Course “An Introduction to SAT and SMT”
Chapter 2: Satisfiability Modulo Theories

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Outline

1. Introduction
   - What is a Theory?
   - Satisfiability Modulo Theories
   - Motivations and Goals of SMT
2. Efficient SMT solving
   - Combining SAT with Theory Solvers
   - Theory Solvers for Theories of Interest (hints)
   - SMT for Combinations of Theories
3. Beyond Solving: Advanced SMT Functionalities
   - Proofs and Unsatisfiable Cores
   - Interpolants
   - All-SMT & Predicate Abstraction (hints)
   - SMT with Cost Optimization (Optimization Modulo Theories)
4. Conclusions & Current Research Directions

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FOL Theories

Traditional Definition (FOL)

Given a FOL signature Σ, a Σ-Theory $\mathcal{T}$ (hereafter simply "theory") is a (possibly infinite) set of FOL closed formulas (axioms)

- Typically used to provide some intended interpretation to the symbols in the signature Σ
- FOL formulas deduces from these axioms via inference rules
- Definition used by logicians,
- Very low practical use in AR & Formal Verification

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Example: A FOL Theory of Positive Integer Numbers (aka "Peano Arithmetic", $\mathcal{P}$)

- Signature
  - (basic) unary predicate symbol: NatNum ("natural number")
  - (basic) unary function symbol: S ("successor")
  - (basic) constant symbol: 0
  - (derived) binary function symbols: +,* (infix)
  - (derived) constant symbols: 1,2,3,4,5,6,...
- Axioms
  1. $NatNum(0)$
  2. $\forall x.(NatNum(x) \rightarrow NatNum(S(x)))$
  3. $\forall x.(NatNum(x) \rightarrow (0 \neq S(x)))$
  4. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x \neq y) \rightarrow (S(x) \neq S(y))))$
  5. $\forall x.(NatNum(x) \rightarrow (x = (0 + x)))$
  6. $\forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow (S(x) + y) = S(x + y))$
  7. $1 = S(0),\ 2 = S(1),\ 3 = S(2),\ ...$
- Formulas deduced
  - ex: $\mathcal{P} \vdash NatNum(25)$
  - ex: $\mathcal{P} \vdash \forall x, y.((NatNum(x) \wedge NatNum(y)) \rightarrow ((x + y) = (y + x)))$

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Given a FOL signature Σ, a Σ-Theory $\mathcal{T}$ (hereafter simply "theory") is one (or more) model(s) constraining the interpretations of Σ

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  - predicate symbols mapped into relations on domain elements
    - ex: ". $<$." mapped into the arithmetical relation "less then"
  - function symbols mapped into functions on domain elements
    - ex: "$S(.)$" mapped into the arithmetical function "successor of"

  These symbols are called interpreted
- Compliant with previous definition: model(s) satisfying all axioms
- Ad hoc "$\mathcal{T}$-aware" decision procedures for reasoning on formulas
- Very effective in practical applications

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- Domain: integer numbers
- Numerical constants interpreted as numbers
  - ex: “1”, “1346231” mapped directly into the corresponding number
- function and predicates interpreted as arithmetical operations
  - “+” as addiction, “*” as multiplication, “$<$” as less-then, . etc.
- ILP solvers used to do logical reasoning
  - ex: $(3x - 2y \leq 3) \wedge (4y - 2z < -7) \models (6x - 2z < -1)$

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Satisfiability, Validity, Entailment (Modulo a Theory $\mathcal{T}$)

Definitions

- Idea: We restrict to models satisfying $\mathcal{T}$ ("$\mathcal{T}$-models")
- A formula is satisfiable in $\mathcal{T}$ (aka "φ is $\mathcal{T}$-satisfiable") iff some model satisfying $\mathcal{T}$ satisfies also φ
  - ex: $(x < 3)$ satisfiable in $\mathcal{LIA}$
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Properties

- φ is $\mathcal{T}$-valid iff $\neg\varphi$ is $\mathcal{T}$-unsatisfiable
- $\varphi \models_{\mathcal{T}} \psi$ iff $\varphi \rightarrow \psi$ is $\mathcal{T}$-valid

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The problem of deciding the satisfiability of (typically quantifier-free) formulas in some decidable first-order theory $\mathcal{T}$

- $\mathcal{T}$ can also be a combination of theories $\bigcup_i \mathcal{T}_i$.

SMT($\mathcal{T}$): Theories of Interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions ($\mathcal{EUF}$):
  $((x = y) \wedge (y = f(z))) \rightarrow (g(x) = g(f(z)))$
- Difference logic ($\mathcal{DL}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
- UTVPI ($\mathcal{UTVPI}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x + z \leq 6)$
- Linear arithmetic over the rationals ($\mathcal{LRA}$):
  $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
- Linear arithmetic over the integers ($\mathcal{LIA}$): $(x = x_l + 2^{16} x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
- Arrays ($\mathcal{AR}$): $(i = j) \vee read(write(a, i, e), j) = read(a, j)$
- Bit vectors ($\mathcal{BV}$): $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$
- Non-Linear arithmetic over the reals ($\mathcal{NLA}(\mathbb{R})$) :
  $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = b + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
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Some theories of interest (e.g., for formal verification)

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Satisfiability Modulo Theories (SMT($\mathcal{T}$)): Example

Example: SMT($\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$)

$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge$
$((f(d) = f(0)) \rightarrow (read(write(V, i, x), i + d) = x + 1))$

- involves arithmetical, arrays, and uninterpreted function/predicate symbols, plus Boolean operators
  - Is it satisfiable?
  - No:

$$
\begin{array}{ll}
 & \varphi \\
\Longrightarrow_{\mathcal{LIA}} & (d = 0) \\
\Longrightarrow_{\mathcal{EUF}} & (f(d) = f(0)) \\
\Longrightarrow_{Bool} & (read(write(V, i, x), i + d) = x + 1) \\
\Longrightarrow_{\mathcal{LIA}} & (read(write(V, i, x), i) = x + 1) \\
\Longrightarrow_{\mathcal{LIA}} & \neg(read(write(V, i, x), i) = x) \\
\Longrightarrow_{\mathcal{AR}} & \bot
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SMT and SMT solvers

Common fact about SMT problems from various applications

SMT requires capabilities for heavy Boolean reasoning combined with capabilities for reasoning in expressive decidable F.O. theories

- SAT alone not expressive enough
- standard automated theorem proving inadequate (e.g., arithmetic)
- may involve also numerical computation (e.g., simplex)

Modern SMT solvers

- combine SAT solvers with $\mathcal{T}$-specific decision procedures (theory solvers or $\mathcal{T}$-solvers)
  - contributions from SAT, Automated Theorem Proving (ATP), formal verification (FV) and operational research (OR)

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Notational remark (1): most/all examples in $\mathcal{LRA}$

For better readability, in most/all the examples of this presentation we will use the theory of linear arithmetic on rational numbers ($\mathcal{LRA}$) because of its intuitive semantics. E.g.:

$$(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$$

Nevertheless, analogous examples can be built with all other theories of interest.

Notational remark (2): “constants” vs. “variables”

- Consider, e.g., the formula:
  $(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$
- How do we call A_1, A_2?:
  (*a*) Boolean/propositional variables?
  (*b*) uninterpreted 0-ary predicates?
- How do we call x_1, x_2, x_3?:
  (*a*) domain variables?
  (*b*) uninterpreted Skolem constants/0-ary uninterpreted functions?
- Hint:
  (*a*) typically used in SAT, CSP and OR communities
  (*b*) typically used in logic & ATP communities

Hereafter we call A_1, A_2 “Boolean/propositional variables” and x_1, x_2, x_3 "domain variables” (logic purists, please forgive me!)

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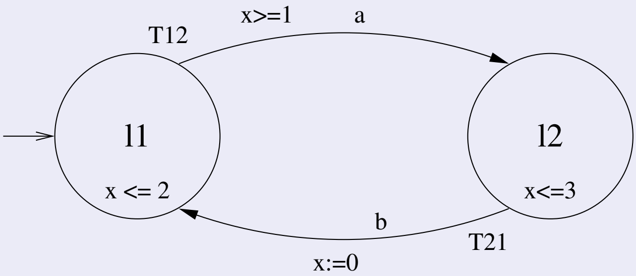
Outline

Some Motivating Applications

Interest in SMT triggered by some real-word applications

- Verification of Hybrid & Timed Systems
- Verification of RTL Circuit Designs & of Microcode
- SW Verification
- Planning with Resources
- Temporal reasoning
- Scheduling
- Compiler optimization
- ...

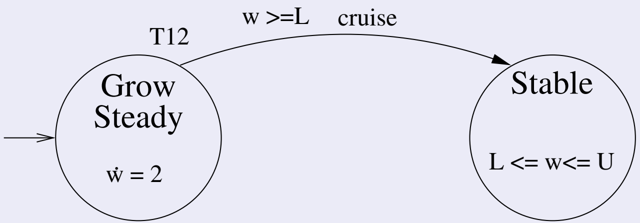
Verification of Timed Systems



- Model checking of Timed Systems [6, 35, 58], ...
- Timed Automata encoded into $\mathcal{T}$-formulas:
  - discrete information (locations, transitions, events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \leq 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$ or $\mathcal{LRA}$ required

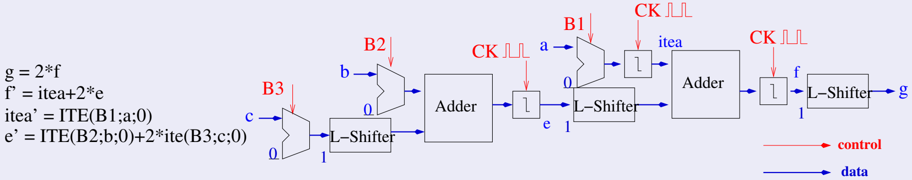
Verification of Hybrid Systems ...



- Model checking of Hybrid Systems [5],...
- Hybrid Automata encoded into $\mathcal{L}$-formulas:
  - discrete information (locs, trans., events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \leq 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$
  - Evolution of Physical Variables (e.g., speed, pressure) with linear $(\omega_4 = 2\omega_3)$ and non-linear constraints $(P_1 V_1 = 4 T_1)$ on $\mathbb{Q}$
- Undecidable under simple hypotheses!

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$, $\mathcal{LRA}$ or $\mathcal{NLA}(\mathbb{R})$ required

Verification of HW circuit designs & microcode



- SAT/SMT-based Model Checking & Equiv. Checking of RTL designs, symbolic simulation of μ-code [25, 22, 42]
- Control paths handled by Boolean reasoning
- Data paths information abstracted into theory-specific terms
  - words (bit-vectors, integers, $\mathcal{EUF}$ vars, ...): $\underline{a}[31:0]$, a
  - word operations: ($\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, $\mathcal{LIA}$, $\mathcal{NLA}(\mathbb{Z})$ operators) $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$, $(a = a_L + 2^{16} a_H)$, $(m_1 = store(m_0, l_0, v_0))$, ...
- Trades heavy Boolean reasoning ($\approx 2^{64}$ factors) with $\mathcal{T}$-solving

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, modulo-$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Verification of SW systems

```
...
10. i = 0;
11. acc = 0.0;
12. while (i<dim) {
13.   acc += V[i];
14.   i++;
15. }
...
```

....

$(pc = 10) \rightarrow ((i' = 0) \wedge (pc' = 11))$

$(pc = 11) \rightarrow ((acc' = 0.0) \wedge (pc' = 12))$

$(pc = 12) \rightarrow ((i < dim) \rightarrow \wedge (pc' = 13))$

$(pc = 12) \rightarrow (\neg(i < dim) \rightarrow \wedge (pc' = 16))$

$(pc = 13) \rightarrow ((acc' = acc + read(V, i)) \wedge (pc' = 14))$

$(pc = 14) \rightarrow (i' = i + 1) \wedge (pc' = 15))$

$(pc = 15) \rightarrow (pc' = 16))$

...

- Verification of SW code
  - BMC, K-induction, Check of proof obligations, interpolation-based model checking, symbolic simulation, concolic testing, ...

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, (modulo-)$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Planning with Resources [81]

- SAT-bases planning augmented with numerical constraints
- Straightforward to encode into into SMT($\mathcal{LRA}$)

Example (sketch) [81]

$(Deliver)$	$\wedge$ // goal
$(MaxLoad)$	$\wedge$ // load constraint
$(MaxFuel)$	$\wedge$ // fuel constraint
$(Move \rightarrow MinFuel)$	$\wedge$ // move requires fuel
$(Move \rightarrow Deliver)$	$\wedge$ // move implies delivery
$(GoodTrip \rightarrow Deliver)$	$\wedge$ // a good trip requires
$(GoodTrip \rightarrow AllLoaded)$	$\wedge$ // a full delivery
$(MaxLoad \rightarrow (load \leq 30))$	$\wedge$ // load limit
$(MaxFuel \rightarrow (fuel \leq 15))$	$\wedge$ // fuel limit
$(MinFuel \rightarrow (fuel \geq 7 + 0.5 load))$	$\wedge$ // fuel constraint
$(AllLoaded \rightarrow (load = 45))$	//

(Disjunctive) Temporal Reasoning [78, 2]

- Temporal reasoning problems encoded as disjunctions of difference constraints

$$
\begin{array}{llll}
((x_1 - x_2 \leq 6) & \vee (x_3 - x_4 \leq -2)) & \wedge \\
((x_2 - x_3 \leq -2) & \vee (x_4 - x_5 \leq 5)) & \wedge \\
((x_2 - x_1 \leq 4) & \vee (x_3 - x_7 \leq -6)) & \wedge \\
\ldots & &
\end{array}
$$

- Straightforward to encode into into SMT($\mathcal{DL}$)

Goal

Provide an overview of standard "lazy" SMT:

- foundations
- SMT-solving techniques
- beyond solving: advanced SMT functionalities
- ongoing research

We do not cover related approaches like:

- Eager SAT encodings
- Rewrite-based approaches

We refer to [71, 10] for an overview and references.

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A prominent "lazy" approach [45, 2, 81, 3, 8, 35] (aka "DPLL($\mathcal{T}$)")

- a CDCL SAT solver is used to enumerate truth assignments μ_i for (the Boolean abstraction φ^p of) the input formula φ
  - the Boolean abstraction φ^p of φ maps theory atoms in φ into fresh Boolean variables
- a theory-specific solver *$\mathcal{T}$-solver* checks the $\mathcal{T}$-satisfiability of the set of $\mathcal{T}$-literals corresponding to each assignment

- Built on top of modern SAT CDCL solvers
  - benefit for free from all modern CDCL techniques (e.g., Boolean preprocessing, backjumping & learning, restarts,...)
  - benefit for free from all state-of-the-art data structures and implementation tricks (e.g., two-watched literals,...)
- Many techniques to maximize the benefits of integration [71, 10]
- Many lazy SMT tools available (Barcelogic, CVC4, MathSAT, OpenSMT, Yices, Z3, . . .)

