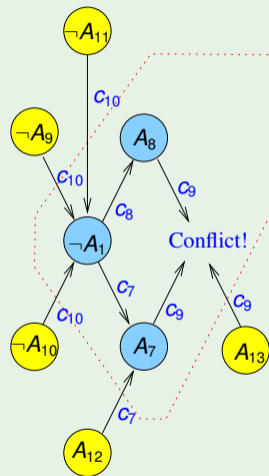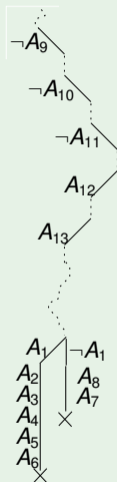
$\implies$ conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.

$\implies$ learn $C_{11} := A_9 \lor A_{10} \lor A_{11} \lor \neg A_{12} \lor \neg A_{13}$

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$
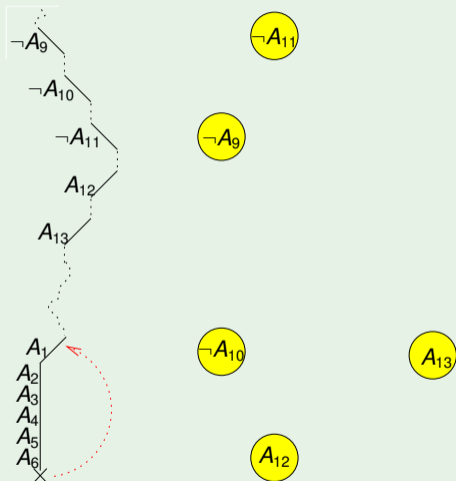
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

...

$\Longrightarrow$ backtrack to A_{13} $\Longrightarrow$ Lots of search saved!

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$

$c_6 : \neg A_5 \lor \neg A_6$

$c_7 : A_1 \lor A_7 \lor \neg A_{12}$

$c_8 : A_1 \lor A_8$

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$ ✓

$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ ✓

$c_{11} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_{12} \lor \neg A_{13}$ ✓

...
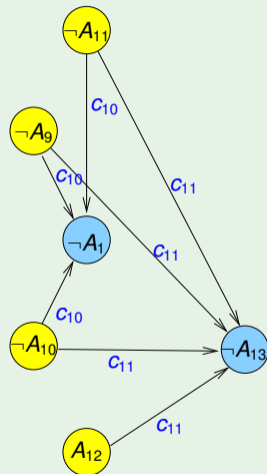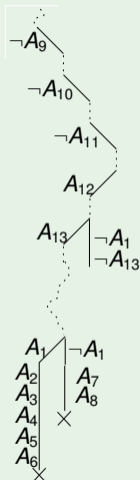
$\implies$ backtrack to A_{13}, then set A_{13} and A_1 to $\bot$,...

State-of-the-art backjumping and learning [44]

- Idea: when a branch μ fails,
  - (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \overset{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the 1^{st}UIP strategy
  - (ii) learning: add the conflict clause C to the clause set
  - (iii) backjumping: backtrack to the highest branching point s.t. the stack contains all-but-one literals in η, and then unit-propagate the unassigned literal on C

$c_1 : \neg A_1 \lor A_2$ ✓
$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓
$c_6 : \neg A_5 \lor \neg A_6$ ✗
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓
$c_8 : A_1 \lor A_8$ ✓
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
...

$\implies$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \lor A_{11} \lor \neg A_4$

- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$
$\implies$ backtrack to A_{11}, then assign $\neg A_4$

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

$c_6 : \neg A_5 \vee \neg A_6$ ✗

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...

$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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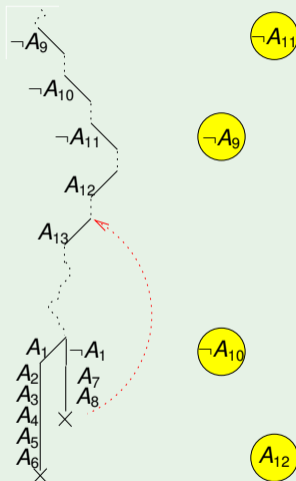$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$
...

