Course "Introduction to SAT & SMT"
TEST

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[COPY WITH SOLUTIONS]

1

Let φ be a generic Boolean formula, and let $\varphi^{tree}_{nnf} \stackrel{\text{def}}{=} NNF^{tree}(\varphi)$ and $\varphi^{dag}_{nnf} \stackrel{\text{def}}{=} NNF^{dag}(\varphi)$, s.c. $NNF()^{tree}$ and $NNF()^{dag}$ are the conversion into negative normal form using a tree and a DAG representation of the formulas respectively.

Let $|\varphi|$, $|\varphi^{tree}_{nnf}|$ and $|\varphi^{dag}_{nnf}|$ denote the size of φ, φ^{tree}_{nnf} and φ^{dag}_{nnf} respectively.

For each of the following sentences, say if it is true or false.

(a) $|\varphi^{tree}_{nnf}|$ is in worst-case polynomial in size wrt. $|\varphi|$. [Solution: False.]

(b) $|\varphi^{dag}_{nnf}|$ is in worst-case polynomial in size wrt. $|\varphi|$. [Solution: True.]

(c) φ^{dag}_{nnf} has the same number of distinct Boolean variables as φ has. [Solution: True.]

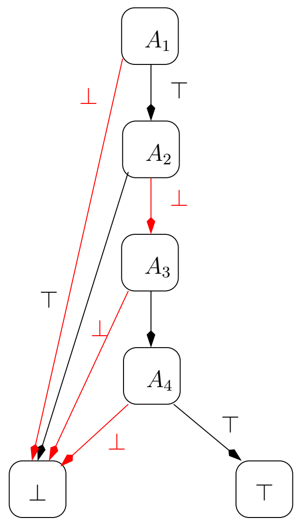
(d) A model for φ^{dag}_{nnf} (if any) is also a model for φ, and vice versa. [Solution: True.]

2

Using the variable ordering " A_1, A_2, A_3, A_4", draw the OBDD corresponding to the following formulas:

$$A_1 \wedge (\neg A_1 \vee \neg A_2) \wedge (A_2 \vee A_3) \wedge (\neg A_3 \vee A_4)$$

[Solution:
SOLUTION: The formula is represented by the following OBDD:



]

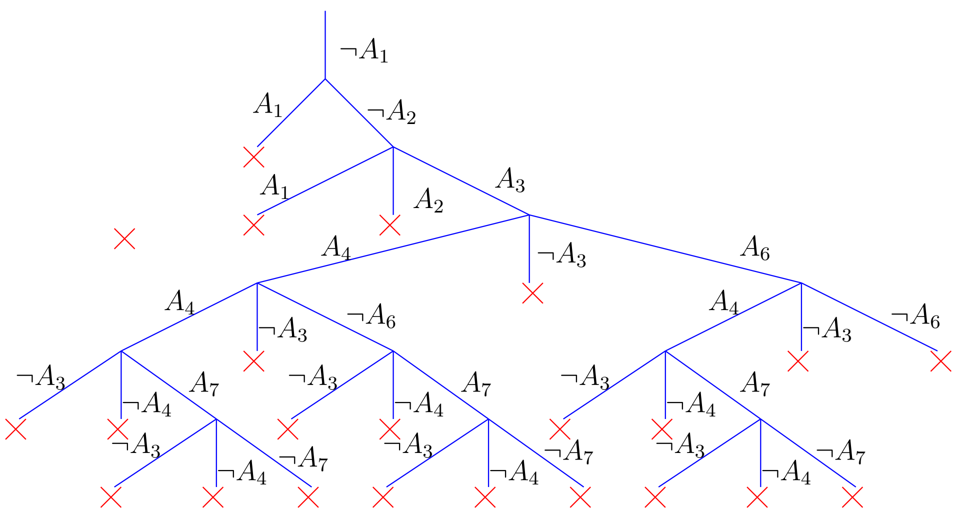
3

Using the semantic tableaux algorithm, decide whether the following formula is satisfiable or not. (Write the search tree.)

$$
\begin{array}{rcrcrl}
 & & & & (\neg A_1) & \wedge \\
 & & (\quad A_1 & \vee & \neg A_2) & \wedge \\
(\quad A_1 & \vee & A_2 & \vee & A_3) & \wedge \\
(\quad A_4 & \vee & \neg A_3 & \vee & A_6) & \wedge \\
(\quad A_4 & \vee & \neg A_3 & \vee & \neg A_6) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & A_7) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & \neg A_7) &
\end{array}
$$

(Literal-selection criteria to your choice.)