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Basic schema: example

$\varphi =$

$c_1: \quad \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2: \quad \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3: \quad (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4: \quad \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5: \quad A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6: \quad (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7: \quad A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

true, *false*

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$

$\Longrightarrow$ unsatisfiable in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

Basic schema: example

$\varphi =$

$c_1: \quad \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2: \quad \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3: \quad (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4: \quad \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5: \quad A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6: \quad (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7: \quad A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

true, *false*

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$

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$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

true, *false*

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

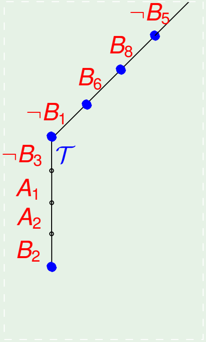
$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$



$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$

$\Longrightarrow$ unsatisfiable in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

Basic schema: example

$\varphi =$
$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$
$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$
$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$
$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$
$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$
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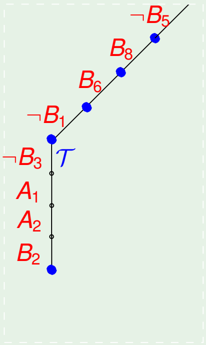
$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

true, *false*

$\varphi^p =$
$\neg B_1 \vee A_1$
$\neg A_2 \vee B_2$
$B_3 \vee A_2$
$\neg B_4 \vee \neg B_5 \vee \neg A_1$
$A_1 \vee B_3$
$B_6 \vee B_7 \vee \neg A_1$
$A_1 \vee B_8 \vee A_2$

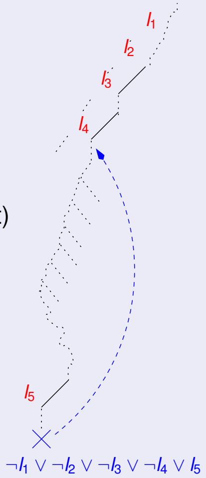


$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$
$$\mu = \{\underline{\neg(3v_1 - v_3 \leq 6)}, \underline{(v_3 = 3v_5 + 4)}, (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), \underline{(v_1 - v_5 \leq 1)}\}$$

$\Longrightarrow$ unsatisfiable in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning [50, 81, 3, 8, 35]

- Similar to Boolean backjumping & learning
- important property of $\mathcal{T}$-solver:
  - extraction of $\mathcal{T}$-conflict sets: if μ is $\mathcal{T}$-unsatisfiable, then $\mathcal{T}$*-solver* (μ) returns the subset η of μ causing the $\mathcal{T}$-unsatisfiability of μ ($\mathcal{T}$-conflict set)
- If so, the $\mathcal{T}$-conflict clause $C := \neg\eta$ is used to drive the backjumping & learning mechanism of the SAT solver
  $\Longrightarrow$ lots of search saved
- the less redundant is η, the more search is saved



$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example

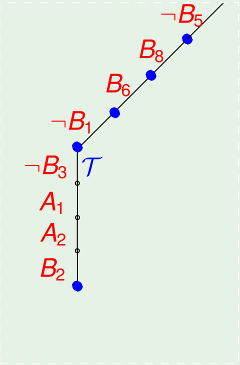
$\varphi =$

$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$
$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$
$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$
$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$
$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$
$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$
$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$
$c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$

$\varphi^p =$

$\neg B_1 \vee A_1$
$\neg A_2 \vee B_2$
$B_3 \vee A_2$
$\neg B_4 \vee \neg B_5 \vee \neg A_1$
$A_1 \vee B_3$
$B_6 \vee B_7 \vee \neg A_1$
$A_1 \vee B_8 \vee A_2$
$B_5 \vee \neg B_8 \vee \neg B_2$

true, *false*



$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$
$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$
$$\eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\}$$
$$\eta^p = \{\neg B_5, B_8, B_2\}$$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example

$\varphi =$

$c_1: \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2: \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3: (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4: \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5: A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6: (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7: A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$c_8: (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

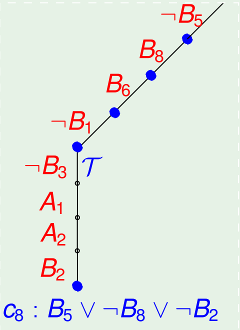
$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

$B_5 \vee \neg B_8 \vee \neg B_2$

true, *false*



$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$

$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$

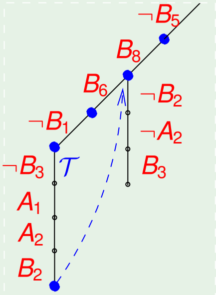
$\eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\}$

$\eta^p = \{\neg B_5, B_8, B_2\}$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example

$\varphi =$		$\varphi^p =$
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$

true, *false*

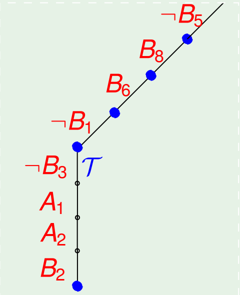


$c_8 : B_5 \vee \neg B_8 \vee \neg B_2$

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$
$$\mu = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1)\}$$
$$\eta = \{\neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4), (v_1 - v_5 \leq 1)\}$$
$$\eta^p = \{\neg B_5, B_8, B_2\}$$

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

	$\varphi =$	$\varphi^p =$
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \le 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \le 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \ge 5) \vee \neg(3v_1 - v_3 \le 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \le 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \le 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8' :	$(3v_1 - v_3 \le 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee B_1$
c_8 :	$(3v_1 - v_3 \le 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$
	true, *false*	



c_8: *theory conflicting clause*

$$\dfrac{\dfrac{\dfrac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_8} \quad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2}(B_2) \quad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3}(\neg A_2) \quad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}}}{\underbrace{B_5 \vee \neg B_8 \vee B_1}_{c_8'}}(B_3)$$

c_8': *mixed Boolean+theory conflict clause*

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

$\varphi =$

$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

$c_8' : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$

$c_8 : (3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$

true, *false*

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

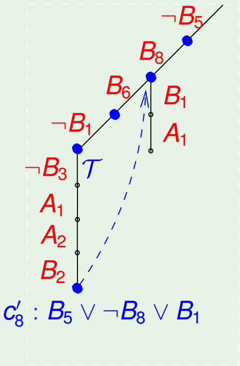
$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

$B_5 \vee \neg B_8 \vee B_1$

$B_5 \vee \neg B_8 \vee \neg B_2$



c_8: *theory conflicting clause*: $B_5 \vee \neg B_8 \vee \neg B_2$; c_2: $\neg A_2 \vee B_2$

$$\frac{B_5 \vee \neg B_8 \vee \neg B_2 \qquad \neg A_2 \vee B_2}{B_5 \vee \neg B_8 \vee \neg A_2}\ (B_2)$$

c_3: $B_3 \vee A_2$

$$\frac{B_5 \vee \neg B_8 \vee \neg A_2 \qquad B_3 \vee A_2}{B_5 \vee \neg B_8 \vee B_3}\ (\neg A_2)$$

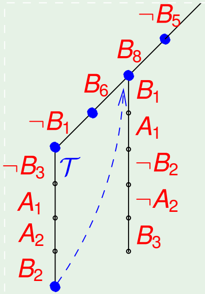
$c_{\mathcal{T}}$: $B_5 \vee B_1 \vee \neg B_3$

$$\frac{B_5 \vee \neg B_8 \vee B_3 \qquad B_5 \vee B_1 \vee \neg B_3}{B_5 \vee \neg B_8 \vee B_1}\ (B_3)$$

c_8': *mixed Boolean+theory conflict clause*

$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)

	$\varphi =$	$\varphi^p =$
c_1 :	$\neg(2v_2 - v_3 > 2) \vee A_1$	$\neg B_1 \vee A_1$
c_2 :	$\neg A_2 \vee (v_1 - v_5 \leq 1)$	$\neg A_2 \vee B_2$
c_3 :	$(3v_1 - 2v_2 \leq 3) \vee A_2$	$B_3 \vee A_2$
c_4 :	$\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$	$\neg B_4 \vee \neg B_5 \vee \neg A_1$
c_5 :	$A_1 \vee (3v_1 - 2v_2 \leq 3)$	$A_1 \vee B_3$
c_6 :	$(v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$	$B_6 \vee B_7 \vee \neg A_1$
c_7 :	$A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$	$A_1 \vee B_8 \vee A_2$
c_8' :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee B_1$
c_8 :	$(3v_1 - v_3 \leq 6) \vee \neg(v_3 = 3v_5 + 4) \vee \ldots$	$B_5 \vee \neg B_8 \vee \neg B_2$
	true, *false*	



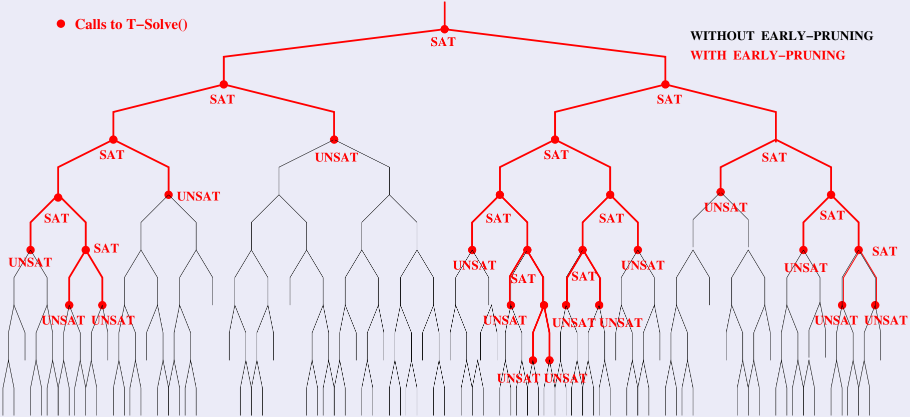
c_8' : $B_5 \vee \neg B_8 \vee B_1$

c_8 : $B_5 \vee \neg B_8 \vee \neg B_2$

$$\frac{\dfrac{\dfrac{\overbrace{B_5 \vee \neg B_8 \vee \neg B_2}^{c_8:\ \text{theory conflicting clause}} \qquad \overbrace{\neg A_2 \vee B_2}^{c_2}}{B_5 \vee \neg B_8 \vee \neg A_2}\ (B_2) \qquad \overbrace{B_3 \vee A_2}^{c_3}}{B_5 \vee \neg B_8 \vee B_3}\ (\neg A_2) \qquad \overbrace{B_5 \vee B_1 \vee \neg B_3}^{c_{\mathcal{T}}}}{\underbrace{B_5 \vee \neg B_8 \vee B_1}_{c_8':\ \text{mixed Boolean+theory conflict clause}}}\ (B_3)$$

Early Pruning [45, 2, 81]

- Introduce a $\mathcal{T}$-satisfiability test on intermediate assignments:
  if $\mathcal{T}$*-solver* returns UNSAT, the procedure backtracks.
  - benefit: prunes drastically the Boolean search
  - Drawback: possibly many useless calls to $\mathcal{T}$*-solver*



Early pruning: example

$$\varphi = \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge$$
$$\{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge$$
$$\{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge$$
$$\{\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1\} \wedge$$
$$\{A_1 \vee (3v_1 - 2v_2 \leq 3)\} \wedge$$
$$\{(v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge$$
$$\{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\}.$$

$$\varphi^p = \{\neg B_1 \vee A_1\} \wedge$$
$$\{\neg A_2 \vee B_2\} \wedge$$
$$\{B_3 \vee A_2\} \wedge$$
$$\{\neg B_4 \vee \neg B_5 \vee \neg A_1\} \wedge$$
$$\{A_1 \vee B_3\} \wedge$$
$$\{B_6 \vee B_7 \vee \neg A_1\} \wedge$$
$$\{A_1 \vee B_8 \vee A_2\}.$$

- Suppose it is built the intermediate assignment:

$$\mu'^p = \neg B_1 \wedge \neg A_2 \wedge B_3 \wedge \neg B_5.$$

  corresponding to the following set of $\mathcal{T}$-literals

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

- If $\mathcal{T}$-*solver* is invoked on μ', then it returns UNSAT, and DPLL backtracks without exploring any extension of μ'.

Early Pruning [45, 2, 81] (cont.)

- Different strategies for interleaving Boolean search steps and $\mathcal{T}$-*solver* calls
  - Eager E.P. [81, 11, 79, 44]): invoke $\mathcal{T}$-*solver* every time a new $\mathcal{T}$-atom is added to the assignment (unit propagations included)
  - Selective E.P.: Do not call $\mathcal{T}$-*solver* if the have been added only literals which hardly cause any $\mathcal{T}$-conflict with the previous assignment (e.g., Boolean literals, disequalities $(x - y \neq 3)$, $\mathcal{T}$-literals introducing new variables $(x - z = 3)$)
  - Weakened E.P.: for intermediate checks only, use weaker but faster versions of $\mathcal{T}$-*solver* (e.g., check μ on $\mathbb{R}$ rather than on $\mathbb{Z}$): $\{(x - y \leq 4), (z - x \leq -6), (z = y), (3x + 2y - 3z = 4)\}$

Early pruning: remark

Incrementality & Backtrackability of $\mathcal{T}$-*solvers*

- With early pruning, lots of incremental calls to $\mathcal{T}$-*solver*:

$\mathcal{T}$-*solver* (μ_1)	$\Rightarrow$ *Sat*	Undo μ_4, μ_3, μ_2		
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2)$	$\Rightarrow$ *Sat*	$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2')$	$\Rightarrow$ *Sat*
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2 \cup \mu_3)$	$\Rightarrow$ *Sat*	$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2' \cup \mu_3')$	$\Rightarrow$ *Sat*
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2 \cup \mu_3 \cup \mu_4)$	$\Rightarrow$ *Unsat*	...	

$\Longrightarrow$ Desirable features of $\mathcal{T}$-*solvers*:

- incrementality: $\mathcal{T}$-*solver*$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}$-*solver*(μ_1) without restarting from scratch
- backtrackability (resettability): $\mathcal{T}$-*solver* can efficiently undo steps and return to a previous status on the stack

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Early pruning: remark

Incrementality & Backtrackability of $\mathcal{T}$-*solvers*

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$\mathcal{T}$-*solver* (μ_1)	$\Rightarrow$ *Sat*	Undo μ_4, μ_3, μ_2		
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$\mathcal{T}$-Propagation [2, 3, 44]

- strictly related to early pruning
- important property of $\mathcal{T}$-*solver*:
  - $\mathcal{T}$-deduction: when a partial assignment μ is $\mathcal{T}$-satisfiable, $\mathcal{T}$-*solver* may be able to return also an assignment η to some unassigned atom occurring in φ s.t. $\mu \models_{\mathcal{T}} \eta$.
- If so:
  - the literal η is then unit-propagated;
  - optionally, a $\mathcal{T}$-deduction clause $C := \neg\mu' \vee \eta$ can be learned, μ' being the subset of μ which caused the deduction ($\mu' \models_{\mathcal{T}} \eta$)
  - lazy explanation: compute C only if needed for conflict analysis

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$\mathcal{T}$-propagation: example

$\varphi =$

$c_1: \quad \neg(2v_2 - v_3 > 2) \vee A_1$
$c_2: \quad \neg A_2 \vee (v_1 - v_5 \leq 1)$
$c_3: \quad (3v_1 - 2v_2 \leq 3) \vee A_2$
$c_4: \quad \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$
$c_5: \quad A_1 \vee (3v_1 - 2v_2 \leq 3)$
$c_6: \quad (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$
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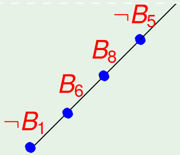
$\varphi^p =$

$\neg B_1 \vee A_1$
$\neg A_2 \vee B_2$
$B_3 \vee A_2$
$\neg B_4 \vee \neg B_5 \vee \neg A_1$
$A_1 \vee B_3$
$B_6 \vee B_7 \vee \neg A_1$
$A_1 \vee B_8 \vee A_2$



true, *false*

$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1\}$$
$$\mu = \{\underline{\neg(3v_1 - v_3 \leq 6)}, (v_3 = 3v_5 + 4), (v_2 - v_4 \leq 6), \underline{\neg(2v_2 - v_3 > 2)}\}$$
$$\models_{\mathcal{LRA}} \underbrace{\neg(3v_1 - 2v_2 \leq 3)}_{\neg B_3}$$

$\Longrightarrow$ propagate $\neg B_3$ [and learn the deduction clause $B_5 \vee B_1 \vee \neg B_3$]

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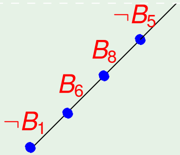

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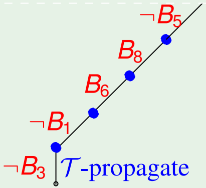

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Pure-literal filtering [81, 3, 17]

Property

If we have non-Boolean $\mathcal{T}$-atoms occurring only positively [negatively] in the original formula φ (learned clauses are not considered), we can drop every negative [positive] occurrence of them from the assignment to be checked by $\mathcal{T}$*-solver* (and from the $\mathcal{T}$-deducible ones).

- increases the chances of finding a model
- reduces the effort for the $\mathcal{T}$*-solver*
- eliminates unnecessary "nasty" negated literals (e.g. negative equalities like $\neg(3v_1 - 9v_2 = 3)$ in $\mathcal{LIA}$ force splitting: $(3v_1 - 9v_2 > 3) \vee (3v_1 - 9v_2 < 3)$).
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\varphi = \; & \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\
& \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\
& \{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge \\
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$\Longrightarrow$ Sat: $v_1 = v_2 = v_3 = 0, v_5 = -4/3$ is a solution

Note

- $(3v_1 - v_3 \leq -2)$ "filtered out" from μ' because it occurs only negatively in the original formula φ
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Preprocessing atoms [45, 50, 4]

Source of inefficiency:

Semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other]

$\Longrightarrow$ they may be assigned different [resp. identical] truth values.

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Rewrite a priori trivially-equivalent atoms/literals into the same atom/literal.

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Preprocessing atoms (cont.)

- Sorting: $(v_1 + v_2 \leq v_3 + 1), (v_2 + v_1 \leq v_3 + 1), (v_1 + v_2 - 1 \leq v_3) \Longrightarrow (v_1 + v_2 - v_3 \leq 1))$;
- Rewriting dual operators: $(v_1 < v_2), (v_1 \geq v_2) \Longrightarrow (v_1 < v_2), \neg(v_1 < v_2)$
- Exploiting associativity: $(v_1 + (v_2 + v_3) = 1), ((v_1 + v_2) + v_3) = 1) \Longrightarrow (v_1 + v_2 + v_3 = 1)$;
- Factoring $(v_1 + 2.0v_2 \leq 4.0), (-2.0v_1 - 4.0v_2 \geq -8.0), \Longrightarrow (0.25v_1 + 0.5v_2 \leq 1.0)$;
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Static Learning [2, 4]

- Often possible to quickly detect a priori short and "obviously unsatisfiable" pairs or triplets of literals occurring in φ.
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  - substitution $\{(x = y), (2x - 3z \leq 3), \neg(2y - 3z \leq 3)\}$
  - ...
- Preprocessing step: detect these literals and add blocking clauses to the input formula: (e.g., $\neg(x = 0) \vee \neg(x = 1)$)

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Other optimization techniques

- $\mathcal{T}$-deduced-literal filtering
- Ghost-literal filtering
- $\mathcal{T}$-*solver* layering
- $\mathcal{T}$-*solver* clustering
- ...