$\Longrightarrow$ backtrack up to $A_{11} \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}\}$

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

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$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$ ✓

...
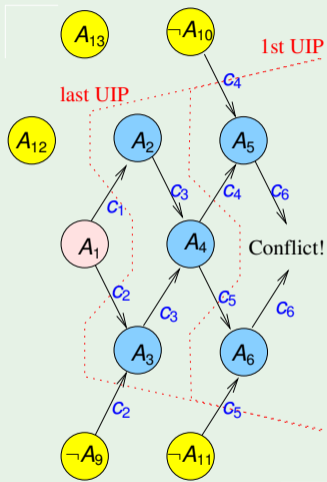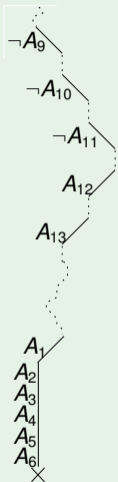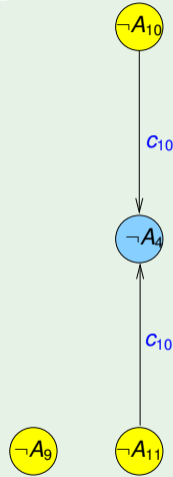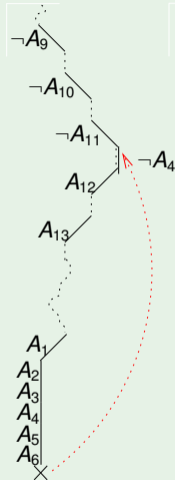
$\implies$ unit propagate $\neg A_4 \implies \{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_4\}...$

1st UIP strategy and backjumping – intuition

- An UIP is a single reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
  - does not enlarge the conflict
  - requires less resolution steps to compute *C*
  - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
  - jump higher
  - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [2, 38]

Idea: When a conflict set η is revealed, then $C \overset{\text{def}}{=} \neg\eta$ is added to φ
$\Longrightarrow$ the solver will no more generate an assignment containing η:
when $|\eta| - 1$ literals in η are assigned, the other is set $\bot$ by unit-propagation on C
$\Longrightarrow$ Drastic pruning of the search!

Learning – example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
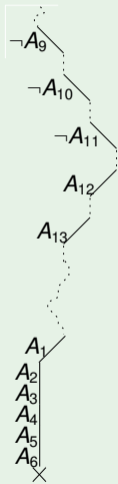
$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ ✓

...

$\Longrightarrow$ Unit: $\{\neg A_1, \neg A_{13}\}$

Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their activity.

A clause is currently active if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

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- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [30])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.

$\implies$ CDCL solvers require polynomial space

"Lazy" Strategy

- when a clause is involved in conflict analisis, increase its activity
- when needed, drop the least-active clauses

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- Backjumping: allows for climbing up to many decision levels in the stack
  - intuition: "go back to the oldest decision where you'd have done something different if only you had known C"
  - $\implies$ may avoid lots of redundant search
- Learning: in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
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Remark: the "quality" of conflict sets

- Different ideas of "good" conflict set
  - Backjumping: if causes the highest backjump ("local" role)
  - Learning: if causes the maximum pruning ("global" role)
- Many different strategies implemented (see, e.g., [2, 38, 44])

Outline

Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
  - detect and remove subsumed clauses
  - detect & collapse equivalent literals
  - apply basic resolution steps
  - ...

Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee l_3 \vee l_1) \wedge \varphi_3$$
$$\Downarrow$$
$$\varphi_1 \wedge (l_1 \vee l_2) \wedge \varphi_2 \wedge \varphi_3$$

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [7]

Repeat:

(i) build the implication graph induced by binary clauses

(ii) detect strongly connected cycles $\implies$ equivalence classes of literals

(iii) perform substitutions

(iv) perform unit and pure literal.

Until (no more simplification is possible).

- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$
$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

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$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [7]

Repeat:

(i) build the implication graph induced by binary clauses

(ii) detect strongly connected cycles $\Longrightarrow$ equivalence classes of literals

(iii) perform substitutions

(iv) perform unit and pure literal.

Until (no more simplification is possible).

