

Automated Reasoning and Formal Verification

Module I: Automated Reasoning

Ch. 01: **Propositional Satisfiability (SAT)**

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M.S. in Computer Science, Mathematics, & Artificial Intelligence Systems
Academic year 2024-2025

last update: Friday 21st February, 2025, 11:19

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Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

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Propositional Logic (aka Boolean Logic)



Basic Definitions

- **Propositional formula** (aka **Boolean formula**)
 - \top, \perp are formulas
 - a **propositional atom** A_1, A_2, A_3, \dots is a formula;
 - if φ_1 and φ_2 are formulas, then
 $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$
are formulas.
- Ex: $\varphi \stackrel{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4)))$
- **Atoms**(φ): the set $\{A_1, \dots, A_N\}$ of atoms occurring in φ .
 - Ex: $\text{Atoms}(\varphi) = \{A_1, A_2, A_3, A_4\}$
- **Literal**: a propositional atom A_i (**positive literal**) or its negation $\neg A_i$ (**negative literal**)
 - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- **Clause**: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee \dots)$)
- **Cube**: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge \dots)$)

Semantics of Boolean operators

Truth Table

α	β	$\neg\alpha$	$\alpha\wedge\beta$	$\alpha\vee\beta$	$\alpha\rightarrow\beta$	$\alpha\leftarrow\beta$	$\alpha\leftrightarrow\beta$	$\alpha\oplus\beta$
\perp	\perp	T	\perp	\perp	T	T	T	\perp
\perp	T	T	\perp	T	T	\perp	\perp	T
T	\perp	\perp	\perp	T	\perp	T	\perp	T
T	T	\perp	T	T	T	T	T	\perp

English meaning of connectives

English	Logic
A and B	$A \wedge B$
A if B A when B A whenever B	$A \leftarrow B$
if A, then B A implies B A forces B A requires B	$A \rightarrow B$
A precisely when B A if and only if B	$A \leftrightarrow B$
A or B (or both) A unless B	$A \vee B$ (logical or)
either A or B (but not both)	$A \oplus B$ (exclusive or)

Semantics of Boolean operators (cont.)

Note

- \wedge , \vee , \leftrightarrow and \oplus are commutative:

$$(\alpha \wedge \beta) \iff (\beta \wedge \alpha)$$

$$(\alpha \vee \beta) \iff (\beta \vee \alpha)$$

$$(\alpha \leftrightarrow \beta) \iff (\beta \leftrightarrow \alpha)$$

$$(\alpha \oplus \beta) \iff (\beta \oplus \alpha)$$

- \wedge , \vee , \leftrightarrow and \oplus are associative:

$$((\alpha \wedge \beta) \wedge \gamma) \iff (\alpha \wedge (\beta \wedge \gamma)) \iff (\alpha \wedge \beta \wedge \gamma)$$

$$((\alpha \vee \beta) \vee \gamma) \iff (\alpha \vee (\beta \vee \gamma)) \iff (\alpha \vee \beta \vee \gamma)$$

$$((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) \iff (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) \iff (\alpha \leftrightarrow \beta \leftrightarrow \gamma)$$

$$((\alpha \oplus \beta) \oplus \gamma) \iff (\alpha \oplus (\beta \oplus \gamma)) \iff (\alpha \oplus \beta \oplus \gamma)$$

- \rightarrow , \leftarrow are neither commutative nor associative:

$$(\alpha \rightarrow \beta) \not\iff (\beta \rightarrow \alpha)$$

$$((\alpha \rightarrow \beta) \rightarrow \gamma) \not\iff (\alpha \rightarrow (\beta \rightarrow \gamma))$$

Equivalences with Boolean Operators

$\neg\neg\alpha$	\iff	α
$(\alpha \vee \beta)$	\iff	$\neg(\neg\alpha \wedge \neg\beta)$
$\neg(\alpha \vee \beta)$	\iff	$(\neg\alpha \wedge \neg\beta)$
$(\alpha \wedge \beta)$	\iff	$\neg(\neg\alpha \vee \neg\beta)$
$\neg(\alpha \wedge \beta)$	\iff	$(\neg\alpha \vee \neg\beta)$
$(\alpha \rightarrow \beta)$	\iff	$(\neg\alpha \vee \beta)$
$\neg(\alpha \rightarrow \beta)$	\iff	$(\alpha \wedge \neg\beta)$
$(\alpha \leftarrow \beta)$	\iff	$(\alpha \vee \neg\beta)$
$\neg(\alpha \leftarrow \beta)$	\iff	$(\neg\alpha \wedge \beta)$
$(\alpha \leftrightarrow \beta)$	\iff	$((\alpha \rightarrow \beta) \wedge (\alpha \leftarrow \beta))$
	\iff	$((\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta))$
$\neg(\alpha \leftrightarrow \beta)$	\iff	$(\neg\alpha \leftrightarrow \beta)$
	\iff	$(\alpha \leftrightarrow \neg\beta)$
	\iff	$((\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta))$
$(\alpha \oplus \beta)$	\iff	$\neg(\alpha \leftrightarrow \beta)$

Boolean logic can be expressed in terms of $\{\neg, \wedge\}$ (or $\{\neg, \vee\}$) only!

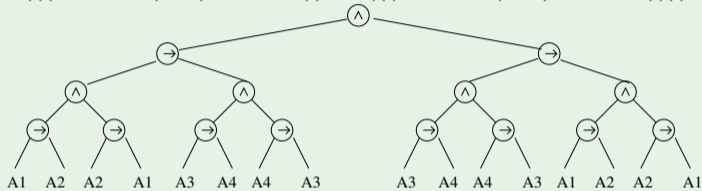
Tree & DAG Representations of Formulas

- Formulas can be represented either as **trees** or as **DAGS** (Directed Acyclic Graphs)
- **DAG representation can be up to exponentially smaller**
 - in particular, when \leftrightarrow 's are involved

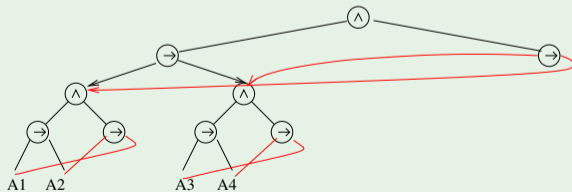
$$\begin{aligned} & (A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\ & \quad \Downarrow \\ & (((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge \\ & ((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2))) \\ & \quad \Downarrow \\ & (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge \\ & (((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)))) \end{aligned}$$

Tree & DAG Representations of Formulas: Example

$((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \wedge$
 $((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)))$



Tree Representation



DAG Representation

Semantics: Basic Definitions

- **Total truth assignment** μ for φ :
 $\mu : \mathit{Atoms}(\varphi) \mapsto \{\top, \perp\}$.
 - represents a **possible world** or a **possible state of the world**
- **Partial Truth assignment** μ for φ :
 $\mu : \mathcal{A} \mapsto \{\top, \perp\}, \mathcal{A} \subset \mathit{Atoms}(\varphi)$.
 - represents 2^k total assignments, k is # unassigned variables
- **Notation: set and formula representations of an assignment**
 - μ can be represented **as a set of literals**:
EX: $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies \{A_1, \neg A_2\}$
 - μ can be represented **as a formula (cube)**:
EX: $\{\mu(A_1) := \top, \mu(A_2) := \perp\} \implies (A_1 \wedge \neg A_2)$

Semantics: Basic Definitions [cont.]

- A **total** truth assignment μ **satisfies** φ (μ is a model of φ , $\mu \models \varphi$):

$$\mu \models A_i \iff \mu(A_i) = \top$$

$$\mu \models \neg\varphi \iff \text{not } \mu \models \varphi$$

$$\mu \models \alpha \wedge \beta \iff \mu \models \alpha \text{ and } \mu \models \beta$$

$$\mu \models \alpha \vee \beta \iff \mu \models \alpha \text{ or } \mu \models \beta$$

$$\mu \models \alpha \rightarrow \beta \iff \text{if } \mu \models \alpha, \text{ then } \mu \models \beta$$

$$\mu \models \alpha \leftrightarrow \beta \iff \mu \models \alpha \text{ iff } \mu \models \beta$$

$$\mu \models \alpha \oplus \beta \iff \mu \models \alpha \text{ iff not } \mu \models \beta$$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$ (the set of models of φ)
- A **partial** truth assignment μ **satisfies** φ iff all total assignments extending μ satisfy φ
 - Ex: $\{A_1\} \models (A_1 \vee A_2)$ because both $\{A_1, A_2\} \models (A_1 \vee A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$
- φ is **satisfiable** iff $\mu \models \varphi$ for some μ (i.e. $M(\varphi) \neq \emptyset$)
- α **entails** β ($\alpha \models \beta$): $\alpha \models \beta$ iff $\mu \models \alpha \implies \mu \models \beta$ for all μ s (i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is **valid** ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for all μ s (i.e., $\mu \in M(\varphi)$ for all μ s)

Properties & Results

Property

φ is valid iff $\neg\varphi$ is not satisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

Equivalence and Equi-Satisfiability

- α and β are **equivalent** iff, for every μ , $\mu \models \alpha$ iff $\mu \models \beta$
(i.e., if $M(\alpha) = M(\beta)$)
- α and β are **equi-satisfiable** iff exists μ_1 s.t. $\mu_1 \models \alpha$ iff exists μ_2 s.t. $\mu_2 \models \beta$
(i.e., if $M(\alpha) \neq \emptyset$ iff $M(\beta) \neq \emptyset$)
- α, β equivalent
 $\Downarrow \Updownarrow$
 α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
 $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α : $\beta \stackrel{\text{def}}{=} T(\alpha)$:
 - T is **validity-preserving** [resp. **satisfiability-preserving**] iff
 $T(\alpha)$ and α are equivalent [resp. equi-satisfiable]

Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

- Intuition:

- $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \perp\}$ s.t. $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \perp\}$, $\mu \cup \{v := \text{truthvalue}\} \models \varphi$

- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

NP-Completeness of SAT

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is **NP-complete**

⇒ The most important logical problems (**validity**, **inference**, **entailment**, **equivalence**, ...) can be straightforwardly reduced to **(un)satisfiability**, and are thus **(co)NP-complete**.



No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- **Positive/negative occurrences**

- φ occurs positively in φ ;
- if $\neg\varphi_1$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ
- if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ ,
then φ_1 and φ_2 occur positively [negatively] in φ ;
- if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ ,
then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ ;
- if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ ,
then φ_1 and φ_2 occur positively and negatively in φ ;

Negative Normal Form (NNF)

- φ is in **Negative normal form** iff it is given only by the recursive applications of \wedge, \vee to literals.
- **every φ can be reduced into NNF:**
 - (i) substituting all \rightarrow 's and \leftrightarrow 's:

$$\begin{aligned}\alpha \rightarrow \beta &\implies \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta &\implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)\end{aligned}$$

- (ii) pushing down negations recursively:

$$\begin{aligned}\neg(\alpha \wedge \beta) &\implies \neg\alpha \vee \neg\beta \\ \neg(\alpha \vee \beta) &\implies \neg\alpha \wedge \neg\beta \\ \neg\neg\alpha &\implies \alpha\end{aligned}$$

- Every non-atomic subformula in $NNF(\varphi)$ **occurs with positive polarity only**
 \implies a subformula ψ occurring with both polarities in φ is encoded as both $NNF(\psi)$ and $NNF(\neg\psi)$
- The reduction is **linear** if a DAG representation is used.
- Preserves the **equivalence** of formulas.

NNF: Example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

↓

$$\begin{aligned} & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \wedge \\ & (((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))) \end{aligned}$$

↓

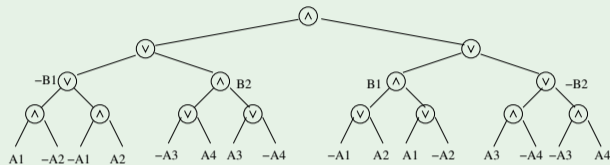
$$\begin{aligned} & ((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2))) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \end{aligned}$$

↓

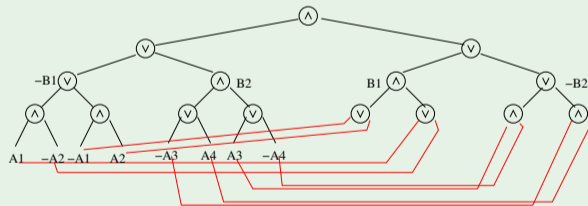
$$\begin{aligned} & (((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge \\ & (((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))) \end{aligned}$$

NNF: Example [cont.]

Note



Tree Representation



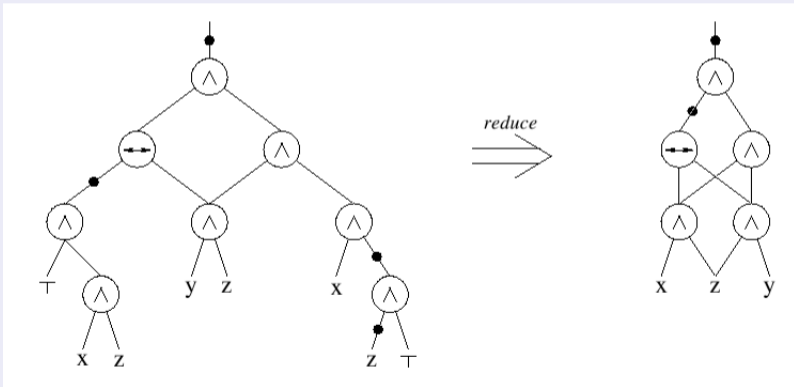
DAG Representation

For each non-literal subformula φ , φ and $\neg\varphi$ have different representations \implies they are not shared.

Optimized polynomial representations

And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations:
{ \wedge , \leftrightarrow , \neg }-only, negations on arcs, sorting of subformulae, lifting of \neg 's over \leftrightarrow 's,...