(see [71, 10] for an overview)

Other SAT-solving techniques for SMT?

Frequently-asked question:

Are CDCL SAT solvers the only suitable Boolean Engines for SMT?

Some previous attempts:

- Ordered Binary Decision Diagrams (OBDDs) [82, 60, 1]
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CDCL based currently much more efficient.

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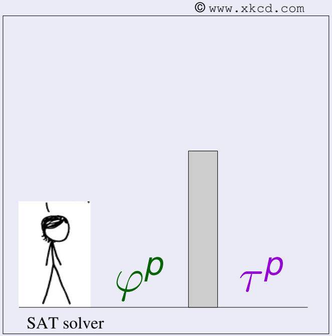

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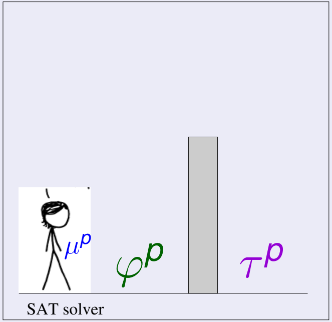

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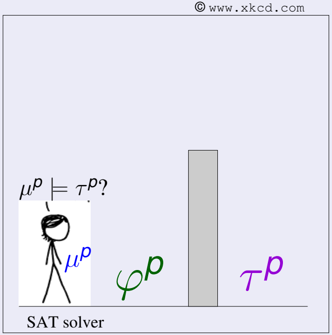

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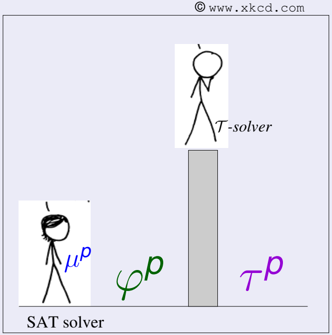

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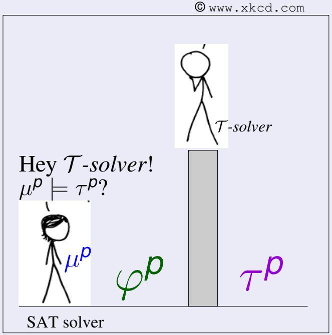

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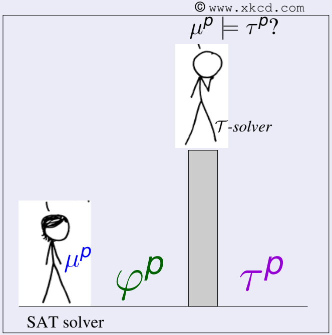

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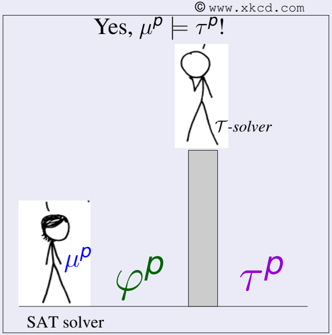

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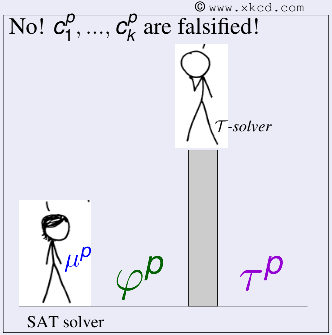

Example

φ :
c_1 : $\{A_1\}$
c_2 : $\{\neg A_1 \vee (x - z > 4)\}$
c_3 : $\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$
c_4 : $\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$
c_5 : $\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$
c_6 : $\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$
c_7 : $\{A_3 \vee \neg(x + y = 0) \vee A_2\}$
c_8 : $\{\neg A_3 \vee (z + y = 2)\}$
$\mathcal{T}$: (all possible $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of φ)
c_9 : $\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$
c_{10} : $\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$
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φ^p :
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c_5 : $\{B_3 \vee \neg A_4 \vee A_5\}$
c_6 : $\{\neg B_4 \vee B_5 \vee \neg A_5\}$
c_7 : $\{A_3 \vee \neg B_6 \vee A_2\}$
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τ^p :
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... ...

μ_1^p : $\{A_1, B_1, \neg A_2, A_3, \neg A_4, \neg A_5, \neg B_6, B_5, B_3, B_4, B_7, \neg B_2\}$
μ_1 : $\{(x - z > 4), \neg(x + y = 0), (x + y = 1), (x - y \leq 3), (y - z \leq 1),$
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satisfies φ^p, but violates both c_{10} and c_{12} in τ^p.

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Exercise

Consider the following formula in the theory $\mathcal{EUF}$.

$$
\begin{aligned}
\varphi = \quad & \{(f(x) = f(f(y))) \ \vee A_2\} \quad \wedge \\
& \{\neg(h(x, f(y)) = h(g(x), y)) \ \vee \underline{\neg(h(x, g(z) = h(f(x), y)))} \ \vee \neg A_1\} \ \wedge \\
& \{A_1 \vee (h(x, y) = h(y, x))\} \quad \wedge \\
& \{\underline{(x = f(x))} \ \vee A_3 \vee \neg A_1\} \ \wedge \\
& \{\underline{\neg(w(x) = g(f(y)))} \ \vee A_1\} \wedge \\
& \{\underline{\neg A_2} \vee (w(g(x)) = w(f(x)))\} \wedge \\
& \{A_1 \vee \underline{(y = g(z))} \ \vee A_2\}
\end{aligned}
$$

and consider the partial truth assignment μ given by the underlined literals above:

$$\{\neg(w(x) = g(f(y))), \neg A_2, \neg(h(x, g(z) = h(f(x), y))), (x = f(x)), (y = g(z))\}.$$

1. Does (the Boolean abstraction of) μ propositionally satisfy (the Boolean abstraction of) φ?
2. Is μ satisfiable in $\mathcal{EUF}$?
   1. If no, find a minimal conflict set for μ and the corresponding conflict clause C.
   2. If yes, show one unassigned literal which can be deduced from μ, and show the corresponding deduction clause C.

Outline

Summary: desirable properties for $\mathcal{T}$*-solver*

- Correctness & Completeness: be correct & complete
- Time efficiency: be fast
- Incrementality & backtrackability: $\mathcal{T}$*-solver*$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}$*-solver*(μ_1)
- Diagnosis capabilities: $\mathcal{T}$*-solver* able to produce conflict sets
- Deduction capabilities: $\mathcal{T}$*-solver* able to deduce assignments

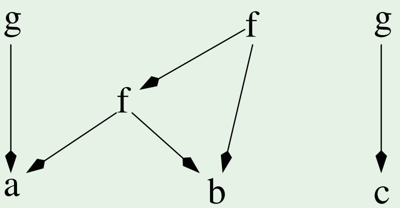
$\mathcal{T}$-solvers for Equality and Uninterpreted Functions ($\mathcal{EUF}$)

- Typically used as a "core" $\mathcal{T}$*-solver*
- $\mathcal{EUF}$ polynomial: $O(n \cdot log(n))$
- Fully incremental and backtrackable (stack-based)
- Uses a congruence closure data structures (E-Graphs) [39, 64, 34],
  - based on the Union-Find data-structure for equivalence classes
- Supports efficient $\mathcal{T}$-propagation
  - Exhaustive for positive equalities
  - Incomplete for disequalities
- Supports Lazy explanations and conflict generation
  - However, minimality not guaranteed
- Supports efficient extensions
  (e.g., Integer offsets, Bit-vector slicing and concatenation)

$\mathcal{T}$-solvers for $\mathcal{EUF}$: Example

Idea (sketch):
given the set of terms occurring in the formula represented as nodes in a DAG (aka term bank):

- if $(t = s)$, then merge the eq. classes of t and s
  - e.g. use the union-find data structure
- if $\forall i \in 1...k$, t_i and s_i pairwise belong to the same eq. classes, then merge the eq. classes of $f(t_1, ..., t_k)$ and $f(s_1, ..., s_k)$
- if $(t \neq s)$ and t and s belong to the same eq. class, then conflict



$$f(a,b) = a$$
$$f(f(a,b),b) = c$$
$$g(a) \neq g(c)$$

Example borrowed from [39].

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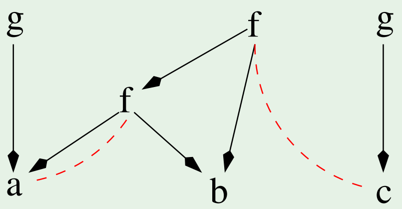

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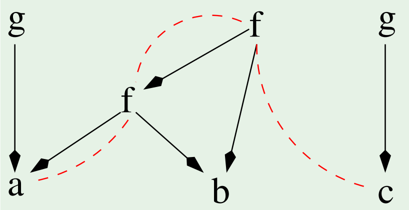

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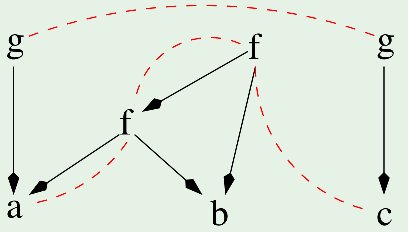

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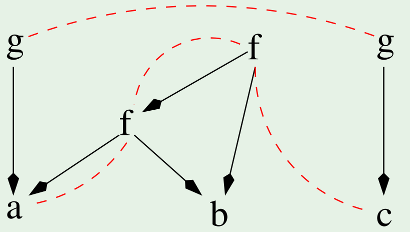

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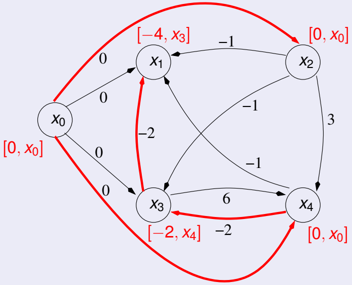
$\mathcal{T}$-solvers for Difference logic ($\mathcal{DL}$)

- $\mathcal{DL}$ polynomial: $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 33]
- Ex:

$$\{(x_1 - x_2 \le -1), (x_1 - x_4 \le -1), (x_1 - x_3 \le -2), (x_2 - x_1 \le 2),$$
$$(x_3 - x_4 \le -2), (x_3 - x_2 \le -1), (x_4 - x_2 \le 3), (x_4 - x_3 \le 6)\}$$



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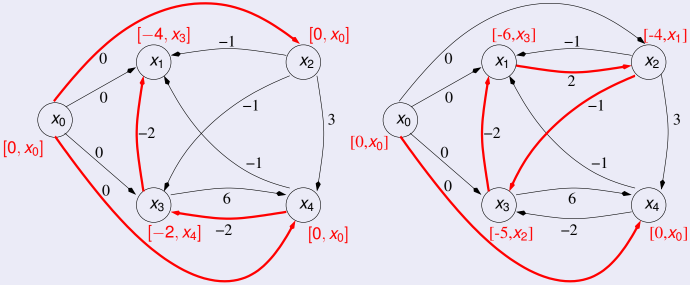
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 33]
- Ex:

$$\{(x_1 - x_2 \leq -1), (x_1 - x_4 \leq -1), (x_1 - x_3 \leq -2), (x_2 - x_1 \leq 2), (x_3 - x_4 \leq -2), (x_3 - x_2 \leq -1), (x_4 - x_2 \leq 3), (x_4 - x_3 \leq 6)\}$$



⟹ Sat ⟹ Unsat

$\mathcal{T}$-solvers for Linear arithmetic over the rationals ($\mathcal{LRA}$)

- EX: $\{(s_1 - s_2 \leq 5.2), (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0), \neg(s_1 = s_0)\}$
- $\mathcal{LRA}$ polynomial
- variants of the simplex LP algorithm [41]
- [41] allows for detecting conflict sets & performing $\mathcal{T}$-propagation
- strict inequalities $t < 0$ rewritten as $t + \epsilon \leq 0$, ϵ treated symbolically

$$\overbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix}}^{\mathcal{B}} = \begin{bmatrix} \ldots A_{1j} \ldots \\ \vdots \\ A_{i1} \ldots A_{ij} \ldots A_{iM} \\ \vdots \\ \ldots A_{Nj} \ldots \end{bmatrix} \overbrace{\begin{bmatrix} x_{N+1} \\ \vdots \\ x_j \\ \vdots \\ x_{N+M} \end{bmatrix}}^{\mathcal{N}};$$

Invariant: $\beta(x_j) \in [l_j, u_j] \; \forall x_j \in \mathcal{N}$

Remark: infinite precision arithmetic

In order to avoid incorrect results due to numerical errors and to overflows, all $\mathcal{T}$*-solvers* for $\mathcal{LRA}$, $\mathcal{LIA}$ and their subtheories which are based on numerical algorithms must be implemented on top of infinite-precision-arithmetic software packages.

$\mathcal{T}$-solvers for Linear arithmetic over the integers ($\mathcal{LIA}$)

- EX: $\{(x := x_l + 2^{16}x_h), (x \geq 0), (x \leq 2^{16} - 1)\}$
- $\mathcal{LIA}$ NP-complete
- combination of many techniques: simplex, branch&bound, cutting planes, ... [41, 47]

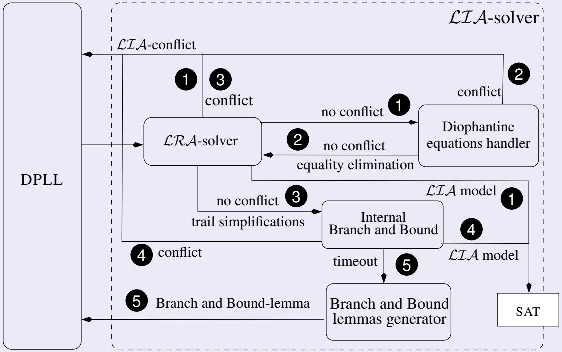


Figure courtesy of A. Griggio [47]

$\mathcal{T}$-solvers for Arrays ($\mathcal{AR}$)

- EX: $(write(A, i, v) = write(B, i, w)) \land \neg(v = w)$
- NP-complete
- congruence closure ($\mathcal{EUF}$) plus on-the-fly instantiation of array's axioms:

$$\forall a.\forall i.\forall e.\ (read(write(a, i, e), i) = e), \tag{1}$$

$$\forall a.\forall i.\forall j.\forall e.\ ((i \neq j) \rightarrow read(write(a, i, e), j) = read(a, j)), \tag{2}$$

$$\forall a.\forall b.\ (\forall i.(read(a, i) = read(b, i)) \rightarrow (a = b)). \tag{3}$$

- EX:

$$\begin{array}{ll} Input: & (write(A, i, v) = write(B, i, w)) \land \neg(v = w) \\ inst.\ (1): & (read(write(A, i, v), i) = v) \\ & (read(write(B, i, w), i) = w) \\ \models_{\mathcal{EUF}} & (v = w) \\ \models_{Bool} & \bot \end{array}$$

- many strategies discussed in the literature (e.g., [39, 46, 20, 38])

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$)

Bit vectors ($\mathcal{BV}$)

- EX: $\{(x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[16]}[3:0]), ...\}$
- NP-hard
- involve complex word-level operations: word partition/concat, modulo-2^N arithmetic, shifts, bitwise-operations, multiplexers, ...
- $\mathcal{T}$-solving: combination of rewriting & simplification techniques with either:
  - final encoding into $\mathcal{LIA}$ [19, 22]
  - final encoding into SAT (lazy bit-blasting) [25, 43, 21, 42]

Eager approach

Most solvers use an eager approach for $\mathcal{BV}$ (e.g., [21]):

- Heavy preprocessing, based on rewriting rules
- bit-blasting

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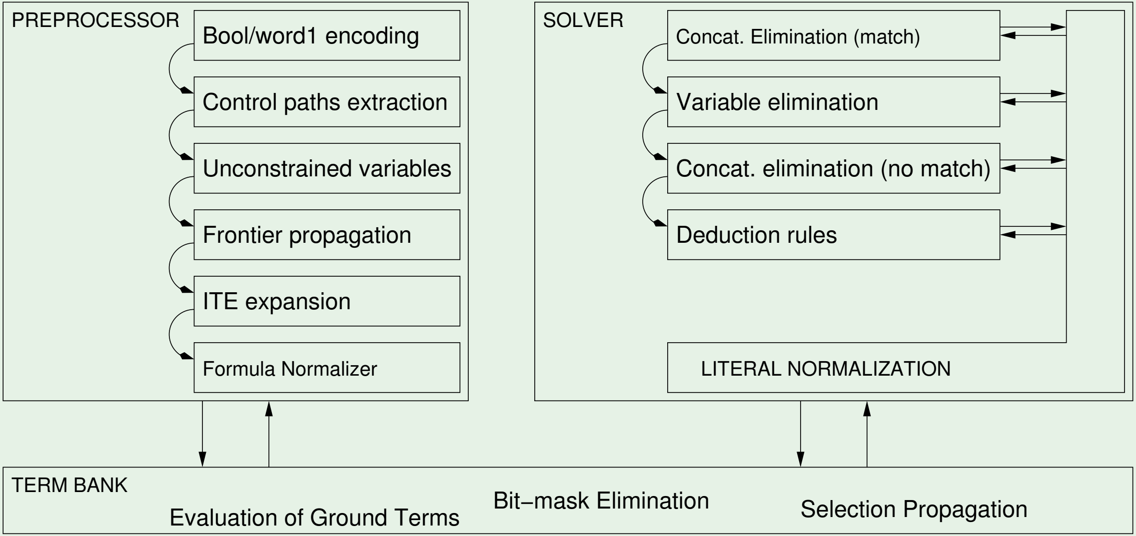
$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$) [cont.]