- Ex:

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Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$
$$\Downarrow_{resolve}$$
$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$
$$\Downarrow_{unit-propagate}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- Branch is the source of non-determinism for DPLL
  $\implies$ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

Some Heuristics

- MOMS heuristics (DPLL): pick the literal occurring **m**ost **o**ften in the **m**inimal **s**ize clauses
  $\implies$ fast and simple, many variants
- Jeroslow-Wang (DPLL): choose the literal with maximum

$$score(l) := \Sigma_{l \in c \ \& \ c \in \varphi} \ 2^{-|c|}$$

  $\implies$ estimates l's contribution to the satisfiability of φ
- Satz [21] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
  $\implies$ maximizes the effects of unit propagation
- VSIDS [28] (CDCL+): **v**ariable **s**tate **i**ndependent **d**ecaying **s**um
  - "static": scores updated only at the end of a branch
  - "local": privileges variable in recently learned clauses

Restarts [16]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- avoid getting stuck in certain areas of the search space
- $\implies$ may significantly reduce the overall search space

Outline

SAT under assumptions: $SAT(\varphi, \{l_1, ..., l_n\})$ [12]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, ..., l_n\}$: $SAT(\varphi, \{l_1, ..., l_n\})$
  - $SAT(\varphi, \{l_1, ..., l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^{n} l_i)$
  - often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1)$, $SAT(\varphi, \mathcal{A}_2)$, ...
- Idea:
  - $l_1, ..., l_n$ "decided" at decision level 0 before starting the search
  - if backjump to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT

Property

If the "decision" strategy for conflict analysis is used,
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Selection of sub-formulas

Idea: select clauses [12, 23]

Let φ be $\bigwedge_{i=1}^{n} C_i$.

- let $S_1...S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \overset{\text{def}}{=} \{S_{i_1}, ..., S_{i_K}\} \subseteq \{S_1, ..., S_n\}$
- $\implies$ SAT($\bigwedge_{i=1}^{n}(\neg S_i \vee C_i), \mathcal{A}$): same as SAT($\bigwedge_{i=i_1}^{i_k}(C_i)$)
- if S_i is not assumed, then $\neg S_i \vee C_i$ does not contribute to search
- $\implies$ "Select" (activate) only a subset of the clauses in φ at each call.

Generalised Idea: select blocks of clauses

Let φ be $\bigwedge_{i=1}^{n}(\bigwedge_{j=1}^{n_i} C_{ij})$.

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Example

- Initial formula φ:
  $$(\quad A_1 \quad \vee \neg A_2 \quad \vee \neg A_3 \quad) \wedge \quad // \text{ group } 1$$
  $$(\neg A_3 \quad \vee \quad A_2 \quad \vee \neg A_5 \quad) \wedge \quad // \text{ group } 1$$
  $$(\neg A_2 \quad \vee \quad A_5 \quad \vee \quad A_7 \quad) \wedge \quad // \text{ group } 2$$
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  $$(\neg A_1 \quad \vee \neg A_3 \quad \vee \quad A_8 \quad) \wedge \quad // \text{ group } 3$$

- Augmented formula φ':
  $$(\neg S_1 \quad \vee \quad A_1 \quad \vee \neg A_2 \quad \vee \neg A_3 \quad) \wedge \quad // \text{ group } 1 \text{ inactive}$$
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- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
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Incremental SAT solving [12, 11]

- Many CDCL solvers provide a stack-based incremental interface
  - it is possible to push/pop ϕ_i into a stack of subformulas $\{\phi_1, ..., \phi_k\}$
  - check incrementally the satisfiability of $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} \phi_i$.
- Maintains the status of the search from one call to the other
  - in particular it records the learned clauses (plus other information)
  - $\implies$ reuses search from one call to another
- Very useful in many applications (in particular in FV)

- Idea: incremental calls $SAT(\varphi' \wedge A_1), SAT(\varphi' \wedge A_2), ...$
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  - $\Longrightarrow$ C contains the vars selecting the subformulas it is derived from
  - $\Longrightarrow$ in $SAT(\varphi', \mathcal{A})$, if some $S_i \notin \mathcal{A}$, then C is not active

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    - in $SAT(\varphi', \mathcal{A})$, if some $S_i \not\in \mathcal{A}$, then C is not active

Incremental SAT solving [12, 11]

- Many CDCL solvers provide a stack-based incremental interface
  - it is possible to push/pop ϕ_i into a stack of subformulas $\{\phi_1, ..., \phi_k\}$
  - check incrementally the satisfiability of $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} \phi_i$.
- Maintains the status of the search from one call to the other
  - in particular it records the learned clauses (plus other information)
  - $\implies$ reuses search from one call to another
- Very useful in many applications (in particular in FV)

- Idea: incremental calls $SAT(\varphi', \mathcal{A}_1)$, $SAT(\varphi', \mathcal{A}_2)$,...
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Example

- Initial formula φ:

