Course “An Introduction to SAT and SMT”
Chapter 1: Propositional Satisfiability (SAT)

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Outline

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Propositional Logic (aka Boolean Logic)

Basic Definitions

- Propositional formula (aka Boolean formula)
  - $\top, \bot$ are formulas
  - a propositional atom $A_1, A_2, A_3, ...$ is a formula;
  - if φ_1 and φ_2 are formulas, then
    $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2, \varphi_1 \oplus \varphi_2$
    are formulas.
- Ex: $\varphi \stackrel{\text{def}}{=} (\neg(A_1 \rightarrow A_2)) \wedge (A_3 \leftrightarrow (\neg A_1 \oplus (A_2 \vee \neg A_4)))$
- *Atoms*(φ): the set $\{A_1, ..., A_N\}$ of atoms occurring in φ.
  - Ex: $Atoms(\varphi) = \{A_1, A_2, A_3, A_4\}$
- Literal: a propositional atom A_i (positive literal) or its negation $\neg A_i$ (negative literal)
  - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- Clause: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee ...)$)
- Cube: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge ...)$)

Semantics of Boolean operators

Truth Table

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \rightarrow \beta$	$\alpha \leftarrow \beta$	$\alpha \leftrightarrow \beta$	$\alpha \oplus \beta$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

Semantics of Boolean operators (cont.)

Note

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are commutative:

$$
\begin{array}{lcl}
(\alpha \wedge \beta) & \iff & (\beta \wedge \alpha) \\
(\alpha \vee \beta) & \iff & (\beta \vee \alpha) \\
(\alpha \leftrightarrow \beta) & \iff & (\beta \leftrightarrow \alpha) \\
(\alpha \oplus \beta) & \iff & (\beta \oplus \alpha)
\end{array}
$$

- $\wedge$, $\vee$, $\leftrightarrow$ and $\oplus$ are associative:

$$
\begin{array}{lclcl}
((\alpha \wedge \beta) \wedge \gamma) & \iff & (\alpha \wedge (\beta \wedge \gamma)) & \iff & (\alpha \wedge \beta \wedge \gamma) \\
((\alpha \vee \beta) \vee \gamma) & \iff & (\alpha \vee (\beta \vee \gamma)) & \iff & (\alpha \vee \beta \vee \gamma) \\
((\alpha \leftrightarrow \beta) \leftrightarrow \gamma) & \iff & (\alpha \leftrightarrow (\beta \leftrightarrow \gamma)) & \iff & (\alpha \leftrightarrow \beta \leftrightarrow \gamma) \\
((\alpha \oplus \beta) \oplus \gamma) & \iff & (\alpha \oplus (\beta \oplus \gamma)) & \iff & (\alpha \oplus \beta \oplus \gamma)
\end{array}
$$

- $\rightarrow$, $\leftarrow$ are neither commutative nor associative:

$$
\begin{array}{lcl}
(\alpha \rightarrow \beta) & \not\iff & (\beta \rightarrow \alpha) \\
((\alpha \rightarrow \beta) \rightarrow \gamma) & \not\iff & (\alpha \rightarrow (\beta \rightarrow \gamma))
\end{array}
$$

Syntactic Properties of Boolean Operators

$$
\begin{array}{rcl}
\neg\neg\alpha & \Longleftrightarrow & \alpha \\
(\alpha \vee \beta) & \Longleftrightarrow & \neg(\neg\alpha \wedge \neg\beta) \\
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\neg(\alpha \leftarrow \beta) & \Longleftrightarrow & (\neg\alpha \wedge \beta) \\
(\alpha \leftrightarrow \beta) & \Longleftrightarrow & ((\alpha \rightarrow \beta) \wedge (\alpha \leftarrow \beta)) \\
 & \Longleftrightarrow & ((\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)) \\
\neg(\alpha \leftrightarrow \beta) & \Longleftrightarrow & (\neg\alpha \leftrightarrow \beta) \\
 & \Longleftrightarrow & (\alpha \leftrightarrow \neg\beta) \\
 & \Longleftrightarrow & ((\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta)) \\
(\alpha \oplus \beta) & \Longleftrightarrow & \neg(\alpha \leftrightarrow \beta)
\end{array}
$$

Boolean logic can be expressed in terms of $\{\neg, \wedge\}$ (or $\{\neg, \vee\}$) only!

Syntactic Properties of Boolean Operators

$$
\begin{array}{rcl}
\neg\neg\alpha & \Longleftrightarrow & \alpha \\
(\alpha \lor \beta) & \Longleftrightarrow & \neg(\neg\alpha \land \neg\beta) \\
\neg(\alpha \lor \beta) & \Longleftrightarrow & (\neg\alpha \land \neg\beta) \\
(\alpha \land \beta) & \Longleftrightarrow & \neg(\neg\alpha \lor \neg\beta) \\
\neg(\alpha \land \beta) & \Longleftrightarrow & (\neg\alpha \lor \neg\beta) \\
(\alpha \rightarrow \beta) & \Longleftrightarrow & (\neg\alpha \lor \beta) \\
\neg(\alpha \rightarrow \beta) & \Longleftrightarrow & (\alpha \land \neg\beta) \\
(\alpha \leftarrow \beta) & \Longleftrightarrow & (\alpha \lor \neg\beta) \\
\neg(\alpha \leftarrow \beta) & \Longleftrightarrow & (\neg\alpha \land \beta) \\
(\alpha \leftrightarrow \beta) & \Longleftrightarrow & ((\alpha \rightarrow \beta) \land (\alpha \leftarrow \beta)) \\
 & \Longleftrightarrow & ((\neg\alpha \lor \beta) \land (\alpha \lor \neg\beta)) \\
\neg(\alpha \leftrightarrow \beta) & \Longleftrightarrow & (\neg\alpha \leftrightarrow \beta) \\
 & \Longleftrightarrow & (\alpha \leftrightarrow \neg\beta) \\
 & \Longleftrightarrow & ((\alpha \lor \beta) \land (\neg\alpha \lor \neg\beta)) \\
(\alpha \oplus \beta) & \Longleftrightarrow & \neg(\alpha \leftrightarrow \beta)
\end{array}
$$

Boolean logic can be expressed in terms of $\{\neg, \land\}$ (or $\{\neg, \lor\}$) only!

Exercises

1. For every pair of formulas $\alpha \iff \beta$ below, show that α and β can be rewritten into each other by applying the syntactic properties of the previous slide
   - $(A_1 \wedge A_2) \vee A_3 \iff (A_1 \vee A_3) \wedge (A_2 \vee A_3)$
   - $(A_1 \vee A_2) \wedge A_3 \iff (A_1 \wedge A_3) \vee (A_2 \wedge A_3)$
   - $A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow A_4)) \iff (A_1 \wedge A_2 \wedge A_3) \rightarrow A_4$
   - $A_1 \rightarrow (A_2 \wedge A_3) \iff (A_1 \rightarrow A_2) \wedge (A_1 \rightarrow A_3)$
   - $(A_1 \vee A_2) \rightarrow A_3 \iff (A_1 \rightarrow A_3) \wedge (A_2 \rightarrow A_3)$
   - $A_1 \oplus A_2 \iff (A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$
   - $\neg A_1 \leftrightarrow \neg A_2 \iff A_1 \leftrightarrow A_2$
   - $A_1 \leftrightarrow A_2 \leftrightarrow A_3 \iff A_1 \oplus A_2 \oplus A_3$

Tree & DAG Representations of Formulas

- Formulas can be represented either as trees or as DAGS (Directed Acyclic Graphs)
- DAG representation can be up to exponentially smaller
  - in particular, when $\leftrightarrow$'s are involved

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

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$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$
$$\Downarrow$$
$$(((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge$$
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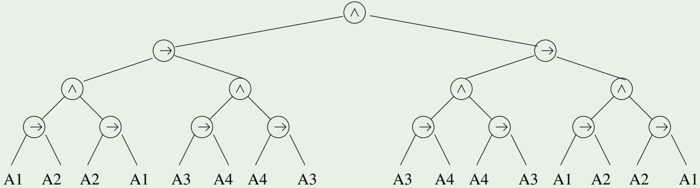
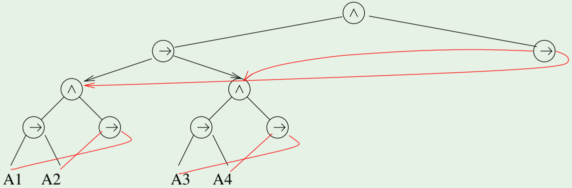
$$(((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1))))$$

Tree & DAG Representations of Formulas: Example

$$(((A_1 \to A_2) \land (A_2 \to A_1)) \to ((A_3 \to A_4) \land (A_4 \to A_3))) \land$$
$$(((A_3 \to A_4) \land (A_4 \to A_3)) \to (((A_1 \to A_2) \land (A_2 \to A_1))))$$



Tree Representation



DAG Representation

Semantics: Basic Definitions

- Total truth assignment μ for φ:
  $\mu : Atoms(\varphi) \longmapsto \{\top, \bot\}$.
  - represents a possible world or a possible state of the world
- Partial Truth assignment μ for φ:
  $\mu : \mathcal{A} \longmapsto \{\top, \bot\}$, $\mathcal{A} \subset Atoms(\varphi)$.
  - represents 2^k total assignments, k is $\#$ unassigned variables
- Notation: set and formula representations of an assignment
  - μ can be represented as a set of literals:
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies \{A_1, \neg A_2\}$
  - μ can be represented as a formula (cube):
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies (A_1 \wedge \neg A_2)$

Semantics: Basic Definitions [cont.]

- A total truth assignment μ satisfies φ (μ is a model of φ, $\mu \models \varphi$):

$$\begin{aligned}
&\mu \models A_i \iff \mu(A_i) = \top \\
&\mu \models \neg\varphi \iff \textit{not}\ \mu \models \varphi \\
&\mu \models \alpha \wedge \beta \iff \mu \models \alpha\ \textit{and}\ \mu \models \beta \\
&\mu \models \alpha \vee \beta \iff \mu \models \alpha\ \textit{or}\ \mu \models \beta \\
&\mu \models \alpha \rightarrow \beta \iff \ \textit{if}\ \mu \models \alpha,\ \textit{then}\ \mu \models \beta \\
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\end{aligned}$$

- $M(\varphi) \stackrel{\text{def}}{=} \{\mu \mid \mu \models \varphi\}$ (the set of models of φ)
- A partial truth assignment μ satisfies φ iff all total assignments extending μ satisfy φ
  - Ex: $\{A_1\} \models (A_1 \vee A_2)$ because both $\{A_1, A_2\} \models (A_1 \vee A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$
- φ is satisfiable iff $\mu \models \varphi$ for some μ (i.e. $M(\varphi) \neq \emptyset$)
- α entails β ($\alpha \models \beta$): $\alpha \models \beta$ iff $\mu \models \alpha \Longrightarrow \mu \models \beta$ for all μs (i.e., $M(\alpha) \subseteq M(\beta)$)
- φ is valid ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for all μs (i.e., $\mu \in M(\varphi)$ for all μs)

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Properties & Results

Property

φ is valid iff $\neg\varphi$ is not satisfiable

Deduction Theorem

$\alpha \models \beta$ iff $\alpha \rightarrow \beta$ is valid ($\models \alpha \rightarrow \beta$)

Corollary

$\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ is not satisfiable

Validity and entailment checking can be straightforwardly reduced to (un)satisfiability checking!

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Equivalence and Equi-Satisfiability

- α and β are equivalent iff, for every μ, $\mu \models \alpha$ iff $\mu \models \beta$ (i.e., if $M(\alpha) = M(\beta)$)
- α and β are equi-satisfiable iff exists μ_1 s.t. $\mu_1 \models \alpha$ iff exists μ_2 s.t. $\mu_2 \models \beta$ (i.e., if $M(\alpha) \neq \emptyset$ iff $M(\beta) \neq \emptyset$)
- α, β equivalent
  $\Downarrow \quad \not\Uparrow$
  α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
  $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α: $\beta \stackrel{\text{def}}{=} T(\alpha)$:
  - T is validity-preserving [resp. satisfiability-preserving] iff $T(\alpha)$ and α are equivalent [resp. equi-satisfiable]

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  (i.e., if $M(\alpha) \neq \emptyset$ iff $M(\beta) \neq \emptyset$)
- α, β equivalent
  $\Downarrow \quad \not\Uparrow$
  α, β equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
  $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when β is the result of applying some transformation T to α: $\beta \stackrel{\text{def}}{=} T(\alpha)$:
  - T is validity-preserving [resp. satisfiability-preserving] iff $T(\alpha)$ and α are equivalent [resp. equi-satisfiable]

Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi \;:=\; \varphi|_{v=\bot} \vee \varphi|_{v=\top}$$
$$\forall v.\varphi \;:=\; \varphi|_{v=\bot} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \ldots, w_n).\varphi \;:=\; \exists w_1 \ldots \exists w_n.\varphi$

- Intuition:
  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \bot\}$ s.t. $\mu \cup \{v := truthvalue\} \models \varphi$
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Complexity

NP-Completeness of SAT

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is NP-complete

$\Longrightarrow$ The most important logical problems (validity, inference, entailment, equivalence, ...) can be straightforwardly reduced to (un)satisfiability, and are thus (co)NP-complete.

$$\Downarrow$$

No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
  - φ occurs positively in φ;
  - if $\neg\varphi_1$ occurs positively [negatively] in φ,
    then φ_1 occurs negatively [positively] in φ
  - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ,
    then φ_1 and φ_2 occur positively [negatively] in φ;
  - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ,
    then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
  - if $\varphi_1 \leftrightarrow \varphi_2$ or $\varphi_1 \oplus \varphi_2$ occurs in φ,
    then φ_1 and φ_2 occur positively and negatively in φ;

Negative Normal Form (NNF)

- φ is in Negative normal form iff it is given only by the recursive applications of $\land, \lor$ to literals.
- every φ can be reduced into NNF:
  (i) substituting all $\rightarrow$'s and $\leftrightarrow$'s:

$$
\begin{array}{lll}
\alpha \rightarrow \beta & \Longrightarrow & \neg\alpha \lor \beta \\
\alpha \leftrightarrow \beta & \Longrightarrow & (\neg\alpha \lor \beta) \land (\alpha \lor \neg\beta)
\end{array}
$$

  (ii) pushing down negations recursively:

$$
\begin{array}{lll}
\neg(\alpha \land \beta) & \Longrightarrow & \neg\alpha \lor \neg\beta \\
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$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

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$$\Downarrow$$

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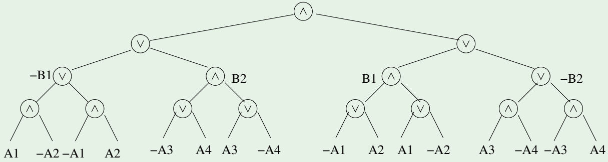
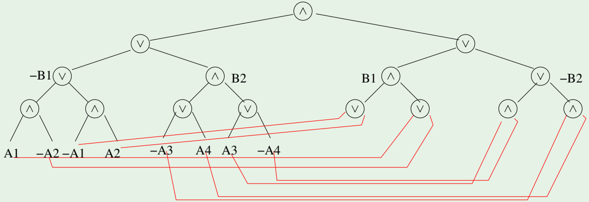
NNF: Example [cont.]

Note



Tree Representation



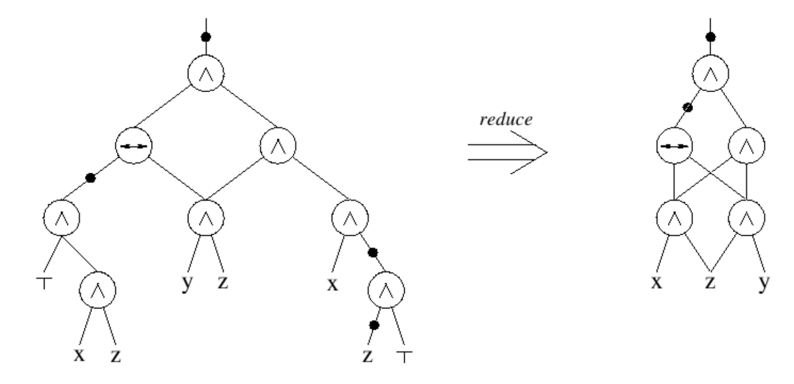
DAG Representation

For each non-literal subformula φ, φ and $\neg\varphi$ have different representations $\Longrightarrow$ they are not shared.

Optimized polynomial representations

And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations:
  $\{\wedge, \leftrightarrow, \neg\}$-only, negations on arcs, sorting of subformulae, lifting of $\neg$'s over $\leftrightarrow$'s,...