Example borrowed from [22]

Lazy bit-blasting

- Two nested SAT solvers
- bit-blast each $\mathcal{BV}$ atom ψ_i
  $\Longrightarrow \Phi \stackrel{\text{def}}{=} \bigwedge_i (A_i \leftrightarrow BB(\psi_i))$,
  A_i fresh variables labeling $\mathcal{BV}$-atoms ψ_i in φ
  $\Longrightarrow \varphi$ $\mathcal{BV}$-satisfiable iff $\varphi^p \wedge \Phi$ satisfiable
- Exploit SAT under assumptions
  - let μ^p an assignment for φ^p, s.t. $\mu^p \stackrel{\text{def}}{=} \{[\neg]A_1, ..., [\neg]A_n\}$
  - $\mathcal{T}$-solver for $\mathcal{BV}$: $SAT_{assumption}(\Phi, \mu^p)$
  - If UNSAT, generate the unsat core $\eta^p \subseteq \mu^p$

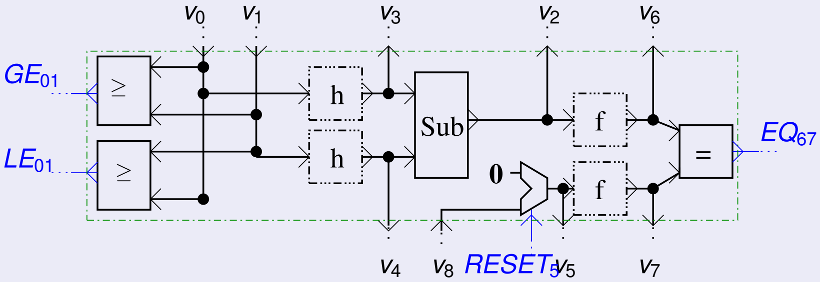
  $\Longrightarrow$ $\neg\eta^p$ used as blocking clause

Outline

1. Introduction
   - What is a Theory?
   - Satisfiability Modulo Theories
   - Motivations and Goals of SMT
2. Efficient SMT solving
   - Combining SAT with Theory Solvers
   - Theory Solvers for Theories of Interest (hints)
   - SMT for Combinations of Theories
3. Beyond Solving: Advanced SMT Functionalities
   - Proofs and Unsatisfiable Cores
   - Interpolants
   - All-SMT & Predicate Abstraction (hints)
   - SMT with Cost Optimization (Optimization Modulo Theories)
4. Conclusions & Current Research Directions

SMT for combined theories: $SMT(\bigcup_i \mathcal{T}_i)$

Problem: Many problems can be expressed as SMT problems only in combination of theories



$\bigcup_i \mathcal{T}_i$ — $SMT(\bigcup_i \mathcal{T}_i)$

$\mathcal{LIA}$: $(GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge$

$\mathcal{EUF}$: $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge$

$\mathcal{LIA}$: $(v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$

$\mathcal{EUF}$ or $\mathcal{LIA}$: $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge$

$\mathcal{EUF}$: $(v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$

$\mathcal{EUF}$ or $\mathcal{LIA}$: $(EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge$ ....

SMT for combined theories: SMT($\mathcal{T}_1 \cup \mathcal{T}_2$)

- Combined theories may be much harder to decide [Pratt'77]
- Solvers have to be combined
- Standard approach for combining $\mathcal{T}_i$-*solver*s: (deterministic) Nelson-Oppen/Shostak (N.O.) [61, 63, 76]
  - based on deduction and exchange of equalities on shared variables
  - combined $\mathcal{T}_i$-*solver*s integrated with a SAT tool
- SMT-specific approaches: Delayed Theory Combination [15, 14] and Model-Based Theory Combination [36]
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Background: Pure Formulas

Consider two theories $\mathcal{T}_1$, $\mathcal{T}_2$ with equality and disjoint signatures Σ_1, Σ_2

- W.l.o.g. we assume all input formulas $\phi \in \mathcal{T}_1 \cup \mathcal{T}_2$ are pure.
  - A formula ϕ is pure iff every atom in ϕ is i-pure for some $i \in \{1, 2\}$.
  - An atom/literal ψ in ϕ is *i*-pure if only $=$, variables and symbols from Σ_i can occur in ψ

Purification:

Maps a formula into an equisatisfiable pure formula by labeling terms with fresh variables

$$(f(\underbrace{x + 3y}_{w}) = g(\underbrace{2x - y}_{t})) \qquad [\textit{not pure}]$$

$$\Downarrow$$

$$(w = x + 3y) \wedge (t = 2x - y) \wedge (f(w) = g(t)) \qquad [\textit{pure}]$$

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Exercise

- Purify the following $\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$-formula (see beginning of chapter):

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \land (d < 1) \land$$
$$((f(d) = f(0)) \rightarrow (read(write(V, i, x), i + d) = x + 1))$$

Background: Interface equalities

Interface variables & equalities

- A variable v occurring in a pure formula ϕ is an interface variable iff it occurs in both 1-pure and 2-pure atoms of ϕ.
- An equality $(v_i = v_j)$ is an interface equality for ϕ iff v_i, v_j are interface variables for ϕ.
- We denote the interface equality $v_i = v_j$ by "e_{ij}"

Example:

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\end{array}
$$

$v_0, v_1, v_2, v_3, v_4, v_5$ are interface variables, v_6, v_7, v_8 are not

$\Longrightarrow (v_0 = v_1)$ is an interface equality, $(v_0 = v_6)$ is not.

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A Σ-theory $\mathcal{T}$ is stably-infinite iff every quantifier-free $\mathcal{T}$-satisfiable formula is satisfiable in an infinite model of $\mathcal{T}$.

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$, $\mathcal{LIA}$ are stably-infinite
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Intuition: a variable can be given an infinite amount of distinct values

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A Σ-theory $\mathcal{T}$ is convex iff, for every collection $l_1, ..., l_k, l', l''$ of literals in $\mathcal{T}$ s.t. l', l'' are in the form $(x = y)$, x, y being variables, we have that:

$\{l_1, ..., l_k\} \models_{\mathcal{T}} (l' \vee l'') \iff \{l_1, ..., l_k\} \models_{\mathcal{T}} l'$ *or* $\{l_1, ..., l_k\} \models_{\mathcal{T}} l''$

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$ are convex
- $\mathcal{LIA}$ is not convex:
  $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \models ((v = v_0) \vee (v = v_1)),$
  $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \not\models (v = v_0)$
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Intuition: non-convexity produces "case splits"

Background: Stably-infinite & Convex Theories

Stably-infinite Theories

A Σ-theory $\mathcal{T}$ is stably-infinite iff every quantifier-free $\mathcal{T}$-satisfiable formula is satisfiable in an infinite model of $\mathcal{T}$.

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$, $\mathcal{LIA}$ are stably-infinite
- (fixed-width) bit-vector theories are not stably-infinite

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Intuition: non-convexity produces “case splits”

$SMT(\bigcup_i \mathcal{T}_i)$ via "classic" Nelson-Oppen

Main Problem

- One predicate shared between distinct theories $\mathcal{T}_i$: equality "="
- Given $\mu \stackrel{\text{def}}{=} \bigcup_i \mu_i$ s.t. each μ_i contains i-pure literals
  - distinct $\mathcal{T}_i$-solver can be invoked separately on each μ_i...
  - ...producing distinct $\mathcal{T}_i$-specific models $\mathcal{M}_i$
- Problem: all models must agree on interface equalities:

$$\mathcal{M}_i \models_{\mathcal{T}_i} (v_k = v_l) \;\; iff \;\; \mathcal{M}_j \models_{\mathcal{T}_j} (v_k = v_l),$$

  for every pair of shared variables v_k, v_l

Main idea

Combine two or more $\mathcal{T}_i$*-solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 77]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
- important improvements and evolutions [69, 7, 39]

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Schema of N.O. combination of $\mathcal{T}$-solvers: no($\mathcal{T}_1, \mathcal{T}_2$)

For $i \in \{1, 2\}$, let $\mathcal{T}_i$ be a stably infinite theory admitting a satisfiability $\mathcal{T}_i$-solver, and μ_i a set of i-pure literals.
We want to to decide the $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiability of $\mu_1 \cup \mu_2$

- each $\mathcal{T}_i$-solver, in turn
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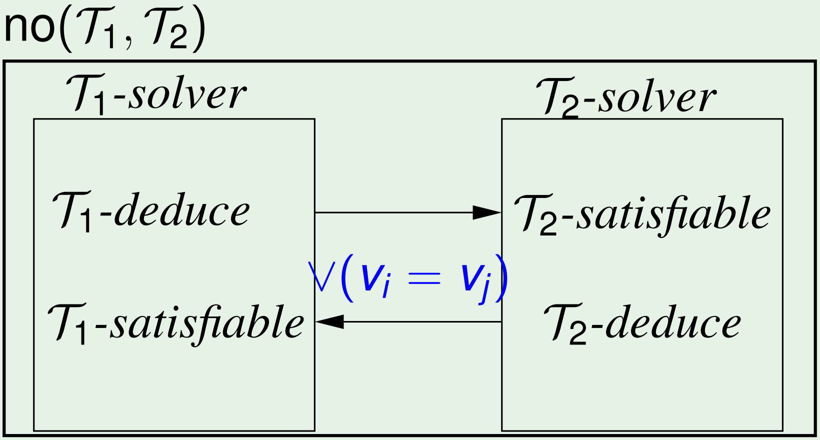

N.O. Example (Convex Theory)

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\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
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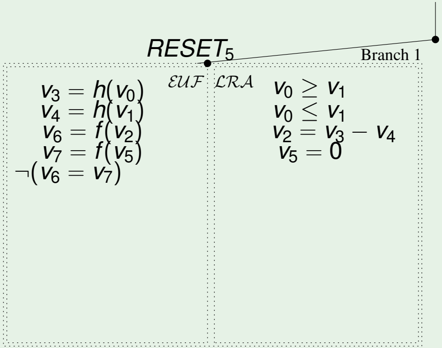

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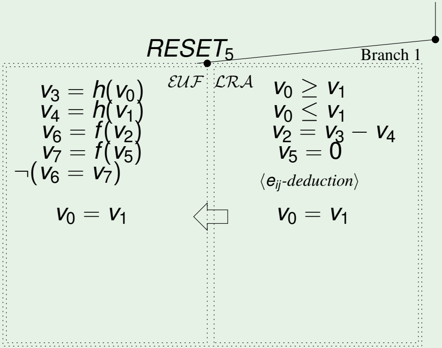

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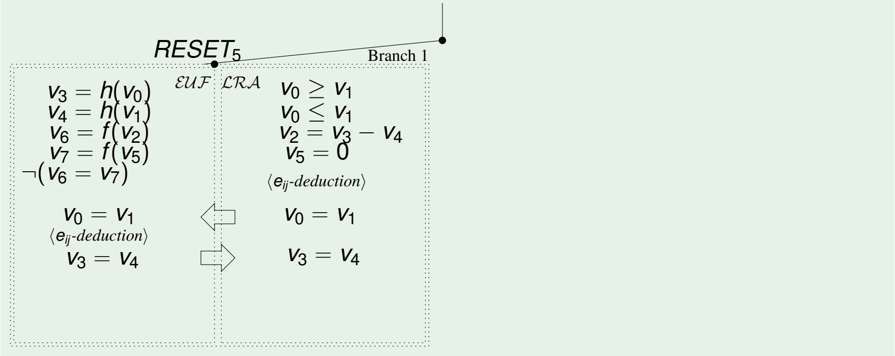

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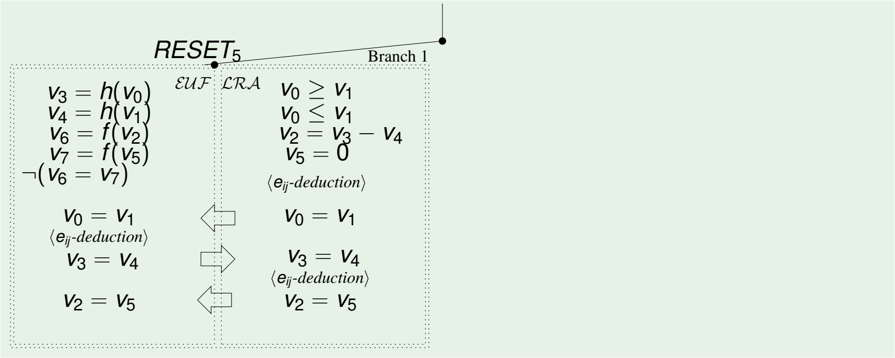

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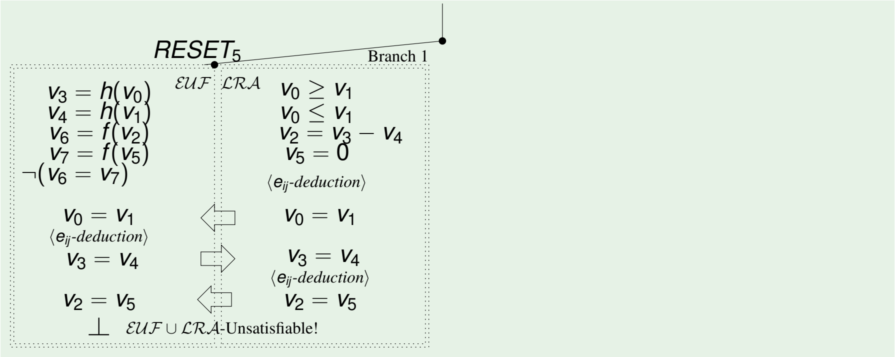

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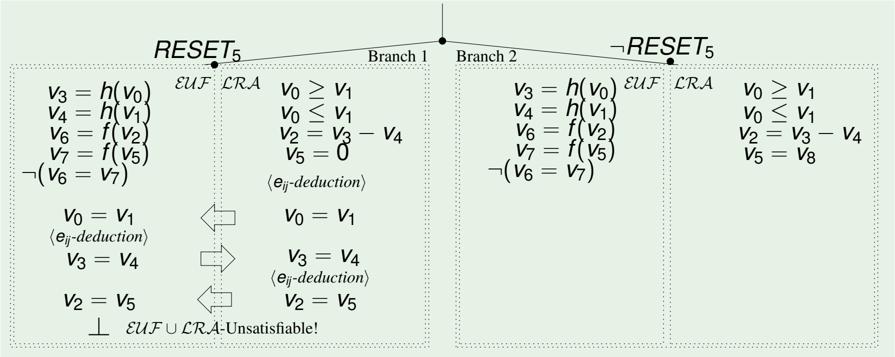

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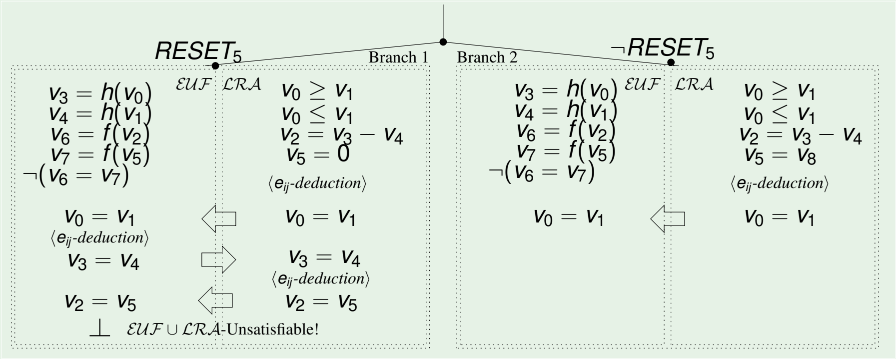