Conjunctive Normal Form (CNF)

- φ is in **Conjunctive normal form** iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^L \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called **clauses**
- Easier to handle: list of lists of literals.
 \implies no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,

(i) expanding implications and equivalences:

$$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$

(ii) pushing down negations recursively:

$$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \implies \neg\alpha \wedge \neg\beta$$

$$\neg\neg\alpha \implies \alpha$$

\implies Negation Normal Form, NNF (see previous slides)

(iii) applying recursively the DeMorgan's Rule: $(\alpha \wedge \beta) \vee \gamma \implies (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$

- Resulting formula worst-case **exponential**:

- ex: $\|CNF(\bigvee_{i=1}^N (l_{i1} \wedge l_{i2}))\| = \| (l_{11} \vee l_{21} \vee \dots \vee l_{N1}) \wedge (l_{12} \vee l_{21} \vee \dots \vee l_{N1}) \wedge \dots \wedge (l_{12} \vee l_{22} \vee \dots \vee l_{N2}) \| = 2^N$

- $Atoms(CNF(\varphi)) = Atoms(\varphi)$

- $CNF(\varphi)$ is **equivalent** to φ .

- Rarely used in practice.

Labeling CNF conversion $CNF_{label}(\varphi)$

Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Tseitin's CNF-ization)

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$

$$\varphi \implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j))$$

$$\varphi \implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$$

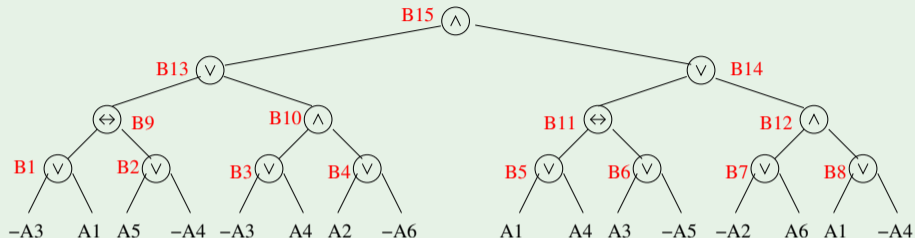
l_i, l_j being literals and B being a “new” variable.

- Worst-case **linear**!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$ is **equi-satisfiable** (but not equivalent) to φ .
 - moreover: $\exists B_1, \dots, B_k. CNF_{label}(\varphi)$ equivalent to φ , s.t. B_1, \dots, B_k all fresh variables introduced
- Much more used than classic conversion in practice

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	\iff	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	\iff	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	\iff	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF Conversion CNF_{label} – Example



$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \wedge$$

... \wedge

$$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \wedge$$

$$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \wedge$$

... \wedge

$$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \wedge$$

$$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \wedge$$

$$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \wedge$$

$$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \wedge$$

B_{15}

$$(\neg B_1 \vee \neg A_3 \vee A_1) \wedge (B_1 \vee A_3) \wedge (B_1 \vee \neg A_1) \wedge$$

... \wedge

$$(\neg B_8 \vee A_1 \vee \neg A_4) \wedge (B_8 \vee \neg A_1) \wedge (B_8 \vee A_4) \wedge$$

$$(\neg B_9 \vee \neg B_1 \vee B_2) \wedge (\neg B_9 \vee B_1 \vee \neg B_2) \wedge$$

$$(B_9 \vee B_1 \vee B_2) \wedge (B_9 \vee \neg B_1 \vee \neg B_2) \wedge$$

= ... \wedge

$$(B_{12} \vee \neg B_7 \vee \neg B_8) \wedge (\neg B_{12} \vee B_7) \wedge (\neg B_{12} \vee B_8) \wedge$$

$$(\neg B_{13} \vee B_9 \vee B_{10}) \wedge (B_{13} \vee \neg B_9) \wedge (B_{13} \vee \neg B_{10}) \wedge$$

$$(\neg B_{14} \vee B_{11} \vee B_{12}) \wedge (B_{14} \vee \neg B_{11}) \wedge (B_{14} \vee \neg B_{12}) \wedge$$

$$(B_{15} \vee \neg B_{13} \vee \neg B_{14}) \wedge (\neg B_{15} \vee B_{13}) \wedge (\neg B_{15} \vee B_{14}) \wedge$$

B_{15}

Improved Labeling CNF conversion CNF_{label}

Polarity-based Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Plaisted&Greenbaum CNF-ization)

- As in the previous case, applying instead the rules:

$$\begin{aligned}\varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \rightarrow (l_i \vee l_j)) && \text{if } (l_i \vee l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \vee l_j)|B] \wedge CNF((l_i \vee l_j) \rightarrow B) && \text{if } (l_i \vee l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \rightarrow (l_i \wedge l_j)) && \text{if } (l_i \wedge l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \wedge l_j)|B] \wedge CNF((l_i \wedge l_j) \rightarrow B) && \text{if } (l_i \wedge l_j) \text{ neg.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \rightarrow (l_i \leftrightarrow l_j)) && \text{if } (l_i \leftrightarrow l_j) \text{ pos.} \\ \varphi &\implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF((l_i \leftrightarrow l_j) \rightarrow B) && \text{if } (l_i \leftrightarrow l_j) \text{ neg.}\end{aligned}$$

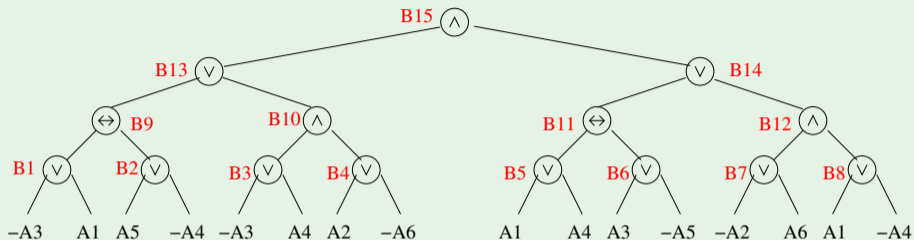
- Smaller in size:

$$\begin{aligned}CNF(B \rightarrow (l_i \vee l_j)) &= (\neg B \vee l_i \vee l_j) \\ CNF(((l_i \vee l_j) \rightarrow B)) &= (\neg l_i \vee B) \wedge (\neg l_j \vee B)\end{aligned}$$

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	\iff	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	\iff	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	\iff	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	\iff	$(B \vee \neg l_i \neg l_j)$
$CNF(B \rightarrow (l_i \leftrightarrow l_j))$	\iff	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j)$
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	\iff	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Improved Labeling CNF conversion CNF_{label} – example



Basic

$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$$

...

$$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \quad \wedge$$

$$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$$

...

$$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \quad \wedge$$

$$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \quad \wedge$$

$$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \quad \wedge$$

$$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \quad \wedge$$

B_{15}

Improved

$$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$$

...

$$CNF(B_8 \rightarrow (A_1 \vee \neg A_4)) \quad \wedge$$

$$CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$$

...

$$CNF(B_{12} \rightarrow (B_7 \wedge B_8)) \quad \wedge$$

$$CNF(B_{13} \rightarrow (B_9 \vee B_{10})) \quad \wedge$$

$$CNF(B_{14} \rightarrow (B_{11} \vee B_{12})) \quad \wedge$$

$$CNF(B_{15} \rightarrow (B_{13} \wedge B_{14})) \quad \wedge$$

B_{15}

Labeling CNF conversion CNF_{label} – further improvements

- Do not apply CNF_{label} when not necessary:
(e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \implies CNF_{label}(\varphi_1) \wedge \varphi_2$, if φ_2 already in CNF)
- Apply DeMorgan's rules where it is more effective:
(e.g., $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \implies CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$)
- Exploit the associativity of \wedge 's and \vee 's:
$$\dots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \dots \implies \dots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3)) \dots$$
- Before applying CNF_{label} , rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Exercises

- 1 Consider the following Boolean formula φ :

$$\neg(((\neg A_1 \rightarrow A_2) \wedge (\neg A_3 \rightarrow A_4)) \vee ((A_5 \rightarrow A_6) \wedge (A_7 \rightarrow \neg A_8)))$$

Compute the Negative Normal Form of φ

- 2 Consider the following Boolean formula φ :

$$((\neg A_1 \wedge A_2) \vee (A_7 \wedge A_4) \vee (\neg A_3 \wedge \neg A_2) \vee (A_5 \wedge \neg A_4))$$

- 1 Produce the CNF formula $CNF(\varphi)$.
- 2 Produce the CNF formula $CNF_{label}(\varphi)$.
- 3 Produce the CNF formula $CNF_{label}(\varphi)$ (improved version)

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 - Generalities
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 - Tableaux
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Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
 - AI, formal verification, circuit synthesis, operational research,....
- Important in AI: $KB \models \alpha$: entail fact α from knowledge base KB (aka **Model Checking**: $M(KB) \subseteq M(\alpha)$)
 - typically $|KB| \gg |\alpha|$
- All propositional reasoning tasks reduced to **satisfiability (SAT)**
 - $KB \models \alpha \implies \text{SAT}(KB \wedge \neg\alpha) = \text{false}$
 - input formula CNF-ized and fed to a **SAT solver**
- **Current SAT solvers dramatically efficient:**
 - handle industrial problems with $10^6 - 10^7$ variables & clauses!
 - used as backend engines in a variety of systems

Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
\perp	\perp	\perp	\perp	\top	\top
\perp	\top	\perp	\top	\top	\perp
\top	\perp	\perp	\top	\perp	\perp
\top	\top	\top	\top	\top	\top

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|\text{Atoms}(\varphi)|}$ lines.
- Never used in practice.

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The Resolution Rule

- **Resolution**: deduction of a new clause from a pair of clauses with exactly one incompatible variable (**resolvent**):

$$\frac{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l}_{\text{resolvent}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'}) \quad (\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{\neg l}_{\text{resolvent}} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}{(\underbrace{l_1 \vee \dots \vee l_k}_{\text{common}} \vee \underbrace{l'_{k+1} \vee \dots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \dots \vee l''_n}_{C''})}$$

- Ex:
$$\frac{(A \vee B \vee C \vee D \vee E) \quad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$$

- Note: many standard inference rules subcases of resolution:
(recall that $\alpha \rightarrow \beta \iff \neg\alpha \vee \beta$)

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \text{ (trans.)} \quad \frac{A \quad A \rightarrow B}{B} \text{ (m. ponens)} \quad \frac{\neg B \quad A \rightarrow B}{\neg A} \text{ (m. tollens)}$$

Improvements: Subsumption & Unit Propagation

Alternative “set” notation (Γ clause set):

$$\frac{\Gamma, \phi_1, \dots, \phi_n}{\Gamma, \phi'_1, \dots, \phi'_n} \quad \left(\text{e.g., } \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2} \right)$$

- Removal of valid clauses:

$$\frac{\Gamma \wedge (p \vee \neg p \vee C)}{\Gamma}$$

- Clause Subsumption (C clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

$$\frac{\Gamma \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma \wedge (l) \wedge (\bigvee_i l_i)}$$

- Unit Subsumption:

$$\frac{\Gamma \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma \wedge (l)}$$

- Unit Propagation = Unit Resolution + Unit Subsumption

“Deterministic” rule: applied **before** other “non-deterministic” rules!

Remark

What happens with more than 1 resolvent?

- Common mistake: the following is not a correct application of the resolution rule:

$$\frac{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2), (C_1 \vee C_2)}$$

- Rather, a correct application would be:

$$\frac{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2)}{\Gamma, (C_1 \vee l_1 \vee l_2), (C_2 \vee \neg l_1 \vee \neg l_2), (C_1 \vee l_2 \vee C_2 \vee \neg l_2)}$$

... but $(C_1 \vee l_2 \vee C_2 \vee \neg l_2)$ is valid and should be removed

⇒ no clause is produced

Basic Propositional Inference: Resolution [33, 10]

- Assume input formula in CNF
 - if not, apply Tseitin CNF-ization first
- ⇒ φ is represented as a set of clauses
- **Search** for a refutation of φ (is φ unsatisfiable?)
 - recall: $\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ unsatisfiable
- Basic idea: **apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either**
 - a false clause is generated, or
 - the resolution rule is no more applicable
- **Correct:** if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- **Complete:** if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- **Time-inefficient**
- **Very Memory-inefficient (exponential in memory)**
- Many different strategies