Conjunctive Normal Form (CNF)

- φ is in Conjunctive normal form iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
- Easier to handle: list of lists of literals.
  $\Longrightarrow$ no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,
  (i) expanding implications and equivalences:
  $$\alpha \rightarrow \beta \implies \neg\alpha \vee \beta$$
  $$\alpha \leftrightarrow \beta \implies (\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$$
  (ii) pushing down negations recursively:
  $$\neg(\alpha \wedge \beta) \implies \neg\alpha \vee \neg\beta$$
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- Resulting formula worst-case exponential:
  - ex: $||\mathrm{CNF}(\bigvee_{i=1}^{N}(l_{i1} \wedge l_{i2})|| = ||(l_{11} \vee l_{21} \vee ... \vee l_{N1}) \wedge (l_{12} \vee l_{21} \vee ... \vee l_{N1}) \wedge ... \wedge (l_{12} \vee l_{22} \vee ... \vee l_{N2})|| = 2^N$
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Labeling CNF conversion $CNF_{label}(\varphi)$

Labeling CNF conversion $CNF_{label}(\varphi)$ (aka Tseitin's CNF-ization)

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$
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  l_i, l_j being literals and B being a "new" variable.
- Worst-case linear!
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$
- $CNF_{label}(\varphi)$ is equi-satisfiable (but not equivalent) to φ.
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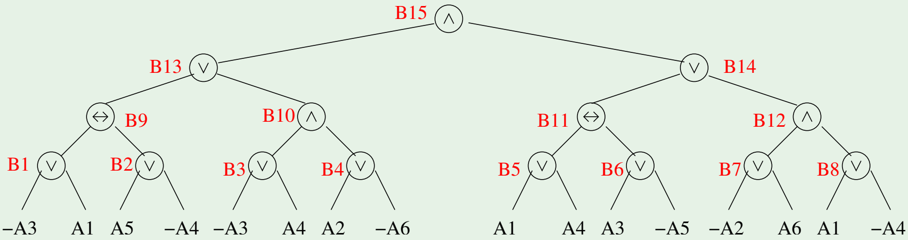
Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF Conversion CNF_{label} – Example



$$
\begin{array}{lcl}
CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))\wedge & & (\neg B_1 \vee \neg A_3 \vee A_1) \wedge (B_1 \vee A_3) \wedge (B_1 \vee \neg A_1)\wedge \\
\ldots\wedge & & \ldots\wedge \\
CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4))\wedge & & (\neg B_8 \vee A_1 \vee \neg A_4) \wedge (B_8 \vee \neg A_1) \wedge (B_8 \vee A_4)\wedge \\
CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2))\wedge & & (\neg B_9 \vee \neg B_1 \vee B_2) \wedge (\neg B_9 \vee B_1 \vee \neg B_2)\wedge \\
 & & (B_9 \vee B_1 \vee B_2) \wedge (B_9 \vee \neg B_1 \vee \neg B_2)\wedge \\
\ldots\wedge & = & \ldots\wedge \\
CNF(B_{12} \leftrightarrow (B_7 \wedge B_8))\wedge & & (B_{12} \vee \neg B_7 \vee \neg B_8) \wedge (\neg B_{12} \vee B_7) \wedge (\neg B_{12} \vee B_8)\wedge \\
CNF(B_{13} \leftrightarrow (B_9 \vee B_{10}))\wedge & & (\neg B_{13} \vee B_9 \vee B_{10}) \wedge (B_{13} \vee \neg B_9) \wedge (B_{13} \vee \neg B_{10})\wedge \\
CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12}))\wedge & & (\neg B_{14} \vee B_{11} \vee B_{12}) \wedge (B_{14} \vee \neg B_{11}) \wedge (B_{14} \vee \neg B_{12})\wedge \\
CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14}))\wedge & & (B_{15} \vee \neg B_{13} \vee \neg B_{14}) \wedge (\neg B_{15} \vee B_{13}) \wedge (\neg B_{15} \vee B_{14})\wedge \\
B_{15} & & B_{15}
\end{array}
$$

Labeling CNF conversion CNF_{label} (improved)

- As in the previous case, applying instead the rules:

$$
\begin{array}{llll}
\varphi \Longrightarrow \varphi[(l_i \vee l_j)|B] & \wedge\, CNF(B \rightarrow (l_i \vee l_j)) & \text{if } (l_i \vee l_j) \text{ pos.} \\
\varphi \Longrightarrow \varphi[(l_i \vee l_j)|B] & \wedge\, CNF((l_i \vee l_j) \rightarrow B) & \text{if } (l_i \vee l_j) \text{ neg.} \\
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\end{array}
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- Smaller in size:

$$
\begin{array}{ll}
CNF(B \rightarrow (l_i \vee l_j)) & = (\neg B \vee l_i \vee l_j) \\
CNF(((l_i \vee l_j) \rightarrow B)) & = (\neg l_i \vee B) \wedge (\neg l_j \vee B)
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Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i \neg l_j)$
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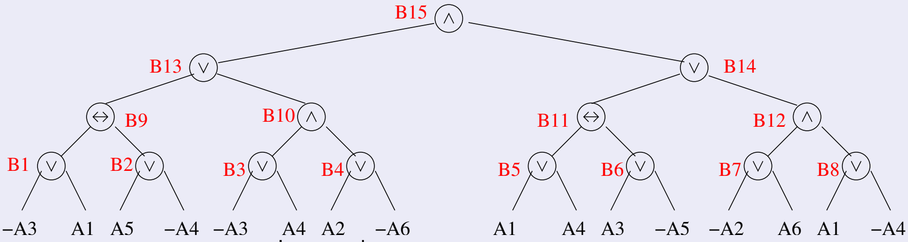
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF conversion CNF_{label} – example



Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))\quad \wedge$
$\ldots \quad \wedge$
$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4))\quad \wedge$
$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2))\quad \wedge$
$\ldots \quad \wedge$
$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8))\quad \wedge$
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B_{15}

Improved

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Labeling CNF conversion CNF_{label} – further improvements

- Do not apply CNF_{label} when not necessary:
  (e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \Longrightarrow CNF_{label}(\varphi_1) \wedge \varphi_2$, if φ_2 already in CNF)
- Apply DeMorgan's rules where it is more effective:
  (e.g., $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \Longrightarrow CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$)
- Exploit the associativity of $\wedge$'s and $\vee$'s:
  $\ldots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \ldots \Longrightarrow \ldots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3))\ldots$
- Before applying CNF_{label}, rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Exercises

1. Consider the following Boolean formula φ:
   $\neg(((\neg A_1 \rightarrow \neg A_2) \wedge (\neg A_3 \rightarrow A_4)) \vee ((A_5 \rightarrow A_6) \wedge (A_7 \rightarrow \neg A_8)))$
   Compute the Negative Normal Form of φ
2. Consider the following Boolean formula φ:
   $((\neg A_1 \wedge \neg A_2) \vee (A_7 \wedge A_4) \vee (\neg A_3 \wedge A_2) \vee (A_5 \wedge \neg A_4))$
   1. Produce the CNF formula $CNF(\varphi)$.
   2. Produce the CNF formula $CNF_{label}(\varphi)$.
   3. Produce the CNF formula $CNF_{label}(\varphi)$ (improved version)

Outline

Propositional Reasoning: Generalities

- Automated Reasoning in Propositional Logic fundamental task
  - AI, formal verification, circuit synthesis, operational research,....
- Important in AI: $KB \models \alpha$: entail fact α from knowledge base KB (aka Model Checking: $M(KB) \subseteq M(\alpha)$)
  - typically $KB >> \alpha$
- All propositional reasoning tasks reduced to satisfiability (SAT)
  - $KB \models \alpha \Longrightarrow$ SAT($KB \wedge \neg\alpha$)=false
  - input formula CNF-ized and fed to a SAT solver
- Current SAT solvers dramatically efficient:
  - handle industrial problems with $10^6 - 10^7$ variables & clauses!
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Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|Atoms(\varphi)|}$ lines.
- Never used in practice.

Outline

The Resolution Rule

- Resolution: deduction of a new clause from a pair of clauses with exactly one incompatible variable (resolvent):

$$\frac{(\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{l}^{resolvent} \vee \overbrace{l'_{k+1} \vee \ldots \vee l'_m}^{C'}) \quad (\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{\neg l}^{resolvent} \vee \overbrace{l''_{k+1} \vee \ldots \vee l''_n}^{C''})}{(\underbrace{l_1 \vee \ldots \vee l_k}_{common} \vee \underbrace{l'_{k+1} \vee \ldots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \ldots \vee l''_n}_{C''})}$$

- Ex: $\frac{(A \vee B \vee C \vee D \vee E) \quad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$

- Note: many standard inference rules subcases of resolution:
(recall that $\alpha \rightarrow \beta \iff \neg\alpha \vee \beta$)

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}\ (trans.) \qquad \frac{A \quad A \rightarrow B}{B}\ (m.\ ponens) \qquad \frac{\neg B \quad A \rightarrow B}{\neg A}\ (m.\ tollens)$$

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Improvements: Subsumption & Unit Propagation

Alternative "set" notation (Γ clause set):

$$\frac{\Gamma, \phi_1, ..\phi_n}{\Gamma, \phi'_1, ..\phi'_{n'}} \quad \left(e.g., \frac{\Gamma, C_1 \vee p, C_2 \vee \neg p}{\Gamma, C_1 \vee p, C_2 \vee \neg p, C_1 \vee C_2,} \right)$$

- Clause Subsumption (C clause):

$$\frac{\Gamma \wedge C \wedge (C \vee \bigvee_i l_i)}{\Gamma \wedge (C)}$$

- Unit Resolution:

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- Unit Subsumption:

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- Unit Propagation = Unit Resolution + Unit Subsumption

"Deterministic" rule: applied before other "non-deterministic" rules!

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Basic Propositional Inference: Resolution [47, 14]

- Assume input formula in CNF
  - if not, apply Tseitin CNF-ization first

$\Longrightarrow$ φ is represented as a set of clauses

- Search for a refutation of φ (is φ unsatisfiable?)
  - recall: $\alpha \models \beta$ iff $\alpha \wedge \neg\beta$ unsatisfiable
- Basic idea: apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either
  - a false clause is generated, or
  - the resolution rule is no more applicable
- Correct: if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- Complete: if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- Time-inefficient
- Very Memory-inefficient (exponential in memory)
- Many different strategies

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- Basic idea: apply iteratively the resolution rule to pairs of clauses with a conflicting literal, producing novel clauses, until either
  - a false clause is generated, or
  - the resolution rule is no more applicable
- Correct: if returns an empty clause, then φ unsat ($\alpha \models \beta$)
- Complete: if φ unsat ($\alpha \models \beta$), then it returns an empty clause
- Time-inefficient
- Very Memory-inefficient (exponential in memory)
- Many different strategies

Basic Propositional Inference: Resolution [47, 14]

- Assume input formula in CNF
  - if not, apply Tseitin CNF-ization first

$\Longrightarrow$ φ is represented as a set of clauses

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- Time-inefficient
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- Many different strategies

Resolution: basic strategy [14]

```
function DP(Γ)
    if ⊥ ∈ Γ                                        /* unsat */
        then return False;
    if (Resolve() is no more applicable to Γ)       /* sat      */
        then return True;
    if {a unit clause (l) occurs in Γ}              /* unit      */
        then  Γ := Unit_Propagate(l, Γ));
        return   DP(Γ)
    A := select-variable(Γ);                        /* resolve      */
    Γ = Γ ∪ ⋃_{A∈C′,¬A∈C″} {Resolve(C′, C″)} \ ⋃_{A∈C′,¬A∈C″} {C′, C″}};
    return   DP(Γ)
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$(A_1 \vee A_2)\ \ (A_1 \vee \neg A_2)\ \ (\neg A_1 \vee A_2)\ \ (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2)\ \ (A_2 \vee \neg A_2)\ \ (\neg A_2 \vee A_2)\ \ (\neg A_2)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A_1 \vee A_2)\ (A_1 \vee \neg A_2)\ (\neg A_1 \vee A_2)\ (\neg A_1 \vee \neg A_2)$$
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$$(A_2)\ (A_2 \vee \neg A_2)\ (\neg A_2 \vee A_2)\ (\neg A_2)$$
$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A_1 \vee A_2) \; (A_1 \vee \neg A_2) \; (\neg A_1 \vee A_2) \; (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2) \; (A_2 \vee \neg A_2) \; (\neg A_2 \vee A_2) \; (\neg A_2)$$

$$\Downarrow$$

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Resolution: Examples

$$(A_1 \vee A_2)\ \ (A_1 \vee \neg A_2)\ \ (\neg A_1 \vee A_2)\ \ (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2)\ \ (A_2 \vee \neg A_2)\ \ (\neg A_2 \vee A_2)\ \ (\neg A_2)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples (cont.)

$$(A \lor B \lor C) \;\; (B \lor \neg C \lor \neg F) \;\; (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \;\; (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
$$(A \lor E \lor \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$(A \lor B \lor C)\ \ (B \lor \neg C \lor \neg F)\ \ (\neg B \lor E)$$

$$\Downarrow$$

$$(A \lor C \lor E)\ \ (\neg C \lor \neg F \lor E)$$

$$\Downarrow$$

$$(A \lor E \lor \neg F)$$

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Resolution: Examples (cont.)

$$(A \lor B \lor C) \;\; (B \lor \neg C \lor \neg F) \;\; (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \;\; (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
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$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$(A \lor B \lor C) \;\; (B \lor \neg C \lor \neg F) \;\; (\neg B \lor E)$$
$$\Downarrow$$
$$(A \lor C \lor E) \;\; (\neg C \lor \neg F \lor E)$$
$$\Downarrow$$
$$(A \lor E \lor \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples

$$(A \vee B) \;\; (A \vee \neg B) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(A) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(C) \;\; (\neg C)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A \lor B) \;\; (A \lor \neg B) \;\; (\neg A \lor C) \;\; (\neg A \lor \neg C)$$

$$\Downarrow$$

$$(A) \;\; (\neg A \lor C) \;\; (\neg A \lor \neg C)$$

$$\Downarrow$$

$$(C) \;\; (\neg C)$$

$$\Downarrow$$

$$\bot$$

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$$(A \vee B) \;\; (A \vee \neg B) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(A) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(C) \;\; (\neg C)$$

$$\Downarrow$$

$$\bot$$

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$$(A \vee B) \;\; (A \vee \neg B) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(A) \;\; (\neg A \vee C) \;\; (\neg A \vee \neg C)$$
$$\Downarrow$$
$$(C) \;\; (\neg C)$$
$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
  $\Longrightarrow$ May require exponential space
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

Outline

Semantic tableaux [53]

- Search for an assignment satisfying φ
- applies recursively elimination rules to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i, the branch is closed, otherwise it is open.
- if no rule can be applied to an open branch μ, then $\mu \models \varphi$;
- if all branches are closed, the formula is not satisfiable;

Tableau elimination rules

$$\frac{\Gamma, (\varphi_1 \wedge \varphi_2)}{\Gamma, \varphi_1, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \vee \varphi_2)}{\Gamma, \neg\varphi_1, \neg\varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \rightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2} \qquad (\wedge\text{-elimination})$$

$$\frac{\Gamma, \neg\neg\varphi}{\Gamma, \varphi} \qquad (\neg\neg\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \vee \varphi_2)}{\Gamma, \varphi_1 \qquad \Gamma, \varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \wedge \varphi_2)}{\Gamma, \neg\varphi_1 \qquad \Gamma, \neg\varphi_2} \qquad \frac{\Gamma, (\varphi_1 \rightarrow \varphi_2)}{\Gamma, \neg\varphi_1 \qquad \Gamma, \varphi_2} \qquad (\vee\text{-elimination})$$

$$\frac{\Gamma, (\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \varphi_2 \qquad \Gamma, \neg\varphi_1 \neg\varphi_2} \qquad \frac{\Gamma, \neg(\varphi_1 \leftrightarrow \varphi_2)}{\Gamma, \varphi_1, \neg\varphi_2 \qquad \Gamma, \neg\varphi_1 \varphi_2} \qquad (\leftrightarrow\text{ -elimination}).$$

Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

Semantic Tableaux – Example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

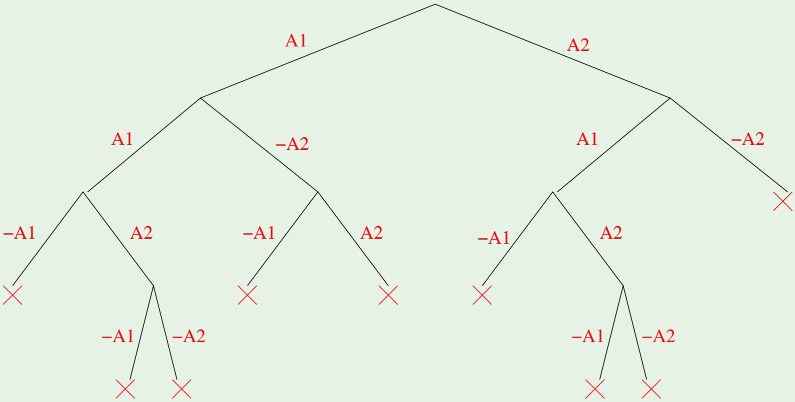


Tableau algorithm