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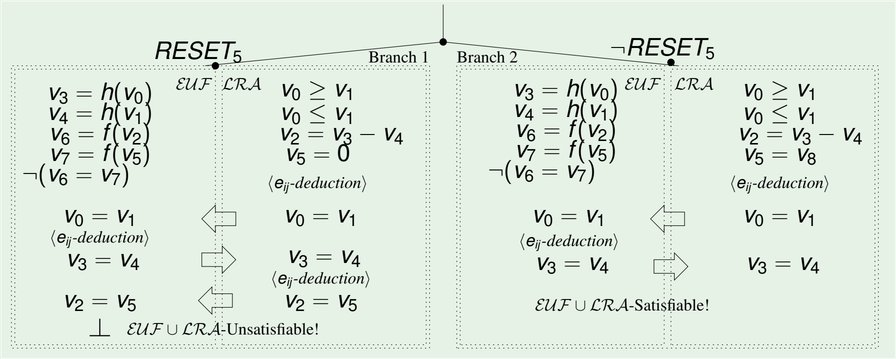
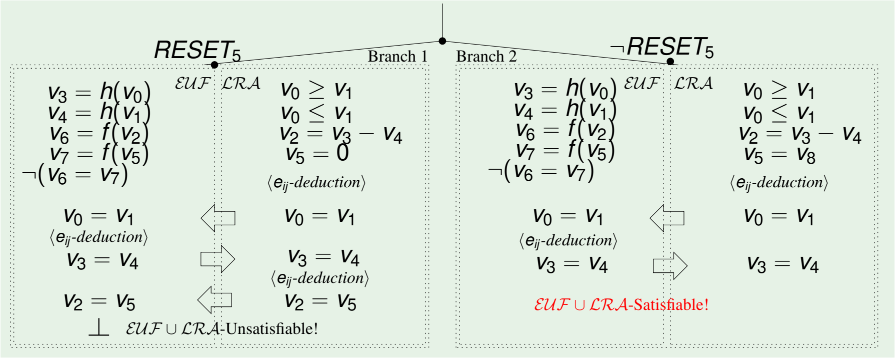
$Both:$ $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).$

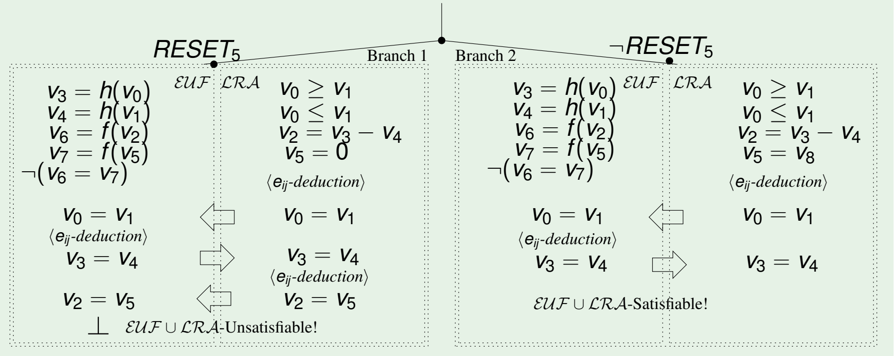


N.O. Example (Convex Theory)

$$\begin{array}{ll} \mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{array}$$



N.O.: example (convex theory) [cont.]



$\mathcal{EUF}$-conflict : $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_5)) \rightarrow \bot$

$\mathcal{LRA}$-deduction : $((v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$

$\mathcal{EUF}$-deduction : $((v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$

$\mathcal{LRA}$-deduction : $((v_0 \geq v_1) \wedge (v_0 \leq v_1)) \rightarrow (v_0 = v_1)$

$\Longrightarrow$

$\mathcal{EUF} \cup \mathcal{LRA}$-conflict : $((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_3 - v_4) \wedge$
$(v_5 = 0) \wedge (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 \geq v_1)) \rightarrow \bot.$

Exercises

For the previous N.O. example:

- write the (minimal) clauses corresponding to each e_{ij}-deduction
- find the final conflict clauses by resolving the e_{ij}-deduction clauses

Exercises

For the previous N.O. example:

- write the (minimal) clauses corresponding to each e_{ij}-deduction
- find the final conflict clauses by resolving the e_{ij}-deduction clauses

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

N.O.: example (non-convex theory)

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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
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N.O.: example (non-convex theory)

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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}\text{-}deduction \rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\Rightarrow$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
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$$v_1 = v_3$$

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$\langle e_{ij}\textit{-deduction}\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

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$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
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$$v_1 = v_3$$

$\langle e_{ij}\textit{-deduction}\rangle$

$$v_5 = v_6$$

N.O.: example (non-convex theory)

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$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$$v_1 = v_3$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_5 = v_6$$

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$$v_2 \leq v_6 + 1$$

⟨e_{ij}-deduction⟩

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

⟨e_{ij}-deduction⟩

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$$v_1 = v_3$$

⟨e_{ij}-deduction⟩

$$v_5 = v_6$$

$$v_2 = v_3$$
$$\bot$$

N.O.: example (non-convex theory)

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$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
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$$v_2 \leq v_6 + 1$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$$v_5 = v_6$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$$v_1 = v_3$$

$\langle e_{ij}\text{-deduction}\rangle$

$$v_5 = v_6$$

$$v_2 = v_3 \qquad v_2 = v_4$$
$$\bot \qquad \bot$$

N.O.: example (non-convex theory)

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$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\Rightarrow$

$v_1 = v_3$ | $v_1 = v_4$

$\langle e_{ij}$*-deduction*$\rangle$ | SAT!

$v_5 = v_6$

$\Leftarrow$

$$v_5 = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\Rightarrow$

$v_2 = v_3$ | $v_2 = v_4$

$\bot$ | $\bot$

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
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$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_1 = v_3 \vee v_1 = v_4$$

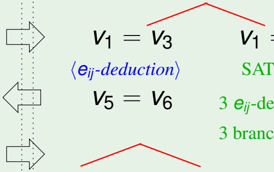
$$v_5 = v_6$$

$\langle e_{ij}$*-deduction*$\rangle$

$$v_2 = v_3 \vee v_2 = v_4$$

$\mu_{\mathcal{EUF}}$

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$



$v_1 = v_3$ $\qquad$ $v_1 = v_4$

$\langle e_{ij}$*-deduction*$\rangle$ $\qquad$ SAT!

$v_5 = v_6$ $\qquad$ 3 e_{ij}-deductions, 3 branches

$v_2 = v_3$ $\qquad$ $v_2 = v_4$

$\bot$ $\qquad$ $\bot$

$SMT(\bigcup_i \mathcal{T}_i)$ via "classic" Nelson-Oppen

Main idea

Combine two or more $\mathcal{T}_i$*-solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 77]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
- important improvements and evolutions [69, 7, 39]
- drawbacks [23, 24]:
  - require (possibly expensive) deduction capabilities from $\mathcal{T}_i$*-solvers*
  - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij}'s
  - generate (typically long) $(\bigcup_i \mathcal{T}_i)$-lemmas, without interface equalities $\Longrightarrow$ no backjumping & learning from e_{ij}-reasoning

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Combine two or more $\mathcal{T}_i$-*solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 77]

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  - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij}'s
  - generate (typically long) $(\bigcup_i \mathcal{T}_i)$-lemmas, without interface equalities
    $\Longrightarrow$ no backjumping & learning from e_{ij}-reasoning

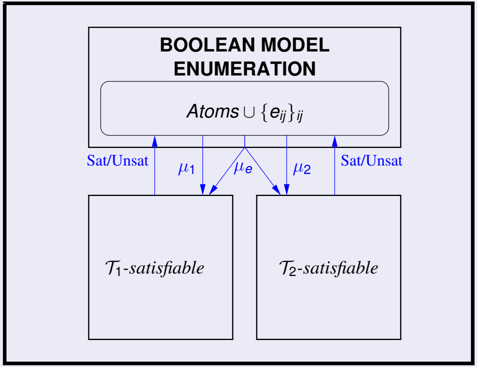
$SMT(\bigcup_i \mathcal{T}_i)$ via Delayed Theory Combination (DTC)

Main idea

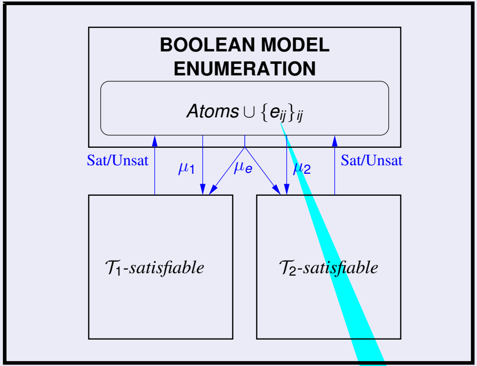
Delegate to the CDCL SAT solver part/most of the (possibly very expensive) reasoning effort on interface equalities previously due to the $\mathcal{T}_i$-*solvers* (e_{ij}-deduction, case-split). [15, 16, 24]

- based on Boolean reasoning on interface equalities via CDCL (plus $\mathcal{T}$-propagation)
- important improvements and evolutions [36, 9]
- feature wrt N.O. [23, 24]
  - do not require (possibly expensive) deduction capabilities from $\mathcal{T}_i$-*solvers*
  - with non-convex theories, case-splits on e_{ij}'s handled by SAT
  - generate $\mathcal{T}_i$-lemmas with interface equalities $\Longrightarrow$ backjumping & learning from e_{ij}-reasoning

DTC: Basic schema

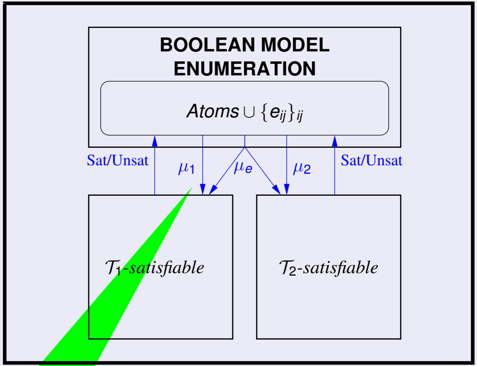


DTC: Basic schema



The boolean solver assigns values not only to atoms in $Atoms(\phi)$, but also to interface equalities $\{(v_i = v_j)\}_{ij}$:
$\mu = \mu_1 \cup \mu_2 \cup \mu_e, \quad \mu_e := \{[\neg](v_i = v_j) | v_i, v_j \in \mu_1 \cup \mu_2\}$

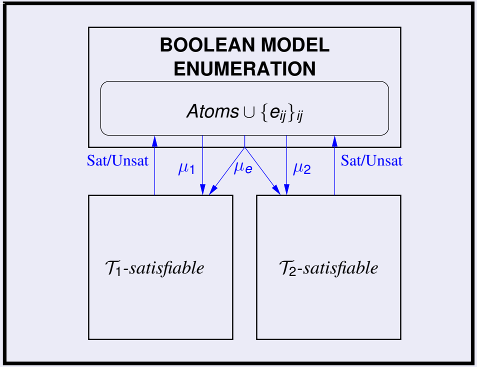
DTC: Basic schema



Each $\mathcal{T}_i$-*solver* interacts only with the boolean solver

- receives $\mu_i' := \mu_i \cup \mu_e$ from Bool
- checks the $\mathcal{T}_i$-satisfiability of μ_i'

DTC: Basic schema



...until either:

- some μ propositionally satisfies ϕ and both $\mu_i' := \mu_i \cup \mu_e$ are T_i-consistent
  $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-sat)
- no more assignment μ are available
  $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-unsat)

DTC: enhanced schema

- CDCL-based assignment enumeration on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$,
  $\Longrightarrow$ benefits of state-of-the-art SAT techniques
- Early pruning: invoke the $\mathcal{T}_i$-*solver*'s before every Boolean decision
  $\Longrightarrow$ total assignments generated only when strictly necessary
- Branching: branching on e_{ij}'s postponed
  $\Longrightarrow$ Boolean search on e_{ij}'s performed only when strictly necessary
- Theory-Backjumping & Learning: e_{ij}'s are involved in conflicts
  $\Longrightarrow$ e_{ij}'s can be assigned by unit propagation
- Theory-deduction & learning: if $\mathcal{T}_i$-*solver* deduces unassigned literals l on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$
  - l is passed back to the Boolean solver, which unit-propagates it
  - the deduction $\mu' \models l$ is learned as a clause $\mu' \rightarrow l$ (deduction clause)
- ...

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

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$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$\neg(v_1 = v_3)$

$\mathcal{LIA}$-unsat, C_{13}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \to ((v_1 = v_3) \lor (v_1 = v_4))$$

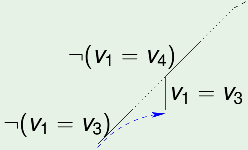
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ | $v_5 = v_4 - 1$
$v_1 \leq 1$ | $v_3 = 0$
$v_2 \geq v_6$ | $v_4 = 1$
$v_2 \leq v_6 + 1$



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

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$f(v_3) = v_5$
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$\mu_{\mathcal{LIA}}$:

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$v_1 \leq 1$ $\quad v_3 = 0$
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$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$\neg(v_1 = v_3)$

$\neg(v_5 = v_6)$

$\mathcal{EUF}$-unsat, C_{56}

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
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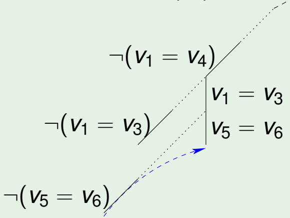


$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

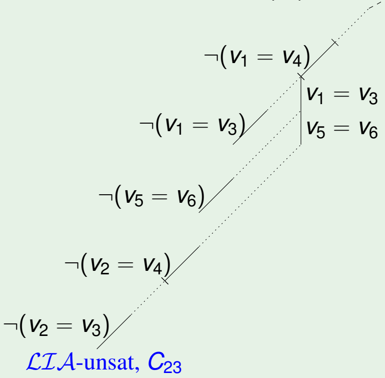
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$



$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
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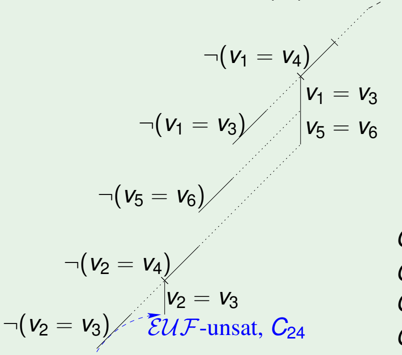
$$v_2 \leq v_6 + 1$$



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$

$\neg(f(v_2) = f(v_4))$

$f(v_3) = v_5$

$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$

$v_1 \leq 1$ $\quad v_3 = 0$

$v_2 \geq v_6$ $\quad v_4 = 1$

$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$\neg(v_1 = v_3)$

$v_5 = v_6$

$v_2 = v_4$

$\neg(v_5 = v_6)$

$\mathcal{EUF}$-unsat, C_{14}

$\neg(v_2 = v_4)$

$v_2 = v_3$

$\neg(v_2 = v_3)$

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$

$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$

$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$

$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$

$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

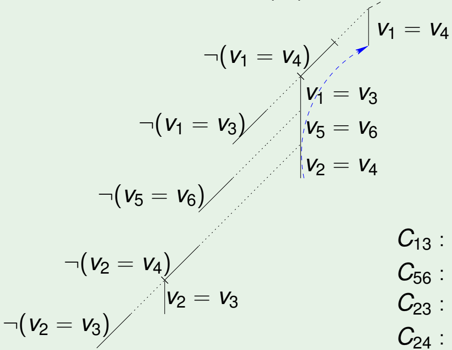
DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
$$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$v_1 = v_4$ $\quad$ $\neg(v_1 = v_4)$

SAT!

$v_1 = v_3$ $\quad$ $\neg(v_1 = v_3)$

$v_5 = v_6$ $\quad$ $\neg(v_5 = v_6)$

$v_2 = v_4$ $\quad$ $\neg(v_2 = v_4)$

$v_2 = v_3$ $\quad$ $\neg(v_2 = v_3)$

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$
$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
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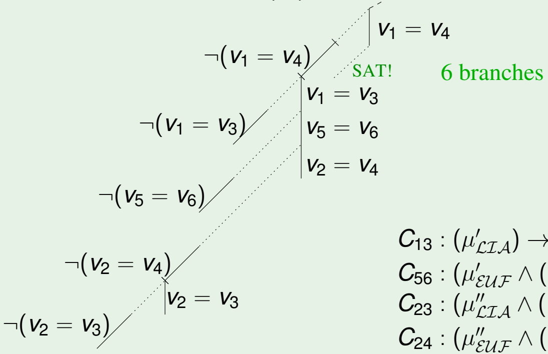

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
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$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
$$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$$

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$v_1 = v_4$

$\neg(v_1 = v_4)$

SAT!