  ... $\wedge$
  $(\ A_1 \quad \vee \neg A_2 \quad \vee \neg A_3 \ )\wedge \quad // \ \phi_1$
  $(\neg A_3 \quad \vee \ A_2 \quad \vee \neg A_5 \ )\wedge \quad // \ \phi_1$

- Augmented formula φ':

  ... $\wedge$
  $(\neg S_1 \quad \vee \ A_1 \quad \vee \neg A_2 \quad \vee \neg A_3 \ )\wedge \quad // \ \phi_1$
  $(\neg S_1 \quad \vee \neg A_3 \quad \vee \ A_2 \quad \vee \neg A_5 \ )\wedge \quad // \ \phi_1$

[push(S_1)]: $SAT(\varphi', \{..., S_1\})$: ϕ_1 active $\implies$ learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

  $$
  \begin{array}{ll}
  ... & \wedge \\
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  \end{array}
  $$

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  \begin{array}{ll}
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  (\neg S_1\ \vee\ A_1\ \vee\neg A_2\ \vee\neg A_3\ )\wedge & // \phi_1 \\
  (\neg S_1\ \vee\neg A_3\ \vee\ A_2\ \vee\neg A_5\ )\wedge & // \phi_1
  \end{array}
  $$

  $$
  (\neg S_1\ \vee\ A_1\ \vee\neg A_3\ \vee\neg A_5\ )\wedge \quad // \textit{learned } C_1
  $$

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Example

- Initial formula φ:

```
...                          ∧
(  A₁   ∨¬A₂   ∨¬A₃  )∧   // φ₁
(¬A₃  ∨  A₂   ∨¬A₅  )∧   // φ₁
(¬A₂  ∨  A₅   ∨  A₇  )∧   // φ₂
(¬A₁  ∨¬A₃   ∨¬A₅  )∧   // φ₂
```

- Augmented formula φ':

```
...                                    ∧
(¬S₁  ∨  A₁   ∨¬A₂   ∨¬A₃  )∧   // φ₁
(¬S₁  ∨¬A₃   ∨  A₂   ∨¬A₅  )∧   // φ₁
(¬S₂  ∨¬A₂   ∨  A₅   ∨  A₇  )∧   // φ₂ inactive
(¬S₂  ∨¬A₁   ∨¬A₃   ∨¬A₅  )∧   // φ₂ inactive

(¬S₁  ∨  A₁   ∨¬A₃   ∨¬A₅  )∧   // learned C₁
```

[push(S_2)]: $SAT(\varphi', \{..., S_1, S_2\})$: ϕ_1, ϕ_2 active $\Longrightarrow$ learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

  $$
  \begin{array}{ll}
  ... & \wedge \\
  (\ A_1 \ \vee \neg A_2 \ \vee \neg A_3 \ ) \wedge & // \ \phi_1 \\
  (\neg A_3 \ \vee \ A_2 \ \vee \neg A_5 \ ) \wedge & // \ \phi_1 \\
  (\neg A_2 \ \vee \ A_5 \ \vee \ A_7 \ ) \wedge & // \ \phi_2 \\
  (\neg A_1 \ \vee \neg A_3 \ \vee \neg A_5 \ ) \wedge & // \ \phi_2
  \end{array}
  $$

- Augmented formula φ':

  $$
  \begin{array}{ll}
  ... & \wedge \\
  (\neg S_1 \ \vee \ A_1 \ \vee \neg A_2 \ \vee \neg A_3 \ ) \wedge & // \ \phi_1 \\
  (\neg S_1 \ \vee \neg A_3 \ \vee \ A_2 \ \vee \neg A_5 \ ) \wedge & // \ \phi_1 \\
  (\neg S_2 \ \vee \neg A_2 \ \vee \ A_5 \ \vee \ A_7 \ ) \wedge & // \ \phi_2, \ inactive \\
  (\neg S_2 \ \vee \neg A_1 \ \vee \neg A_3 \ \vee \neg A_5 \ ) \wedge & // \ \phi_2, \ inactive \\
  \\
  (\neg S_1 \ \vee \ A_1 \ \vee \neg A_3 \ \vee \neg A_5 \ ) \wedge & // \ learned \ C_1 \\
  (\neg S_1 \ \vee \neg S_2 \ \vee \neg A_3 \ \vee \neg A_5 \ ) \wedge & // \ learned \ C_2, \ inactive
  \end{array}
  $$