Resolution: basic strategy [10]

```
function  $DP(\Gamma)$ 
  if  $\perp \in \Gamma$                                 /* unsat */
    then return False;
  if (Resolve() is no more applicable to  $\Gamma$ ) /* sat   */
    then return True;
  if {a unit clause ( $l$ ) occurs in  $\Gamma$ }      /* unit   */
    then  $\Gamma := Unit\_Propagate(l, \Gamma)$ ;
    return  $DP(\Gamma)$ 
   $A := select\_variable(\Gamma)$ ;              /* resolve */
   $\Gamma = \Gamma \cup \bigcup_{A \in C', \neg A \in C''} \{Resolve(C', C'')\} \setminus \bigcup_{A \in C', \neg A \in C''} \{C', C''\}$ ;
  return  $DP(\Gamma)$ 
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$\begin{array}{cccc} (A_1 \vee A_2) & (A_1 \vee \neg A_2) & (\neg A_1 \vee A_2) & (\neg A_1 \vee \neg A_2) \\ \Downarrow & & & \\ (A_2) & (A_2 \vee \neg A_2) & (\neg A_2 \vee A_2) & (\neg A_2) \\ \Downarrow & & & \\ \perp & & & \end{array}$$

\Rightarrow UNSAT

Resolution: Examples (cont.)

$$\begin{array}{c} (A \vee B \vee C) \quad (B \vee \neg C \vee \neg F) \quad (\neg B \vee E) \\ \Downarrow \\ (A \vee C \vee E) \quad (\neg C \vee \neg F \vee E) \\ \Downarrow \\ (A \vee E \vee \neg F) \end{array}$$

\Rightarrow SAT

Resolution: Examples

$$(A \vee B) \quad (A \vee \neg B) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(A) \quad (\neg A \vee C) \quad (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(C) \quad (\neg C)$$
$$\Downarrow$$
$$\perp$$

\Rightarrow UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
 - ⇒ May require **exponential space**
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

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Semantic tableaux [39]

- **Search** for an assignment satisfying φ
- applies recursively **elimination rules** to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i , the branch is **closed**, otherwise it is **open**.
- if no rule can be applied to an open branch μ , then $\mu \models \varphi$;
- if all branches are **closed**, the formula is **not satisfiable**;

Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \rightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2}$$

(\wedge -elimination)

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi}$$

($\neg\neg$ -elimination)

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \quad \Gamma, \varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \neg\varphi_2}$$

$$\frac{\Gamma, (\varphi_1 \rightarrow \varphi_2)}{\Gamma, \neg\varphi_1 \quad \Gamma, \varphi_2}$$

(\vee -elimination)

$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \quad \Gamma, \neg\varphi_1, \neg\varphi_2}$$

$$\frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \quad \Gamma, \neg\varphi_1, \varphi_2}$$

(\leftrightarrow -elimination).

Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

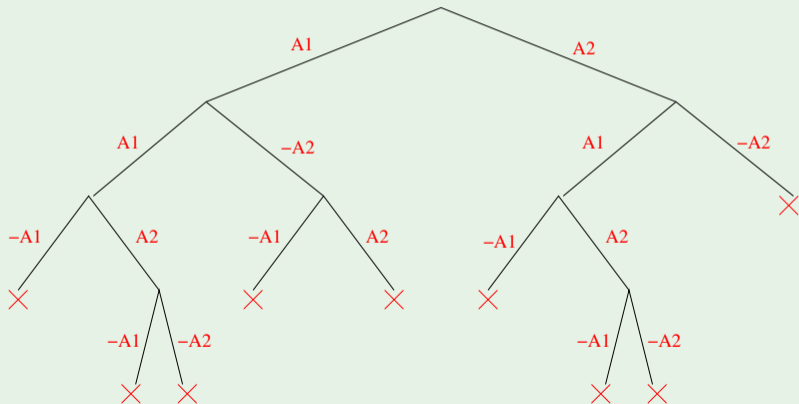
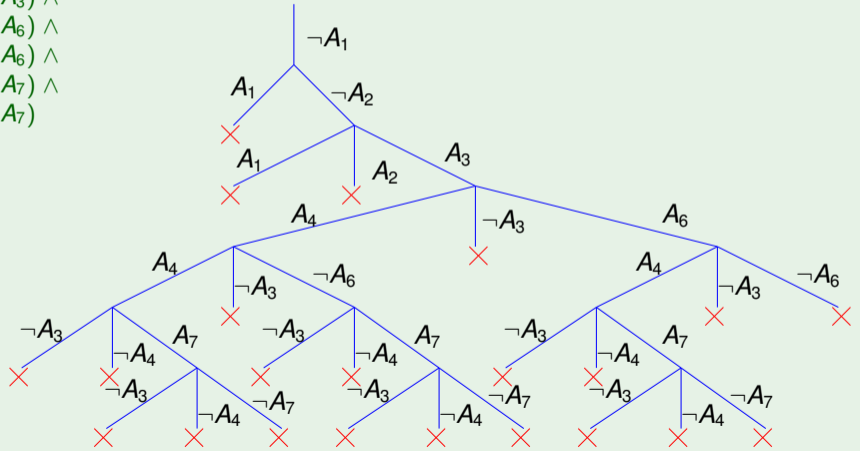


Tableau algorithm

```
function Tableau( $\Gamma$ )  
  if  $A_i \in \Gamma$  and  $\neg A_i \in \Gamma$                                 /* branch closed */  
    then return False;  
  if  $(\varphi_1 \wedge \varphi_2) \in \Gamma$                                     /*  $\wedge$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1, \varphi_2\} \setminus \{(\varphi_1 \wedge \varphi_2)\}$ );  
  if  $(\neg\neg\varphi_1) \in \Gamma$                                         /*  $\neg\neg$ -elimination */  
    then return Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\neg\neg\varphi_1)\}$ );  
  if  $(\varphi_1 \vee \varphi_2) \in \Gamma$                                     /*  $\vee$ -elimination */  
    then return    Tableau( $\Gamma \cup \{\varphi_1\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ ) or  
                   Tableau( $\Gamma \cup \{\varphi_2\} \setminus \{(\varphi_1 \vee \varphi_2)\}$ );  
  ...  
  return True;                                                /* branch expanded */
```

Semantic Tableaux: Example

$(\neg A_1) \wedge$
 $(A_1 \vee \neg A_2) \wedge$
 $(A_1 \vee A_2 \vee A_3) \wedge$
 $(A_4 \vee \neg A_3 \vee A_6) \wedge$
 $(A_4 \vee \neg A_3 \vee \neg A_6) \wedge$
 $(\neg A_3 \vee \neg A_4 \vee A_7) \wedge$
 $(\neg A_3 \vee \neg A_4 \vee \neg A_7)$



\Rightarrow unsat

Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- **Branches on disjunctions**
- **Intuitive, modular, easy to extend**
⇒ loved by logicians.
- **Rather inefficient**
⇒ avoided by computer scientists.
- Requires **polynomial space**

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- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ ;
- At each step assigns a truth value to (all instances of) **one atom**.
- Performs **deterministic choices** first.

$$\frac{\varphi_1 \wedge (I)}{\varphi_1[I|\top]} \text{ (Unit)}$$

$$\frac{\varphi}{\varphi[I|\top]} \text{ (I Pure)}$$

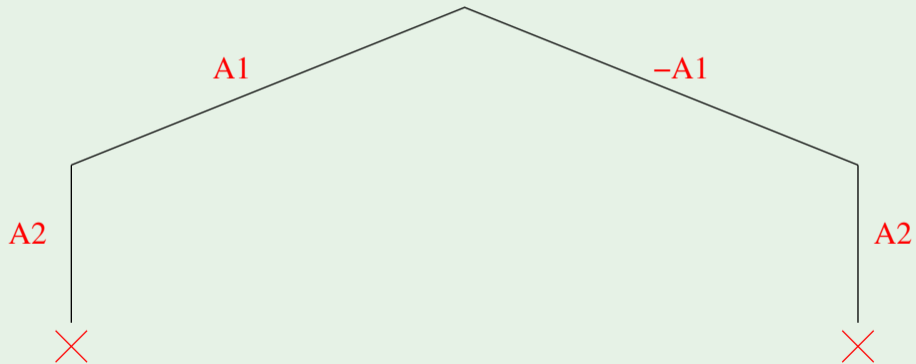
$$\frac{\varphi}{\varphi[I|\top] \quad \varphi[I|\perp]} \text{ (split)}$$

(I is a **pure literal** in φ iff it occurs **only positively**).

- Split applied **if and only if the others cannot be applied**.
- Richer formalisms described in [40, 29, 30]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



DPLL Algorithm