```
function Tableau(Γ)
    if A_i ∈ Γ and ¬A_i ∈ Γ                                  /* branch closed */
        then return False;
    if (φ1 ∧ φ2) ∈ Γ                                         /* ∧-elimination */
        then return Tableau(Γ ∪ {φ1, φ2}\{(φ1 ∧ φ2)});
    if (¬¬φ1) ∈ Γ                                            /* ¬¬-elimination */
        then return Tableau(Γ ∪ {φ1}\{(¬¬φ1)});
    if (φ1 ∨ φ2) ∈ Γ                                         /* ∨-elimination */
        then return     Tableau(Γ ∪ {φ1}\{(φ1 ∨ φ2)})   or
                        Tableau(Γ ∪ {φ2}\{(φ1 ∨ φ2)});
    ...
    return True;                                             /* branch expanded */
```

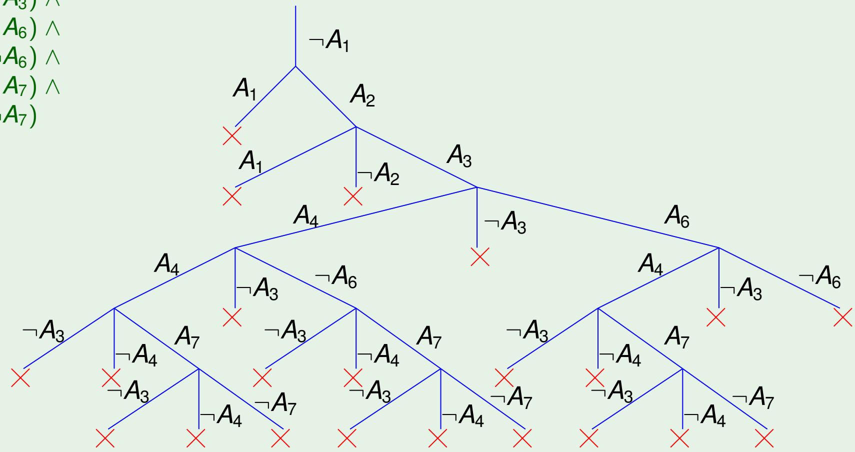
Semantic Tableaux: Example

$$
\begin{array}{rcrcrl}
 & & & & (\neg A_1) & \wedge \\
 & & (\ A_1 & \vee & A_2) & \wedge \\
(\ A_1 & \vee & \neg A_2 & \vee & A_3) & \wedge \\
(\ A_4 & \vee & \neg A_3 & \vee & A_6) & \wedge \\
(\ A_4 & \vee & \neg A_3 & \vee & \neg A_6) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & A_7) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & \neg A_7) &
\end{array}
$$

$\Longrightarrow$ unsat

Semantic Tableaux: Example

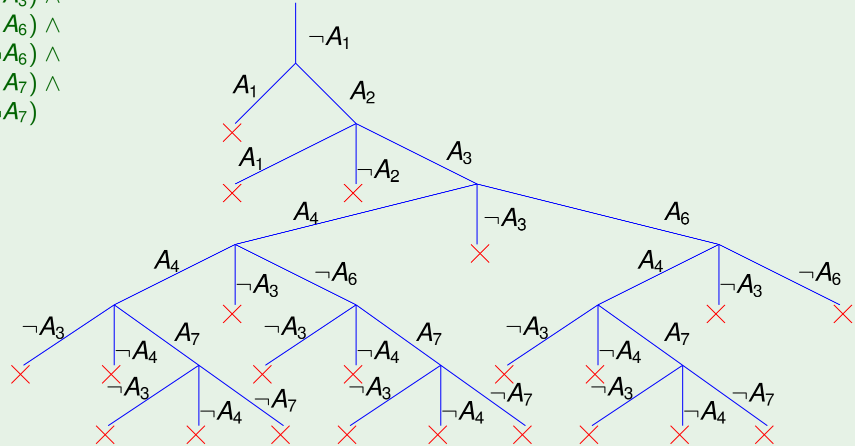
$$
\begin{array}{l}
(\neg A_1) \land \\
(A_1 \lor A_2) \land \\
(A_1 \lor \neg A_2 \lor A_3) \land \\
(A_4 \lor \neg A_3 \lor A_6) \land \\
(A_4 \lor \neg A_3 \lor \neg A_6) \land \\
(\neg A_3 \lor \neg A_4 \lor A_7) \land \\
(\neg A_3 \lor \neg A_4 \lor \neg A_7)
\end{array}
$$



$\Longrightarrow$ unsat

Semantic Tableaux: Example

$$
\begin{array}{rrrrrl}
 & & & & (\neg A_1) & \wedge \\
 & & (\ A_1 & \vee & A_2) & \wedge \\
(\ A_1 & \vee & \neg A_2 & \vee & A_3) & \wedge \\
(\ A_4 & \vee & \neg A_3 & \vee & A_6) & \wedge \\
(\ A_4 & \vee & \neg A_3 & \vee & \neg A_6) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & A_7) & \wedge \\
(\neg A_3 & \vee & \neg A_4 & \vee & \neg A_7) &
\end{array}
$$



$\Longrightarrow$ unsat

Semantic Tableaux – Summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend
  $\Longrightarrow$ loved by logicians.
- Rather inefficient
  $\Longrightarrow$ avoided by computer scientists.
- Requires polynomial space

Outline

DPLL [14, 13]

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ;
- At each step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL rules

$$\frac{\varphi_1 \wedge (l)}{\varphi_1[l|\top]} \ (Unit)$$

$$\frac{\varphi}{\varphi[l|\top]} \ (l\ Pure)$$

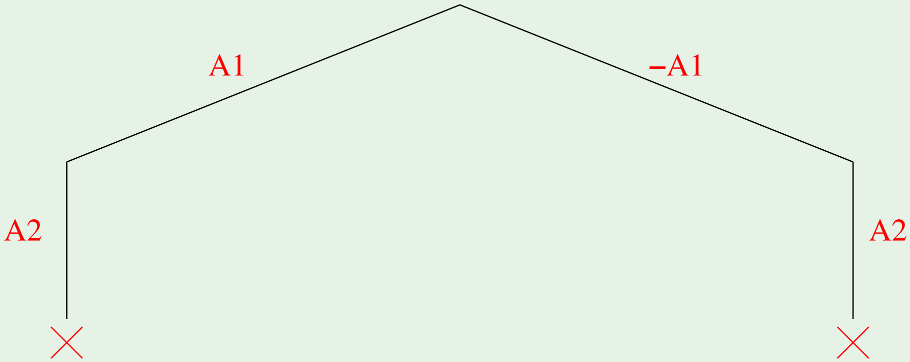
$$\frac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} \ (split)$$

(l is a pure literal in φ iff it occurs only positively).

- Split applied if and only if the others cannot be applied.
- Richer formalisms described in [55, 42, 43]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



DPLL Algorithm

function *DPLL*(φ, μ)
    if $\varphi = \top$ /* base */
        then return *True*;
    if $\varphi = \bot$ /* backtrack */
        then return *False*;
    if {a unit clause (l) occurs in φ}   /* unit */
        then return *DPLL*(*assign*(l, φ), $\mu \wedge l$);
    if {a literal l occurs *pure* in φ}   /* pure */
        then return *DPLL*(*assign*(l, φ), $\mu \wedge l$);
    l := *choose-literal*(φ);   /* split */
    return *DPLL*(*assign*(l, φ), $\mu \wedge l$) **or**
        DPLL(*assign*($\neg l, \varphi$), $\mu \wedge \neg l$);

- The pure-literal rule is nowadays obsolete.
- *choose-literal*(φ) picks only variables still occurring in the formula

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        then return *True*;
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    l := *choose-literal*(φ); /* split */
    return *DPLL*(*assign*(l, φ), $\mu \wedge l$) **or**
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- The pure-literal rule is nowadays obsolete.
- *choose-literal*(φ) picks only variables still occurring in the formula

DPLL Algorithm

```
function DPLL(φ, μ)
    if φ = ⊤                                         /* base      */
        then return True;
    if φ = ⊥                                         /* backtrack */
        then return False;
    if {a unit clause (l) occurs in φ}               /* unit      */
        then return DPLL(assign(l, φ), μ ∧ l);
    if {a literal l occurs pure in φ}                /* pure      */
        then return DPLL(assign(l, φ), μ ∧ l);
    l := choose-literal(φ);                          /* split     */
    return  DPLL(assign(l, φ), μ ∧ l)  or
            DPLL(assign(¬l, φ), μ ∧ ¬l);
```

- The pure-literal rule is nowadays obsolete.
- *choose-literal*(φ) picks only variables still occurring in the formula

DPLL – example

DPLL (without pure-literal rule)

Here "choose-literal" selects variable in alphabetic, selecting true first.

$$
\begin{array}{l}
(\neg C) \wedge \\
(B \vee A \vee C) \wedge \\
(\neg A \vee D) \wedge \\
(\neg E \vee \neg A \vee F) \wedge \\
(\neg E \vee \neg F \vee \neg A) \wedge \\
(G \vee \neg A \vee E) \wedge \\
(E \vee \neg G \vee \neg A) \wedge \\
(A \vee H \vee C) \wedge \\
(\neg H \vee \neg I \vee A) \wedge \\
(I \vee L \vee M) \wedge \\
(\neg L \vee C \vee \neg M) \wedge \\
(A \vee \neg L \vee M) \wedge \\
(L \vee N \vee \neg H) \wedge \\
(I \vee L \vee \neg N)
\end{array}
$$

$\Longrightarrow$ UNSAT

DPLL – example

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Here “choose-literal” selects variable in alphabetic, selecting true first.

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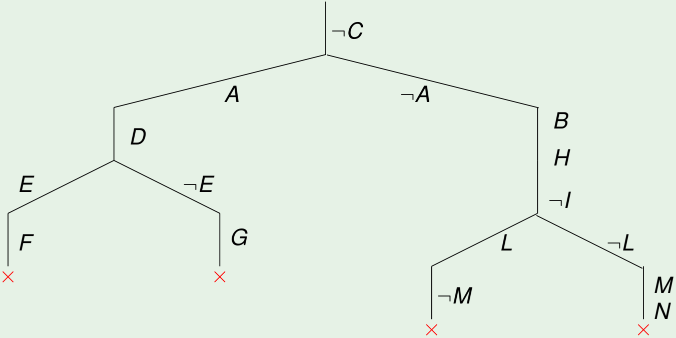
&(G \vee \neg A \vee E) \wedge \\
&(E \vee \neg G \vee \neg A) \wedge \\
&(A \vee H \vee C) \wedge \\
&(\neg H \vee \neg I \vee A) \wedge \\
&(I \vee L \vee M) \wedge \\
&(\neg L \vee C \vee \neg M) \wedge \\
&(A \vee \neg L \vee M) \wedge \\
&(L \vee N \vee \neg H) \wedge \\
&(I \vee L \vee \neg N)
\end{aligned}
$$



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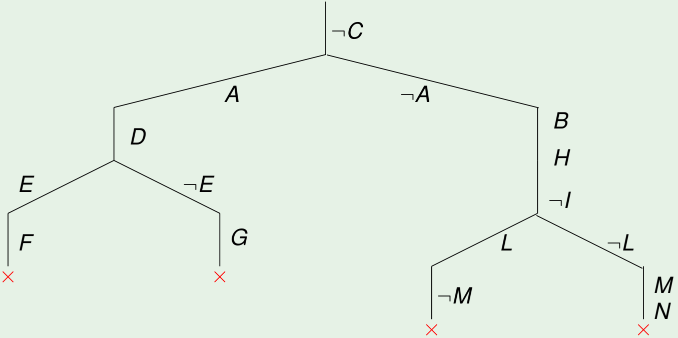
(E \vee \neg G \vee \neg A) \wedge \\
(A \vee H \vee C) \wedge \\
(\neg H \vee \neg I \vee A) \wedge \\
(I \vee L \vee M) \wedge \\
(\neg L \vee C \vee \neg M) \wedge \\
(A \vee \neg L \vee M) \wedge \\
(L \vee N \vee \neg H) \wedge \\
(I \vee L \vee \neg N)
\end{array}
$$



$\Longrightarrow$ UNSAT

DPLL – summary

- Handles CNF formulas (non-CNF variant known [2, 24]).
- Branches on truth values
  $\Longrightarrow$ all instances of an atom assigned simultaneously
- Postpones branching as much as possible.
- Mostly ignored by logicians.
- (The grandfather of) the most efficient SAT algorithms
  $\Longrightarrow$ loved by computer scientists.
- Requires polynomial space
- Choose_literal() critical!
- Many very efficient implementations [59, 52, 3, 41].

Outline

Stochastic Local Search SAT techniques: GSAT, WSAT [51, 50]

- Hill-Climbing techniques: GSAT, WSAT
- looks for a complete assignment;
- starts from a random assignment;
- Greedy search: looks for a better “neighbor” assignment
- Avoid local minima: restart & random walk

The GSAT algorithm [51]

```
function GSAT(φ)
    for i := 1 to Max-tries do
        μ := rand-assign(φ);
        for j := 1 to Max-flips do
            if (score(φ, μ) = 0)
                then return True;
                else Best-flips := hill-climb(φ, μ);
                    A_i := rand-pick(Best-flips);
                    μ := flip(A_i, μ);
        end
    end
    return "no satisfying assignment found".
```

The WalkSAT algorithm(s) [50]