[Solution:



$\Longrightarrow$ the formula is inconsistent.
]

4

Consider the following piece of a much bigger formula, which has been fed to a CDCL SAT solver:

$$
\begin{array}{lrrr}
c_1: & \neg A_7 \vee & A_2 & \\
c_2: & A_4 \vee & A_1 \vee & A_{11} \\
c_3: & A_8 \vee & \neg A_6 \vee & \neg A_4 \\
c_4: & \neg A_5 \vee & \neg A_1 & \\
c_5: & A_7 \vee & \neg A_8 & \\
c_6: & A_7 \vee & A_6 \vee & A_9 \\
c_7: & \neg A_7 \vee & A_3 \vee & \neg A_{12} \\
c_8: & A_4 \vee & A_5 \vee & A_{10} \\
\ldots & & &
\end{array}
$$

Suppose the solver has decided, in order, the following literals (possibly interleaved by others not occurring in the above clauses):

$\{\ldots, \neg A_9, \ldots \neg A_{10}, \ldots \neg A_{11}, \ldots\ A_{12}, \ldots\ A_{13}, \ldots, \neg A_7\}$

(*a*) List the sequence of unit-propagations following after the last decision, each literal tagged (in square brackets) by its antecedent clause

[Solution:

$$
\begin{array}{rl}
\neg A_8 & [c_5] \\
A_6 & [c_6] \\
\neg A_4 & [c_3] \\
A_5 & [c_8] \\
A_1 & [c_2]
\end{array}
$$

]

(*b*) Derive the conflict clause via conflict analysis by means of the 1st-UIP technique

[Solution:

$$
\dfrac{A_4 \vee A_5 \vee A_{10} \qquad \dfrac{A_4 \vee A_1 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_1}^{Conflicting\ cl.}}{A_4 \vee \neg A_5 \vee A_{11}}\ (A_1)}{\underbrace{A_4}_{1st\ UIP} \vee A_{10} \vee A_{11}}\ (A_5)
$$

]

(*c*) Using the 1st-UIP backjumping strategy, update the list of literals above after the backjumping step and the unit-propagation of the UIP

[Solution: $\{\ldots, \neg A_9, \ldots \neg A_{10}, \ldots \neg A_{11},\ A_4\}$]

5

Consider the following CNF formula:

$$
\begin{array}{lcccccl}
(& A_7 & & & & &) \wedge \\
(& A_8 & \vee & \neg A_7 & & &) \wedge \\
(& \neg A_4 & \vee & \neg A_7 & \vee & \neg A_5 &) \wedge \\
(& \neg A_8 & \vee & \neg A_6 & \vee & A_1 &) \wedge \\
(& A_2 & \vee & \neg A_7 & \vee & \neg A_6 &) \wedge \\
(& \neg A_8 & \vee & A_6 & \vee & \neg A_2 &) \wedge \\
(& \neg A_1 & \vee & \neg A_5 & \vee & \neg A_2 &) \wedge \\
(& A_2 & \vee & \neg A_6 & \vee & \neg A_8 &) \wedge \\
(& \neg A_1 & \vee & \neg A_5 & \vee & A_8 &) \wedge \\
(& A_2 & \vee & A_7 & \vee & \neg A_6 &) \wedge \\
(& \neg A_6 & \vee & A_4 & \vee & \neg A_6 &) \wedge \\
(& A_3 & \vee & A_8 & \vee & \neg A_7 &)
\end{array}
$$

Decide *quickly* if it is satisfiable or not, and briefly explain why.

[Solution: After unit-propagating A_7, A_8:

$$
\begin{array}{lcccccl}
(& A_7 & & & & &) \wedge \\
(& A_8 & & & & &) \wedge \\
(& \neg A_4 & \vee & & & \neg A_5 &) \wedge \\
(& & & \neg A_6 & \vee & A_1 &) \wedge \\
(& A_2 & \vee & & & \neg A_6 &) \wedge \\
(& & & A_6 & \vee & \neg A_2 &) \wedge \\
(& \neg A_1 & \vee & \neg A_5 & \vee & \neg A_2 &) \wedge \\
(& A_2 & \vee & \neg A_6 & & &) \wedge \\
(& \neg A_1 & \vee & \neg A_5 & \vee & A_8 &) \wedge \\
(& A_2 & \vee & A_7 & \vee & \neg A_6 &) \wedge \\
(& \neg A_6 & \vee & A_4 & \vee & \neg A_6 &) \wedge \\
(& A_3 & \vee & A_8 & & &)
\end{array}
$$

the result is a Horn formula with no positive unit clauses.
Therefore, the assignment $\{\neg A_i\}_{i=1}^{8}$ is a model.]