$v_1 = v_3$

$\neg(v_1 = v_3)$

$v_5 = v_6$

$v_2 = v_4$

$\neg(v_5 = v_6)$

$\neg(v_2 = v_4)$

$v_2 = v_3$

$\neg(v_2 = v_3)$

Mimics the e_{ij}-deduction
$\mu'_{\mathcal{LIA}} \models_{\mathcal{LIA}} ((v_1 = v_3) \vee (v_1 = v_4))$
and the two branches $(v_1 = v_3), (v_1 = v_4)$

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$
$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$
$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$
$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$
$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$\mathcal{LIA}$-deduce $(v_1 = v_4) \vee (v_1 = v_3)$, C_{13}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

$\neg(v_1 = v_4)$

$v_1 = v_3$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$$\neg(v_1 = v_4)$$

$$v_1 = v_3$$

$\mathcal{EUF}$-deduce $(v_5 = v_6)$, C_{56}

$$v_5 = v_6$$

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$

$\neg(f(v_2) = f(v_4))$

$f(v_3) = v_5$

$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$

$v_1 \leq 1$ $\quad v_3 = 0$

$v_2 \geq v_6$ $\quad v_4 = 1$

$v_2 \leq v_6 + 1$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$v_5 = v_6$ $\quad$ $\mathcal{LIA}$-deduce $(v_2 = v_4) \vee (v_2 = v_3)$, C_{23}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$

$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$

$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$$\neg(v_1 = v_4)$$
$$v_1 = v_3$$
$$v_5 = v_6$$
$$\neg(v_2 = v_4)$$
$$v_2 = v_3$$

$\mathcal{EUF}$-unsat, C_{24}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$

$\neg(v_1 = v_4)$

$v_1 = v_3$

$v_5 = v_6$

$\neg(v_2 = v_4)$ $v_2 = v_4$

$v_2 = v_3$ $\mathcal{EUF}$-unsat, C_{14}

$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
$$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$$

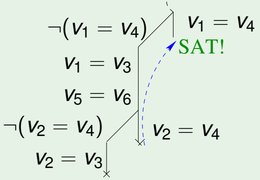
DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$$\neg(f(v_1) = f(v_2))$$
$$\neg(f(v_2) = f(v_4))$$
$$f(v_3) = v_5$$
$$f(v_1) = v_6$$

$\mu_{\mathcal{LIA}}$:

$$v_1 \geq 0 \qquad v_5 = v_4 - 1$$
$$v_1 \leq 1 \qquad v_3 = 0$$
$$v_2 \geq v_6 \qquad v_4 = 1$$
$$v_2 \leq v_6 + 1$$



$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
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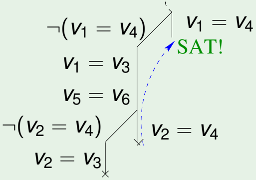
DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$
$v_1 \leq 1$
$v_2 \geq v_6$
$v_2 \leq v_6 + 1$
$v_5 = v_4 - 1$
$v_3 = 0$
$v_4 = 1$



3 e_{ij}-deductions
3 branches

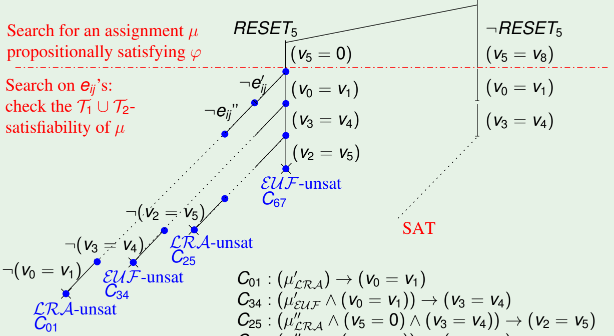
$$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v_3) \vee (v_1 = v_4))$$
$$C_{56} : (\mu'_{\mathcal{EUF}} \wedge (v_1 = v_3)) \rightarrow (v_5 = v_6)$$
$$C_{23} : (\mu''_{\mathcal{LIA}} \wedge (v_5 = v_6)) \rightarrow ((v_2 = v_3) \vee (v_2 = v_4))$$
$$C_{24} : (\mu''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_3)) \rightarrow \bot$$
$$C_{14} : (\mu'''_{\mathcal{EUF}} \wedge (v_1 = v_3) \wedge (v_2 = v_4)) \rightarrow \bot$$

DTC: example without $\mathcal{T}$-propagation (convex theory)

$$
\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}
$$

$\mu_{\mathcal{EUF}}$:
$\{(v_3 = h(v_0)), (v_4 = h(v_1)), \neg(v_6 = v_7),$
$(v_6 = f(v_2)), (v_7 = f(v_5))\}$

$\mu_{\mathcal{LRA}}$:
$\{(v_0 \geq v_1), (v_0 \leq v_1),$
$(v_2 = v_3 - v_4)\}$



$C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1)$
$C_{34} : (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$
$C_{25} : (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$
$C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)$

DTC: example with $\mathcal{T}$-propagation (convex theory)

$$
\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
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\end{array}
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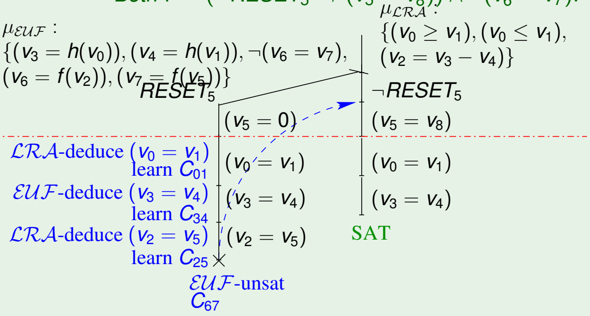

$$
\begin{array}{l}
C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1) \\
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C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)
\end{array}
$$

DTC + Model-based heuristic (aka Model-Based Theory Combination) [36]

- Initially, no interface equalities generated
- When a model is found, check against all the possible interface equalities
  - If $\mathcal{T}_1$ and $\mathcal{T}_2$ agree on the implied equalities, then return SAT
  - Otherwise, branch on equalities implied by $\mathcal{T}_1$-model but not by $\mathcal{T}_2$-model
- "Optimistic" approach, similar to axiom instantiation

Exercises

For each of the previous DTC examples:

- write the (minimal) clauses corresponding to each e_{ij}-deduction (as clauses rather than as implications)
- compute the conflict-analysis steps leading to the backjumping steps in the figures.

For each of the previous DTC examples, draw the case in which the $\mathcal{EUF}$-solver has deduction capabilities and the $\mathcal{LRA}$-solver (resp. the $\mathcal{LIA}$-solver) does not.

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Outline

Beyond Solving: advanced SAT & SMT functionalities

Advanced SMT functionalities (very important in FV):

- Building proofs of $\mathcal{T}$-unsatisfiability
- Extracting $\mathcal{T}$-unsatisfiable Cores
- Computing Craig interpolants
- Performing All-SMT and Predicate Abstraction
- Deciding/optimizing SMT problems with costs

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Outline

Building (Resolution) Proofs of $\mathcal{T}$-Unsatisfiability

Resolution proof of $\mathcal{T}$-unsatisfiability

Very similar to building proofs with plain SAT:

- resolution proofs whose leaves are original clauses and $\mathcal{T}$-lemmas returned by the $\mathcal{T}$-*solver* (i.e., $\mathcal{T}$-conflict and $\mathcal{T}$-deduction clauses)
- built by backward traversal of implication graphs, as in CDCL SAT
- Sub-proofs of $\mathcal{T}$-lemmas can be built in some $\mathcal{T}$-specific deduction framework if requested

Important for:

- certifying $\mathcal{T}$-unsatisfiability results
- computing unsatisfiable cores
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Important for:

- certifying $\mathcal{T}$-unsatisfiability results
- computing unsatisfiable cores
- computing interpolants

Building Proofs of $\mathcal{T}$-Unsatisfiability: example

$$(x=0 \vee \neg(x=1) \vee A_1) \wedge (x=0 \vee x=1 \vee A_2) \wedge (\neg(x=0) \vee x=1 \vee A_2) \wedge$$
$$(\neg A_2 \vee y=1) \wedge (\neg A_1 \vee x+y>3) \wedge (y<0) \wedge (A_2 \vee x-y=4) \wedge (y=2 \vee \neg A_1) \wedge (x \geq 0),$$

$(\neg(x=0) \vee \neg(x=1))_{\mathcal{LIA}}$ $\quad (x=1 \vee \neg(x=0) \vee A_2)$ $\quad (x=0 \vee \neg(x=1) \vee A_1)$ $\quad (x=1 \vee x=0 \vee A_2)$

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$(\neg A_1 \vee y=2)$ $\quad (A_1 \vee A_2)$

$(y=2 \vee A_2)$ $\quad (\neg(y=2) \vee \neg(y<0))_{\mathcal{LIA}}$

$(A_2 \vee \neg(y<0))$ $\quad (\neg A_2 \vee y=1)$

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$(y<0)$ $\quad (\neg(y<0))$

$\bot$

relevant original clauses, irrelevant original clauses, $\mathcal{T}$-lemmas

Example: proof on non-strict $\mathcal{LRA}$ inequalities

- A proof of unsatisfiability for a set of non-strict $\mathcal{LRA}$ inequalities can be obtained by building a linear combination of such inequalities, each time eliminating one or more variables, until you get a contradictory inequality on constant values.
- Example:

$$\varphi \stackrel{\text{def}}{=} (0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2), (0 \le x_3 - 2x_1 - 3), (0 \le 1 - 2x_3).$$

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Extraction of $\mathcal{T}$-unsatisfiable cores

The problem

Given a $\mathcal{T}$-unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) $\mathcal{T}$-unsatisfiable subset ($\mathcal{T}$-unsatisfiable core)

- Wide literature in SAT
- Some implementations, very few literature for SMT [28, 56]
- We recognize three approaches:
  - Proof-based approach (CVC4, MathSAT):
    byproduct of finding a resolution proof
  - Assumption-based approach (Yices):
    use extra variables labeling clauses, as in the plain Boolean case
  - Lemma-Lifting approach [28] :
    use an external (possibly-optimized) Boolean unsat-core extractor

The proof-based approach to $\mathcal{T}$-unsat cores

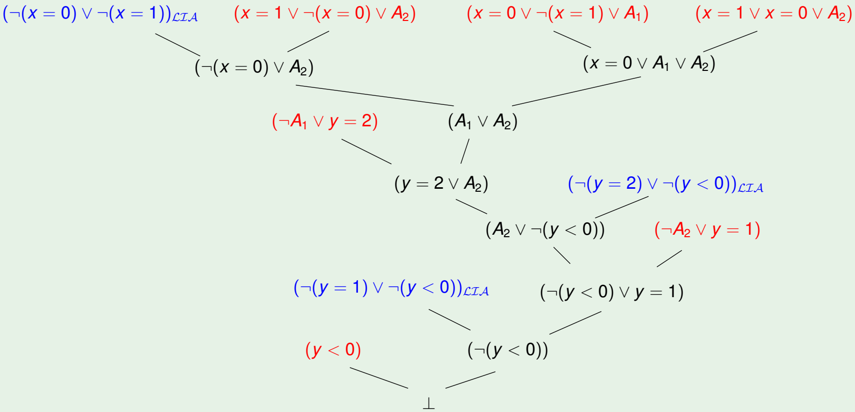
Idea (adapted from [83])

Unsatisfiable core of φ:

- in SAT: the set of leaf clauses of a resolution proof of unsatisfiability of φ
- in SMT($\mathcal{T}$): the set of leaf clauses of a resolution proof of $\mathcal{T}$-unsatisfiability of φ, minus the $\mathcal{T}$-lemmas

The proof-based approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \lor \neg(x = 1) \lor A_1) \land (x = 0 \lor x = 1 \lor A_2) \land (\neg(x = 0) \lor x = 1 \lor A_2) \land$$
$$(\neg A_2 \lor y = 1) \land (\neg A_1 \lor x + y > 3) \land (y < 0) \land (A_2 \lor x - y = 4) \land (y = 2 \lor \neg A_1) \land (x \geq 0),$$



The Assumption-based approach to $\mathcal{T}$-unsat cores

Idea (adapted from [57])

Let φ be $\bigwedge_{i=1}^{n} C_i$ s.t. φ unsatisfiable.

1. each clause C_i in φ is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh "selector" variable
2. the resulting formula is checked for satisfiability under the assumption of all S_i's
3. final conflict clause at dec. level 0: $\bigvee_j \neg S_j$
   $\Longrightarrow \{C_j\}_j$ is the unsat core

Extends straightforwardly to SMT($\mathcal{T}$).

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Extends straightforwardly to SMT($\mathcal{T}$).

The assumption-based approach to $\mathcal{T}$-unsat cores: Example

$$
\begin{aligned}
&(S_1 \rightarrow (x = 0 \vee \neg(x = 1) \vee A_1)) \wedge (S_2 \rightarrow (x = 0 \vee x = 1 \vee A_2)) \wedge \\
&\quad (S_3 \rightarrow (\neg(x = 0) \vee x = 1 \vee A_2)) \wedge (S_4 \rightarrow (\neg A_2 \vee y = 1)) \wedge \\
&\quad\quad (S_5 \rightarrow (\neg A_1 \vee x + y > 3)) \wedge (S_6 \rightarrow y < 0) \wedge \\
&\quad\quad\quad (S_7 \rightarrow (A_2 \vee x - y = 4)) \wedge (S_8 \rightarrow (y = 2 \vee \neg A_1)) \wedge (S_9 \rightarrow x \geq 0)
\end{aligned}
$$

Conflict analysis (Yices 1.0.6) returns:

$$\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_6 \vee \neg S_7 \vee \neg S_8,$$

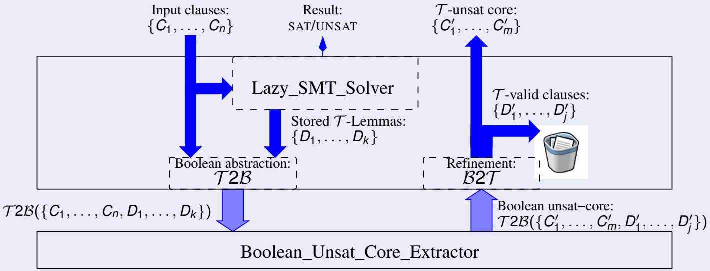
corresponding to the unsat core in red.

The lemma-lifting approach to $\mathcal{T}$-unsat cores

Idea [28, 32]

(i) The $\mathcal{T}$-lemmas D_i are valid in $\mathcal{T}$

(ii) The conjunction of φ with all the $\mathcal{T}$-lemmas $D_1, \ldots, D_k$ is propositionally unsatisfiable: $\mathcal{T}2\mathcal{B}(\varphi \wedge \bigwedge_{i=1}^{n} D_i) \models \bot$.



- interfaces with an external Boolean Unsat-core Extractor

$\Longrightarrow$ benefits for free of all state-of-the-art size-reduction techniques

The lemma-lifting approach to $\mathcal{T}$-unsat cores (cont.)

```
⟨SatValue,Clause_set⟩ 𝒯-Unsat_Core(Clause_set φ) {
   // φ is {C_1,...,C_n}
   if (Lazy_SMT_Solver(φ) == SAT)
      then return ⟨SAT,∅⟩;
   // D_1,...,D_k are the 𝒯-lemmas stored by Lazy_SMT_Solver
   ψ^p=Boolean_Core_Extractor(𝒯2ℬ({C_1,...,C_n,D_1,...,D_k}));
   // ψ^p is 𝒯2ℬ({C'_1,...,C'_m,D'_1,...,D'_j}));
   return ⟨UNSAT,{C'_1,...,C'_m}⟩;
}
```

The lemma-lifting approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$

1 The SMT solver generates the following set of $\mathcal{LIA}$-lemmas:

$$\{(\neg(x = 1) \vee \neg(x = 0)), \quad (\neg(y = 2) \vee \neg(y < 0)), \quad (\neg(y = 1) \vee \neg(y < 0))\}.$$

2 The following formula is passed to the external Boolean core extractor

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$$
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which returns the unsat core in red.

3 The unsat-core is mapped back, the three $\mathcal{T}$-lemmas are removed
$\Longrightarrow$ the final $\mathcal{T}$-unsat core (in red above).

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$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
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Exercise

Consider the following set of clauses φ in $\mathcal{EUF}$.

$$\left\{\begin{array}{l} (\neg(x = y) \vee (f(x) = f(y))), \\ (\neg(x = y) \vee \neg(f(x) = f(y))), \\ ((x = y) \vee (f(x) = f(y))), \\ ((x = y) \vee \neg(f(x) = f(y))) \end{array}\right\}$$

Find a minimal $\mathcal{EUF}$-unsatisfiable core.

Outline

Computing (Craig) Interpolants in SMT

Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models_{\mathcal{T}} \bot$, a *Craig interpolant* is a formula I s.t.:

a) $A \models_{\mathcal{T}} I$,

b) $I \wedge B \models_{\mathcal{T}} \bot$,

c) $I \preceq A$ and $I \preceq B$.

"$I \preceq A$" meaning that all non-interpreted (in $\mathcal{T}$) symbols in I occur in A (including variables)

- Important in some FV applications
- A few works presented for various theories:
  - $\mathcal{EUF}$ [59, 70], $\mathcal{DL}$ [29, 31], $\mathcal{UTVPI}$ [30, 31], $\mathcal{LRA}$ [59, 70, 29, 31], $\mathcal{LIA}$ [51, 18, 48], $\mathcal{BV}$ [52], ...