```
function WalkSAT(φ)
    for i := 1 to Max-tries do
        μ := rand-assign(φ);
        for j := 1 to Max-flips do
            if (score(φ, μ) = 0)
                then return True;
                else C := randomly-pick-clause(unsat-clauses(φ, μ));
                    A_i := heuristically-select-variable(C);
                    μ := flip(A_i, μ);
        end
    end
    return "no satisfying assignment found".
```

- many variants available [26, 56, 4]

SLS SAT solvers – summary

- Handle only CNF formulas.
- Incomplete
- Extremely efficient for some (satisfiable) problems.
- Require polynomial space
- Used in Artificial Intelligence (e.g., planning)
- Lots of variants (see e.g. [30])
- Non-CNF Variants: [48, 49, 5]

Outline

Ordered Binary Decision Diagrams (OBDDs) [11]]

Canonical representation of Boolean formulas

- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top, \bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

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Note

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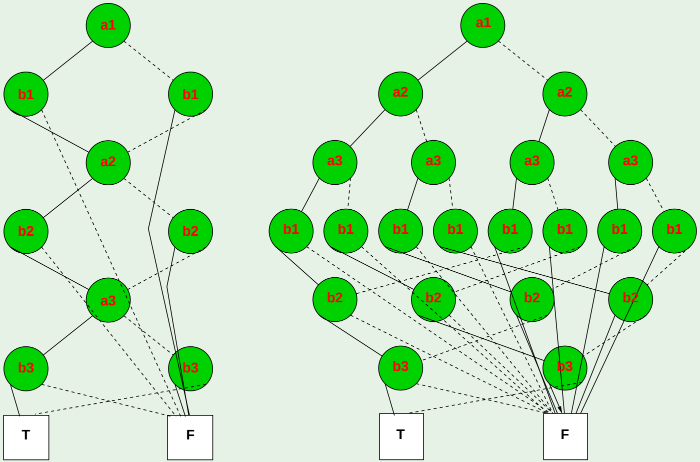
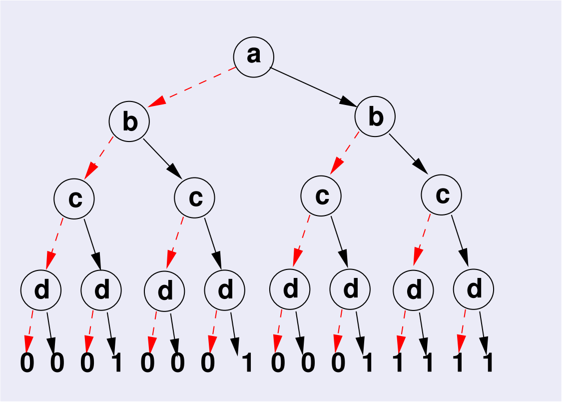
OBDD - Examples



OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- Ordered Decision Tree:
  from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$



From Ordered Decision Trees to OBDD's: reductions

- Recursive applications of the following reductions:
  - share subnodes: point to the same occurrence of a subtree (via hash consing)
  - remove redundancies: nodes with same left and right children can be eliminated: "if A then B else B" $\Longrightarrow$ "B"

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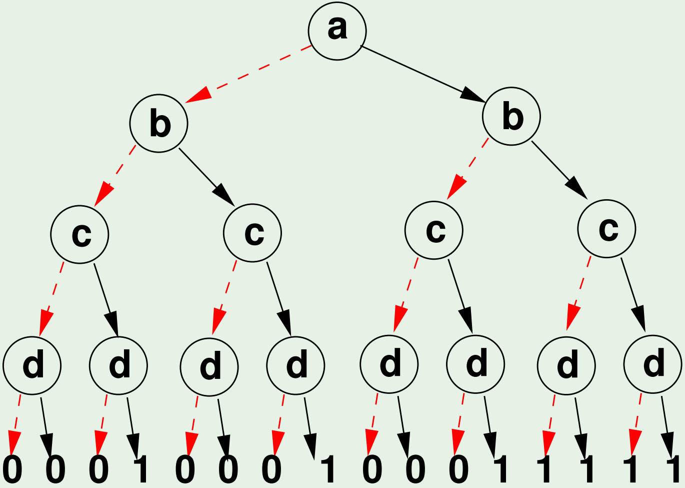
Reduction: example

$\varphi \stackrel{\text{def}}{=} (a \wedge b) \vee (c \wedge d)$



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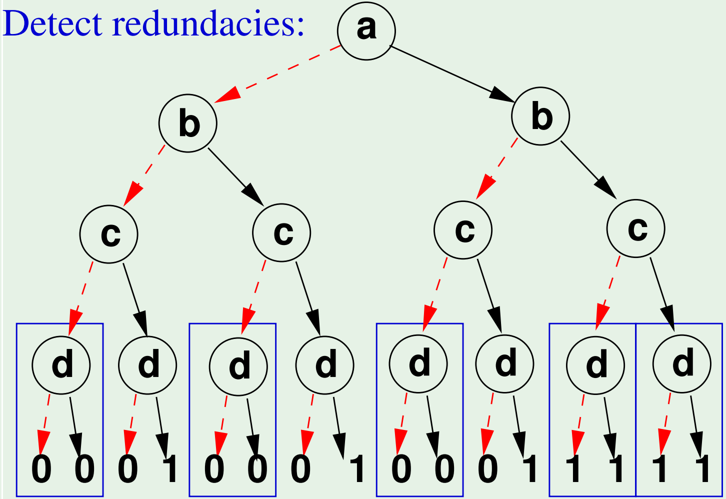
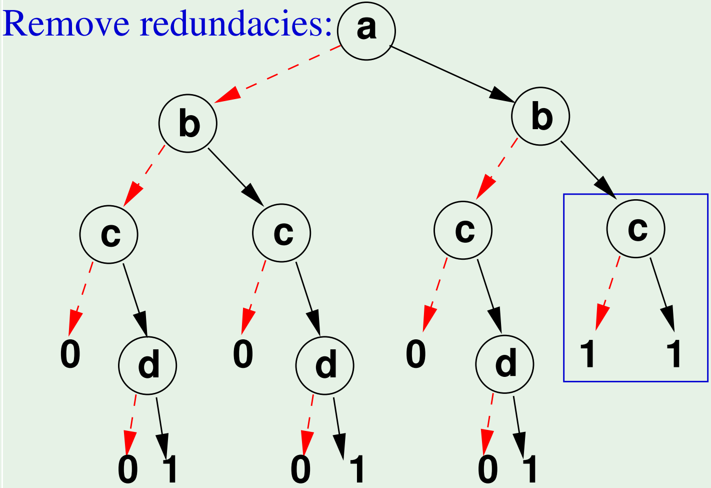
Detect redundacies:



Reduction: example

$\varphi \stackrel{\text{def}}{=} (a \land b) \lor (c \land d)$



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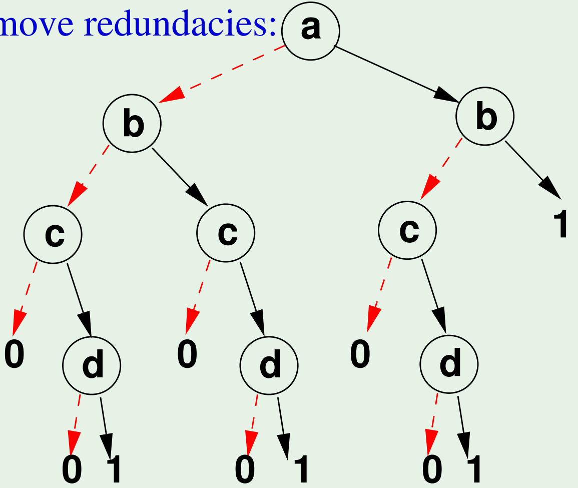

Reduction: example

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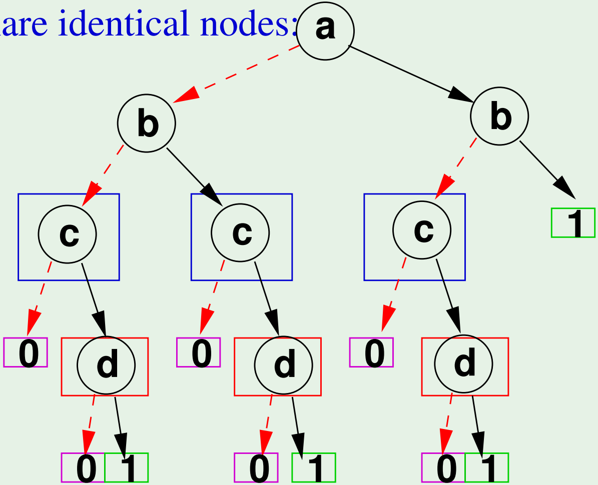
Share identical nodes:



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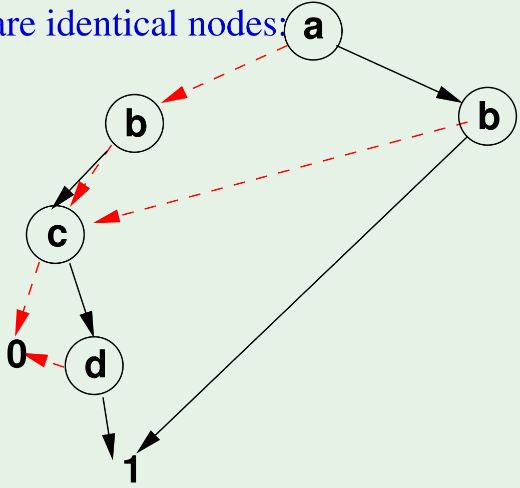

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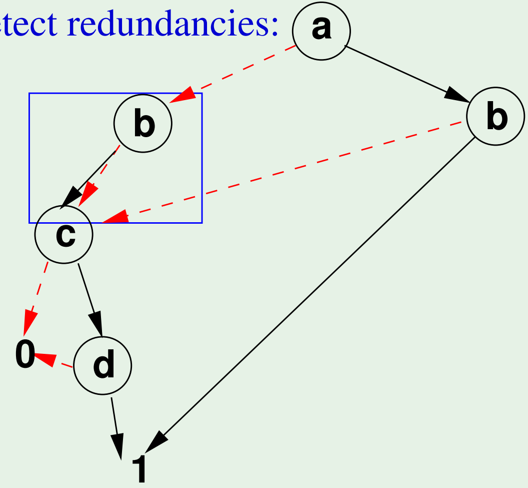

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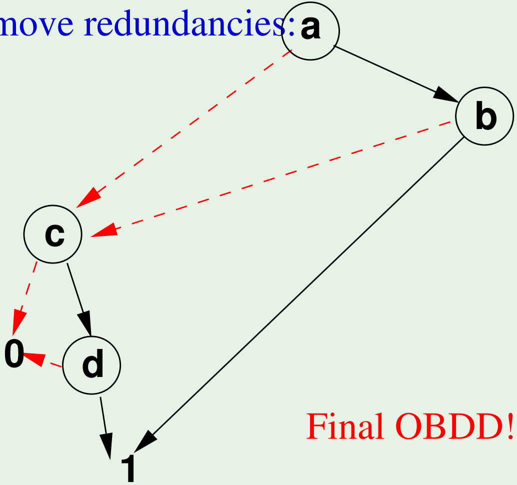

Final OBDD!

If-Then-Else Operators: “*ite*(...)”

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- $ite(\phi, \varphi^{\top}, \varphi^{\bot})$: “If ϕ Then $\varphi^{\top}$ Else $\varphi^{\bot}$”
- $ite(\phi, \varphi^{\top}, \varphi^{\bot}) \stackrel{\text{def}}{=} ((\neg\phi \vee \varphi^{\top}) \wedge (\phi \vee \varphi^{\bot}) \iff ((\phi \wedge \varphi^{\top}) \vee (\neg\phi \wedge \varphi^{\bot}))$
- properties:

$$
\begin{aligned}
\neg ite(\phi, \varphi^{\top}, \varphi^{\bot}) &= ite(\phi, \neg\varphi^{\top}, \neg\varphi^{\bot}) \\
ite(\phi, \varphi_1^{\top}, \varphi_1^{\bot})\ op\ ite(\phi, \varphi_2^{\top}, \varphi_2^{\bot}) &= ite(\phi, (\varphi_1^{\top}\ op\ \varphi_2^{\top}), (\varphi_1^{\bot}\ op\ \varphi_2^{\bot})) \\
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&\qquad (\varphi_1^{\bot}\ op\ ite(\phi_2, \varphi_2^{\top}, \varphi_2^{\bot}))) \\
&= ite(\phi_2, (ite(\phi_1, \varphi_1^{\top}, \varphi_1^{\bot})op\ \varphi_2^{\top}), \\
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\end{aligned}
\qquad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}
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$$ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot) = ite(\phi_1, (\varphi_1^\top\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot)), (\varphi_1^\bot\ op\ ite(\phi_2, \varphi_2^\top, \varphi_2^\bot))) = ite(\phi_2, (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ \varphi_2^\top), (ite(\phi_1, \varphi_1^\top, \varphi_1^\bot)\ op\ \varphi_2^\bot))$$

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\end{aligned}$$

$$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$$

Recursive structure of an OBDD

Assume the variable ordering $A_1, A_2, ..., A_n$:

$$
\begin{aligned}
OBDD(\top, \{A_1, A_2, ..., A_n\}) &= 1 \\
OBDD(\bot, \{A_1, A_2, ..., A_n\}) &= 0 \\
OBDD(\varphi, \{A_1, A_2, ..., A_n\}) &= \textit{ if } A_1 \\
&\quad \textit{then } OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\}) \\
&\quad \textit{else } OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\})
\end{aligned}
$$

Incrementally building an OBDD

- $obdd\_build(\top, \{...\}) := \top$,
- $obdd\_build(\bot, \{...\}) := \bot$,
- $obdd\_build(A_i, \{...\}) := ite(A_i, \top, \bot)$,
- $obdd\_build((\neg\varphi), \{A_1, ..., A_n\}) := apply(\neg, obdd\_build(\varphi, \{A_1, ..., A_n\}))$
- $obdd\_build((\varphi_1 \ op\ \varphi_2), \{A_1, ..., A_n\}) :=$

  $reduce($
  $\quad apply(\quad op,$
  $\qquad obdd\_build(\varphi_1, \{A_1, ..., A_n\}), \quad op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
  $\qquad obdd\_build(\varphi_2, \{A_1, ..., A_n\})$
  $))$

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- *obdd_build*$(\top, \{...\}) := \top$,
- *obdd_build*$(\bot, \{...\}) := \bot$,
- *obdd_build*$(A_i, \{...\}) :=$ *ite*$(A_i, \top, \bot)$,
- *obdd_build*$((\neg\varphi), \{A_1, ..., A_n\}) :=$ *apply*$(\neg,$ *obdd_build*$(\varphi, \{A_1, ..., A_n\}))$
- *obdd_build*$((\varphi_1$ *op* $\varphi_2), \{A_1, ..., A_n\}) :=$
  reduce(
  apply(*op*,
  obdd_build$(\varphi_1, \{A_1, ..., A_n\})$, *op* $\in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$
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  $))$

Incrementally building an OBDD (cont.)

- $apply\ (op,\ O_i, O_j) := (O_i\ op\ O_j)$ **if** $(O_i \in \{\top, \bot\}$ or $O_j \in \{\top, \bot\})$
- $apply\ (\neg,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot)) :=$
  $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\bot))$
- $apply\ (op,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot),\ ite(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$
  if $(A_i = A_j)$ **then** $ite(A_i,\ apply\ (op, \varphi_i^\top, \varphi_j^\top),\ apply\ (op, \varphi_i^\bot, \varphi_j^\bot))$
  if $(A_i < A_j)$ **then** $ite(A_i,\ apply\ (op, \varphi_i^\top, ite(A_j, \varphi_j^\top, \varphi_j^\bot)),\ apply\ (op, \varphi_i^\bot, ite(A_j, \varphi_j^\top, \varphi_j^\bot)))$
  if $(A_i > A_j)$ **then** $ite(A_j,\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top),\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))$

$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD (cont.)

- *apply* $(op,\ O_i, O_j) := (O_i\ op\ O_j)$ **if** $(O_i \in \{\top, \bot\}$ or $O_j \in \{\top, \bot\})$
- *apply* $(\neg,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot)) :=$
  $ite(A_i, apply(\neg, \varphi_i^\top), apply(\neg, \varphi_i^\bot))$
- *apply* $(op,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot),\ ite(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$
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  if $(A_i > A_j)$ **then** $ite(A_j,\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top),\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))$

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Incrementally building an OBDD (cont.)

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$op \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow, \oplus\}$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$
\begin{aligned}
& apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\
= \;& ite(A_1,\; apply(\vee, \top, ite(A_2, \top, \bot)),\; apply(\vee, \bot, ite(A_2, \top, \bot))\;) \\
= \;& ite(A_1,\; \top,\; ite(A_2, \top, \bot)\;)
\end{aligned}
$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$
\begin{aligned}
& apply(\wedge, \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)}, \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)}, \\
= \;& ite(A_1,\; apply(\wedge, \top, \top),\; apply(\wedge,\; ite(A_2, \top, \bot),\; ite(A_2, \bot, \top)) \\
= \;& ite(A_1,\; \top,\; ite(A_2,\; apply(\wedge, \top, \bot),\; apply(\wedge, \bot, \top))) \\
= \;& ite(A_1,\; \top,\; ite(A_2,\; \bot,\; \bot)) \\
= \;& ite(A_1,\; \top,\; \bot)
\end{aligned}
$$

Incrementally building an OBDD: Examples

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$
\begin{aligned}
& apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\
= \; & ite(A_1, \; apply(\vee, \top, ite(A_2, \top, \bot)), \; apply(\vee, \bot, ite(A_2, \top, \bot)) \;) \\
= \; & ite(A_1, \; \top, \; ite(A_2, \top, \bot) \;)
\end{aligned}
$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$
\begin{aligned}
& apply(\wedge, \; \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)}, \; \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)}, \\
= \; & ite(A_1, \; apply(\wedge, \top, \top), \; apply(\wedge, \; ite(A_2, \top, \bot), \; ite(A_2, \bot, \top)) \\
= \; & ite(A_1, \; \top, \; ite(A_2, \; apply(\wedge, \top, \bot), \; apply(\wedge, \bot, \top))) \\
= \; & ite(A_1, \; \top, \; ite(A_2, \; \bot, \; \bot)) \\
= \; & ite(A_1, \; \top, \; \bot)
\end{aligned}
$$

OBBD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

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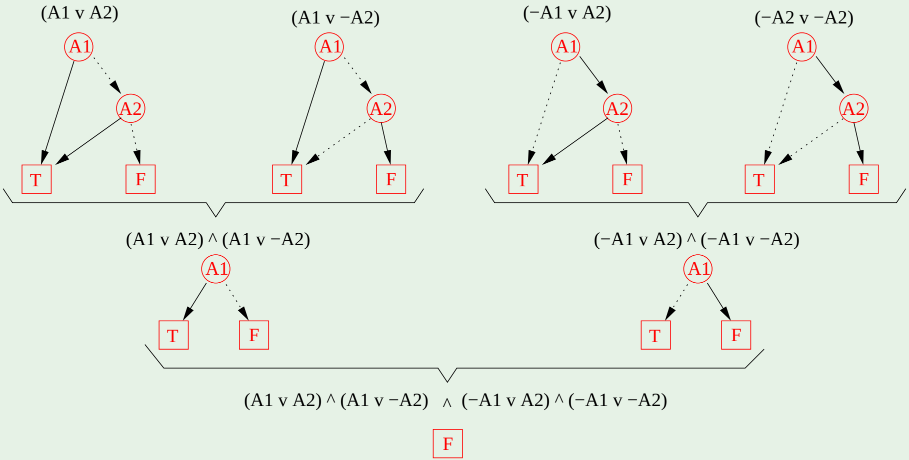
$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$

Linear size

Exponential size

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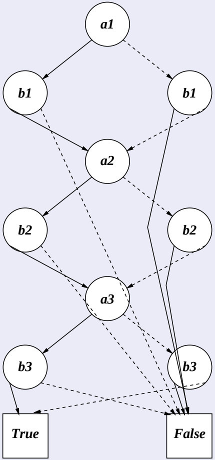

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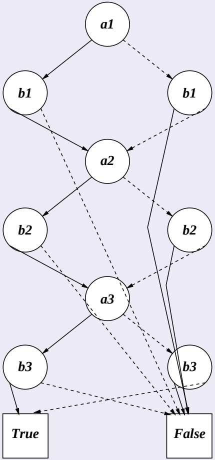

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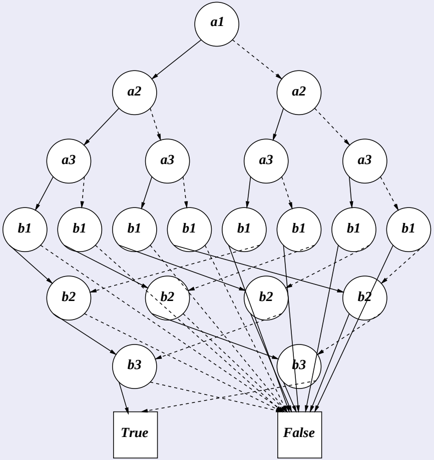

Exponential size

OBDD's as canonical representation of Boolean formulas

- An OBDD is a canonical representation of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff OBDD(\varphi_1) = OBDD(\varphi_2)$$

- equivalence check requires constant time!
  $\Longrightarrow$validity check requires constant time! ($\varphi \leftrightarrow \top$)
  $\Longrightarrow$(un)satisfiability check requires constant time! ($\varphi \leftrightarrow \bot$)
- the set of the paths from the root to 1 represent all the models of the formula
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Exponentiality of OBDD's

- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless P = co-NP)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

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The size of intermediate OBDD's may be bigger than that of the final one (e.g., inconsistent formula)

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Useful Operations over OBDDs

- the equivalence check between two OBDDs is simple
  - are they the same OBDD? ($\Longrightarrow$ constant time)
- the size of a Boolean composition is up to the product of the size of the operands: $|f\ op\ g| = O(|f| \cdot |g|)$

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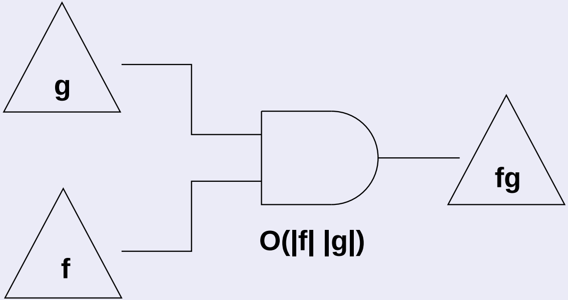

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[Recall] Boolean Quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then

$$\exists v.\varphi \quad := \quad \varphi|_{v=\bot} \vee \varphi|_{v=\top}$$
$$\forall v.\varphi \quad := \quad \varphi|_{v=\bot} \wedge \varphi|_{v=\top}$$

- v does no more occur in $\exists v.\varphi$ and $\forall v.\varphi$!!
- Multi-variable quantification: $\exists(w_1, \ldots, w_n).\varphi \ := \ \exists w_1 \ldots \exists w_n.\varphi$

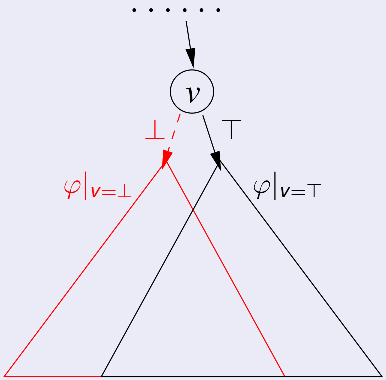
- Intuition:
  - $\mu \models \exists v.\varphi$ iff exists *truthvalue* $\in \{\top, \bot\}$ s.t. $\mu \cup \{v := \mathit{truthvalue}\} \models \varphi$
  - $\mu \models \forall v.\varphi$ iff forall *truthvalue* $\in \{\top, \bot\}$, $\mu \cup \{v := \mathit{truthvalue}\} \models \varphi$
- Example: $\exists(b, c).((a \wedge b) \vee (c \wedge d)) \ = \ a \vee d$

Note

Naive expansion of quantifiers to propositional logic may cause a blow-up in size of the formulae

OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if f is a sub-OBDD labeled by variable v, then $\varphi|_{v=\top}$ and $\varphi|_{v=\bot}$ are the "then" and "else" branches of f



$\Longrightarrow$ lots of sharing of subformulae!

Example

Let $\varphi \stackrel{\text{def}}{=} (A \land (B \lor C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.\varphi$. Using the variable ordering "A, B, C", draw the OBDD corresponding to the formulas φ and φ'.

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$\varphi \stackrel{\text{def}}{=} (A \land (B \lor C))$

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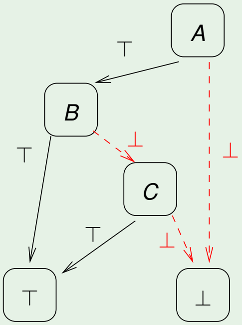
Let $\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$ and $\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. \varphi$. Using the variable ordering "A, B, C", draw the OBDD corresponding to the formulas φ and φ'.

$\varphi \stackrel{\text{def}}{=} (A \wedge (B \vee C))$



Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A. \forall B. (A \wedge (B \vee C))$

which corresponds to the following OBDD:

Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \wedge (B \vee C))$

$$
\begin{array}{rlcll}
\varphi' \stackrel{\text{def}}{=} & \exists A.\forall B.\varphi & & & \\
= & \forall B.(A \wedge (B \vee C)))[A := \top] & & \vee & (\forall B.(A \wedge (B \vee C)))[A := \bot] \\
= & \forall B.(B \vee C)) & & \vee & \forall B.\bot \\
= & ((B \vee C)[B := \top] & \wedge \quad (B \vee C)[B := \bot]) & \vee & \bot \\
= & (\top & \wedge \quad C) & & \\
= & C & & &
\end{array}
$$

which corresponds to the following OBDD:

Example (cont.)

$\varphi' \stackrel{\text{def}}{=} \exists A.\forall B.(A \land (B \lor C))$

$$
\begin{array}{rllll}
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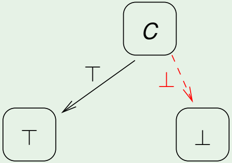
= & \forall B.(A \land (B \lor C)))[A := \top] & & \lor & (\forall B.(A \land (B \lor C)))[A := \bot] \\
= & \forall B.(B \lor C)) & & \lor & \forall B.\bot \\
= & ((B \lor C)[B := \top] & \land \quad (B \lor C)[B := \bot]) & \lor & \bot \\
= & (\top & \land \quad C) & & \\
= & C & & &
\end{array}
$$

which corresponds to the following OBDD:



OBDD – summary

- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
- Canonical representation of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents all models and counter-models of the formula.
- Require exponential space in worst-case
- Very efficient for some practical problems (circuits, symbolic model checking).

Outline

Outline

DPLL: “Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment l
- l is toggled, is labeled as “closed”, and the search proceeds.

DPLL Chronological Backtracking: Drawbacks

Chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible
$\Longrightarrow$ lots of useless search!

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
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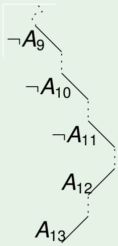
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$\ldots$

DPLL Chronological Backtracking: Example



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$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(initial assignment)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \lor A_2$
$c_2 : \neg A_1 \lor A_3 \lor A_9$
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
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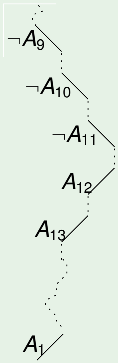
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (branch on A_1)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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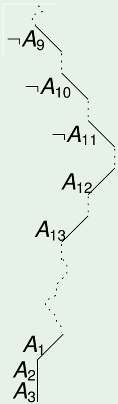

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$
(unit A_2, A_3)

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$ √
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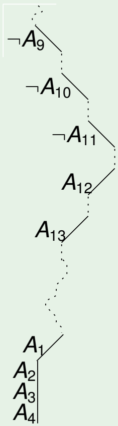

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4\}$
(unit A_4)

DPLL Chronological Backtracking: Example

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$c_6 : \neg A_5 \lor \neg A_6$ $\times$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ $\surd$
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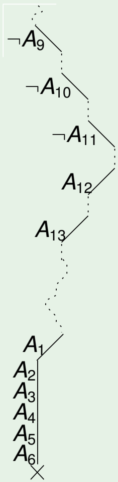
...



$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{1\neg A_4 1}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) $\Longrightarrow$ conflict

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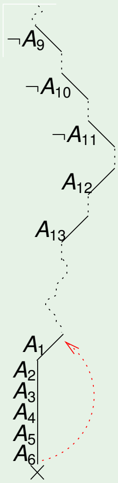

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

DPLL Chronological Backtracking: Example

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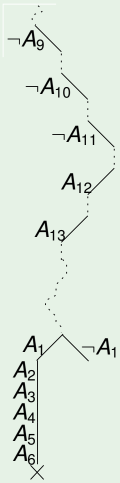
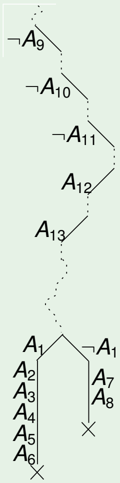


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

DPLL Chronological Backtracking: Example



$c_1 : \neg A_1 \vee A_2$ $\surd$

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$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

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$c_8 : A_1 \vee A_8$ $\surd$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ $\times$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1, A_7, A_8\}$

(unit A_7, A_8) $\Longrightarrow$ conflict

DPLL Chronological Backtracking: Example

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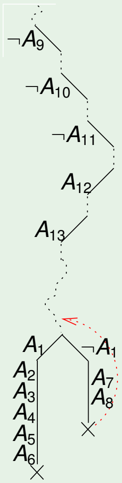
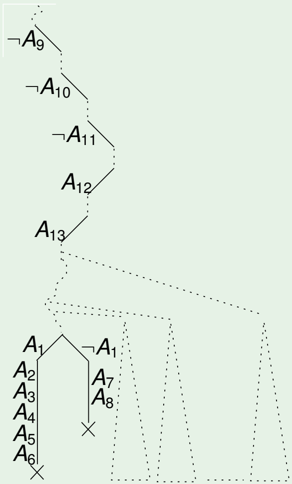


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack to the most recent open branching point

DPLL Chronological Backtracking: Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ lots of useless search before backtracking up to A_{13}!

Outline

Modern Conflict-Driven Clause-Learning SAT Solvers

- Non-recursive, stack-based implementations
- Based on Conflict-Driven Clause-Learning (CDCL) schema
  - inspired to conflict-driven backjumping and learning in CSPs
  - learns implied clauses as nogoods
- Random restarts
  - abandon the current search tree and restart on top level
  - previously-learned clauses maintained
- Smart literal selection heuristics (ex: VSIDS)
  - "static": scores updated only at the end of a branch
  - "local": privileges variable in recently learned clauses
- Smart preprocessing/inprocessing technique to simplify formulas
- Smart indexing techniques (e.g. 2-watched literals)
  - efficiently do/undo assignments and reveal unit clauses
- Allow Incremental Calls (stack-based interface)
  - allow for reusing previous search on "similar" problems

Can handle industrial problems with $10^6 - 10^7$ variables and clauses!

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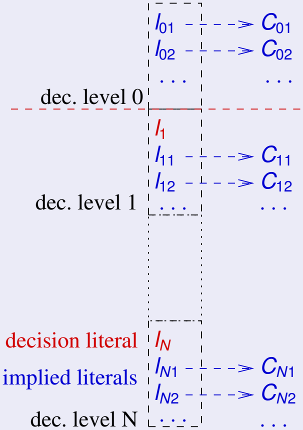
Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into decision levels:
  - one decision literal
  - its implied literals
  - each implied literal tagged with the clause causing its unit-propagation (antecedent clause)
- equivalent to an implication graph



Implication graph

- An implication graph is a DAG s.t.:
  - each node represents a variable assignment (literal)
  - each edge $l_i \overset{c}{\longmapsto} l$ is labeled with a clause
  - the node of a decision literal has no incoming edges
  - all edges incoming into a node l are labeled with the same clause c, s.t. $l_1 \overset{c}{\longmapsto} l, ..., l_n \overset{c}{\longmapsto} l$ iff $c = \neg l_1 \vee ... \vee \neg l_n \vee l$
    (c is said to be the antecedent clause of l)
  - when both l and $\neg l$ occur in the graph, we have a conflict.
- Intuition:
  - representation of the dependencies between literals in μ
  - the graph contains $l_1 \overset{c}{\longmapsto} l, ..., l_n \overset{c}{\longmapsto} l$ iff l has been obtained from $l_1, ..., l_n$ by unit propagation on c
  - a partition of the graph with all decision literals on one side and the conflict on the other represents a conflict set

Implication graph - example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

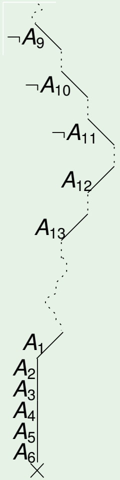
$c_6 : \neg A_5 \vee \neg A_6$ ×

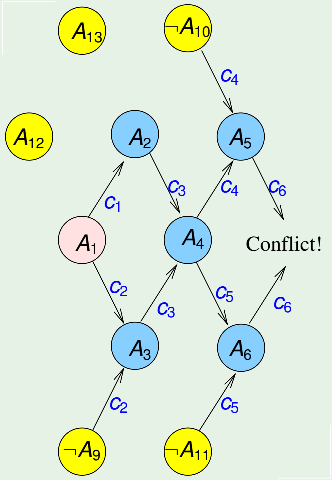
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





Unique implication point - UIP [61]

- A node l in an implication graph is an unique implication point (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through l.
  - the most recent decision node is an UIP (last UIP)
  - all other UIP's have been assigned after the most recent decision

Unique implication point - UIP - example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

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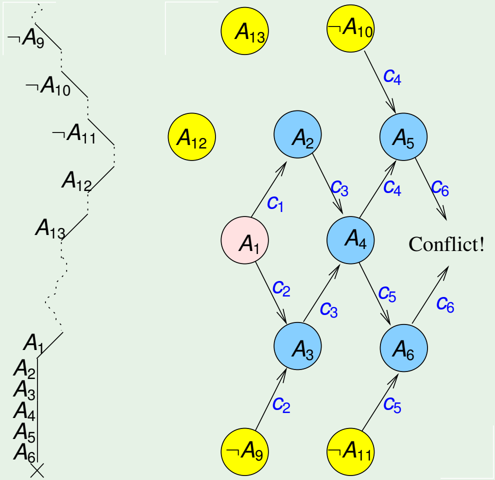
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...



- A_1 is the last UIP
- A_4 is the 1^{st} UIP

Schema of a CDCL DPLL solver [52, 62]