6

Consider the following Boolean formulas:

$$
\begin{array}{rl}
\varphi_1 \stackrel{\text{def}}{=} & (\neg A_7 \vee \neg A_3) \quad \wedge \\
& (\ \ A_7 \vee \neg A_3) \quad \wedge \\
& (\ \ A_2) \qquad\qquad\quad \wedge \\
& (\neg A_2 \vee \neg A_4)
\end{array}
$$

$$
\begin{array}{rl}
\varphi_2 \stackrel{\text{def}}{=} & (\ \ A_3 \vee \ \ A_5) \quad \wedge \\
& (\ \ A_4 \vee \neg A_1) \quad \wedge \\
& (\neg A_5 \vee \ \ A_1)
\end{array}
$$

which are such that $\varphi_1 \wedge \varphi_2 \models \perp$. For each of the following formulas, say if it is a Craig interpolant for (φ_1, φ_2) or not.

[Solution: Recall that a Craig interpolant for (φ_1, φ_2) s.t. $\varphi_1 \wedge \varphi_2 \models \perp$ is a formula ψ s.t.

1. $\varphi_1 \models \psi$
2. $\psi \wedge \varphi_2 \models \perp$
3. all atoms in ψ occur in both φ_1 and φ_2.

]

(a)

$$
\begin{array}{l}
(\neg A_7 \vee \neg A_3) \quad \wedge \\
(\ \ A_7 \vee \neg A_3) \quad \wedge \\
(\neg A_4)
\end{array}
$$

[Solution: No, it is not a solution because it does not verify condition 3.]

(b)

$$(\neg A_4)$$

[Solution: No, it is not a solution because it does not verify condition 2.]

(c)

$$(\neg A_3 \wedge \neg A_4)$$

[Solution: Yes]

7

Consider the following formula in the theory $\mathcal{EUF}$ of linear arithmetic on the Rationals.

$$\begin{array}{ll} \varphi = & \{(f(x) = f(f(y))) \ \vee A_2\} \quad \wedge \\ & \{\neg(h(x, f(y)) = h(g(x), y)) \ \vee \underline{\neg(h(x, g(z) = h(f(x), y)))} \ \vee \neg A_1\} \ \wedge \\ & \{A_1 \vee (h(x, y) = h(y, x))\} \quad \wedge \\ & \{\underline{(x = f(x))} \ \vee A_3 \vee \neg A_1\} \ \wedge \\ & \{\underline{\neg(w(x) = g(f(y)))} \ \vee A_1\} \wedge \\ & \{\underline{\neg A_2} \vee (w(g(x)) = w(f(x)))\} \wedge \\ & \{A_1 \vee \underline{(y = g(z))} \ \vee A_2\} \end{array}$$

and consider the partial truth assignment μ given by the underlined literals above:

$$\{\neg(w(x) = g(f(y))), \neg A_2, \neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\}.$$

1. Does (the Boolean abstraction of) μ propositionally satisfy (the Boolean abstraction of) φ? [Solution: No, since there are two clauses which are not satisfied.]

2. Is μ satisfiable in $\mathcal{EUF}$?

   (a) If no, find a minimal conflict set for μ and the corresponding conflict clause C.

   (b) If yes, show one unassigned literal which can be deduced from μ, and show the corresponding deduction clause C.

[Solution:
No, because it contains the following $\mathcal{EUF}$ conflict set:

$$\{\neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\},$$

which corresponds to the following conflict clause:

$$(h(x, g(z) = h(f(x), y))) \vee \neg(x = f(x)), \neg(y = g(z))$$

]

8

Consider the following set of clauses φ in the theory of linear arithmetic on the Integers $\mathcal{EUF}$.

$$\left\{ \begin{array}{l} (\neg(x = y) \vee \quad (f(x) = f(y))), \\ (\neg(x = y) \vee \neg(f(x) = f(y))), \\ (\quad (x = y) \vee \quad (f(x) = f(y))), \\ (\quad (x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

Say which of the following sets is a $\mathcal{EUF}$-unsatisfiable core of φ and which is not. For each one, explain why.

(a)

$$\left\{ \begin{array}{l} (\neg(x = y) \vee \neg(f(x) = f(y))), \\ (\quad (x = y) \vee \quad (f(x) = f(y))), \\ (\quad (x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

[Solution: yes, because it is a subset of φ and it is inconsistent in $\mathcal{EUF}$.]

(b)

$$\left\{ \begin{array}{l} (\neg(x = y) \vee \quad (f(x) = f(y))), \\ (\quad (x = y) \vee \quad (f(x) = f(y))), \\ (\quad (x = y) \vee \neg(f(x) = f(y))) \end{array} \right\}$$

[Solution: no, because is not inconsistent in $\mathcal{EUF}$ (e.g., $\{ \ (x = y), \ (f(x) = f(y))\}$ is $\mathcal{EUF}$-consistent solution).]