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A General Algorithm

Algorithm: Interpolant generation for SMT($\mathcal{T}$) [68, 59]

(i) Generate a resolution proof of $\mathcal{T}$-unsatisfiability $\mathcal{P}$ for $A \wedge B$.

(ii) ...

(iii) For every leaf clause C in $\mathcal{P}$,

- set $I_C \stackrel{\text{def}}{=} C \downarrow B$ if $C \in A$,
- set $I_C \stackrel{\text{def}}{=} \top$ if $C \in B$.

(iv) For every inner node C of $\mathcal{P}$ obtained by resolution from $C_1 \stackrel{\text{def}}{=} p \vee \phi_1$ and $C_2 \stackrel{\text{def}}{=} \neg p \vee \phi_2$,

- set $I_C \stackrel{\text{def}}{=} I_{C_1} \wedge I_{C_2}$ if p occurs in B,
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(v) Output $I_\perp$ as an interpolant for (A, B).

"$\eta \setminus B$" [resp. "$\eta \downarrow B$"] is the set of literals in η whose atoms do not [resp. do] occur in B.

- row 2. only takes place where $\mathcal{T}$ comes in to play

$\Longrightarrow$ Reduced to the problem of finding an interpolant for two sets of $\mathcal{T}$-literals (Boolean and $\mathcal{T}$-specific component decoupled)

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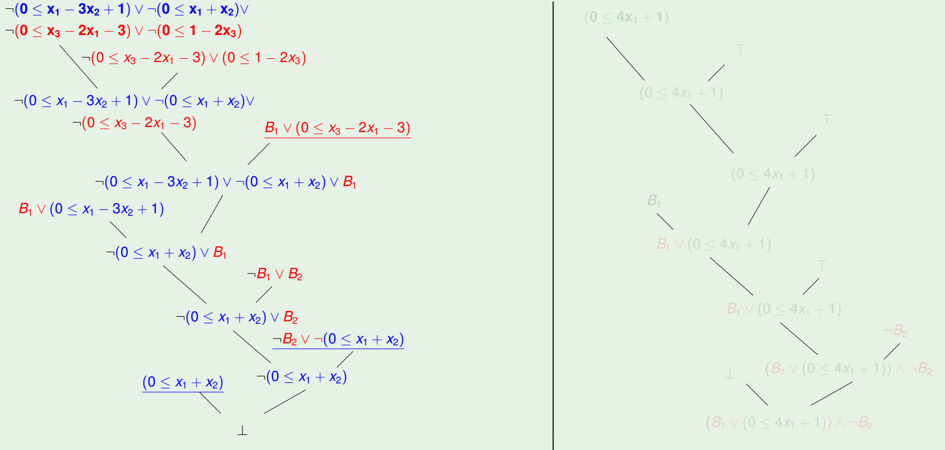
Computing Craig Interpolants in SMT: example

$$A \stackrel{\text{def}}{=} (B_1 \vee (0 \leq x_1 - 3x_2 + 1)) \wedge (0 \leq x_1 + x_2) \wedge (\neg B_2 \vee \neg(0 \leq x_1 + x_2))$$

$$B \stackrel{\text{def}}{=} (\neg(0 \leq x_3 - 2x_1 - 3) \vee (0 \leq 1 - 2x_3)) \wedge (\neg B_1 \vee B_2) \wedge (B_1 \vee (0 \leq x_3 - 2x_1 - 3))$$



original proof **interpolant proof**

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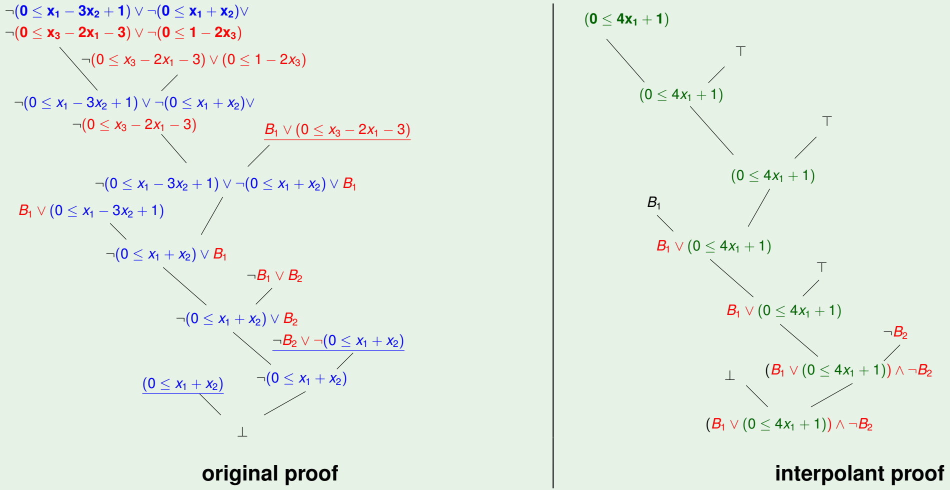

McMillan's algorithm for non-strict $\mathcal{LRA}$ inequalities

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2\}$$
$$B \stackrel{\text{def}}{=} \{(0 \le x_3 - 2x_1 - 3), (0 \le 1 - 2x_3)\}.$$

A proof of unsatisfiability P for $A \wedge B$ is the following:

$$\dfrac{\dfrac{(0 \le x_1 - 3x_2 + 1) \quad (0 \le x_1 + x_2)}{\textsc{Comb}\ (0 \le 4x_1 + 1)\ \textit{with c. 1 and 3}} \qquad \dfrac{(0 \le x_3 - 2x_1 - 3) \quad (0 \le 1 - 2x_3)}{\textsc{Comb}\ (0 \le -4x_1 - 5)\ \textit{with c. 2 and 1}}}{\textsc{Comb}\ (0 \le -4)\ \textit{with c. 1 and 1}}$$

By replacing inequalities in B with $(0 \le 0)$, we obtain the proof P':

$$\dfrac{\dfrac{(0 \le x_1 - 3x_2 + 1) \quad (0 \le x_1 + x_2)}{\textsc{Comb}\ (0 \le 4x_1 + 1)} \qquad \dfrac{(0 \le 0) \quad (0 \le 0)}{\textsc{Comb}\ (0 \le 0)}}{\textsc{Comb}\ (0 \le 4x_1 + 1)}$$

Thus, the interpolant obtained is $(0 \le 4x_1 + 1)$.

$$A \stackrel{\text{def}}{=} \{(0 \leq x_1 - 3x_2 + 1), (0 \leq x_1 + x_2\}$$
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$$\dfrac{\dfrac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\textsc{Comb}\ \ (0 \leq 4x_1 + 1)\ \textit{with c. 1 and 3}} \qquad \dfrac{(0 \leq x_3 - 2x_1 - 3) \quad (0 \leq 1 - 2x_3)}{\textsc{Comb}\ \ (0 \leq -4x_1 - 5)\ \textit{with c. 2 and 1}}}{\textsc{Comb}\ \ (0 \leq -4)\ \textit{with c. 1 and 1}}$$

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$$\dfrac{\dfrac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\textsc{Comb}\ \ (0 \leq 4x_1 + 1)} \qquad \dfrac{(0 \leq 0) \quad (0 \leq 0)}{\textsc{Comb}\ \ (0 \leq 0)}}{\textsc{Comb}\ \ (0 \leq 4x_1 + 1)}$$

Thus, the interpolant obtained is $(0 \leq 4x_1 + 1)$.

McMillan's algorithm for non-strict $\mathcal{LRA}$ inequalities

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2)\}$$
$$B \stackrel{\text{def}}{=} \{(0 \le x_3 - 2x_1 - 3), (0 \le 1 - 2x_3)\}.$$

A proof of unsatisfiability P for $A \wedge B$ is the following:

$$\dfrac{\dfrac{(0 \le x_1 - 3x_2 + 1) \quad (0 \le x_1 + x_2)}{\text{COMB } (0 \le 4x_1 + 1) \textit{ with c. 1 and 3}} \quad \dfrac{(0 \le x_3 - 2x_1 - 3) \quad (0 \le 1 - 2x_3)}{\text{COMB } (0 \le -4x_1 - 5) \textit{ with c. 2 and 1}}}{\text{COMB } (0 \le -4) \textit{ with c. 1 and 1}}$$

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Example: Interpolation Algorithms for Difference Logic

An inference-based algorithm [59]

$$A \stackrel{\text{def}}{=} \{(0 \le x_1 - x_2 + 1), (0 \le x_2 - x_3), (0 \le x_4 - x_5 - 1)\}$$

$$B \stackrel{\text{def}}{=} \{(0 \le x_5 - x_1), (0 \le x_3 - x_4 - 1)\}.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(0 \le x_1 - x_2 + 1) \quad (0 \le x_2 - x_3)}{\text{COMB} \quad (0 \le x_1 - x_3 + 1)} \quad (0 \le x_4 - x_5 - 1)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)} \quad (0 \le x_5 - x_1)}{\text{COMB} \quad (0 \le -x_3 + x_4)} \quad (0 \le x_3 - x_4 - 1)}{\text{COMB} \quad (0 \le -1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(0 \le x_1 - x_2 + 1) \quad (0 \le x_2 - x_3)}{\text{COMB} \quad (0 \le x_1 - x_3 + 1)} \quad (0 \le x_4 - x_5 - 1)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)} \quad (0 \le 0)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)} \quad (0 \le 0)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)}$$

$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + x_4 - x_5)$ (not in $\mathcal{DL}$, and weaker).

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$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + x_4 - x_5)$ (not in $\mathcal{DL}$, and weaker).

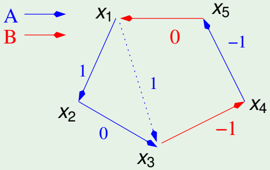
Example: Interpolation Algorithms for Difference Logic

A graph-based algorithm [29, 31]

Noticing that $(0 \leq x_i - x_j + c) \Longleftrightarrow (x_j - x_i \leq c)$:

$$A \stackrel{\text{def}}{=} \{\overbrace{(0 \leq x_1 - x_2 + 1), (0 \leq x_2 - x_3)}^{\textit{Chord}:\ (0 \leq x_1 - x_3 + 1)}, (0 \leq x_4 - x_5 - 1)\}$$

$$B \stackrel{\text{def}}{=} \{(0 \leq x_5 - x_1), (0 \leq x_3 - x_4 - 1)\}.$$



$\Longrightarrow$ Interpolant: $(0 \leq x_1 - x_3 + 1) \land (0 \leq x_4 - x_5 - 1)$ (still in $\mathcal{DL}$)

Exercise

Consider the following formulas in difference logic ($\mathcal{DL}$):

$$
\begin{array}{rcll}
\varphi_1 & \stackrel{\text{def}}{=} & (x_2 - x_3 \leq -4) & \wedge \\
 & & (x_3 - x_4 \leq -6) & \wedge \\
 & & (x_5 - x_6 \leq 4) & \wedge \\
 & & (x_6 - x_1 \leq 2) & \wedge \\
 & & (x_6 - x_7 \leq -2) & \wedge \\
 & & (x_7 - x_8 \leq 1) &
\end{array}
$$

$$
\begin{array}{rcll}
\varphi_2 & \stackrel{\text{def}}{=} & (x_4 - x_9 \leq 2) & \wedge \\
 & & (x_9 - x_5 \leq 0) & \wedge \\
 & & (x_1 - x_2 \leq 1) &
\end{array}
$$

which are such that $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \perp$. Compute an interpolant for $\langle \varphi_1, \varphi_2 \rangle$, using both methods presented in previous slides.

Outline

All-SAT/All-SMT (hints)

- All-SAT: enumerate all truth assignments satisfying φ
- All-SMT: enumerate all $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ
- All-SMT over an "important" subset of atoms $\Gamma \stackrel{\text{def}}{=} \{\gamma_i\}_i$: enumerate all assignments over Γ which can be extended to $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ
  $\Longrightarrow$ can compute predicate abstraction
- Algorithms:
  - BCLT [53]
    each time a $\mathcal{T}$-satisfiable assignment $\{l_1, ..., l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \Gamma$ belonged to the clause set
  - MathSAT/NuSMV [26]
    As above, plus the Boolean search of the SMT solver is driven by an OBDD.

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Predicate Abstraction

Predicate abstraction

if $\varphi(\mathbf{v})$ is a SMT formula over the domain variables $\mathbf{v} \stackrel{\text{def}}{=} \{v_j\}_j$, $\{\gamma_i\}_i$ is a set of "relevant" predicates over $\mathbf{v}$, and $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$ a set of fresh Boolean labels, then:

$$
\begin{aligned}
& \mathit{PredAbs}_{\mathbf{P}}(\varphi) \\
\stackrel{\text{def}}{=} \quad & \exists \mathbf{v}.(\ \varphi(\mathbf{v}) \wedge \bigwedge_i P_i \leftrightarrow \gamma_i(\mathbf{v})\ ) \\
= \quad & \bigvee \left\{ \mu \;\middle|\; \begin{array}{c} \mu \text{ truth assignment on } \mathbf{P} \\ \text{s.t. } \mu \wedge \varphi \wedge \bigwedge_i (P_i \leftrightarrow \gamma_i) \text{ is } \mathcal{T}\text{-satisfiable} \end{array} \right\}
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- important step in FV: extracts finite-state abstractions from a infinite state space

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- important step in FV: extracts finite-state abstractions from a infinite state space

Predicate Abstraction: example

$$\varphi \stackrel{\text{def}}{=} (v_1 + v_2 > 12)$$
$$\gamma_1 \stackrel{\text{def}}{=} (v_1 + v_2 = 2)$$
$$\gamma_2 \stackrel{\text{def}}{=} (v_1 - v_2 < 10)$$

$$\Downarrow$$

$$PreAbs(\varphi)_{\{P_1,P_2\}} \stackrel{\text{def}}{=} \exists\, v_1\ v_2\ .\left(\begin{array}{ll}(v_1 + v_2 > 12) & \wedge\\ (P_1 \leftrightarrow (v_1 + v_2 = 2)) & \wedge\\ (P_2 \leftrightarrow (v_1 - v_2 < 10)) & \end{array}\right)$$
$$= (\neg P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2)$$
$$= \neg P_1 .$$

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Outline

Optimization Modulo Theories: General Case

Ingredients: $\langle \varphi, cost \rangle$

- a SMT formula φ in some background theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$
  - $\bigcup_i \mathcal{T}_i$ may be empty
  - $\mathcal{T}_{\preceq}$ has a predicate $\preceq$ representing a total order
- a $\mathcal{T}_{\preceq}$-variable/term "*cost*" occurring in φ

Optimization Modulo $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ (OMT($\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$))

The problem of finding a model $\mathcal{M}$ for φ whose value of *cost* is minimum according to $\preceq$.

- maximization is dual

Note

The cost term can be rewritten as a variable

$$\langle \varphi, term \rangle \implies \langle \varphi \wedge (cost = term), cost \rangle, \quad cost\ fresh$$

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Optimization Modulo Theories with $\mathcal{LA}$ costs

Ingredients

- an SMT formula φ on $\mathcal{LA} \cup \mathcal{T}$
  - $\mathcal{LA}$ can be $\mathcal{LRA}$, $\mathcal{LIA}$ or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
  - $\mathcal{LA}$ and $\mathcal{T}_i$ Nelson-Oppen theories
    (i.e. signature-disjoint infinite-domain theories)
- a $\mathcal{LA}$ variable [term] "*cost*" occurring in φ
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t.
  lb $\leq$ *cost* $<$ ub (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{LA}$ costs (OMT($\mathcal{LA} \cup \mathcal{T}$))

Find a model for φ whose value of *cost* is minimum.

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We first restrict to the case $\mathcal{LA} = \mathcal{LRA}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{LRA}$)).

Optimization Modulo Theories with $\mathcal{LRA}$ costs

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Solving OMT($\mathcal{LRA}$) [72, 73]

General idea

Combine standard SMT and LP minimization techniques.