```
Function CDCL-SAT (formula: φ, assignment & μ) {
        status := preprocess(φ,μ);
        while (1) {
            while (1) {
                status := deduce(φ,μ);
                if (status == Sat)
                    return Sat;
                if (status == Conflict) {
                    ⟨blevel,η⟩ := analyze_conflict(φ,μ);
                    //η is a conflict set
                    if (blevel == 0)
                        return Unsat;
                    else backtrack(blevel,φ,μ);
                }
                else break;
            }
            decide_next_branch(φ,μ);
}       }
```

Schema of a CDCL DPLL solver [52, 62] (cont.)

- `preprocess`(φ, μ) simplifies φ into an easier equisatisfiable formula, updating μ.
- `decide_next_branch`(φ, μ) chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce`(φ, μ) performs all deterministic assignments (unit-propagations plus others), and updates φ, μ accordingly.
- `analyze_conflict`(φ, μ) Computes the subset η of μ causing the conflict (conflict set), and returns the "wrong-decision" level suggested by η ("0" means that η is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack`$(\texttt{blevel}, \varphi, \mu)$ undoes the branches up to blevel, and updates φ, μ accordingly

Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
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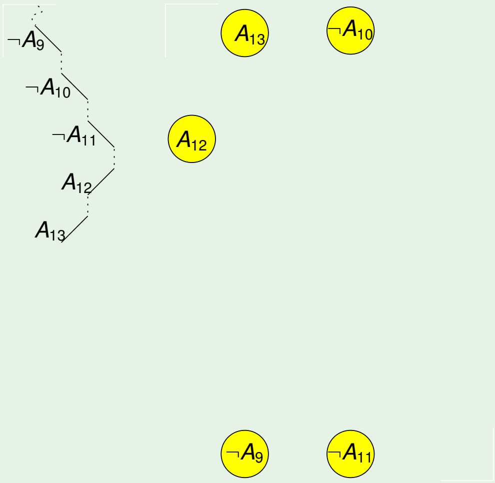

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(Initial assignment. Note: c_1, ..., c_9 inconsistent.)

Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ $\surd$
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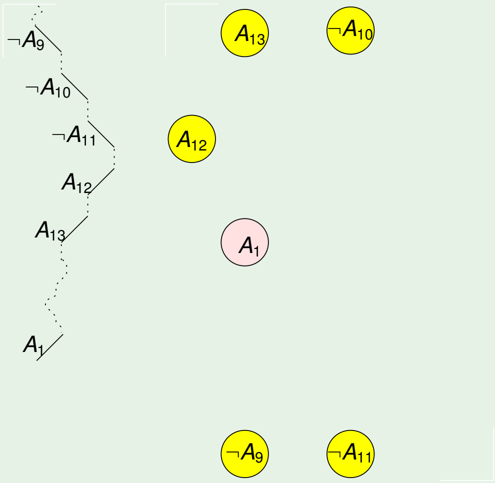

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (decide A_1)

Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

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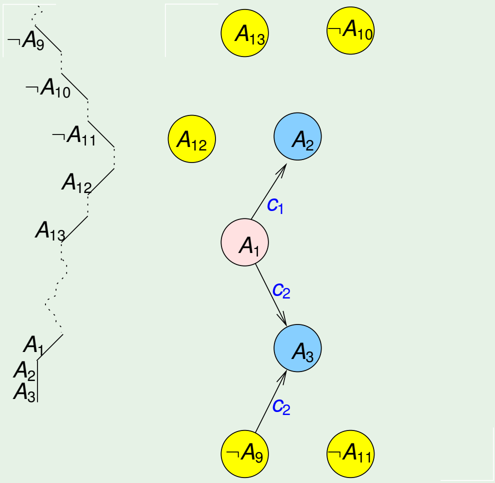

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3\}$

(unit A_2, A_3)

Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
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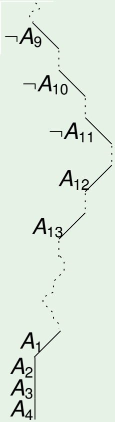

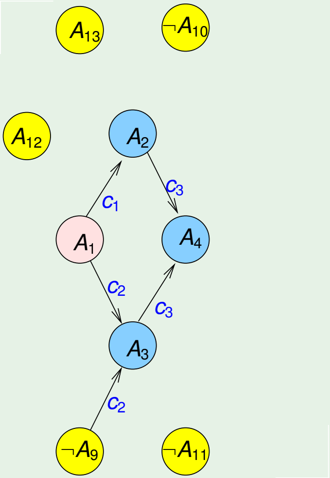

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4\}$
(unit A_4)

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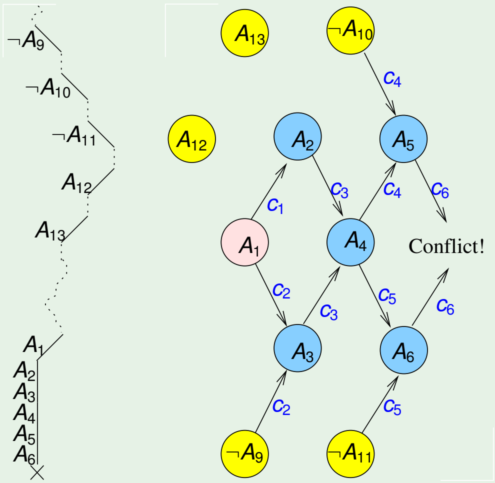

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{1\neg A_4 1}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) $\Longrightarrow$ conflict

Backjumping and learning: general ideas [3, 52]

- When a branch μ fails:
  (i) conflict analysis: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (conflict set η)
  (ii) learning: add the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ to the clause set
  (iii) backjumping: use η to decide the point where to backtrack
- may jump back up much more than one decision level in the stack
  $\Longrightarrow$ may avoid lots of redundant search!!.
- we illustrate two main backjumping & learning strategies:
  - the original strategy presented in [52]
  - the state-of-the-art 1^{st}UIP strategy of [61]

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

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criterium: decision

...until C contains only decision literals

Conflicting cl.: $\neg A_5 \vee \neg A_6$

$$\frac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{\text{Conflicting cl.}}}{\neg A_4 \vee \neg A_5 \vee A_{11}} \; (A_6)$$

$$\frac{\neg A_4 \vee A_5 \vee A_{10} \qquad \neg A_4 \vee \neg A_5 \vee A_{11}}{\neg A_4 \vee A_{10} \vee A_{11}} \; (A_5)$$

$$\frac{\neg A_2 \vee \neg A_3 \vee A_4 \qquad \neg A_4 \vee A_{10} \vee A_{11}}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}} \; (A_4)$$

$$\frac{\neg A_1 \vee A_3 \vee A_9 \qquad \neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}{\neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \; (A_3)$$

$$\frac{\neg A_1 \vee A_2 \qquad \neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}{\neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \; (A_2)$$

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criterium: last UIP

... until C contains only one literal assigned at current decision level: the decision literal (last UIP)

Conflicting cl.

$$\frac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{\text{Conflicting cl.}}}{\neg A_4 \vee \neg A_5 \vee A_{11}} \; (A_6)$$

$$\frac{\neg A_4 \vee A_5 \vee A_{10} \qquad \neg A_4 \vee \neg A_5 \vee A_{11}}{\neg A_4 \vee A_{10} \vee A_{11}} \; (A_5)$$

$$\frac{\neg A_2 \vee \neg A_3 \vee A_4 \qquad \neg A_4 \vee A_{10} \vee A_{11}}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}} \; (A_4)$$

$$\frac{\neg A_1 \vee A_3 \vee A_9 \qquad \neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}{\neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \; (A_3)$$

$$\frac{\neg A_1 \vee A_2 \qquad \neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}{\neg A_1 \vee A_9 \vee A_{10} \vee A_{11}} \; (A_2)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterium: 1st UIP

... until C contains only one literal assigned at current decision level (1st UIP)

$$\dfrac{\neg A_4 \vee A_5 \vee A_{10} \qquad \dfrac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{Conflicting\ cl.}}{\neg A_4 \vee \neg A_5 \vee A_{11}}\ (A_6)}{\underbrace{\neg A_4}_{1st\ UIP} \vee A_{10} \vee A_{11}}\ (A_5)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

Note:

$\varphi \models C$, so that C can be safely added to φ.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

Note:

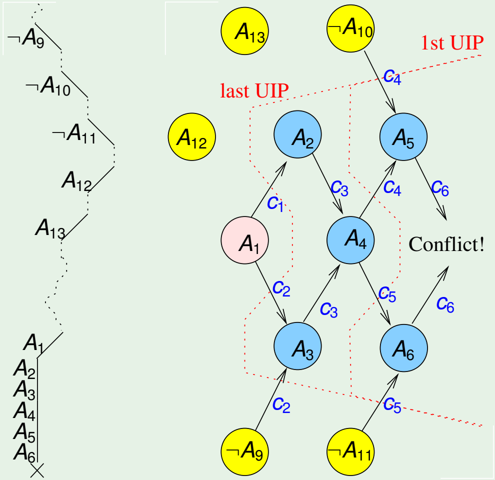
$\varphi \models C$, so that C can be safely added to φ.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis and implication graph - example

$c_1 : \neg A_1 \vee A_2$ ✓
$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
$c_6 : \neg A_5 \vee \neg A_6$ ×
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
$c_8 : A_1 \vee A_8$ ✓
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



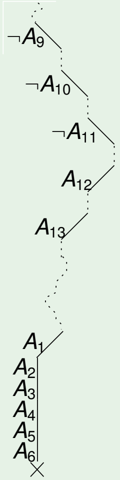
Note: in this case decision and last-UIP criteria produce the same partition

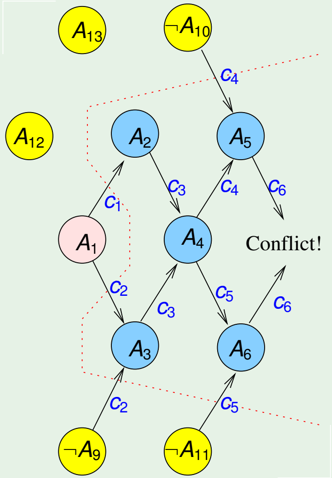
The original backjumping and learning strategy of [52]

- Idea: when a branch μ fails,
  (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the "Decision" criterion;
  (ii) learning: add the conflict clause C to the clause set
  (iii) backjumping: backtrack to the most recent branching point s.t. the stack does not fully contain η, and then unit-propagate the unassigned literal on C

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ √
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ √
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ √
$c_6 : \neg A_5 \vee \neg A_6$ ×
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √
$c_8 : A_1 \vee A_8$ √
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...

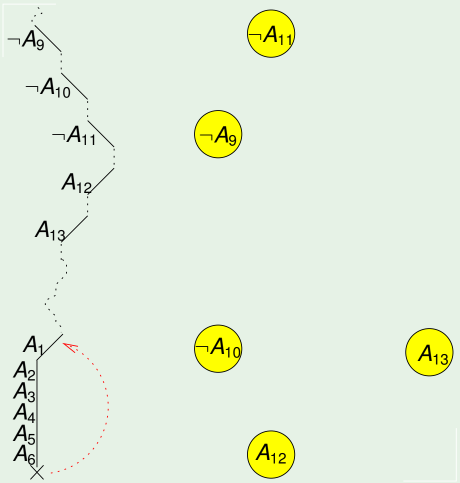




$\Longrightarrow$ Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ ("decision" schema)
$\Longrightarrow$ learn the conflict clause $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

The Original Backjumping Strategy: Example

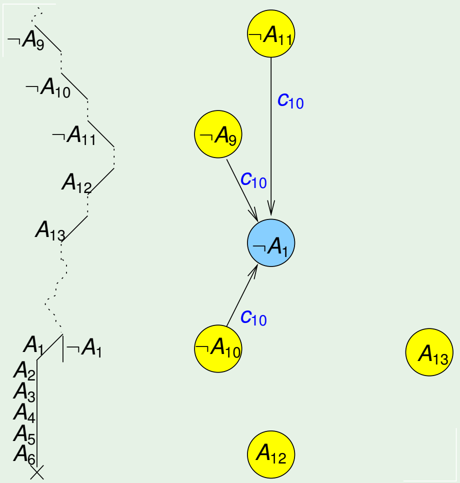
$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

The Original Backjumping Strategy: Example

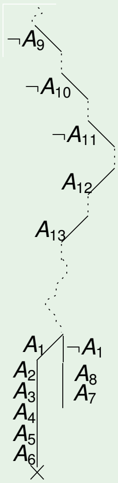
$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √
...

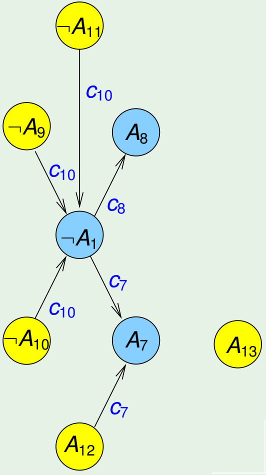


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √
$c_8 : A_1 \vee A_8$ √
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √
...

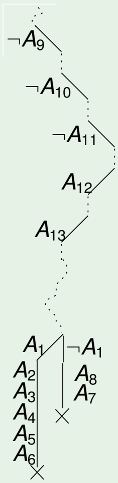


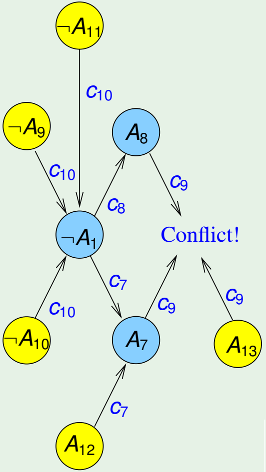


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$
(unit A_7, A_8)

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \lor A_2$ √
$c_2 : \neg A_1 \lor A_3 \lor A_9$ √
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$
$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ √
$c_8 : A_1 \lor A_8$ √
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$ ×
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ √
...

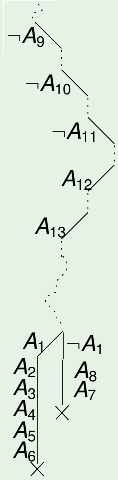


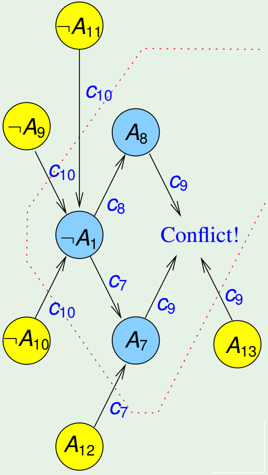


$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1, A_7, A_8\}$
Conflict!

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √
$c_8 : A_1 \vee A_8$ √
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √
...

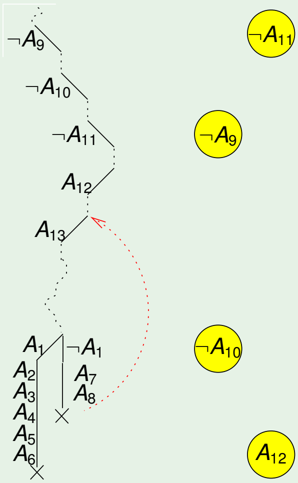




$\Longrightarrow$ conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.
$\Longrightarrow$ learn $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

The Original Backjumping Strategy: Example

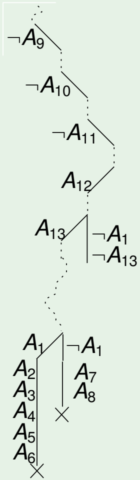
$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$
...

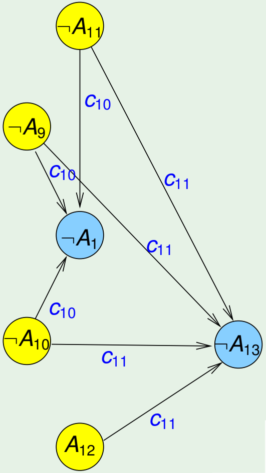


$\Longrightarrow$ backtrack to A_{13} $\Longrightarrow$ Lots of search saved!

The Original Backjumping Strategy: Example

$c_1 : \neg A_1 \vee A_2$ √
$c_2 : \neg A_1 \vee A_3 \vee A_9$ √
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ √
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √
$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ √
...





$\Longrightarrow$ backtrack to A_{13}, then set A_{13} and A_1 to $\perp$,...

State-of-the-art backjumping and learning [61]

- Idea: when a branch μ fails,
  (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the 1^{st}UIP strategy
  (ii) learning: add the conflict clause C to the clause set
  (iii) backjumping: backtrack to the highest branching point s.t. the stack contains all-but-one literals in η, and then unit-propagate the unassigned literal on C

1st UIP strategy – example (7)

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ √

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ √

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ √

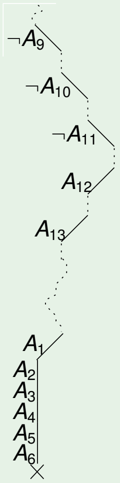
$c_6 : \neg A_5 \vee \neg A_6$ ×

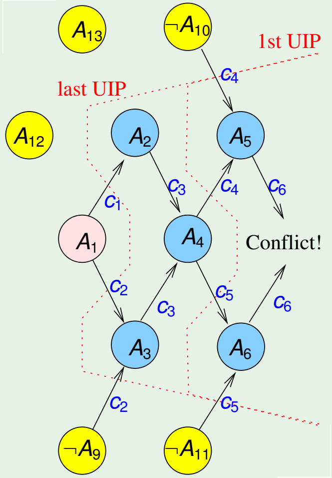
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ √

$c_8 : A_1 \vee A_8$ √

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping [61]

- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$
$\Longrightarrow$ backtrack to A_{11}, then assign $\neg A_4$

1st UIP strategy – example (7)

$c_1 : \neg A_1 \lor A_2$ √

$c_2 : \neg A_1 \lor A_3 \lor A_9$ √

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ √

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ √

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ √

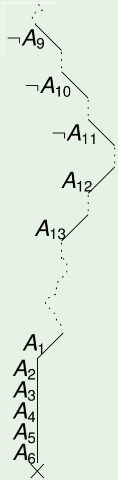
$c_6 : \neg A_5 \lor \neg A_6$ ×

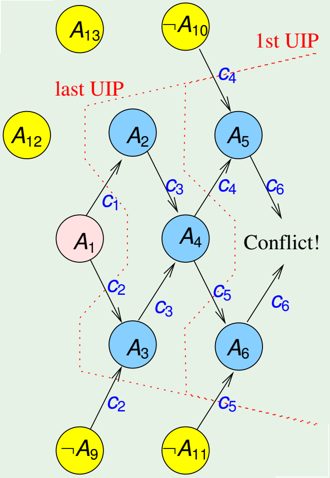
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ √

$c_8 : A_1 \lor A_8$ √

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...

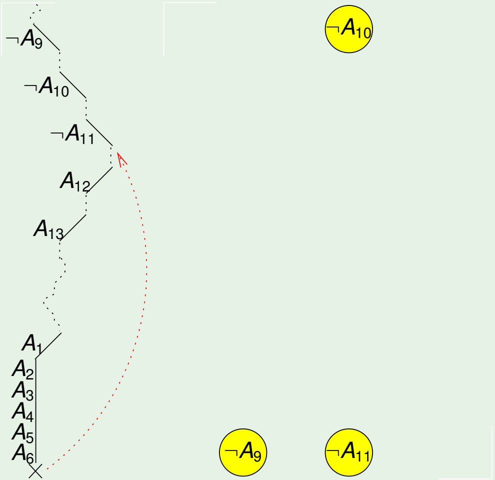




$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \lor A_{11} \lor \neg A_4$

1st UIP strategy – example (8)

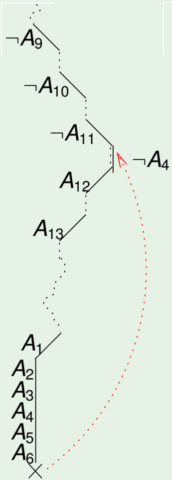
$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$
...

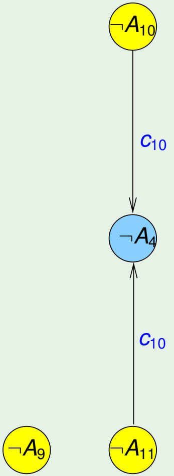


$\Longrightarrow$ backtrack up to $A_{11} \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}\}$

1st UIP strategy – example (9)

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ $\surd$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ $\surd$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$ $\surd$
...





$\Longrightarrow$ unit propagate $\neg A_4 \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_4\}$...

1st UIP strategy and backjumping – intuition

- An UIP is a single reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
  - does not enlarge the conflict
  - requires less resolution steps to compute C
  - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
  - jump higher
  - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [3, 52]

Idea: When a conflict set η is revealed, then $C \stackrel{\text{def}}{=} \neg\eta$ added to φ
$\Longrightarrow$ the solver will no more generate an assignment containing η: when $|\eta| - 1$ literals in η are assigned, the other is set $\perp$ by unit-propagation on C
$\Longrightarrow$ Drastic pruning of the search!

Learning – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
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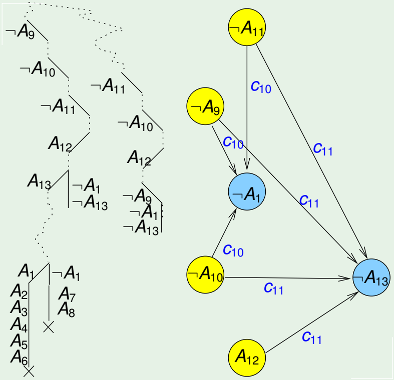
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ √
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √
$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ √
...
$\Longrightarrow$ Unit: $\{\neg A_1, \neg A_{13}\}$



Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their activity.

A clause is currently active if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

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Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [43])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.
$\Longrightarrow$ CDCL solvers require polynomial space

"Lazy" Strategy

- when a clause is involved in conflict analisis, increase its activity
- when needed, drop the least-active clauses

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- when needed, drop the least-active clauses

State-of-the-art backjumping and learning: intuitions

- Backjumping: allows for climbing up to many decision levels in the stack
  - intuition: " go back to the oldest decision where you'd have done something different if only you had known C"

  $\Longrightarrow$ may avoid lots of redundant search
- Learning: in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
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  $\Longrightarrow$ avoid finding the same conflict again

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Remark: the "quality" of conflict sets

- Different ideas of "good" conflict set
  - Backjumping: if causes the highest backjump ("local" role)
  - Learning: if causes the maximum pruning ("global" role)
- Many different strategies implemented (see, e.g., [3, 52, 61])

Outline

Preprocessing/Inprocessing

- Part of `preprocess()` and `deduce()` steps respectively
- Simplify current formula into an equivalently-satisfiable one
- Must be fast (in particular inprocessing)
- Some techniques:
  - detect and remove subsumed clauses
  - detect & collapse equivalent literals
  - apply basic resolution steps
  - ...

Preprocessing/Inprocessing (cont.)

Detect and remove subsumed clauses:

$$\varphi_1 \land (l_2 \lor l_1) \land \varphi_2 \land (l_2 \lor l_3 \lor l_1) \land \varphi_3$$
$$\Downarrow$$
$$\varphi_1 \land (l_1 \lor l_2) \land \varphi_2 \land \varphi_3$$

Preprocessing/Inprocessing (cont.)

Detect & collapse equivalent literals [10]

Repeat:

(i) build the implication graph induced by binary clauses

(ii) detect strongly connected cycles $\Longrightarrow$ equivalence classes of literals

(iii) perform substitutions

(iv) perform unit and pure literal.

Until (no more simplification is possible).

- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$
$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

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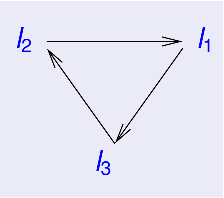

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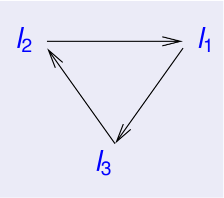

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Preprocessing/Inprocessing (cont.)

Apply some basic steps of resolution (and simplify)

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3$$
$$\Downarrow_{resolve}$$
$$\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3$$
$$\Downarrow_{unit-propagate}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]$$

Literal-Decision Heuristics (aka Branching Heuristics)

- Implemented in `decide_next_branch()`
- Branch is the source of non-determinism for DPLL
  $\Longrightarrow$ critical for efficiency
- Many literal-decision heuristics in literature (for DPLL & CDCL)

Some Heuristics

- MOMS heuristics (DPLL): pick the literal occurring most often in the minimal size clauses
  $\Longrightarrow$ fast and simple, many variants
- Jeroslow-Wang (DPLL): choose the literal with maximum

$$score(l) := \Sigma_{l \in c \ \& \ c \in \varphi} 2^{-|c|}$$

  $\Longrightarrow$ estimates l's contribution to the satisfiability of φ
- Satz [32] (DPLL): selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
  $\Longrightarrow$ maximizes the effects of unit propagation
- VSIDS [41] (CDCL+): variable state independent decaying sum
  - "static": scores updated only at the end of a branch
  - "local": privileges variable in recently learned clauses

Restarts [25]

Idea: (according to some strategy) restart the search

- abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- avoid getting stuck in certain areas of the search space

$\Longrightarrow$ may significantly reduce the overall search space

Outline

SAT under assumptions: $SAT(\varphi, \{l_1, ..., l_n\})$ [17]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, ..., l_n\}$: $SAT(\varphi, \{l_1, ..., l_n\})$
  - $SAT(\varphi, \{l_1, ..., l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^{n} l_i)$
  - often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2), ...$
- Idea:
  - $l_1, ..., l_n$ "decided" at decision level 0 before starting the search
  - if backjump to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT

Property

If the "decision" strategy for conflict analysis is used,
then η is the subset of assumptions causing the inconsistency

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Selection of sub-formulas

Idea: select clauses [17, 34]

Let φ be $\bigwedge_{i=1}^{n} C_i$.

- let $S_1 \ldots S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \stackrel{\text{def}}{=} \{S_{i_1}, ..., S_{i_K}\} \subseteq \{S_1, ..., S_n\}$

$\Longrightarrow$ SAT($\bigwedge_{i=1}^{n}(\neg S_i \vee C_i), \mathcal{A}$): same as SAT($\bigwedge_{i=i_1}^{i_k}(C_i)$)

- if S_i is not assumed, then $\neg S_i \vee C_i$ does not contribute to search

$\Longrightarrow$ "Select" (activate) only a subset of the clauses in φ at each call.

Generalised Idea: select blocks of clauses

Let φ be $\bigwedge_{i=1}^{n}(\bigwedge_{j=1}^{n_i} C_{ij})$.

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Example

- Initial formula φ:
  $(\ A_1 \vee A_2 \vee \neg A_3\ )\wedge$ // *group* 1
  $(\neg A_3 \vee \neg A_2 \vee \neg A_5\ )\wedge$ // *group* 1
  $(\ A_2 \vee A_5 \vee A_7\ )\wedge$ // *group* 2
  $(\ A_3 \vee A_5 \vee \neg A_8\ )\wedge$ // *group* 2
  $(\neg A_1 \vee \neg A_3 \vee A_8\ )\wedge$ // *group* 3
- Augmented formula φ':
  $(\neg S_1 \vee A_1 \vee A_2 \vee \neg A_3\ )\wedge$ // *group* 1, *inactive*
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  $(\neg S_2 \vee A_2 \vee A_5 \vee A_7\ )\wedge$ // *group* 2, *inactive*
  $(\neg S_2 \vee \neg A_2 \vee A_5 \vee \neg A_8\ )\wedge$ // *group* 2, *inactive*
  $(\neg S_3 \vee \neg A_1 \vee \neg A_3 \vee A_8\ )\wedge$ // *group* 3
- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
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\end{array}
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(\neg S_1 & \vee A_1 & \vee A_2 & \vee \neg A_3 &)\wedge & // \textit{ group } 1, \textit{inactive} \\
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\begin{array}{llllll}
(\neg S_1 & \vee A_1 & \vee A_2 & \vee \neg A_3 &)\wedge & // \textit{group } 1, \textit{inactive} \\
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- $SAT(\varphi', \{S_2, S_3\})$: activates group 2,3
- $SAT(\varphi', \{S_1, S_3\})$: activates group 1,3

Incremental SAT solving [17, 16]

- Many CDCL solvers provide a stack-based incremental interface
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- Maintains the status of the search from one call to the other
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Example

- Initial formula φ:

$$\begin{array}{llllll} \ldots & & & & \wedge & \\ (\; A_1 & \vee \; A_2 & \vee \neg A_3 &) \wedge & // \; \phi_1 \\ (\neg A_3 & \vee \neg A_2 & \vee \neg A_5 &) \wedge & // \; \phi_1 \end{array}$$

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$$\begin{array}{llllll} \ldots & & & & \wedge & \\ (\neg S_1 & \vee \; A_1 & \vee \; A_2 & \vee \neg A_3 &) \wedge & // \; \phi_1 \\ (\neg S_1 & \vee \neg A_3 & \vee \neg A_2 & \vee \neg A_5 &) \wedge & // \; \phi_1 \end{array}$$

[push(S_1)]: $SAT(\varphi', \{..., S_1\})$: ϕ_1 active $\Longrightarrow$ learn C_1 from ϕ_1

- C_1 derived from $\phi_1 \Longrightarrow C_1$ active only when ϕ_1 is active
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\end{array}
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 & & & & & \\
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Outline

Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

- Building proofs of unsatisfiability
- Extracting unsatisfiable Cores
- Computing Craig Interpolants
- Enumeration in SAT: AllSAT (hints)
- Optimization in SAT: MaxSAT (hints)

Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
  - to verify the result of the solver
  - to understand a "reason" for unsatisfiability
  - to build unsatisfiable cores and interpolants
- Can be built by keeping track of the resolution steps performed when constructing the conflict clauses.

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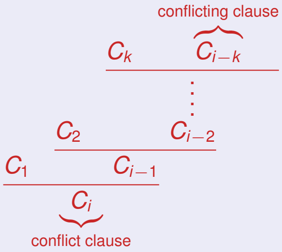
Building Proofs of Unsatisfiability

- Recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal



- C_1, ..., C_k, and C_{i-k} can be either original or learned clauses
- each resolution (sub)proof can be easily tracked:

```
k i-k -> i-k-1
...
2 i-2 -> i-1
1 i-1 -> i
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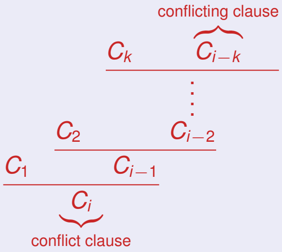

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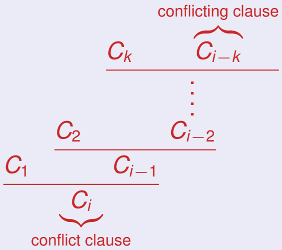

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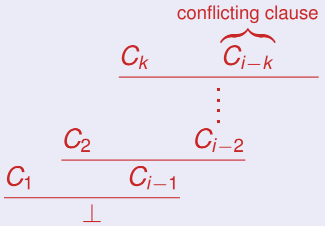
Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces "false" as conflict clause:



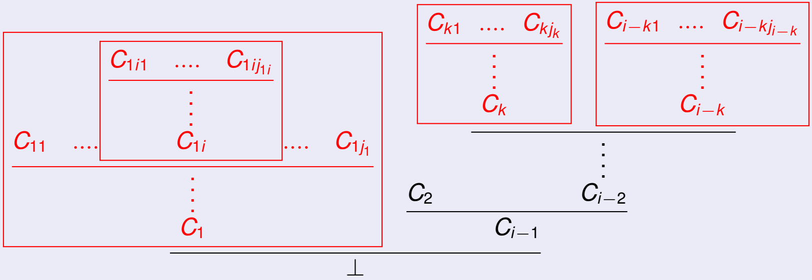
- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- $C_1, ..., C_k$, and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

- for every learned leaf clause C_i, substitute C_i with the resolution proof generating it

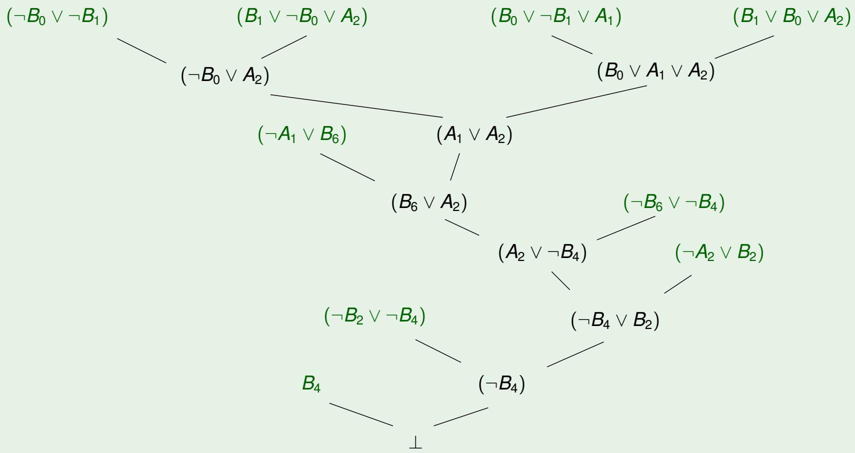
until all leaf clauses are original clauses



$\Longrightarrow$ We obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$$(B_0 \lor \neg B_1 \lor A_1) \land (B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor B_1 \lor A_2) \land (\neg B_0 \lor \neg B_1) \land (\neg B_2 \lor \neg B_4) \land$$
$$(\neg A_2 \lor B_2) \land (\neg A_1 \lor B_3) \land B_4 \land (A_2 \lor B_5) \land (\neg B_6 \lor \neg B_4) \land (B_6 \lor \neg A_1) \land B_7$$



Extraction of unsatisfiable cores

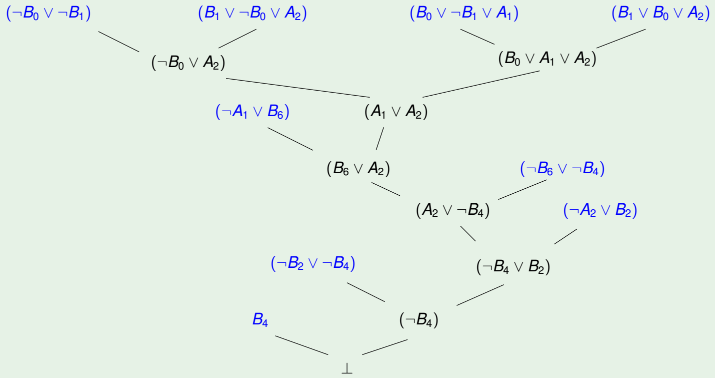
- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset

  $\Longrightarrow$ unsatisfiable cores (aka (Minimal) Unsatisfiable Subsets, (M)US)
- Lots of literature on the topic [63, 35, 37, 44, 60, 27, 21, 9]
- We recognize two main approaches:
  - Proof-based approach [63]: byproduct of finding a resolution proof
  - Assumption-based approach [35]: use extra variables labeling clauses
- Many optimizations for further reducing the size of the core:
  - repeat the process up to fixpoit
  - remove clauses one-by one, until satisfiability is obtained
  - combinations of the two processed above
  - ...

The proof-based approach to core extraction [63]

Unsat core: the set of leaf clauses of a resolution proof

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$



The assumption-based approach to core extraction [35]

Based on the following process:

(i) each clause C_i is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh "selector" variable

(ii) before starting the search each S_i is forced to true.

(iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$

$\Longrightarrow$ $\{C_j\}_j$ is the unsat core!

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The assumption-based approach to core extraction

Example

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge$
$B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

(i) add selector variables: $(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge$
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$(\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7)$

(ii) The conflict analysis returns: $\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11}$,

(iii) corresponding to the unsat core:

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Notation: Let "$X \preceq Y$", X, Y being Boolean formulas, denote the fact that all Boolean atoms in X occur also in Y.

Definition: Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models \bot$, a *Craig interpolant* is a formula I s.t.:

a) $A \models I$,

b) $I \wedge B \models \bot$,

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- Very important in many Formal Verification applications
- A few works presented [45, 36, 38]

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(i) Generate a resolution proof of unsatisfiability $\mathcal{P}$ for $A \wedge B$.

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(iii) For every leaf clause C in $\mathcal{P}$,

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(v) Output $I_\perp$ as an interpolant for (A, B).

"$\eta \downarrow B$" [resp. "$\eta \setminus B$"] is the set of literals in η whose atoms do [resp. do] occur in B.

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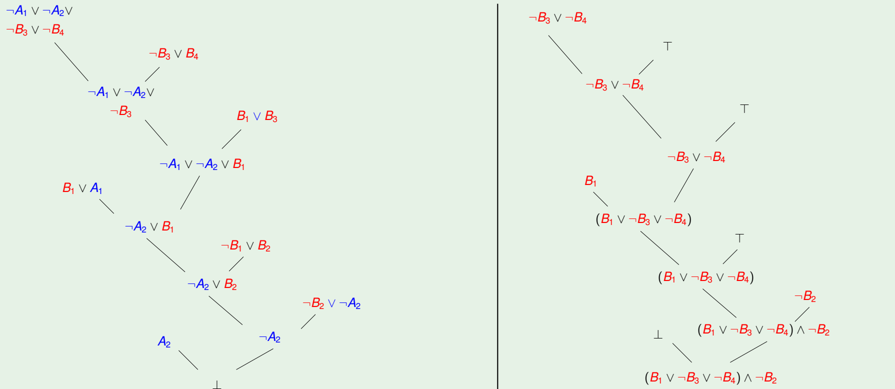
Computing Craig Interpolants in SAT: example

$$A \stackrel{\text{def}}{=} (B_1 \vee A_1) \wedge A_2 \wedge (\neg B_2 \vee \neg A_2) \wedge (\neg A_1 \vee \neg A_2 \vee \neg B_3 \vee \neg B_4)$$

$$B \stackrel{\text{def}}{=} (\neg B_3 \vee B_4) \wedge (\neg B_1 \vee B_2) \wedge (B_1 \vee B_3)$$



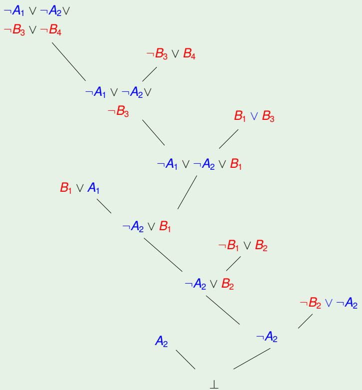
original proof | **interpolant proof**

$\Longrightarrow$ $(B_1 \vee \neg B_3 \vee \neg B_4) \wedge \neg B_2$ is an interpolant

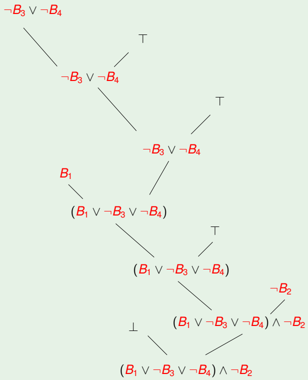
Computing Craig Interpolants in SAT: example

$$A \stackrel{\text{def}}{=} (B_1 \lor A_1) \land A_2 \land (\neg B_2 \lor \neg A_2) \land (\neg A_1 \lor \neg A_2 \lor \neg B_3 \lor \neg B_4)$$

$$B \stackrel{\text{def}}{=} (\neg B_3 \lor B_4) \land (\neg B_1 \lor B_2) \land (B_1 \lor B_3)$$



original proof



interpolant proof

$\Longrightarrow$ $(B_1 \lor \neg B_3 \lor \neg B_4) \land \neg B_2$ is an interpolant

All-SAT (hints)

- All-SAT: enumerate all truth assignments satisfying φ
- All-SAT over an "important" subset of atoms $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$: enumerate all assignments over $\mathbf{P}$ which can be extended to satisfiable truth assignments propositionally satisfying φ
- Algorithms
  - BCLT [Lahiri et al, CAV'06]:
    each time a satisfiable assignment $\{l_1, ..., l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$ belonged to the clause set
  - MathSAT/NuSMV [Cavada et al, FMCAD'07]:
    As above, plus the Boolean search of the SAT solver is driven by an OBDD.

MaxSAT (hints)

- MaxSAT: given a pair of CNF formulas $\langle\varphi_h, \varphi_s\rangle$ s.t. $\varphi_h \wedge \varphi_s \models \bot$, $\varphi_s \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s.
- Weighted MaxSAT: given also the positive integer penalties $\{w_1, ..., w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to optimization
  $\Longrightarrow$ much harder than SAT
- Many different approaches (see e.g. [33])
- EX:

$$\varphi_h \stackrel{\text{def}}{=} (A_1 \vee A_2) \qquad \varphi_s \stackrel{\text{def}}{=} \left(\begin{array}{lcl} (A_1 \vee \neg A_2) & \wedge & [4] \\ (\neg A_1 \vee A_2) & \wedge & [3] \\ (\neg A_1 \vee \neg A_2) & \wedge & [2] \end{array} \right)$$

$\Longrightarrow \mu = \{A_1, A_2\}$ (penalty = 2)

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Outline

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Tractable subclasses of SAT

- SAT in general is an NP-complete problem
- Some subclasses of SAT are tractable
- Two noteworthy tractable subclasses of SAT:
  - Horn Formulas (Horn-SAT)
  - 2-CNF formulas (2-SAT)

Horn Formulas

- A Horn formula is a CNF Boolean formula s.t. each clause contains at most one positive literal.