[push(S_2)]: $SAT(\varphi', \{..., S_1, S_2\})$: ϕ_1, ϕ_2 active $\Longrightarrow$ learn C_2 from ϕ_1, ϕ_2

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Example

- Initial formula φ:

  ```
  ...                          ∧
  (  A₁   ∨¬A₂   ∨¬A₃  )∧    // φ₁
  (¬A₃   ∨  A₂   ∨¬A₅  )∧    // φ₁


  (¬A₁   ∨¬A₃   ∨  A₈  )∧    // φ₃
  ```

- Augmented formula φ':

  ```
  ...                                    ∧
  (¬S₁   ∨  A₁   ∨¬A₂   ∨¬A₃  )∧    // φ₁
  (¬S₁   ∨¬A₃   ∨  A₂   ∨¬A₅  )∧    // φ₁
  (¬S₂   ∨¬A₂   ∨  A₅   ∨  A₇  )∧    // φ₂, inactive
  (¬S₂   ∨¬A₁   ∨¬A₃   ∨¬A₅  )∧    // φ₂, inactive
  (¬S₃   ∨¬A₁   ∨¬A₃   ∨  A₈  )∧    // φ₃
  (¬S₁   ∨  A₁   ∨¬A₃   ∨¬A₅  )∧    // learned C₁
  (¬S₁   ∨¬S₂   ∨¬A₃   ∨¬A₅  )∧    // learned C₂, inactive
  ```

$[pop(S_2); push(S_3)]$: $SAT(\varphi', \{..., S_1, S_3\})$: ϕ_1, ϕ_3 active $\Longrightarrow ...$

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \Longrightarrow C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ:

$$
\begin{array}{lll}
... & & \wedge \\
(\; A_1 & \vee \neg A_2 & \vee \neg A_3 \;)\wedge & // \; \phi_1 \\
(\neg A_3 & \vee \; A_2 & \vee \neg A_5 \;)\wedge & // \; \phi_1 \\
\\
(\neg A_1 & \vee \neg A_3 & \vee \; A_8 \;)\wedge & // \; \phi_3
\end{array}
$$

- Augmented formula φ':

$$
\begin{array}{llll}
... & & & \wedge \\
(\neg S_1 & \vee \; A_1 & \vee \neg A_2 & \vee \neg A_3 \;)\wedge & // \; \phi_1 \\
(\neg S_1 & \vee \neg A_3 & \vee \; A_2 & \vee \neg A_5 \;)\wedge & // \; \phi_1 \\
(\neg S_2 & \vee \neg A_2 & \vee \; A_5 & \vee \; A_7 \;)\wedge & // \; \phi_2, \textit{inactive} \\
(\neg S_2 & \vee \neg A_1 & \vee \neg A_3 & \vee \neg A_5 \;)\wedge & // \; \phi_2, \textit{inactive} \\
(\neg S_3 & \vee \neg A_1 & \vee \neg A_3 & \vee \; A_8 \;)\wedge & // \; \phi_3 \\
(\neg S_1 & \vee \; A_1 & \vee \neg A_3 & \vee \neg A_5 \;)\wedge & // \; \textit{learned } C_1 \\
(\neg S_1 & \vee \neg S_2 & \vee \neg A_3 & \vee \neg A_5 \;)\wedge & // \; \textit{learned } C_2, \textit{inactive}
\end{array}
$$

$[pop(S_2); push(S_3)]$: $SAT(\varphi', \{..., S_1, S_3\})$: ϕ_1, ϕ_3 active $\Longrightarrow$...

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Outline

Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

- Building proofs of unsatisfiability
- Extracting unsatisfiable Cores
- Enumeration in SAT: AllSAT (hints)
- Optimization in SAT: MaxSAT (hints)

Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
    - to verify the result of the solver
    - to understand a "reason" for unsatisfiability
    - to build unsatisfiable cores and interpolants
- Can be built by keeping track of the resolution steps performed when constructing the conflict clauses.

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Building Proofs of Unsatisfiability

- Recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal

- $C_1, ..., C_k$, and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

```
k i-k -> i-k-1

...

2 i-2 -> i-1
1 i-1 -> i
```

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$$\frac{\displaystyle \frac{\displaystyle \frac{\displaystyle \frac{C_k \qquad \overbrace{C_{i-k}}^{\text{conflicting clause}}}{\vdots}}{C_2 \qquad C_{i-2}}}{C_1 \qquad C_{i-1}}}{\underbrace{C_i}_{\text{conflict clause}}}$$

- $C_1, ..., C_k$, and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

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  k i-k -> i-k-1

  ...