Offline Schema

- Minimizer: based on the Simplex $\mathcal{LRA}$-solver by [40]
  - Handles strict inequalities
- Search Strategies:
  - Linear-Search strategy
  - Mixed Linear/Binary strategy

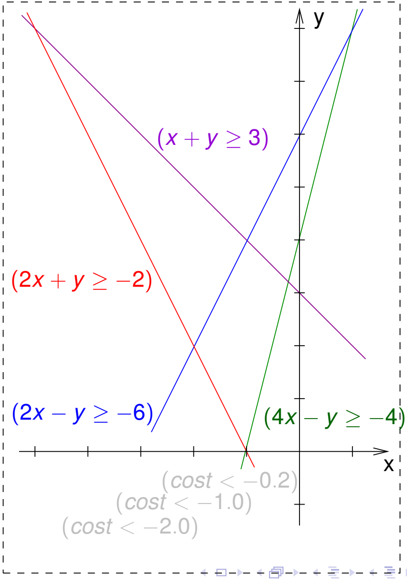
A toy example (linear search)

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rcl}
\varphi & \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
 & \wedge & (\ \ A_1 \vee (x + y \geq 3)) \\
 & \wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
 & \wedge & (\ \ A_2 \vee (2x - y \geq -6)) \\
 & \wedge & (cost < -0.2) \\
 & \wedge & (cost < -1.0) \\
 & \wedge & (cost < -2.0) \\
cost & \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$



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- OMT($\mathcal{LRA}$) problem:

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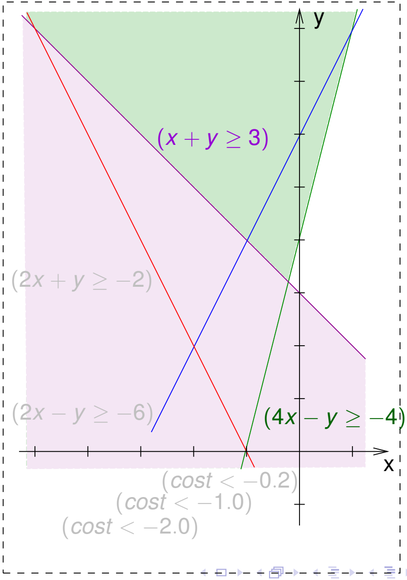
$\Longrightarrow$ SAT, $min = -0.2$



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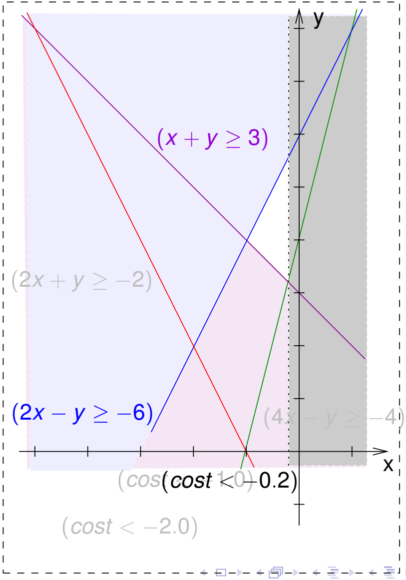

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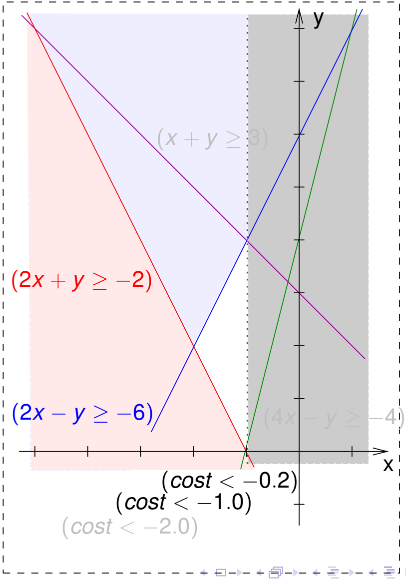
\wedge \; & (cost < -2.0) \\
cost \stackrel{\text{def}}{=} \; & x
\end{aligned}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \;\; A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$

$\Longrightarrow$ SAT, $min = -2.0$



A toy example (linear search)

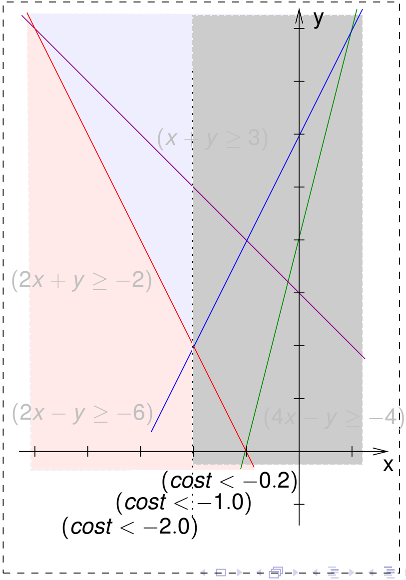
[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{aligned}
\varphi \stackrel{\text{def}}{=} \quad & (\neg A_1 \vee (2x + y \geq -2)) \\
\wedge \quad & (\ \ A_1 \vee (x + y \geq 3)) \\
\wedge \quad & (\neg A_2 \vee (4x - y \geq -4)) \\
\wedge \quad & (\ \ A_2 \vee (2x - y \geq -6)) \\
\wedge \quad & (cost < -0.2) \\
\wedge \quad & (cost < -1.0) \\
\wedge \quad & (cost < -2.0) \\
cost \stackrel{\text{def}}{=} \quad & x
\end{aligned}
$$

- $$\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$$

$\Longrightarrow$ UNSAT, $min = -2.0$

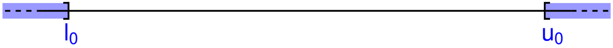


Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}; \text{u} \leftarrow \text{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\};$

while $(\text{l} < \text{u})$ **do**



Offline Schema: Mixed Linear/Binary-Search Strategy

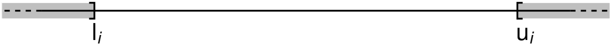
Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}; \text{u} \leftarrow \text{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\};$

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode



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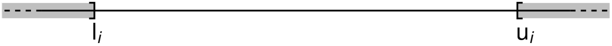
Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$
$\text{l} \leftarrow \text{lb}; \text{u} \leftarrow \text{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\};$
while ($\text{l} < \text{u}$) **do**
  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode
    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}; \text{u} \leftarrow \text{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\};$

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

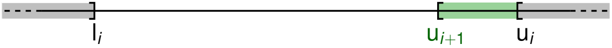
    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  if (res = SAT) **then**

    $\langle \mathcal{M}, \text{u} \rangle \leftarrow \mathcal{LRA}$-Solver.Minimize($cost, \mu$);

    $\varphi \leftarrow \varphi \cup \{(cost < \text{u})\};$

  else {res = UNSAT}



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, cost, \mathsf{lb}, \mathsf{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\mathsf{l} \leftarrow \mathsf{lb}$; $\mathsf{u} \leftarrow \mathsf{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \mathsf{lb}), (cost < \mathsf{ub})\}$;

while ($\mathsf{l} < \mathsf{u}$) **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode

    $\langle \mathsf{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  if ($\mathsf{res} = \textsc{sat}$) **then**

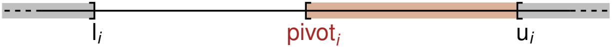
  else {$\mathsf{res} = \textsc{unsat}$}

    $\mathsf{l} \leftarrow \mathsf{u}$;

return $\langle \mathcal{M}, \mathsf{u} \rangle$

Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \mathit{cost}, \mathrm{lb}, \mathrm{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$
$\mathrm{l} \leftarrow \mathrm{lb}; \mathrm{u} \leftarrow \mathrm{ub}; \mathcal{M} \leftarrow \emptyset; \varphi \leftarrow \varphi \cup \{\neg(\mathit{cost} < \mathrm{lb}), (\mathit{cost} < \mathrm{ub})\}$;
while ($\mathrm{l} < \mathrm{u}$) **do**
  if (BinSearchMode()) **then** // Binary-search Mode
    pivot $\leftarrow$ ComputePivot(l, u);
    $\varphi \leftarrow \varphi \cup \{(\mathit{cost} < \mathrm{pivot})\}$;
    $\langle \mathrm{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);
  else // Linear-search Mode



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Input: $\langle \varphi, \textit{cost}, \mathsf{lb}, \mathsf{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\mathsf{l} \leftarrow \mathsf{lb}$; $\mathsf{u} \leftarrow \mathsf{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(\textit{cost} < \mathsf{lb}), (\textit{cost} < \mathsf{ub})\}$;

while $(\mathsf{l} < \mathsf{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    pivot $\leftarrow$ ComputePivot(l, u);

    $\varphi \leftarrow \varphi \cup \{(\textit{cost} < \text{pivot})\}$;

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

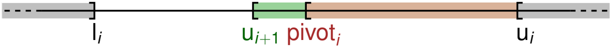
  else // Linear-search Mode

  if (res = SAT) **then**

    $\langle \mathcal{M}, \mathsf{u} \rangle \leftarrow \mathcal{LRA}$-Solver.Minimize($\textit{cost}, \mu$);

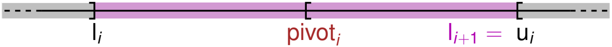
    $\varphi \leftarrow \varphi \cup \{(\textit{cost} < \mathsf{u})\}$;

  else {res = UNSAT}



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle\varphi, cost, \mathrm{lb}, \mathrm{ub}\rangle$ // lb can be $-\infty$, ub can be $+\infty$
$\mathrm{l} \leftarrow \mathrm{lb}$; $\mathrm{u} \leftarrow \mathrm{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \mathrm{lb}), (cost < \mathrm{ub})\}$;
while $(\mathrm{l} < \mathrm{u})$ **do**
  if (BinSearchMode()) **then** // Binary-search Mode
    pivot $\leftarrow$ ComputePivot(l, u);
    $\varphi \leftarrow \varphi \cup \{(cost < \mathrm{pivot})\}$;
    $\langle \mathrm{res}, \mu\rangle \leftarrow$ SMT.IncrementalSolve(φ);
  else // Linear-search Mode
  if (res $=$ SAT) **then**
  else {res $=$ UNSAT}
    if $((cost < \mathrm{pivot}) \notin$ SMT.ExtractUnsatCore(φ)) **then**
      $\mathrm{l} \leftarrow \mathrm{u}$;
    else
return $\langle\mathcal{M}, \mathrm{u}\rangle$



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Input: $\langle \varphi, cost, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(cost < \text{lb}), (cost < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    pivot $\leftarrow$ ComputePivot(l, u);

    $\varphi \leftarrow \varphi \cup \{(cost < \text{pivot})\}$;

    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

  else // Linear-search Mode

  if (res = SAT) **then**

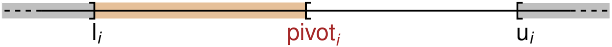
  else {res = UNSAT}

    if $((cost < \text{pivot}) \notin$ SMT.ExtractUnsatCore(φ)) **then**

    else

      $\text{l} \leftarrow \text{pivot}$;

      $\varphi \leftarrow (\varphi \setminus \{(cost < \text{pivot})\}) \cup \{\neg(cost < \text{pivot})\}$;



OMT with Lexicographic Combination of Objectives [13]

The problem

Find one optimal model $\mathcal{M}$ minimizing $\underline{c} \stackrel{\text{def}}{=} cost_1, cost_2, ..., cost_k$ lexicographically.

Solution

- Intuition:
  $\{minimize\ cost_1\}$
  $when$ UNSAT
  $\{substitute\ unit\ clause\ (cost_1 < min_1)\ with\ (cost_1 = min_1)\}$
  $\{minimize\ cost_2\}$
  ...
- improvement:
  - each time UNSAT is found, add $\bigwedge_i (cost_i \leq \mathcal{M}_i(cost_i))$ to φ

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) I

SMT with Pseudo-Boolean Constraints & Weighted MaxSMT

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\ \ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\ \ x_j\ fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

$$MaxSMT: \quad \langle \varphi_h, \bigwedge_j \psi_j \rangle \quad s.t.\ \psi_j\ soft,\ \ w_j = weight(\psi_j),\ \ w_i > 0$$

$$\Downarrow$$

$$minimize\ \sum_j x_j,\ \ x_j, A_j\ fresh$$

$$\varphi_h \wedge \bigwedge_j (A_j \vee \psi_j) \wedge \bigwedge_j (\neg A_j \vee (x_j = w_j)) \wedge (A_j \vee (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$OMT + PB: \quad \sum_j w_j \cdot A_j, \ \ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j, \ \ x_j \ fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$
$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

Range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$" logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i's are assigned :
  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \ \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
  $\Longrightarrow$ drastic pruning of the search
- same for weighted MaxSMT

Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$OMT + PB:$ $\sum_j w_j \cdot A_j,\ w_i > 0\ \ //(\sum_j ite(A_j, w_j, 0))$

$\Downarrow$

$\sum_j x_j,\ x_j\ fresh$

s.t. $\ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$

$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$

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Remark: range constraints “$(x_j \geq 0) \wedge (x_j \leq w_j)$”

$$OMT + PB: \quad \sum_j w_j \cdot A_j, \ \ w_i > 0 \ \ //(\sum_j ite(A_j, w_j, 0))$$
$$\Downarrow$$
$$\sum_j x_j, \ \ x_j \ fresh$$
$$\text{s.t.} \quad ... \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$
$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

Range constraints “$(x_j \geq 0) \wedge (x_j \leq w_j)$” logically redundant, but essential for efficiency:

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  Ex: $w_1 = 4, \ w_2 = 7, \ \sum_{i=1} x_i < 10, \ A_1 = A_2 = \top, \ A_i = * \ \forall i > 2.$
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Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\; w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\; x_j \; fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$
$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j))$$

Range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$" logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i's are assigned :
  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \; \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
  $\Longrightarrow$ drastic pruning of the search
- same for weighted MaxSMT

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) II

OMT with Min-Max [Max-Min] optimization

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$, find a solution which minimizes the maximum value among $\{cost_1, ..., cost_k\}$. (Max-Min dual.)

- Frequent in some applications (e.g. [73, 80])

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge \bigwedge_i (cost_i \leq cost), cost\}$ s.t. *cost* fresh.

OMT with linear combinations of costs

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$ and a set of weights $\{w_1, ..., w_k\}$, find a solution which minimizes $\sum_i w_i \cdot cost_i$.

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge (cost = \sum_i w_i \cdot cost_i), cost\}$ s.t. *cost* fresh.

These objectives can be composed with other OMT($\mathcal{LA}$) objectives.

Other OMT Functionalities [hints]

Incremental interface [13, 75]

Allows for pushing/popping sub-formulas into a stack, and then run OMT incrementally over them, reusing previous search.

- useful in some applications (e.g., BMC with parametric systems)
- straightforward variant of incremental SAT and SMT solvers

Pareto Fronts [13, 12]

- Given $cost_1$, $cost_2$, compute $\mathcal{M}_1, ..., \mathcal{M}_i, ..., \mathcal{M}_j, ...$ s.t.:
  - either $\mathcal{M}_i(cost_1) > \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) > \mathcal{M}_j(cost_2)$ and $\mathcal{M}_i(cost_1) < \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) < \mathcal{M}_j(cost_2)$
  - for each $\mathcal{M}_i$, no $\mathcal{M}'$ dominates $\mathcal{M}_i$
- no objective can be improved without degrading some other one

Some OMT tools

- BCLT [66, 54]
  http://www.cs.upc.edu/~oliveras/bclt-main.html
- OptiMathSAT [72, 73, 75, 74], on top of MathSAT [27]
  http://optimathsat.disi.unitn.it
- Symba [55], on top of Z3 [37]
  https://bitbucket.org/arieg/symba/src
- νZ [13, 12], on top of Z3 [37]
  http://z3.codeplex.com

Outline

Conclusions

- SMT very popular, due to successful application in many domains
- Combines techniques from SAT, ATP and operational research
- Not only satisfiability, but also advanced functionalities

Open/ongoing research directions

- Solving:
  - improve efficiency (e.g. $\mathcal{BV}$, $\mathcal{AR}$, $\mathcal{LIA}$ & their combinations)
    "a never-ending fight against the search-space explosion problem [E. Clarke, Turing-award winner 2007]"
  - develop efficient solvers for other theories ($\mathcal{NLA}(\mathbb{R})$, $\mathcal{NLA}(\mathbb{Z})$)
  - develop efficient solvers more-recent theories (e.g., floating-point arithmetic)
  - ...
- Functionalities
  - Interpolation in some theories ($\mathcal{LIA}$, $\mathcal{BV}$) still very challenging
  - Predicate abstraction (AllSMT) still a bottleneck in SMT-based FV
  - Optimization Modulo theories still in very early stage
  - ...
- Combination of SMT solvers and ATP (SMT with quantifiers)
- Integration & customization of SMT solvers with (FV) tools
- See also [67]

Links I

- survey papers:
  - Roberto Sebastiani: "Lazy Satisfiability Modulo Theories". Journal on Satisfiability, Boolean Modeling and Computation, JSAT. Vol. 3, 2007. Pag 141–224, ©IOS Press.
  - Clark Barrett, Roberto Sebastiani, Sanjit Seshia, Cesare Tinelli "Satisfiability Modulo Theories". Part II, Chapter 26, The Handbook of Satisfiability. 2009. ©IOS press.
  - Leonardo de Moura and Nikolaj Bjørner. “Satisfiability modulo theories: introduction and applications”. Communications of the ACM, 54 (9), 2011. ©ACM press.
- web links:
  - The SMT library SMT-LIB: `http://goedel.cs.uiowa.edu/smtlib/`
  - The SMT Competition SMT-COMP: `http://www.smtcomp.org/`
  - The SAT/SMT Schools `http://satassociation.org/sat-smt-school.html`

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A SAT Based Approach for Solving Formulas over Boolean and Linear Mathematical Propositions.
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The list of references above is by no means intended to be all-inclusive. I apologize both with the authors and with the readers for all the relevant works which are not cited here.


"That's all Folks!"