```
function DPLL( $\varphi, \mu$ )  
  if  $\varphi = \top$                                 /* base */  
    then return True;  
  if  $\varphi = \perp$                                 /* backtrack */  
    then return False;  
  if {a unit clause (l) occurs in  $\varphi$ }        /* unit */  
    then return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ );  
  if {a literal l occurs pure in  $\varphi$ }         /* pure */  
    then return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ );  
  l := choose-literal( $\varphi$ );                    /* split */  
  return DPLL(assign(l,  $\varphi$ ),  $\mu \wedge l$ ) or  
        DPLL(assign( $\neg l$ ,  $\varphi$ ),  $\mu \wedge \neg l$ );
```

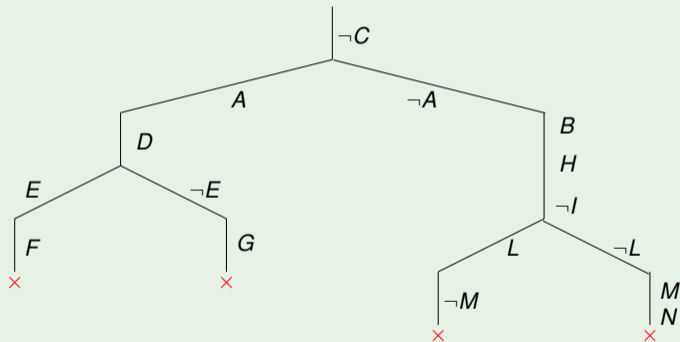
- The pure-literal rule is nowadays obsolete.
- *choose-literal*(φ) picks only variables still occurring in the formula

DPLL – example

DPLL (without pure-literal rule)

Here “choose-literal” selects variable in alphabetic, selecting true first.

$(\neg C \vee A \vee B) \wedge$
 $(B \vee A \vee C) \wedge$
 $(\neg A \vee D \vee E) \wedge$
 $(\neg E \vee \neg A \vee F) \wedge$
 $(\neg E \vee \neg F \vee \neg A) \wedge$
 $(G \vee \neg A \vee E) \wedge$
 $(E \vee \neg G \vee \neg A) \wedge$
 $(A \vee H \vee C) \wedge$
 $(\neg H \vee \neg I \vee A) \wedge$
 $(I \vee L \vee M) \wedge$
 $(\neg L \vee C \vee \neg M) \wedge$
 $(A \vee \neg L \vee M) \wedge$
 $(L \vee N \vee \neg H) \wedge$
 $(I \vee L \vee \neg N)$



⇒ UNSAT

DPLL – summary

- Handles **CNF formulas** (non-CNF variant known [1, 15]).
- **Branches on truth values**
⇒ all instances of an atom assigned simultaneously
- **Postpones branching as much as possible.**
- Mostly ignored by logicians.
- (The grandfather of) **the most efficient SAT algorithms**
⇒ loved by computer scientists.
- Requires **polynomial space**
- **Choose_literal()** critical!
- Many very efficient implementations [42, 38, 2, 28].

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Ordered Binary Decision Diagrams (OBDDs) [8]

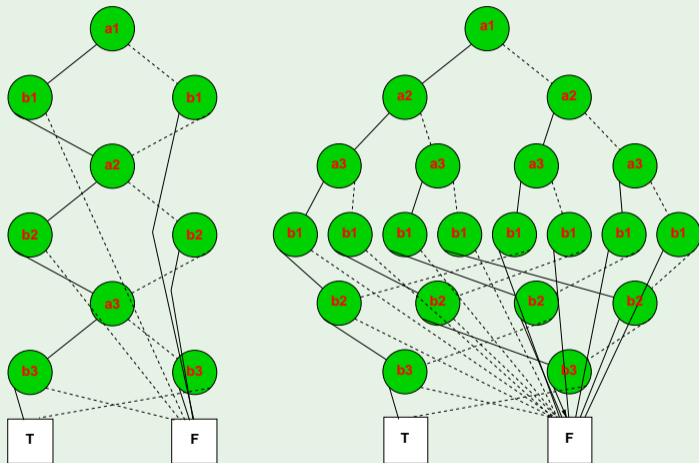
Canonical representation of Boolean formulas

- “If-then-else” binary direct acyclic graphs (DAGs) with one root and two leaves: **1**, **0** (or **T**, **⊥**; or **T**, **F**)
- **Variable ordering** A_1, A_2, \dots, A_n imposed a priori.
- Paths leading to **1** represent **models**
Paths leading to **0** represent **counter-models**

Note

Some authors call them **Reduced** Ordered Binary Decision Diagrams (**ROBDDs**)

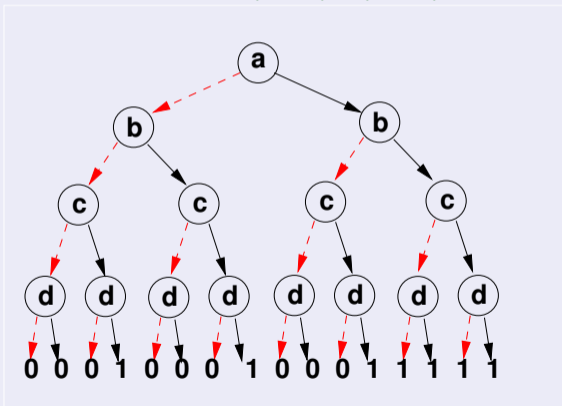
OBDD - Examples



OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- **Ordered Decision Tree:**
from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$

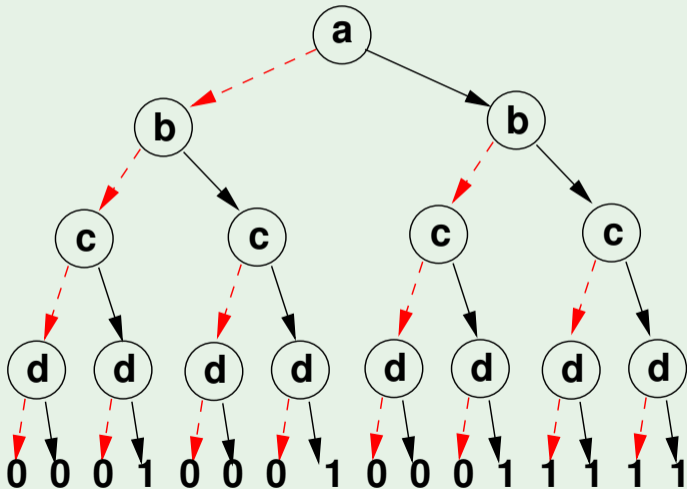


From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following **reductions**:
 - **share subnodes**: point to the same occurrence of a subtree (via **hash consing**)
 - **remove redundancies**: nodes with same left and right children can be eliminated:
"if A then B else B " \implies " B "

Reduction: example

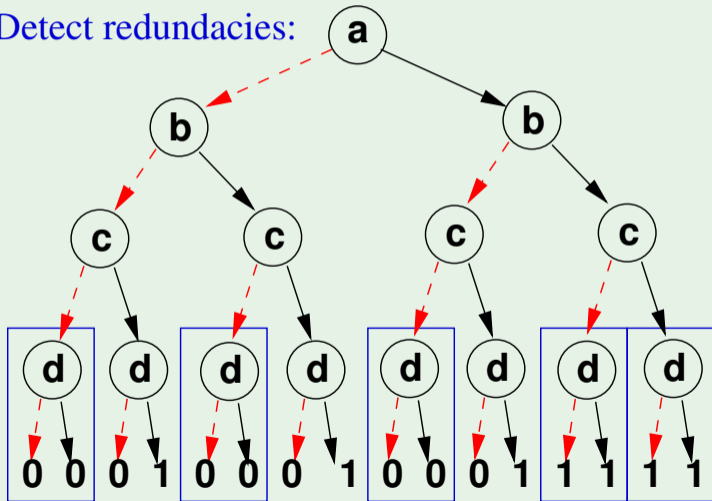
$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

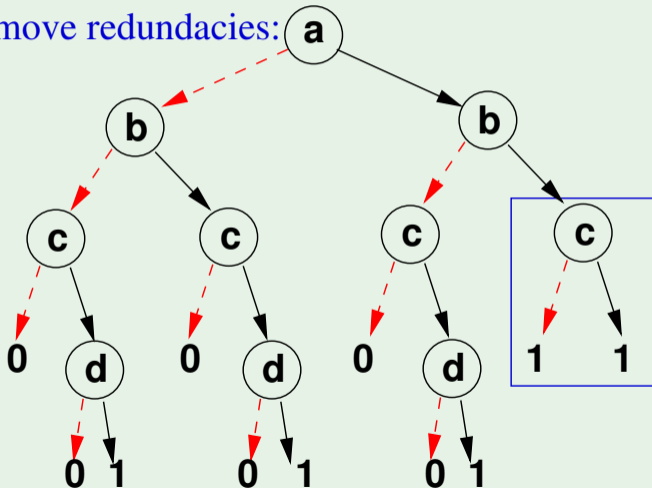
Detect redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

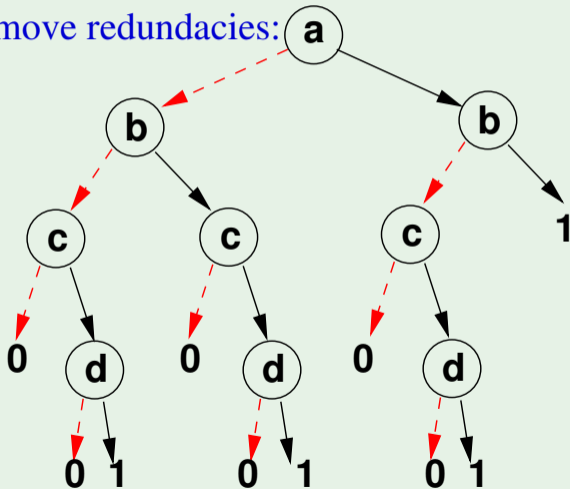
Remove redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

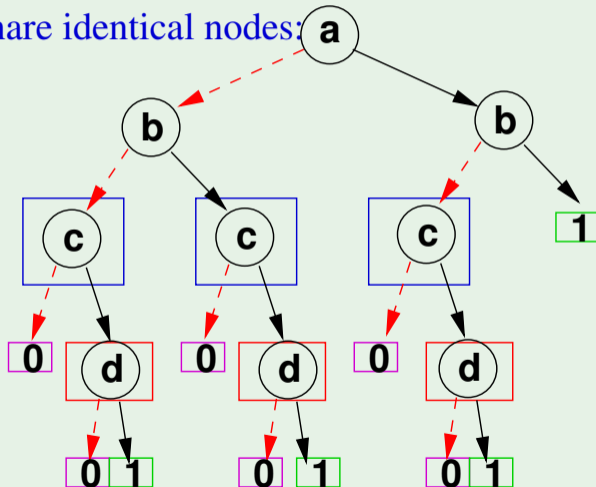
Remove redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

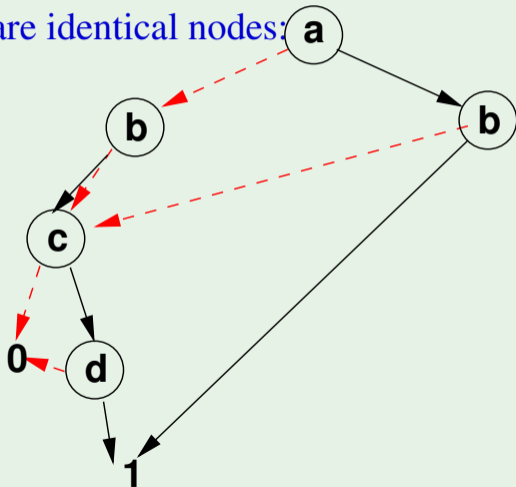
Share identical nodes:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

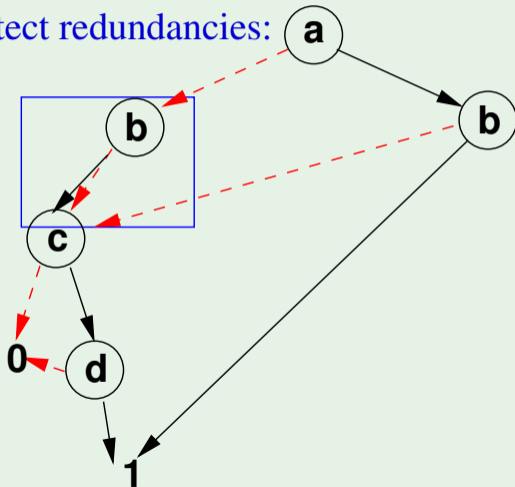
Share identical nodes:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

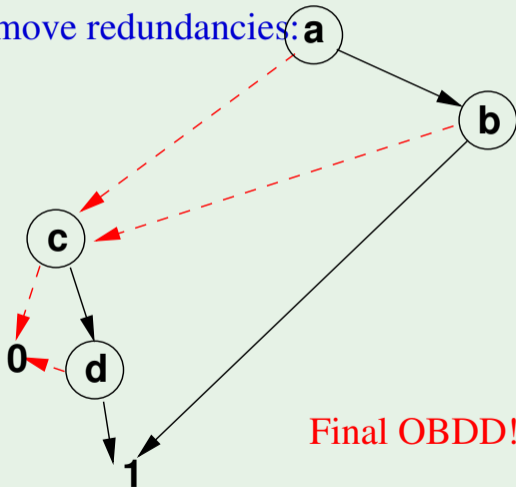
Detect redundancies:



Reduction: example

$$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$$

Remove redundancies:



If-Then-Else Operators: “*ite*(...)”

If-Then-Else Operators: “*ite*(...)”

● *ite*($\phi, \varphi^{\top}, \varphi^{\perp}$): “If ϕ Then φ^{\top} Else φ^{\perp} ”

● *ite*($\phi, \varphi^{\top}, \varphi^{\perp}$) $\stackrel{\text{def}}{=} ((\neg\phi \vee \varphi^{\top}) \wedge (\phi \vee \varphi^{\perp})) \iff ((\phi \wedge \varphi^{\top}) \vee (\neg\phi \wedge \varphi^{\perp}))$

● properties:

ite($\neg\phi, \varphi^{\top}, \varphi^{\perp}$)

= *ite*($\phi, \varphi^{\perp}, \varphi^{\top}$)

\neg *ite*($\phi, \varphi^{\top}, \varphi^{\perp}$)

= *ite*($\phi, \neg\varphi^{\top}, \neg\varphi^{\perp}$)

ite($\phi, \varphi_1^{\top}, \varphi_1^{\perp}$) *op* *ite*($\phi, \varphi_2^{\top}, \varphi_2^{\perp}$) = *ite*($\phi, (\varphi_1^{\top} \text{ op } \varphi_2^{\top}), (\varphi_1^{\perp} \text{ op } \varphi_2^{\perp})$)

ite($\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}$) *op* *ite*($\phi_2, \varphi_2^{\top}, \varphi_2^{\perp}$) = *ite*($\phi_1, (\varphi_1^{\top} \text{ op } \textit{ite}(\phi_2, \varphi_2^{\top}, \varphi_2^{\perp})),$

$(\varphi_1^{\perp} \text{ op } \textit{ite}(\phi_2, \varphi_2^{\top}, \varphi_2^{\perp}))$)

= *ite*($\phi_2, (\textit{ite}(\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}) \text{ op } \varphi_2^{\top}),$

$(\textit{ite}(\phi_1, \varphi_1^{\top}, \varphi_1^{\perp}) \text{ op } \varphi_2^{\perp})$)

op $\in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Recursive structure of an OBDD

Assume the variable ordering A_1, A_2, \dots, A_n :

$$\begin{aligned} \text{OBDD}(\top, \{A_1, A_2, \dots, A_n\}) &= 1 \\ \text{OBDD}(\perp, \{A_1, A_2, \dots, A_n\}) &= 0 \\ \text{OBDD}(\varphi, \{A_1, A_2, \dots, A_n\}) &= \begin{aligned} &\text{if } A_1 \\ &\text{then } \text{OBDD}(\varphi[A_1|\top], \{A_2, \dots, A_n\}) \\ &\text{else } \text{OBDD}(\varphi[A_1|\perp], \{A_2, \dots, A_n\}) \end{aligned} \end{aligned}$$

Incrementally building an OBDD

- $obdd_build(\top, \{\dots\}) := \top$,
- $obdd_build(\perp, \{\dots\}) := \perp$,
- $obdd_build(A_i, \{\dots\}) := ite(A_i, \top, \perp)$,
- $obdd_build((\neg\varphi), \{A_1, \dots, A_n\}) := apply(\neg, obdd_build(\varphi, \{A_1, \dots, A_n\}))$
- $obdd_build((\varphi_1 \text{ op } \varphi_2), \{A_1, \dots, A_n\}) :=$
 $reduce($
 $apply($ $op,$
 $obdd_build(\varphi_1, \{A_1, \dots, A_n\}),$ $op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
 $obdd_build(\varphi_2, \{A_1, \dots, A_n\})$
 $)$
 $)$

Incrementally building an OBDD (cont.)

- $apply(op, O_i, O_j) := (O_i op O_j)$ **if** $(O_i \in \{\top, \perp\}$ or $O_j \in \{\top, \perp\})$
- $apply(\neg, ite(A_i, \varphi_i^\top, \varphi_i^\perp)) :=$
 $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\perp))$
- $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), ite(A_j, \varphi_j^\top, \varphi_j^\perp)) :=$
if $(A_i = A_j)$ **then** $ite(A_i, apply(op, \varphi_i^\top, \varphi_j^\top),$
 $apply(op, \varphi_i^\perp, \varphi_j^\perp))$
if $(A_i < A_j)$ **then** $ite(A_i, apply(op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\perp)),$
 $apply(op, \varphi_i^\perp, ite(A_j, \varphi_j^\top, \varphi_j^\perp)))$
if $(A_i > A_j)$ **then** $ite(A_j, apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\top),$
 $apply(op, ite(A_i, \varphi_i^\top, \varphi_i^\perp), \varphi_j^\perp))$

$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

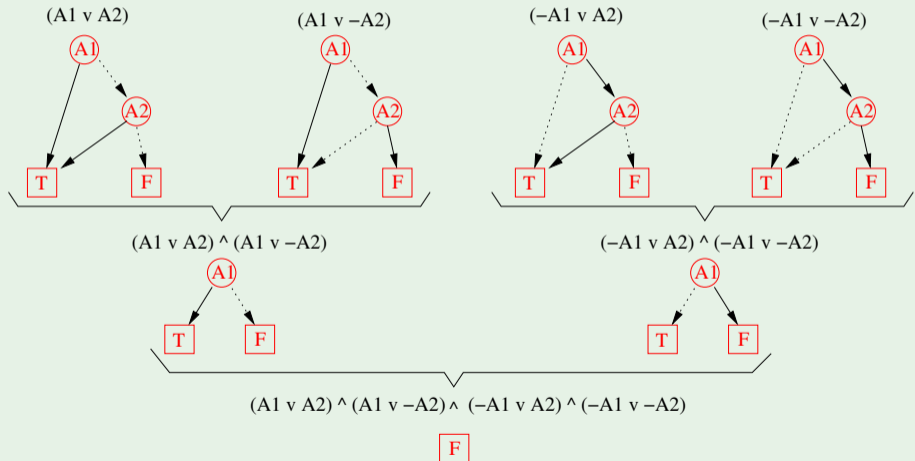
$$\begin{aligned} & \text{apply}(\vee, \overbrace{\text{ite}(A_1, \top, \perp)}^{A_1}, \overbrace{\text{ite}(A_2, \top, \perp)}^{A_2}) \\ &= \text{ite}(A_1, \text{apply}(\vee, \top, \text{ite}(A_2, \top, \perp)), \text{apply}(\vee, \perp, \text{ite}(A_2, \top, \perp))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp)) \end{aligned}$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2), (A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$\begin{aligned} & \text{apply}(\wedge, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \top, \perp))}^{(A_1 \vee A_2)}, \overbrace{\text{ite}(A_1, \top, \text{ite}(A_2, \perp, \top))}^{(A_1 \vee \neg A_2)}), \\ &= \text{ite}(A_1, \text{apply}(\wedge, \top, \top), \text{apply}(\wedge, \text{ite}(A_2, \top, \perp), \text{ite}(A_2, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \text{apply}(\wedge, \top, \perp), \text{apply}(\wedge, \perp, \top))) \\ &= \text{ite}(A_1, \top, \text{ite}(A_2, \perp, \perp)) \\ &= \text{ite}(A_1, \top, \perp) \end{aligned}$$

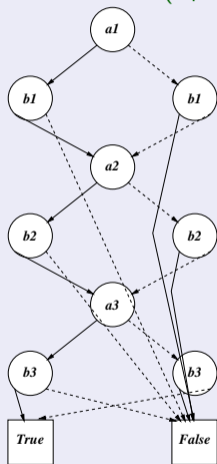
OBDD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

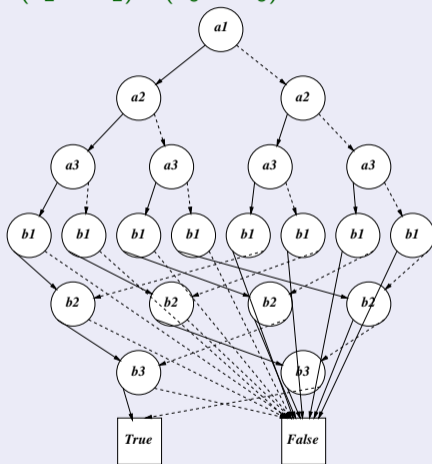


Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$



Linear size



Exponential size

OBDD's as canonical representation of Boolean formulas

- An OBDD is a **canonical representation** of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff \text{OBDD}(\varphi_1) = \text{OBDD}(\varphi_2)$$

- equivalence check requires **constant time!**
 - ⇒ validity check requires constant time! ($\varphi \leftrightarrow \top$)
 - ⇒ (un)satisfiability check requires constant time! ($\varphi \leftrightarrow \perp$)
- the set of the paths from the root to 1 represent all the **models** of the formula
- the set of the paths from the root to 0 represent all the **counter-models** of the formula

Exponentiality of OBDD's

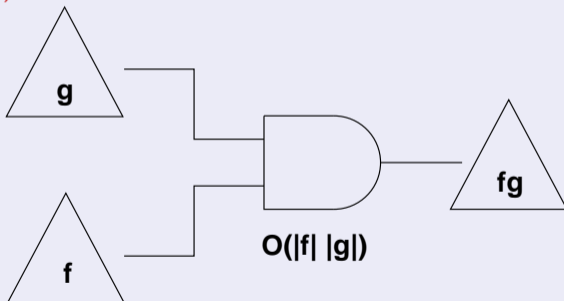
- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless $P = \text{co-NP}$)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

Note

The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

Useful Operations over OBDDs

- the **equivalence check** between two OBDDs is simple
 - are they the same OBDD? (\implies constant time)
- the size of a **Boolean composition** is up to the product of the size of the operands:
 $|f \text{ op } g| = O(|f| \cdot |g|)$



(but typically much smaller on average).

[Recall] Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi := \varphi|_{v=\perp} \vee \varphi|_{v=\top}$$

$$\forall v.\varphi := \varphi|_{v=\perp} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \dots, w_n).\varphi := \exists w_1 \dots \exists w_n.\varphi$

- Intuition:

- $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \perp\}$ s.t. $\mu \cup \{v := \text{truthvalue}\} \models \varphi$
- $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \perp\}$, $\mu \cup \{v := \text{truthvalue}\} \models \varphi$

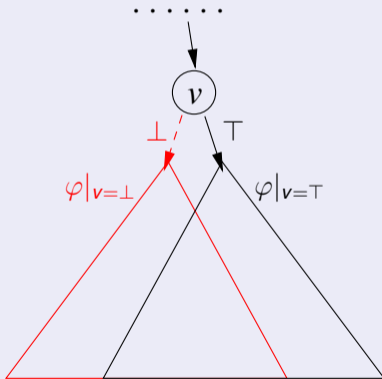
- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) = a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
 - if f is a sub-OBDD labeled by variable v , then $\varphi|_{v=T}$ and $\varphi|_{v=\perp}$ are the “then” and “else” branches of f

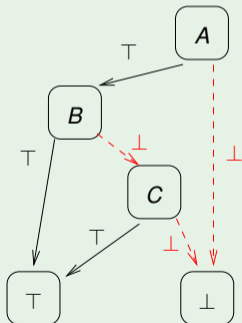


⇒ lots of sharing of subformulae!

Example

Let $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. \varphi$. Using the variable ordering “A, B, C”, draw the OBDD corresponding to the formulas φ and φ' .

$$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$$

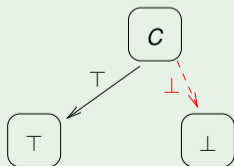


Example (cont.)

$$\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. (A \wedge (B \vee C))$$

$$\begin{aligned} \varphi' &\stackrel{\text{def}}{=} \exists A. \forall B. \varphi \\ &= \forall B. (A \wedge (B \vee C)) [A := \top] && \vee (\forall B. (A \wedge (B \vee C))) [A := \perp] \\ &= \forall B. (B \vee C) && \vee \forall B. \perp \\ &= ((B \vee C) [B := \top] \quad \wedge \quad (B \vee C) [B := \perp]) && \vee \perp \\ &= (\top \quad \wedge \quad C) \\ &= C \end{aligned}$$

which corresponds to the following OBDD:



OBDD – summary

- **Factorize** common parts of the search tree (DAG)
- Require setting a **variable ordering** a priori (**critical!**)
- **Canonical representation** of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents **all** models and counter-models of the formula.
- Require **exponential space** in worst-case
- **Very efficient** for some practical problems (circuits, symbolic model checking).

Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

Outline

- 1 Boolean Logics and SAT
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DPLL: “Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment /
- / is toggled, is labeled as “closed”, and the search proceeds.

DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible

⇒ lots of useless search!

DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

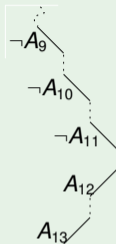
$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

...

DPLL Chronological Backtracking: Example

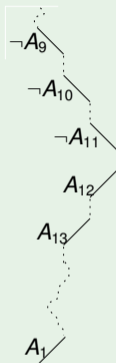
$C_1 : \neg A_1 \vee A_2$
 $C_2 : \neg A_1 \vee A_3 \vee A_9$
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
 $C_6 : \neg A_5 \vee \neg A_6$
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
 $C_8 : A_1 \vee A_8$
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
(initial assignment)

DPLL Chronological Backtracking: Example

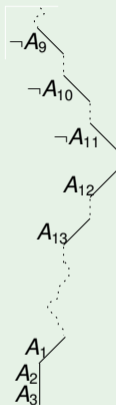
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$
... (branch on A_1)

DPLL Chronological Backtracking: Example

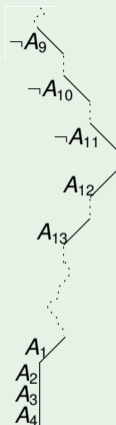
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
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- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$
(unit A_2, A_3)

DPLL Chronological Backtracking: Example

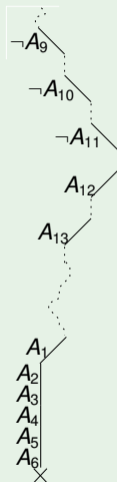
- $C_1 : \neg A_1 \vee A_2 \quad \checkmark$
- $C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4 \quad \checkmark$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$
(unit A_4)

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
 $C_6 : \neg A_5 \vee \neg A_6$ ✗
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
 $C_8 : A_1 \vee A_8$ ✓
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

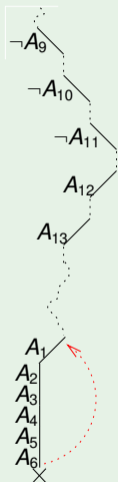


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) \implies conflict

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
 \implies backtrack up to A_1



DPLL Chronological Backtracking: Example

$$C_1 : \neg A_1 \vee A_2 \quad \checkmark$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9 \quad \checkmark$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

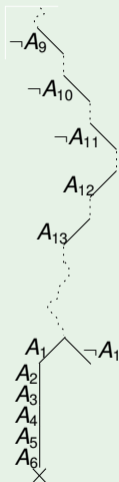
$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$$

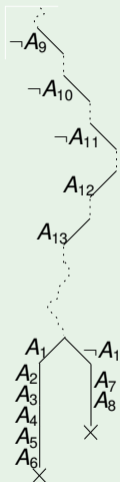
...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1\}$
(unit $\neg A_1$)



DPLL Chronological Backtracking: Example

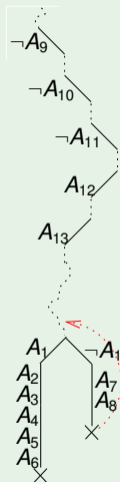
- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
 $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
 $C_6 : \neg A_5 \vee \neg A_6$
 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
 $C_8 : A_1 \vee A_8$ ✓
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✗
...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1, A_7, A_8\}$
(unit A_7, A_8) \implies conflict

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
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- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

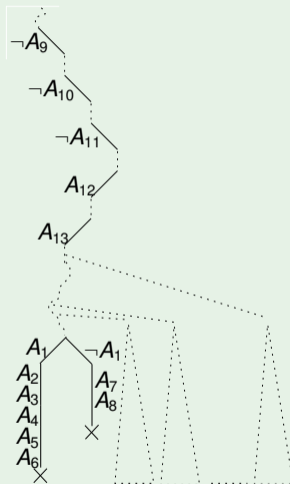
\implies backtrack to the most recent open branching point

DPLL Chronological Backtracking: Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

\Rightarrow lots of useless search before backtracking up to A_{13} !



Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 **Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - **Conflict-Driven Clause-Learning SAT solvers**
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

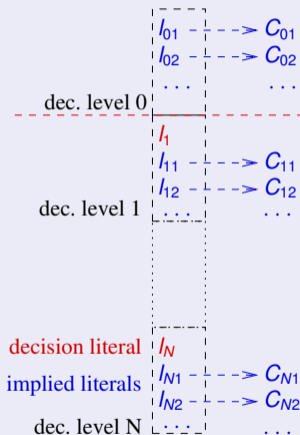
Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
 - inspired to conflict-driven backjumping and learning in CSPs
 - learns implied clauses as nogoods
- Random restarts
 - abandon the current search tree and restart on top level
 - previously-learned clauses maintained
- Smart literal selection heuristics (ex: VSIDS)
 - “static”: scores updated only at the end of a branch
 - “local”: privileges variable in recently learned clauses
- Smart preprocessing/inprocessing technique to simplify formulas
- Smart indexing techniques (e.g. 2-watched literals)
 - efficiently do/undo assignments and reveal unit clauses
- Allow Incremental Calls (stack-based interface)
 - allow for reusing previous search on “similar” problems

Can handle industrial problems with 10^6 – 10^7 variables and clauses!

Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into **decision levels**:
 - one **decision literal**
 - its **implied literals**
 - each implied literal tagged with the clause causing its unit-propagation (**antecedent clause**)
- equivalent to an **implication graph**

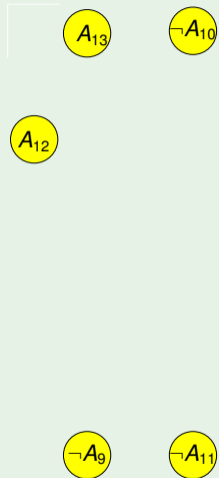
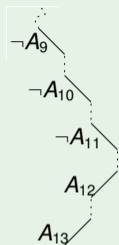


Implication graph

- An **implication graph** is a DAG s.t.:
 - each node represents a variable assignment (literal)
 - each edge $l_i \xrightarrow{c} l$ is labeled with a clause
 - the node of a decision literal has no incoming edges
 - all edges incoming into a node l are labeled with the same clause c , s.t. $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$ iff $c = \neg l_1 \vee \dots \vee \neg l_n \vee l$
(c is said to be the **antecedent clause** of l)
 - when both l and $\neg l$ occur in the graph, we have a **conflict**.
- Intuition:
 - representation of the dependencies between literals in μ
 - the graph contains $l_1 \xrightarrow{c} l, \dots, l_n \xrightarrow{c} l$ iff l has been obtained from l_1, \dots, l_n by unit propagation on c
 - a partition of the graph with all decision literals on one side and the conflict on the other represents a **conflict set**

Example

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...

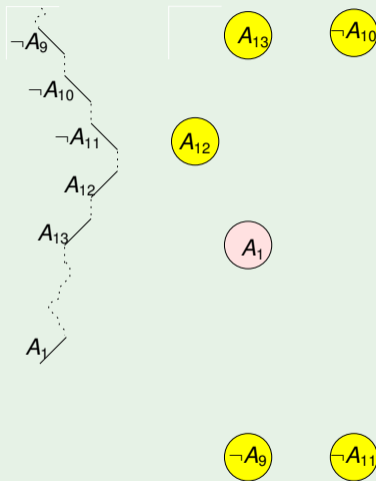


$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$

(Initial assignment. Note: c_1, \dots, c_9 inconsistent.)

Example

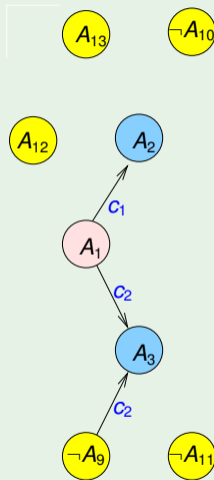
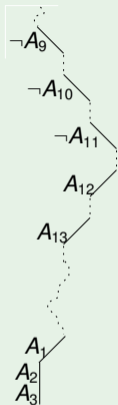
- $C_1 : \neg A_1 \vee A_2$
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- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12} \quad \checkmark$
- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1\}$
... (decide A_1)

Example

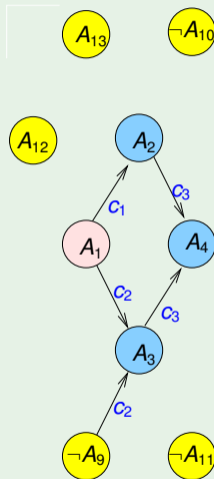
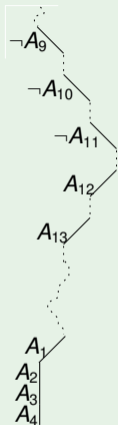
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- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3\}$
(unit A_2, A_3)

Example

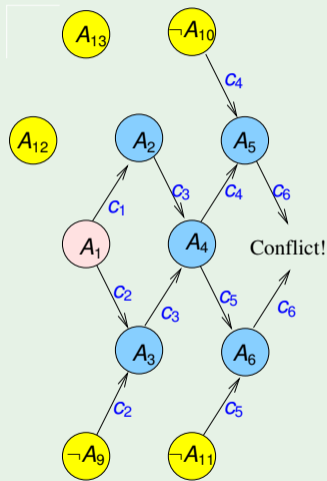
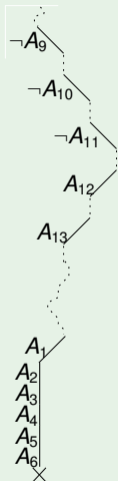
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- $C_8 : A_1 \vee A_8 \quad \checkmark$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4\}$
(unit A_4)

Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, A_1, A_2, A_3, A_4, A_5, A_6\}$
 (unit A_5, A_6) \implies conflict

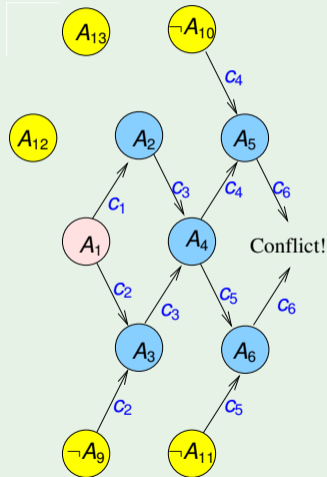
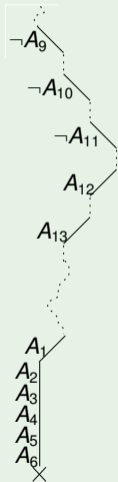
Unique implication point - UIP [44]

- A node l in an implication graph is an **unique implication point** (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through l .
 - the most recent decision node is an UIP (**last UIP**)
 - all other UIP's have been assigned after the most recent decision

Unique implication point - UIP - example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...

- A_1 is the last UIP
- A_4 is the 1st UIP



Schema of a CDCL DPLL solver [38, 45]