(c)

$$\left\{ \begin{array}{l} (\neg(x = y) \vee \neg(f(x) = f(y))), \\ (\quad (x = y) \vee \quad (f(x) = f(y))), \\ (\quad (x = y) \vee \neg(f(x) = f(y))), \\ ((x = f(y))) \end{array} \right\}$$

[Solution: no, because it is not a subset of φ.]

9

Let $\mathcal{LRA}$ be the logic of linear arithmetic over the rationals and $\mathcal{EUF}$ be the logic of equality and uninterpreted functions. Consider the following pure formula φ in the combined logic $\mathcal{LRA} \cup \mathcal{EUF}$:

$$(x = 1.0) \wedge (h = 1.0) \wedge (k = 1.0) \wedge (y = 2h - k) \wedge (z < w) \tag{1}$$
$$(z = f(x)) \wedge (w = f(y)) \tag{2}$$

Say which variables are interface variables, list the interface equalities for this formula (modulo symmetry), and decide whether this formulas is $\mathcal{LRA} \cup \mathcal{EUF}$-satisfiable or not, using either Nelson-Oppen or Delayed Theory Combination.

[Solution: Only x, y, z, w occur both in $\mathcal{LRA}$-atoms (2) and in $\mathcal{EUF}$-atoms (1). Thus x, y, z, w are the interface variables, and $x = y, x = z, x = w, y = z, y = w, z = w$ are the interface equalities.

Nelson-Oppen: From (2) in $\mathcal{LRA}$ we infer the interface equality $(x = y)$. Adding the latter to (1), we infer the interface equality $(z = w)$ in $\mathcal{EUF}$. Adding the latter to (2), we get a contradiction in $\mathcal{LRA}$ against $(z < w)$.

Delayed Theory Combination: By unit-propagation, φ causes only one branch containing all its literals. Then the SAT solver assigns first a negative value to the interface equality (x=y), adding $\neg(x = y)$ to the assignment, which is found inconsistent in $\mathcal{LRA}$:

$$(h = 1.0) \wedge (k = 1.0) \wedge (x = 1.0) \wedge (y = 2h - k) \wedge \neg(x = y). \tag{3}$$

Then the SAT solver backtracks, adding $(x = y)$ to the assignment. Then the SAT solver assigns first a negative value to the interface equality (z=w), adding $\neg(z = w)$ to the assignment, which is found inconsistent in $\mathcal{LRA}$: :

$$(z = f(x)) \wedge (w = f(y)) \wedge (x = y) \wedge \neg(z = w). \tag{4}$$

Thus, with either technique, we can conclude that φ is $\mathcal{LRA} \cup \mathcal{EUF}$-unsatisfiable.]

10

Consider the following formulas in difference logic ($\mathcal{DL}$):

$$\begin{array}{rlr} \varphi_1 \stackrel{\text{def}}{=} & (x_2 - x_3 \le -4) & \wedge \\ & (x_3 - x_4 \le -6) & \wedge \\ & (x_5 - x_6 \le 4) & \wedge \\ & (x_6 - x_1 \le 2) & \wedge \\ & (x_6 - x_7 \le -2) & \wedge \\ & (x_7 - x_8 \le 1) & \end{array}$$

$$\begin{array}{rlr} \varphi_2 \stackrel{\text{def}}{=} & (x_4 - x_9 \le 2) & \wedge \\ & (x_9 - x_5 \le 0) & \wedge \\ & (x_1 - x_2 \le 1) & \end{array}$$

which are such that $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \bot$. For each of the following formulas, say if it is a Craig interpolant in $\mathcal{DL}$ for (φ_1, φ_2), and explain why.

[Solution: Recall that a Craig interpolant for (φ_1, φ_2) s.t. $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \bot$ is a formula ψ s.t.

1. $\varphi_1 \models_{\mathcal{DL}} \psi$
2. $\psi \wedge \varphi_2 \models_{\mathcal{DL}} \bot$
3. all symbols in ψ occur in both φ_1 and φ_2.

]

(a) $(x_2 - x_3 + x_6 - x_1 \le -2)$

[Solution: no, because, e.g., x_3 is not a symbol occurring in φ_2.]

(b) $(x_2 - x_4 \le -10)$

[Solution: No, because it violates condition 2.]

(c) $(x_2 - x_4 \le -10) \wedge$ $(x_5 - x_1 \le 6)$

[Solution: yes, because it is a $\mathcal{DL}$ formula and it verifies all conditions 1., 2., 3.]