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Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces "false" as conflict clause:

- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- $C_1, ..., C_k$, and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

- for every learned leaf clause C_i, substitute C_i with the resolution proof generating it

until all leaf clauses are original clauses
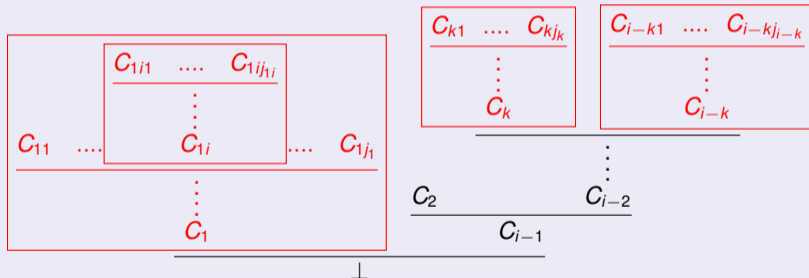
$\implies$ We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$

$(\neg B_0 \vee \neg B_1)$

$(B_1 \vee \neg B_0 \vee A_2)$

$(B_0 \vee \neg B_1 \vee A_1)$

$(B_1 \vee B_0 \vee A_2)$

$(\neg B_0 \vee A_2)$

$(B_0 \vee A_1 \vee A_2)$

$(\neg A_1 \vee B_6)$

$(A_1 \vee A_2)$

$(B_6 \vee A_2)$

$(\neg B_6 \vee \neg B_4)$

$(A_2 \vee \neg B_4)$

$(\neg A_2 \vee B_2)$

$(\neg B_2 \vee \neg B_4)$

$(\neg B_4 \vee B_2)$

B_4

$(\neg B_4)$

$\perp$

Extraction of unsatisfiable cores

- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset
  $\implies$ unsatisfiable cores (aka (Minimal) Unsatisfiable Subsets, (M)US)
- Lots of literature on the topic [46, 24, 26, 31, 43, 19, 13, 6]
- We recognize two main approaches:
  - Proof-based approach [46]: byproduct of finding a resolution proof
  - Assumption-based approach [24]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
  - repeat the process up to fixpoit
  - remove clauses one-by one, until satisfiability is obtained
  - combinations of the two processed above
  - ...

The proof-based approach to core extraction [46]

Unsat core: the set of leaf clauses of a resolution proof

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$
$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

The assumption-based approach to core extraction [24]

Based on the following process:

(i) each clause C_i is substituted by $\neg S_i \lor C_i$, s.t. S_i fresh "selector" variable

(ii) before starting the search each S_i is forced to true.

(iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$

$\implies \{C_j\}_j$ is the unsat core!

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The assumption-based approach to core extraction

Example

$(B_0 \lor \neg B_1 \lor A_1) \land (B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor B_1 \lor A_2) \land$
$(\neg B_0 \lor \neg B_1) \land (\neg B_2 \lor \neg B_4) \land (\neg A_2 \lor B_2) \land (\neg A_1 \lor B_3) \land$
$B_4 \land (A_2 \lor B_5) \land (\neg B_6 \lor \neg B_4) \land (B_6 \lor \neg A_1) \land B_7$

(i) add selector variables:
$(\neg S_1 \lor B_0 \lor \neg B_1 \lor A_1) \land (\neg S_2 \lor B_0 \lor B_1 \lor A_2) \land (\neg S_3 \lor \neg B_0 \lor B_1 \lor A_2) \land$
$(\neg S_4 \lor \neg B_0 \lor \neg B_1) \land (\neg S_5 \lor \neg B_2 \lor \neg B_4) \land (\neg S_6 \lor \neg A_2 \lor B_2) \land$
$(\neg S_7 \lor \neg A_1 \lor B_3) \land (\neg S_8 \lor B_4) \land (\neg S_9 \lor A_2 \lor B_5) \land (\neg S_{10} \lor \neg B_6 \lor \neg B_4) \land$
$(\neg S_{11} \lor B_6 \lor \neg A_1) \land (\neg S_{12} \lor B_7)$

(ii) The conflict analysis returns: $\neg S_1 \lor \neg S_2 \lor \neg S_3 \lor \neg S_4 \lor \neg S_5 \lor \neg S_6 \lor \neg S_8 \lor \neg S_{10} \lor \neg S_{11}$,