```
Function CDCL-SAT (formula:  $\varphi$ , assignment &  $\mu$ ) {
  status := preprocess( $\varphi, \mu$ );
  while (1) {
    while (1) {
      status := deduce( $\varphi, \mu$ );
      if (status == Sat)
        return Sat;
      if (status == Conflict) {
         $\langle \text{blevel}, \eta \rangle := \text{analyze\_conflict}(\varphi, \mu)$ ;
        //  $\eta$  is a conflict set
        if (blevel == 0)
          return Unsat;
        else backtrack(blevel,  $\varphi, \mu$ );
      }
      else break;
    }
    decide_next_branch( $\varphi, \mu$ );
  }
}
```

Schema of a CDCL DPLL solver [38, 45] (cont.)

- `preprocess(φ, μ)` simplifies φ into an easier equisatisfiable formula, updating μ .
- `decide_next_branch(φ, μ)` chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce(φ, μ)` performs all deterministic assignments (unit-propagations plus others), and updates φ, μ accordingly.
- `analyze_conflict(φ, μ)` Computes the subset η of μ causing the conflict (conflict set), and returns the “wrong-decision” level suggested by η (“0” means that η is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack($blevel, \varphi, \mu$)` undoes the branches up to $blevel$, and updates φ, μ accordingly

Backjumping and learning: general ideas [2, 38]

- When a branch μ fails:
 - (i) **conflict analysis**: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (**conflict set** η)
 - (ii) **learning**: add the **conflict clause** $C \stackrel{\text{def}}{=} \neg\eta$ to the clause set
 - (iii) **backjumping**: use η to decide the point where to backtrack
- Jump back up much more than one decision level in the stack
 \implies **may avoid lots of redundant search!!**.
- We illustrate two main backjumping & learning strategies:
 - the original strategy presented in [38]
 - the state-of-the-art 1stUIP strategy of [44]

Conflict analysis

1. $C :=$ falsified clause (**conflicting clause**)
2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in Cuntil C verifies some given termination criteria

Conflict analysis

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2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C
 until C verifies some given termination criteria

criterion: **1st UIP**

... until C contains only one literal assigned at current decision level (**1st UIP**)

$$\begin{array}{c}
 \text{Conflicting cl.} \\
 \overline{\neg A_4 \vee A_6 \vee A_{11}} \quad \overline{\neg A_5 \vee \neg A_6} \quad (A_6) \\
 \hline
 \neg A_4 \vee A_5 \vee A_{10} \quad \neg A_4 \vee \neg A_5 \vee A_{11} \quad (A_5) \\
 \hline
 \underbrace{\neg A_4}_{\text{1st UIP}} \vee A_{10} \vee A_{11}
 \end{array}$$

Conflict analysis

1. $C :=$ falsified clause (**conflicting clause**)
2. repeat
 - (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in Cuntil C verifies some given termination criteria

Note:

$\varphi \models C$, so that C can be safely added to φ .

Note:

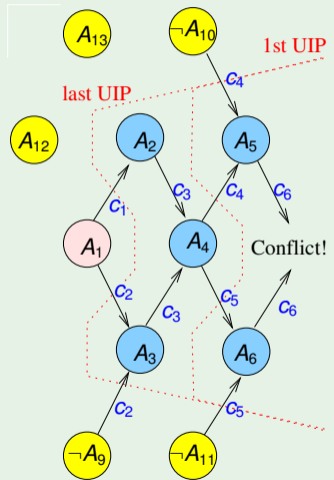
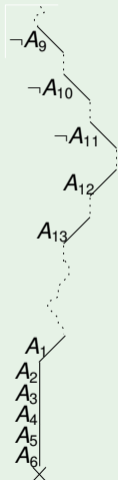
Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis and implication graph - example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...

Note: in

this case decision and last-UIP criteria produce the same partition

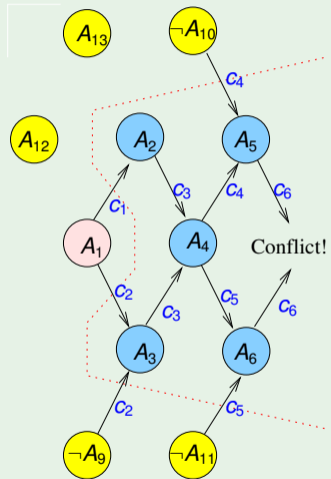
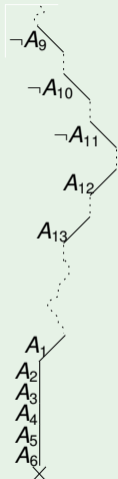


The original backjumping and learning strategy of [38]

- Idea: when a branch μ fails,
 - (i) **conflict analysis**: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the “Decision” criterion;
 - (ii) **learning**: add the conflict clause C to the clause set
 - (iii) **backjumping**: backtrack to the most recent branching point s.t. the stack does not fully contain η , and then unit-propagate the unassigned literal on C

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
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- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...

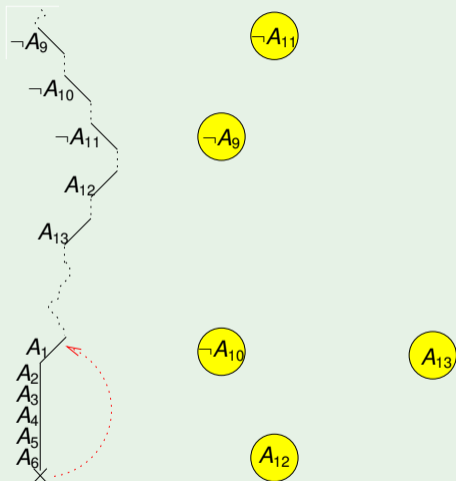


- ⇒ Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ ("decision" schema)
- ⇒ learn the conflict clause $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$
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- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
- ...

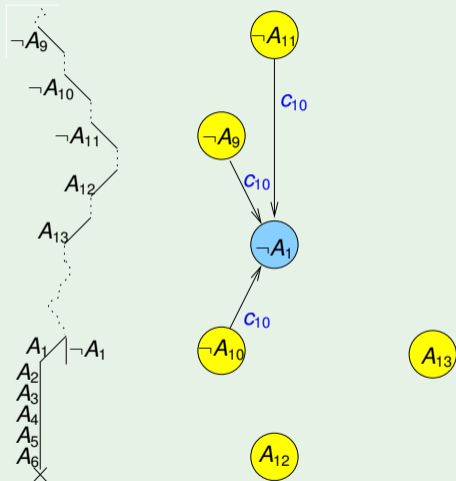
$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots\}$
 \Rightarrow backtrack up to A_1



The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
 $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
 $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
 $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
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 $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
 $C_8 : A_1 \vee A_8$
 $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
 $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
...

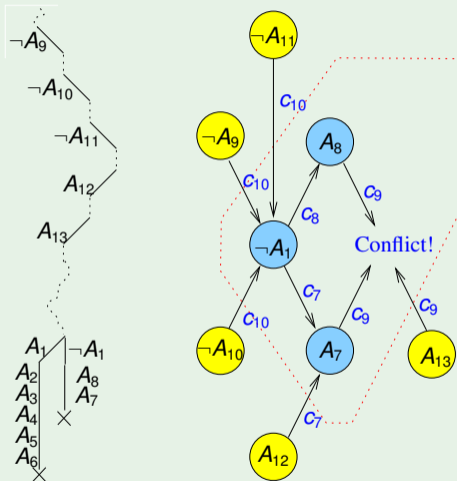
$\{\dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \dots, \neg A_1\}$
(unit $\neg A_1$)



The Original Backjumping Strategy: Example

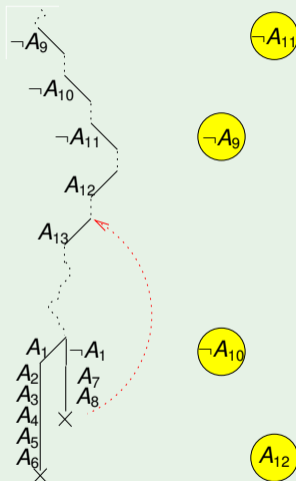
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- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✗
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- ...

\Rightarrow conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.
 \Rightarrow learn $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$



The Original Backjumping Strategy: Example

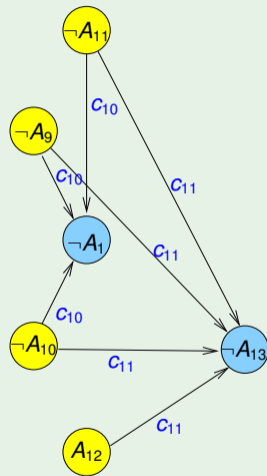
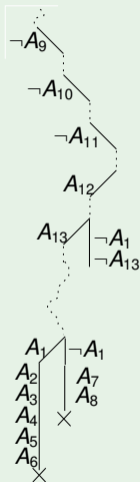
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- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$
- ...



⇒ backtrack to A_{13} ⇒ Lots of search saved!

The Original Backjumping Strategy: Example

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- $C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓
- $C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ ✓
- ...



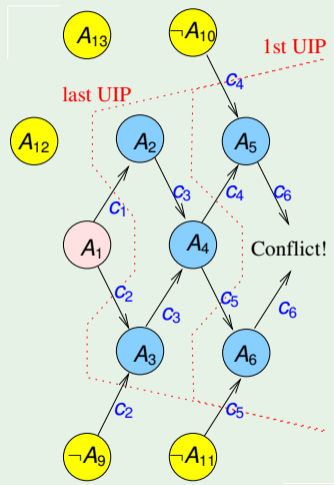
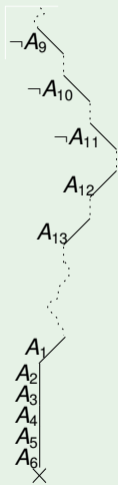
⇒ backtrack to A_{13} , then set A_{13} and A_1 to \perp, \dots

State-of-the-art backjumping and learning [44]

- Idea: when a branch μ fails,
 - (i) **conflict analysis**: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the **1stUIP strategy**
 - (ii) **learning**: add the conflict clause C to the clause set
 - (iii) **backjumping**: **backtrack to the highest branching point s.t. the stack contains all-but-one literals in η , and then unit-propagate the unassigned literal on C**

1st UIP strategy – example (7)

- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
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- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



⇒ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping [44]

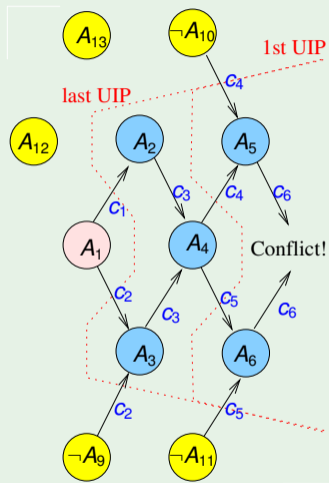
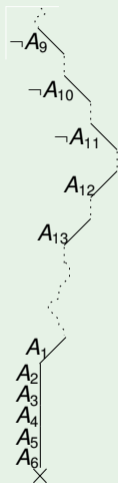
- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

\implies backtrack to A_{11} , then assign $\neg A_4$

1st UIP strategy – example (7)

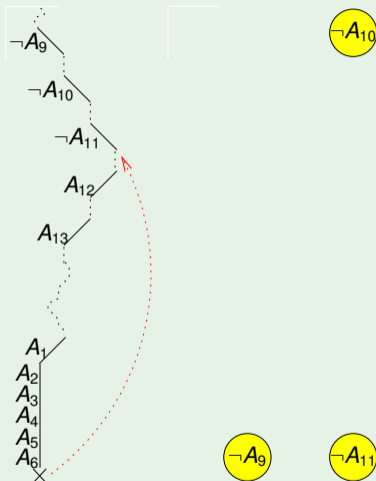
- $C_1 : \neg A_1 \vee A_2$ ✓
- $C_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
- $C_6 : \neg A_5 \vee \neg A_6$ ✗
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
- $C_8 : A_1 \vee A_8$ ✓
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓
- ...



⇒ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy – example (8)

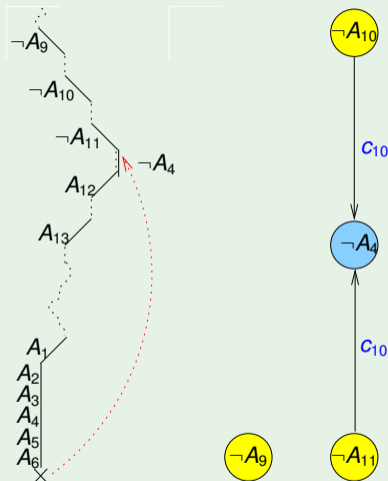
- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10}$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11}$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4$
- ...



\Rightarrow backtrack up to $A_{11} \Rightarrow \{\dots, \neg A_9, \neg A_{10}, \neg A_{11}\}$

1st UIP strategy – example (9)

- $C_1 : \neg A_1 \vee A_2$
- $C_2 : \neg A_1 \vee A_3 \vee A_9$
- $C_3 : \neg A_2 \vee \neg A_3 \vee A_4$
- $C_4 : \neg A_4 \vee A_5 \vee A_{10} \quad \checkmark$
- $C_5 : \neg A_4 \vee A_6 \vee A_{11} \quad \checkmark$
- $C_6 : \neg A_5 \vee \neg A_6$
- $C_7 : A_1 \vee A_7 \vee \neg A_{12}$
- $C_8 : A_1 \vee A_8$
- $C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
- $C_{10} : A_{10} \vee A_{11} \vee \neg A_4 \quad \checkmark$
- ...



\Rightarrow unit propagate $\neg A_4 \Rightarrow \{ \dots, \neg A_9, \neg A_{10}, \neg A_{11}, A_4 \} \dots$

1st UIP strategy and backjumping – intuition

- An UIP is a **single** reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
 - does not enlarge the conflict
 - requires less resolution steps to compute C
 - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
 - jump higher
 - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [2, 38]

Idea: When a conflict set η is revealed, then $C \stackrel{\text{def}}{=} \neg\eta$ is added to φ

\implies the solver will no more generate an assignment containing η :

when $|\eta| - 1$ literals in η are assigned, the other is set \perp by unit-propagation on C

\implies **Drastic pruning of the search!**

Learning – example

$$C_1 : \neg A_1 \vee A_2$$

$$C_2 : \neg A_1 \vee A_3 \vee A_9$$

$$C_3 : \neg A_2 \vee \neg A_3 \vee A_4$$

$$C_4 : \neg A_4 \vee A_5 \vee A_{10}$$

$$C_5 : \neg A_4 \vee A_6 \vee A_{11}$$

$$C_6 : \neg A_5 \vee \neg A_6$$

$$C_7 : A_1 \vee A_7 \vee \neg A_{12}$$

$$C_8 : A_1 \vee A_8$$

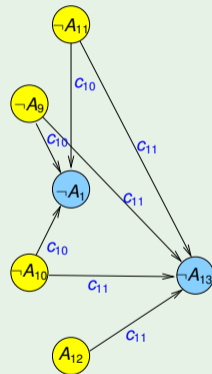
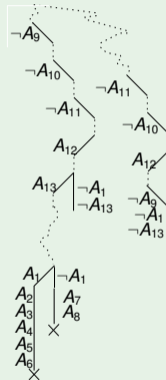
$$C_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13} \quad \checkmark$$

$$C_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1 \quad \checkmark$$

$$C_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13} \quad \checkmark$$

...

\Rightarrow Unit: $\{\neg A_1, \neg A_{13}\}$



Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their **activity**.

A clause is currently **active** if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [30])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.

⇒ CDCL solvers require polynomial space

“Lazy” Strategy

- when a clause is involved in conflict analysis, increase its activity
- when needed, drop the least-active clauses

State-of-the-art backjumping and learning: intuitions

- **Backjumping:** allows for climbing up to many decision levels in the stack
 - intuition: “go back to the oldest decision where you’d have done something different if only you had known C ”
⇒ may avoid lots of redundant search
- **Learning:** in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
 - intuition: “when you’re about to repeat the mistake, do the opposite of the last step”
⇒ avoid finding the same conflict again

Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
 - Backjumping: if causes the highest backjump (“local” role)
 - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented (see, e.g., [2, 38, 44])

Outline

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- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers**
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements**
 - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization

Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
 - detect and remove subsumed clauses
 - detect & collapse equivalent literals
 - apply basic resolution steps
 - ...

Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\begin{aligned} \varphi_1 \wedge (b_2 \vee h_1) \wedge \varphi_2 \wedge (b_2 \vee b_3 \vee h_1) \wedge \varphi_3 \\ \Downarrow \\ \varphi_1 \wedge (h_1 \vee b_2) \wedge \varphi_2 \wedge \varphi_3 \end{aligned}$$

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [7]

Repeat:

- (i) build the implication graph induced by binary clauses
- (ii) detect **strongly connected cycles** \implies **equivalence classes of literals**
- (iii) perform substitutions
- (iv) perform unit and pure literal.

Until (no more simplification is possible).

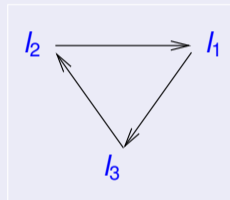
- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$

$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.



Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$

\Downarrow *resolve*

$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$

\Downarrow *unit-propagate*

$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- **Branch** is the source of non-determinism for DPLL
⇒ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

Some Heuristics

- **MOMS** heuristics (DPLL): pick the literal occurring **most** often in the **minimal size** clauses
⇒ fast and simple, many variants
- **Jeroslow-Wang** (DPLL): choose the literal with maximum

$$\text{score}(l) := \sum_{l \in c \ \& \ c \in \varphi} 2^{-|c|}$$

⇒ estimates l 's contribution to the satisfiability of φ

- **Satz** [21] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
⇒ maximizes the effects of unit propagation
- **VSIDS** [28] (CDCL+): **v**ariable **s**tate **i**ndependent **d**ecaying **s**um
 - “static”: scores updated only at the end of a branch
 - “local”: privileges variable in recently learned clauses

Restarts [16]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
 - The clauses learned prior to the restart are still there after the restart and can help pruning the search space
 - avoid getting stuck in certain areas of the search space
- ⇒ may significantly reduce the overall search space

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SAT under assumptions: $SAT(\varphi, \{l_1, \dots, l_n\})$ [12]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, \dots, l_n\}$: $SAT(\varphi, \{l_1, \dots, l_n\})$
 - $SAT(\varphi, \{l_1, \dots, l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^n l_i)$
 - often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2), \dots$
- Idea:
 - l_1, \dots, l_n “decided” at decision level 0 before starting the search
 - if backjump to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT

Property

If the “decision” strategy for conflict analysis is used,
then η is the subset of assumptions causing the inconsistency

Selection of sub-formulas

Idea: select clauses [12, 23]

Let φ be $\bigwedge_{i=1}^n C_i$.

- let $S_1 \dots S_n$ be fresh Boolean atoms (selection variables).

- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$

\Rightarrow $\text{SAT}(\bigwedge_{i=1}^n (\neg S_i \vee C_i), \mathcal{A})$: same as $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (C_i))$

- if S_i is not assumed, then $\neg S_i \vee C_i$ does not contribute to search

\Rightarrow “Select” (activate) only a subset of the clauses in φ at each call.

Generalised Idea: select blocks of clauses

Let φ be $\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} C_{ij})$.

- let $S_1 \dots S_n$ be fresh Boolean atoms (selection variables).

- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, \dots, S_{i_k}\} \subseteq \{S_1, \dots, S_n\}$

- $\text{SAT}(\bigwedge_{i=1}^n (\bigwedge_{j=1}^{n_i} (\neg S_i \vee C_{ij})), \mathcal{A})$: same as $\text{SAT}(\bigwedge_{i=i_1}^{i_k} (\bigwedge_{j=1}^{n_i} C_{ij}))$

\Rightarrow Allows for “selecting” block of clauses at each call.

Example

- Initial formula φ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Example

- Initial formula φ :

$(A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1

$(\neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1

$(\neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2

$(A_3 \vee A_5 \vee \neg A_8) \wedge$ // group 2

$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- Augmented formula φ' :

$(\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge$ // group 1, inactive

$(\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge$ // group 1, inactive

$(\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge$ // group 2, inactive

$(\neg S_2 \vee A_2 \vee A_5 \vee \neg A_8) \wedge$ // group 2, inactive

$(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge$ // group 3

- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3

- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Incremental SAT solving [12, 11]

- Many CDCL solvers provide a **stack-based incremental interface**
 - it is possible to push/pop ϕ_i into a stack of subformulas $\{\phi_1, \dots, \phi_k\}$
 - check incrementally the satisfiability of $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^k \phi_i$.
 - Maintains the **status** of the search from one call to the other
 - in particular it records the **learned clauses** (plus other information)
 - \Rightarrow **reuses search from one call to another**
 - Very useful in many applications (in particular in FV)
-
- Idea: **incremental** calls **$SAT(\varphi', \mathcal{A}_1), SAT(\varphi', \mathcal{A}_2), \dots$**
 - $\varphi' \stackrel{\text{def}}{=} \bigwedge_i (\neg S_i \vee \phi_i)$, $\mathcal{A}_j \subseteq \{S_1, \dots, S_k\}$, $(\neg S_i \vee \bigwedge_j C_{ij}) \stackrel{\text{def}}{=} \bigwedge_j (\neg S_i \vee C_{ij})$
 - push/pop selection variables S_i
 - in practice, also subformulas ϕ_i can be pushed/popped
 - Key efficiency issue: **learned clauses safely reused from call to call (even if assumptions have been popped)**
 - a learned clause $C \stackrel{\text{def}}{=} \bigvee_j \neg S_j \vee C'$ is s.t. $\bigwedge_j (\neg S_j \vee \phi_j) \models C$
 - \Rightarrow C contains the vars selecting the subformulas it is derived from
 - \Rightarrow in $SAT(\varphi', \mathcal{A})$, if some $S_j \notin \mathcal{A}$, then C is not active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

[push(S_1)]: $SAT(\varphi', \{\dots, S_1\})$: ϕ_1 active \implies learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

$$(\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1$$

[push(S_1)]: $SAT(\varphi', \{\dots, S_1\})$: ϕ_1 active \implies learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \\ (\neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \end{array}$$

[push(S_2): SAT(φ' , { \dots , S_1 , S_2 })]: ϕ_1, ϕ_2 active \implies learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \\ (\neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \end{array}$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2 \text{ inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2 \text{ inactive} \\ \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \text{learned } C_2, \text{ inactive} \end{array}$$

[push(S_2): SAT(φ' , $\{\dots, S_1, S_2\}$): ϕ_1, ϕ_2 active \implies learn C_2 from ϕ_1, ϕ_2

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
- C_2 derived from $\phi_1, \phi_2 \implies C_2$ active only when both ϕ_1, ϕ_2 are active

Example

- Initial formula φ :

$$\begin{array}{l} \dots \\ (A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \end{array}$$

$$(\neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3$$

- Augmented formula φ' :

$$\begin{array}{l} \dots \\ (\neg S_1 \vee A_1 \vee \neg A_2 \vee \neg A_3) \wedge // \phi_1 \\ (\neg S_1 \vee \neg A_3 \vee A_2 \vee \neg A_5) \wedge // \phi_1 \\ (\neg S_2 \vee \neg A_2 \vee A_5 \vee A_7) \wedge // \phi_2, \textit{inactive} \\ (\neg S_2 \vee \neg A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \phi_2, \textit{inactive} \\ (\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8) \wedge // \phi_3 \\ (\neg S_1 \vee A_1 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_1 \\ (\neg S_1 \vee \neg S_2 \vee \neg A_3 \vee \neg A_5) \wedge // \textit{learned } C_2, \textit{inactive} \end{array}$$

[pop(S_2);push(S_3): SAT(φ' , { \dots , S_1 , S_3 })]: ϕ_1, ϕ_3 active $\implies \dots$

- C_1 derived from $\phi_1 \implies C_1$ active only when ϕ_1 is active
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Example

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[pop(S_2);push(S_3): SAT(φ' , { \dots , S_1 , S_3 })]: ϕ_1, ϕ_3 active $\implies \dots$

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Outline

- 1 Boolean Logics and SAT
- 2 Basic SAT-Solving Techniques
 - Generalities
 - Resolution
 - Tableaux
 - DPLL
- 3 Ordered Binary Decision Diagrams – OBDDs
- 4 Modern CDCL SAT Solvers
 - Limitations of Chronological Backtracking
 - Conflict-Driven Clause-Learning SAT solvers
 - Further Improvements
 - SAT Under Assumptions & Incremental SAT
- 5 SAT Functionalities: proofs, unsat cores, optimization**

Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

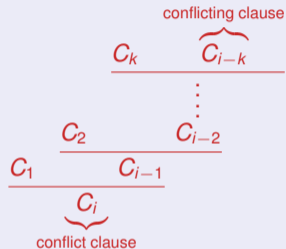
- Building **proofs of unsatisfiability**
- Extracting **unsatisfiable Cores**
- Enumeration in SAT: **AlISAT** (hints)
- Optimization in SAT: **MaxSAT** (hints)

Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
 - to verify the result of the solver
 - to understand a “reason” for unsatisfiability
 - to build unsatisfiable cores and interpolants
- Can be built by **keeping track of the resolution steps performed when constructing the conflict clauses.**

Building Proofs of Unsatisfiability

- Recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal



- C_1, \dots, C_k , and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

$k \quad i-k \quad \rightarrow \quad i-k-1$

\dots

$2 \quad i-2 \quad \rightarrow \quad i-1$

$1 \quad i-1 \quad \rightarrow \quad i$

Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces “false” as conflict clause:

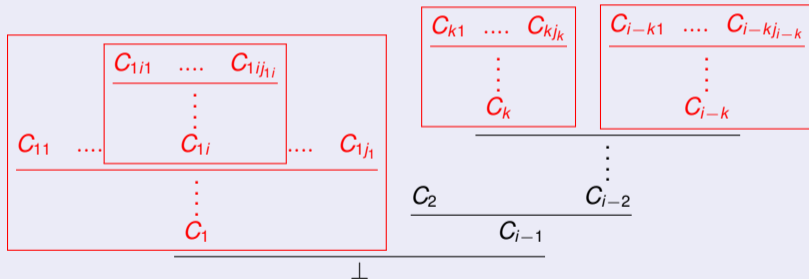
$$\begin{array}{c} \text{conflicting clause} \\ C_k \quad \overbrace{C_{i-k}} \\ \hline \vdots \\ C_2 \quad C_{i-2} \\ \hline C_1 \quad C_{i-1} \\ \hline \perp \end{array}$$

- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- C_1, \dots, C_k , and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

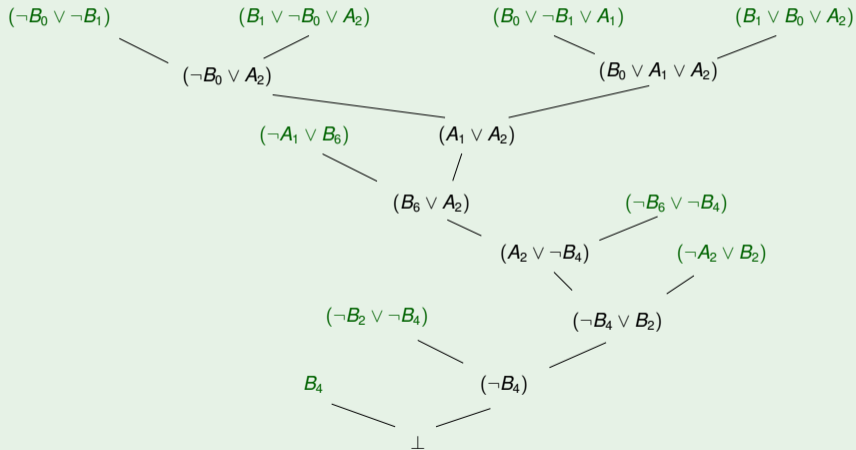
- for every **learned** leaf clause C_i , substitute C_i with the resolution proof generating it until all leaf clauses are original clauses



\Rightarrow We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$
 $(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$



Extraction of unsatisfiable cores

- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset
 - ⇒ **unsatisfiable cores** (aka **(Minimal) Unsatisfiable Subsets, (M)US**)
- Lots of literature on the topic [46, 24, 26, 31, 43, 19, 13, 6]
- We recognize two main approaches:
 - **Proof-based** approach [46]: byproduct of finding a resolution proof
 - **Assumption-based** approach [24]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
 - repeat the process up to fixpoint
 - remove clauses one-by one, until satisfiability is obtained
 - combinations of the two processed above
 - ...

The assumption-based approach to core extraction [24]

Based on the following process:

- (i) each clause C_i is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh “selector” variable
- (ii) before starting the search each S_i is forced to true.
- (iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$
 $\implies \{C_j\}_j$ is the unsat core!

The assumption-based approach to core extraction

Example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge \\ B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$

(i) add selector variables:

$$\begin{aligned} & (\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge \\ & (\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge \\ & (\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge \\ & (\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7) \end{aligned}$$

(ii) The conflict analysis returns: $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$,

(iii) corresponding to the unsat core:

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge \\ (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge \\ B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$$

All-SAT (hints)

All-SAT & Projected All-SAT

- **All-SAT**: enumerate all truth assignments satisfying φ
- **Projected All-SAT**: given an “important” subset of atoms $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$, enumerate all assignments over \mathbf{P} which can be extended to truth assignments satisfying φ
- Algorithms
 - **BCLT** [Lahiri et al, CAV'06]:
each time a satisfiable assignment $\{l_1, \dots, l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$ belonged to the clause set
 - **MathSAT/NuSMV** [Cavada et al, FMCAD'07]:
As above, plus the Boolean search of the SAT solver is driven by an OBDD.

MaxSAT (hints)

- **MaxSAT**: given a pair of CNF formulas $\langle \varphi_h, \varphi_s \rangle$ s.t. $\varphi_h \wedge \varphi_s \models \perp$, $\varphi_s \stackrel{\text{def}}{=} \{C_1, \dots, C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s .
- **Weighted MaxSAT**: given also the positive integer **penalties** $\{w_1, \dots, w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to **optimization**
 \implies much harder than SAT
- Many different approaches (see e.g. [22])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \quad \varphi_s \stackrel{\text{def}}{=} \left(\begin{array}{l} (A_1 \vee \neg A_2) \wedge [4] \\ (\neg A_1 \vee A_2) \wedge [3] \\ (\neg A_1 \vee \neg A_2) \wedge [2] \end{array} \right)$$

$\implies \mu = \{A_1, A_2\}$ (penalty = 2)

References I



A. Armando and E. Giunchiglia.

Embedding Complex Decision Procedures inside an Interactive Theorem Prover.
Annals of Mathematics and Artificial Intelligence, 8(3–4):475–502, 1993.



R. J. Bayardo, Jr. and R. C. Schrag.

Using CSP Look-Back Techniques to Solve Real-World SAT instances.
In *Proc. AAAI'97*, pages 203–208. AAAI Press, 1997.



A. Belov and Z. Stachniak.

Improving variable selection process in stochastic local search for propositional satisfiability.
In *SAT'09*, LNCS. Springer, 2009.



A. Belov and Z. Stachniak.

Improved local search for circuit satisfiability.
In *SAT*, volume 6175 of *LNCS*, pages 293–299. Springer, 2010.



A. Biere, M. J. H. Heule, H. van Maaren, and T. Walsh, editors.

Handbook of Satisfiability.
IOS Press, February 2009.



Booleforce, <http://fmv.jku.at/booleforce/>.



R. Brafman.

A simplifier for propositional formulas with many binary clauses.
In *Proc. IJCAI'01*, 2001.



R. E. Bryant.

Graph-Based Algorithms for Boolean Function Manipulation.
IEEE Transactions on Computers, C-35(8):677–691, Aug. 1986.

References II



M. Davis, G. Longemann, and D. Loveland.

A machine program for theorem proving.
Journal of the ACM, 5(7), 1962.



M. Davis and H. Putnam.

A computing procedure for quantification theory.
Journal of the ACM, 7:201–215, 1960.



N. Eén and N. Sörensson.

Temporal induction by incremental sat solving.
Electr. Notes Theor. Comput. Sci., 89(4):543–560, 2003.



N. Eén and N. Sörensson.

An extensible SAT-solver.
In *Theory and Applications of Satisfiability Testing (SAT 2003)*, volume 2919 of *LNCS*, pages 502–518. Springer, 2004.



R. Gershman, M. Koifman, and O. Strichman.

Deriving Small Unsatisfiable Cores with Dominators.
In *Proc. CAV'06*, volume 4144 of *LNCS*. Springer, 2006.



E. Giunchiglia, M. Narizzano, A. Tacchella, and M. Vardi.

Towards an Efficient Library for SAT: a Manifesto.
In *Proc. SAT 2001*, *Electronics Notes in Discrete Mathematics*. Elsevier Science., 2001.



E. Giunchiglia and R. Sebastiani.

Applying the Davis-Putnam procedure to non-clausal formulas.
In *Proc. AI'IA'99*, volume 1792 of *LNAI*. Springer, 1999.

References III



C. Gomes, B. Selman, and H. Kautz.

Boosting Combinatorial Search Through Randomization.
In *Proceedings of the Fifteenth National Conference on Artificial Intelligence*, 1998.



C. P. Gomes, A. Sabharwal, and B. Selman.

Model Counting, chapter 20, pages 633–654.
In Biere et al. [5], February 2009.



H. H. Hoos and T. Stutzle.

Stochastic Local Search Foundation And Application.
Morgan Kaufmann, 2005.



J. Huang.

MUP: a minimal unsatisfiability prover.
In *Proc. ASP-DAC '05*. ACM Press, 2005.



H. A. Kautz, A. Sabharwal, and B. Selman.

Incomplete Algorithms, chapter 6, pages 185–203.
In Biere et al. [5], February 2009.



C. M. Li and Anbulagan.

Heuristics based on unit propagation for satisfiability problems.
In *Proceedings of the 15th International Joint Conference on Artificial Intelligence (IJCAI-97)*, pages 366–371, 1997.



C. M. Li and F. Manyà.

MaxSAT, Hard and Soft Constraints, chapter 19, pages 613–631.
In Biere et al. [5], February 2009.

References IV



I. Lynce and J. Marques-Silva.
On Computing Minimum Unsatisfiable Cores.
In *7th International Conference on Theory and Applications of Satisfiability Testing*, 2004.



I. Lynce and J. P. Marques-Silva.
On computing minimum unsatisfiable cores.
In *SAT*, 2004.



K. McMillan.
Interpolation and SAT-based model checking.
In *Proc. CAV*, 2003.



K. McMillan and N. Amla.
Automatic abstraction without counterexamples.
In *Proc. of TACAS*, 2003.



K. L. McMillan.
An interpolating theorem prover.
Theor. Comput. Sci., 345(1):101–121, 2005.



M. W. Moskewicz, C. F. Madigan, Y. Z., L. Zhang, and S. Malik.
Chaff: Engineering an efficient SAT solver.
In *Design Automation Conference*, 2001.



R. Nieuwenhuis, A. Oliveras, and C. Tinelli.
Abstract DPLL and abstract DPLL modulo theories.
In F. Baader and A. Voronkov, editors, *Proceedings of the 11th International Conference on Logic for Programming, Artificial Intelligence and Reasoning (LPAR'04)*, Montevideo, Uruguay, volume 3452 of *LNCS*, pages 36–50. Springer, 2005.

References V



R. Nieuwenhuis, A. Oliveras, and C. Tinelli.

Solving SAT and SAT Modulo Theories: from an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).
Journal of the ACM, 53(6):937–977, November 2006.



Y. Oh, M. N. Mneimneh, Z. S. Andraus, K. A. Sakallah, and I. L. Markov.

Amuse: A Minimally-Unsatisfiable Subformula Extractor.
In *Proc. DAC'04. ACM/IEEE*, 2004.



P. Pudlák.

Lower bounds for resolution and cutting planes proofs and monotone computations.
J. of Symb. Logic, 62(3), 1997.



A. Robinson.

A machine-oriented logic based on the resolution principle.
Journal of the ACM, 12:23–41, 1965.



R. Sebastiani.

Applying GSAT to Non-Clausal Formulas.
Journal of Artificial Intelligence Research, 1:309–314, 1994.



B. Selman and H. Kautz.

Domain-Independent Extension to GSAT: Solving Large Structured Satisfiability Problems.
In *Proc. of the 13th International Joint Conference on Artificial Intelligence*, pages 290–295, 1993.



B. Selman, H. Kautz, and B. Cohen.

Local Search Strategies for Satisfiability Testing.
In *Cliques, Coloring, and Satisfiability*, volume 26 of *DIMACS*, pages 521–532, 1996.

References VI



B. Selman, H. Levesque., and D. Mitchell.

A New Method for Solving Hard Satisfiability Problems.

In *Proc. of the 10th National Conference on Artificial Intelligence*, pages 440–446, 1992.



J. P. M. Silva and K. A. Sakallah.

GRASP - A new Search Algorithm for Satisfiability.

In *Proc. ICCAD'96*, 1996.



R. M. Smullyan.

First-Order Logic.

Springer-Verlag, NY, 1968.



C. Tinelli.

A DPLL-based Calculus for Ground Satisfiability Modulo Theories.

In *Proc. JELIA-02*, volume 2424 of *LNAI*, pages 308–319. Springer, 2002.



D. Tompkins and H. Hoos.

UBCSAT: An Implementation and Experimentation Environment for SLS Algorithms for SAT and MAX-SAT.

In *SAT*, volume 3542 of *LNCS*. Springer, 2004.



H. Zhang and M. Stickel.

Implementing the Davis-Putnam algorithm by tries.

Technical report, University of Iowa, August 1994.



J. Zhang, S. Li, and S. Shen.

Extracting Minimum Unsatisfiable Cores with a Greedy Genetic Algorithm.

In *Proc. ACAI*, volume 4304 of *LNCS*. Springer, 2006.

References VII



L. Zhang, C. F. Madigan, M. H. Moskewicz, and S. Malik.

Efficient conflict driven learning in a boolean satisfiability solver.

In *ICCAD '01: Proceedings of the 2001 IEEE/ACM international conference on Computer-aided design*, pages 279–285, Piscataway, NJ, USA, 2001. IEEE Press.



L. Zhang and S. Malik.

The quest for efficient boolean satisfiability solvers.

In *Proc. CAV'02*, number 2404 in LNCS, pages 17–36. Springer, 2002.



L. Zhang and S. Malik.

Extracting small unsatisfiable cores from unsatisfiable boolean formula.

In *Proc. of SAT*, 2003.

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- **SATLive! - Up-to-date links for SAT**
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