(iii) corresponding to the unsat core:
$(B_0 \lor \neg B_1 \lor A_1) \land (B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor B_1 \lor A_2) \land$
$(\neg B_0 \lor \neg B_1) \land (\neg B_2 \lor \neg B_4) \land (\neg A_2 \lor B_2) \land$
$B_4 \land (\neg B_6 \lor \neg B_4) \land (B_6 \lor \neg A_1)$

The assumption-based approach to core extraction

Example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge$
$B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

(i) add selector variables:

$(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge$
$(\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge$
$(\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7)$

(ii) The conflict analysis returns: $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$,

(iii) corresponding to the unsat core:

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge$
$B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$

The assumption-based approach to core extraction

$(B_0 \lor \neg B_1 \lor A_1) \land (B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor B_1 \lor A_2) \land$
$(\neg B_0 \lor \neg B_1) \land (\neg B_2 \lor \neg B_4) \land (\neg A_2 \lor B_2) \land (\neg A_1 \lor B_3) \land$
$B_4 \land (A_2 \lor B_5) \land (\neg B_6 \lor \neg B_4) \land (B_6 \lor \neg A_1) \land B_7$

(i) add selector variables:
$(\neg S_1 \lor B_0 \lor \neg B_1 \lor A_1) \land (\neg S_2 \lor B_0 \lor B_1 \lor A_2) \land (\neg S_3 \lor \neg B_0 \lor B_1 \lor A_2) \land$
$(\neg S_4 \lor \neg B_0 \lor \neg B_1) \land (\neg S_5 \lor \neg B_2 \lor \neg B_4) \land (\neg S_6 \lor \neg A_2 \lor B_2) \land$
$(\neg S_7 \lor \neg A_1 \lor B_3) \land (\neg S_8 \lor B_4) \land (\neg S_9 \lor A_2 \lor B_5) \land (\neg S_{10} \lor \neg B_6 \lor \neg B_4) \land$
$(\neg S_{11} \lor B_6 \lor \neg A_1) \land (\neg S_{12} \lor B_7)$

(ii) The conflict analysis returns: $\neg S_1 \lor \neg S_2 \lor \neg S_3 \lor \neg S_4 \lor \neg S_5 \lor \neg S_6 \lor \neg S_8 \lor \neg S_{10} \lor \neg S_{11}$,

(iii) corresponding to the unsat core:
$(B_0 \lor \neg B_1 \lor A_1) \land (B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor B_1 \lor A_2) \land$
$(\neg B_0 \lor \neg B_1) \land (\neg B_2 \lor \neg B_4) \land (\neg A_2 \lor B_2) \land$
$B_4 \land (\neg B_6 \lor \neg B_4) \land (B_6 \lor \neg A_1)$

The assumption-based approach to core extraction

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge$
$B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

(i) add selector variables:

$(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge$
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$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge$
$B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$

All-SAT (hints)

> ## All-SAT & Projected All-SAT
>
> - All-SAT: enumerate all truth assignments satisfying φ
> - Projected All-SAT: given an "important" subset of atoms $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$, enumerate all assignments over $\mathbf{P}$ which can be extended to truth assignments satisfying φ
> - Algorithms
>   - BCLT [Lahiri et al, CAV'06]:
>   each time a satisfiable assignment $\{l_1, ..., l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$ belonged to the clause set
>   - MathSAT/NuSMV [Cavada et al, FMCAD'07]:
>   As above, plus the Boolean search of the SAT solver is driven by an OBDD.

MaxSAT (hints)

- MaxSAT: given a pair of CNF formulas $\langle \varphi_h, \varphi_s \rangle$ s.t. $\varphi_h \wedge \varphi_s \models \bot$, $\varphi_s \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s.
- Weighted MaxSAT: given also the positive integer penalties $\{w_1, ..., w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to optimization
  $\implies$ much harder than SAT
- Many different approaches (see e.g. [22])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \qquad \varphi_s \stackrel{\text{def}}{=} \left(\begin{array}{ccc} (\quad A_1 \vee \neg A_2) & \wedge & [4] \\ (\neg A_1 \vee \quad A_2) & \wedge & [3] \\ (\neg A_1 \vee \neg A_2) & \wedge & [2] \end{array} \right)$$

$\implies \mu = \{A_1, A_2\}$ (penalty = 2)

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$\implies \mu = \{A_1, A_2\}$ (penalty = 2)

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