  $A_1 \vee \neg A_2$
  $A_2 \vee \neg A_3 \vee \neg A_4$
  $\neg A_5 \vee \neg A_3 \vee \neg A_4$
  A_3

- Intuition: implications between positive Boolean variables:

$$\begin{aligned} A_2 &\rightarrow A_1 \\ (A_3 \wedge A_4) &\rightarrow A_2 \\ (A_5 \wedge A_3 \wedge A_4) &\rightarrow \bot \\ & \quad\; A_3 \end{aligned}$$

Formulas reducible to Horn

- Remark: Some non-Horn formulas can be reduced to Horn by simply renaming literals

$$\begin{array}{l} A_1 \vee A_2 \\ \neg A_2 \vee \neg A_3 \vee \neg A_4 \\ \neg A_5 \vee \neg A_3 \vee \neg A_4 \\ A_3 \end{array} \Longrightarrow_{B := \neg A_2} \begin{array}{l} A_1 \vee \neg B \\ B \vee \neg A_3 \vee \neg A_4 \\ \neg A_5 \vee \neg A_3 \vee \neg A_4 \\ A_3 \end{array}$$

Tractability of Horn Formulas

Property

Checking the satisfiability of Horn formulas requires polynomial time:

- Hint:
  1. Eliminate unit clauses by propagating their value;
  2. If an empty clause is generated, return unsat
  3. Otherwise, every clause contains at least one negative literal

  $\Longrightarrow$ Assign all variables to $\perp$; return the assignment
- Alternatively: run DPLL/CDCL, selecting negative literals first

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  $\Longrightarrow$ Assign all variables to $\bot$; return the assignment
- Alternatively: run DPLL/CDCL, selecting negative literals first

A simple polynomial procedure for Horn-SAT

function *Horn_SAT(formula* φ, *assignment &* μ*)* $\{$
    Unit_Propagate(φ, μ*)*;
    if $(\varphi == \bot)$
        then return *UNSAT*;
    else $\{$
        $\mu := \mu \cup \bigcup_{A_i \notin \mu} \{\neg A_i\}$;
        return *SAT*;
$\}$ $\}$

function *Unit_Propagate(formula &* φ, *assignment &* μ*)*
    while $(\varphi \neq \top$ **and** $\varphi \neq \bot$ **and** {a unit clause (l) occurs in φ}) **do** $\{$
        $\varphi = assign(\varphi, l)$;
        $\mu := \mu \cup \{l\}$;
$\}$ $\}$

Example

$$
\begin{array}{rll}
\neg A_1 & \vee \; A_2 & \vee \neg A_3 \\
A_1 & \vee \neg A_3 & \vee \neg A_4 \\
\neg A_2 & \vee \neg A_4 & \\
A_3 & \vee \neg A_4 & \\
A_4 & &
\end{array}
$$

Example

$$
\begin{array}{lll}
\neg A_1 & \vee\ A_2 & \vee \neg A_3 \\
\ \ A_1 & \vee \neg A_3 & \vee \neg A_4 \\
\neg A_2 & \vee \neg A_4 & \\
\ \ A_3 & \vee \neg A_4 & \\
\ \ A_4 & &
\end{array}
$$

$\mu := \{A_4 := \top\}$

Example

$$
\begin{array}{llll}
\neg A_1 & \vee\ A_2 & \vee \neg A_3 \\
A_1 & \vee \neg A_3 & \vee \neg A_4 \\
\neg A_2 & \vee \neg A_4 & \\
A_3 & \vee \neg A_4 & \\
A_4 & &
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example

$$\begin{array}{llll} \neg A_1 & \vee\ A_2 & \vee \neg A_3 \\ A_1 & \vee \neg A_3 & \vee \neg A_4 \\ \neg A_2 & \vee \neg A_4 & \\ A_3 & \vee \neg A_4 & \\ A_4 & & \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example

$$
\begin{array}{llll}
\neg A_1 & \vee\ A_2 & \vee \neg A_3 & \times \\
A_1 & \vee \neg A_3 & \vee \neg A_4 & \\
\neg A_2 & \vee \neg A_4 & & \\
A_3 & \vee \neg A_4 & & \\
A_4 & & &
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_1 := \top\} \Longrightarrow$ UNSAT

Example 2

$$\begin{array}{lll} A_1 & \vee \neg A_2 & \\ A_2 & \vee \neg A_5 & \vee \neg A_4 \\ A_4 & \vee \neg A_3 & \\ A_3 & & \end{array}$$

Example 2

$$
\begin{array}{lll}
A_1 & \vee \neg A_2 & \\
A_2 & \vee \neg A_5 & \vee \neg A_4 \\
A_4 & \vee \neg A_3 & \\
A_3 & &
\end{array}
$$

$\mu := \{A_3 := \top\}$

Example 2

$$
\begin{array}{lll}
A_1 & \vee \neg A_2 & \\
A_2 & \vee \neg A_5 & \vee \neg A_4 \\
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A_3 & &
\end{array}
$$

$\mu := \{A_3 := \top, A_4 := \top\}$

Example 2

$$
\begin{array}{lll}
A_1 & \vee \neg A_2 & \\
A_2 & \vee \neg A_5 & \vee \neg A_4 \\
A_4 & \vee \neg A_3 & \\
A_3 & &
\end{array}
$$

$\mu := \{A_3 := \top, A_4 := \top\} \Longrightarrow$ SAT

2-CNF Formulas

- A 2-CNF formula is a CNF formula in which each clause has (at most) two literals.
  $A_1 \vee \neg A_2$
  $A_2 \vee \neg A_3$
  $\neg A_5 \vee \neg A_3$
  $A_3 \vee \neg A_1$
  A_5
- SAT with 2-CNF formulas requires polynomial time

Tractability of 2-CNF Formulas

Graph-based approach:

(i) Build the implication graph of the formula

(ii) check if it has a cycle containing both A_i and $\neg A_i$ for some i (e.g., by Tarjan's algorithm)
$\Longrightarrow$ the formula is unsatisfiable iff such cycle exists

- requires linear time

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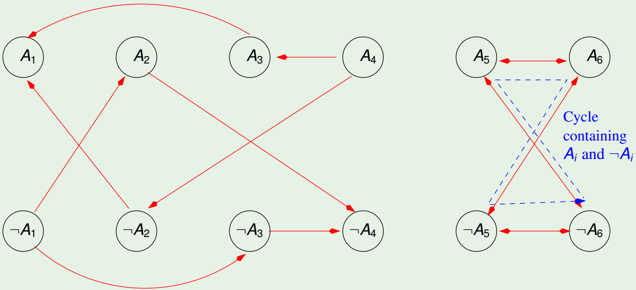
Example:

A_1	$\vee$	A_2
A_1	$\vee\neg$	A_3
$\neg A_2$	$\vee\neg$	A_4
A_3	$\vee\neg$	A_4
A_4		
$\neg A_5$	$\vee$	A_6
A_5	$\vee$	A_6
A_5	$\vee\neg$	A_6
$\neg A_5$	$\vee\neg$	A_6



Tractability of 2-CNF Formulas

Idea

Let φ, l s.t. $var(l) \in \varphi$ and $(\varphi \wedge l) \not\models_{BCP} \bot$.

- φ': clauses remained after BCP
- φ'': clauses removed by BCP

Suppose φ' is UNSAT. Can we conclude anything about φ?

- Case φ is >2-CNF: No!
  - there may be (non-unit) clauses $C \in \varphi'$ s.t. $(\neg l \vee C) \in \varphi$

  $\Longrightarrow$ $\varphi \neq \varphi' \wedge \varphi''$ and $\varphi' \models \bot \not\Longrightarrow \varphi \models \bot$

  $\Longrightarrow$ we must check also $\varphi \wedge \neg l$
- Case φ is 2-CNF: Yes!
  - there cannot be clause $C \in \varphi'$ s.t. $(\neg l \vee C) \in \varphi$

  $\Longrightarrow$ $\varphi = \varphi' \wedge \varphi''$ and $\varphi' \models \bot \Longrightarrow \varphi \models \bot$

  $\Longrightarrow$ φ is UNSAT

Note: we need to check first that $(\varphi \wedge l) \not\models_{BCP} \bot$:
If $(\varphi \wedge l) \models_{BCP} \bot$, then $\varphi' \models \bot \not\Longrightarrow \varphi \models \bot$ (see later Example 2).

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- φ': clauses remained after BCP
- φ'': clauses removed by BCP

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  - there cannot be clause $C \in \varphi'$ s.t. $(\neg l \vee C) \in \varphi$

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Note: we need to check first that $(\varphi \wedge l) \not\models_{BCP} \bot$:
If $(\varphi \wedge l) \models_{BCP} \bot$, then $\varphi' \models \bot \not\Longrightarrow \varphi \models \bot$ (see later Example 2).

A simple polynomial procedure for 2-SAT

function *2_SAT(formula* φ*, assignment &* μ*)* {
    Unit_Propagate(φ, μ);
    if ($\varphi == \bot$) **then return** *UNSAT*;
    if ($\varphi == \top$) **then return** *SAT*;
    while *True* **do** {
        {choose some literal *l* occurring in φ};
        save(φ, μ);
        $\varphi := \varphi \wedge$ *l*;
        Unit_Propagate(φ, μ);
        if ($\varphi == \bot$) **then** {
            retrieve(φ, μ);
            $\varphi = \varphi \wedge \neg$*l*;
            Unit_Propagate(φ, μ); }
        if ($\varphi == \bot$) **then return** *UNSAT*;
        if ($\varphi == \top$) **then return** *SAT*;
} };

Example

$$
\begin{array}{rl}
A_1 & \lor \ A_2 \\
A_1 & \lor \neg A_3 \\
\neg A_2 & \lor \neg A_4 \\
A_3 & \lor \neg A_4 \\
A_4 & \\
\neg A_5 & \lor \ A_6 \\
A_5 & \lor \ A_6 \\
A_5 & \lor \neg A_6 \\
\neg A_5 & \lor \neg A_6
\end{array}
$$

Example

$$
\begin{array}{rl}
A_1 & \lor \; A_2 \\
A_1 & \lor \neg A_3 \\
\neg A_2 & \lor \neg A_4 \\
A_3 & \lor \neg A_4 \\
A_4 & \\
\neg A_5 & \lor \; A_6 \\
A_5 & \lor \; A_6 \\
A_5 & \lor \neg A_6 \\
\neg A_5 & \lor \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top\}$

Example

$$
\begin{array}{rl}
A_1 & \vee \quad A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \quad A_6 \\
A_5 & \vee \quad A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example

$$\begin{array}{rl}
A_1 & \lor \ A_2 \\
A_1 & \lor \neg A_3 \\
\neg A_2 & \lor \neg A_4 \\
A_3 & \lor \neg A_4 \\
A_4 & \\
\neg A_5 & \lor \ A_6 \\
A_5 & \lor \ A_6 \\
A_5 & \lor \neg A_6 \\
\neg A_5 & \lor \neg A_6
\end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example

$$
\begin{array}{rl}
A_1 & \vee \; A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \; A_6 \\
A_5 & \vee \; A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot\}$ (Select $\neg A_6$)

Example

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \quad \times \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot, A_5 := \bot\} \Longrightarrow$ backtrack

Example

$$\begin{array}{rl} A_1 & \vee\ A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee\ A_6 \\ A_5 & \vee\ A_6 \\ A_5 & \vee \neg A_6 \\ \neg A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top\}$ (Select A_6)

Example

$$
\begin{array}{rl}
A_1 & \vee \; A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \; A_6 \\
A_5 & \vee \; A_6 \\
A_5 & \vee \neg A_6 \quad \times \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top, A_5 := \top\} \Longrightarrow$ UNSAT

Example 2

$$\begin{aligned}
A_1 &\lor A_2 \\
A_1 &\lor \neg A_3 \\
\neg A_2 &\lor \neg A_4 \\
A_3 &\lor \neg A_4 \\
A_4 & \\
\neg A_5 &\lor A_6 \\
A_5 &\lor A_6 \\
A_5 &\lor \neg A_6
\end{aligned}$$

Example 2

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top\}$

Example 2

$$
\begin{array}{rcl}
A_1 & \vee & A_2 \\
A_1 & \vee & \neg A_3 \\
\neg A_2 & \vee & \neg A_4 \\
A_3 & \vee & \neg A_4 \\
A_4 & & \\
\neg A_5 & \vee & A_6 \\
A_5 & \vee & A_6 \\
A_5 & \vee & \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example 2

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example 2

$$\begin{array}{rl} A_1 & \vee \; A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee \; A_6 \\ A_5 & \vee \; A_6 \\ A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot\}$ (Select $\neg A_6$)

Example 2

$$\begin{array}{rl} A_1 & \vee \; A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee \; A_6 \\ A_5 & \vee \; A_6 \quad \times \\ A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot, A_5 := \bot\} \Longrightarrow$ backtrack

Example 2

$$
\begin{array}{rl}
A_1 & \vee \; A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \; A_6 \\
A_5 & \vee \; A_6 \\
A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top\}$ (Select A_6)

Example 2

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top, A_5 := \top\} \Longrightarrow$ SAT

Outline

The satisfiability of k-CNF (k-SAT) [19]

- k-CNF: CNF s.t. all clauses have k literals
- the satisfiability of 2-CNF is polynomial
- the satisfiability of k-CNF is NP-complete for $k \geq 3$
- every k-CNF formula can be converted into 3-CNF:

$$l_1 \vee l_2 \vee \ldots \vee l_{k-1} \vee l_k$$

$$\Downarrow$$

$$\begin{array}{c}(l_1 \vee l_2 \vee B_1) \wedge \\ (\neg B_1 \vee l_3 \vee B_2) \wedge \\ \ldots \\ (\neg B_{k-4} \vee l_{k-2} \vee B_{k-3}) \wedge \\ (\neg B_{k-3} \vee l_{k-1} \vee l_k)\end{array}$$

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$$
\begin{gathered}
l_1 \vee l_2 \vee \ldots \vee l_{k-1} \vee l_k \\
\Downarrow \\
(l_1 \vee l_2 \vee B_1) \wedge \\
(\neg B_1 \vee l_3 \vee B_2) \wedge \\
\ldots \\
(\neg B_{k-4} \vee l_{k-2} \vee B_{k-3}) \wedge \\
(\neg B_{k-3} \vee l_{k-1} \vee l_k)
\end{gathered}
$$

Random K-CNF formulas generation

Random k-CNF formulas with N variables and L clauses:
DO

(i) pick with uniform probability a set of k atoms over N
(ii) randomly negate each atom with probability 0.5
(iii) create a disjunction of the resulting literals

UNTIL L different clauses have been generated;

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Random k-SAT plots

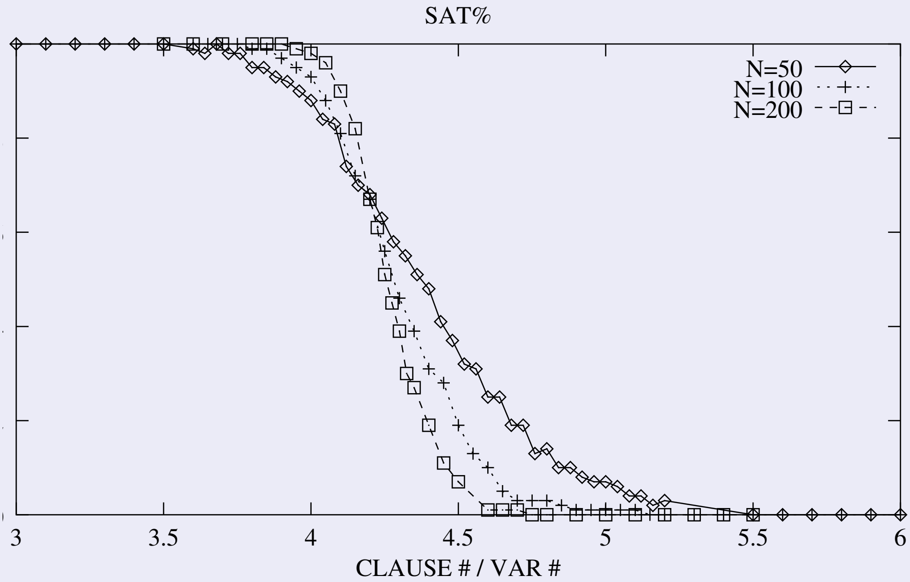
- fix k and N
- for increasing L, randomly generate and solve (500,1000,10000,...) problems with k, L, N
- plot
  - satisfiability percentages
  - median/geometrical mean CPU time/# of steps

  against L/N

The phase transition phenomenon: SAT % Plots [39, 31]

- Increasing L/N we pass from 100% satisfiable to 100% unsatisfiable formulas
- the decay becomes steeper with N
- for $N \rightarrow \infty$, the plot converges to a step in the cross-over point ($L/N \approx 4.28$ for k=3)
- Revealed for many other NP-complete problems
- Many theoretical models [57, 20, 31, 15, 40]
- Strong relation with Thermodynamics

The phase transition phenomenon: SAT % Plots /cont.) [39, 31]

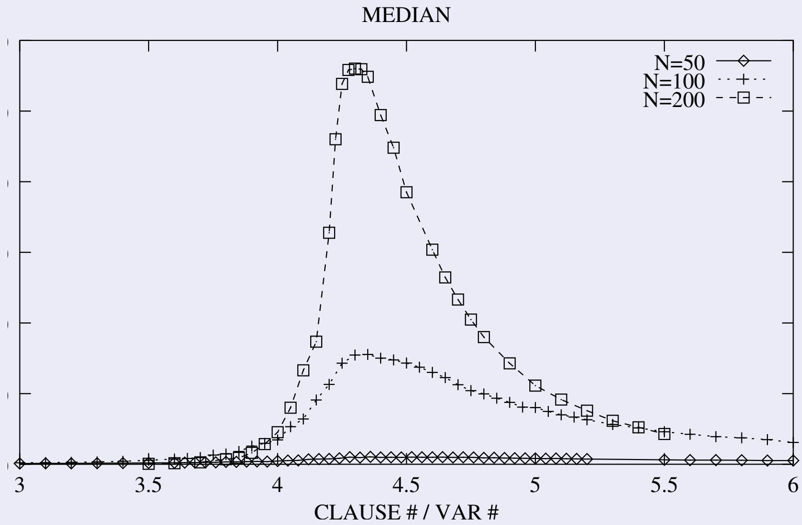


The phase transition phenomenon: CPU times/step

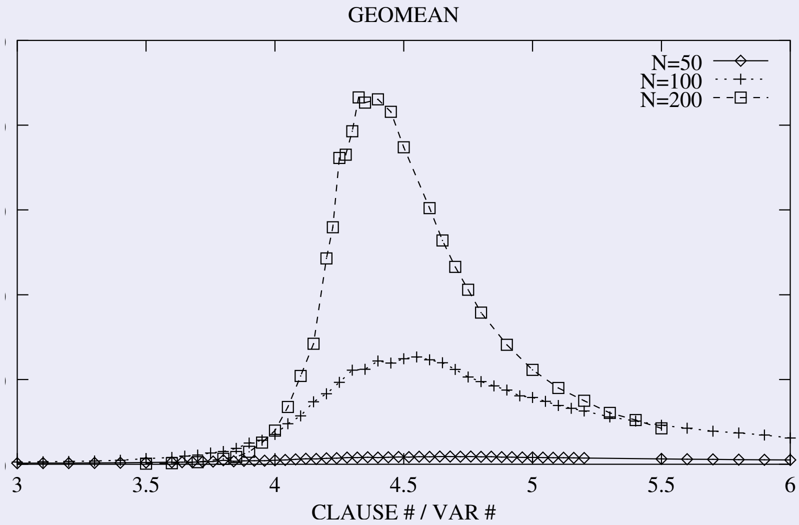
Using search algorithms (DPLL):

- Increasing L/N we pass from easy problems, to very hard problems down to hard problems
- the peak is centered in the 50% satisfiable point
- the decay becomes steeper with N
- for $N \rightarrow \infty$, the plot converges to an impulse in the cross-over point ($L/N \approx 4.28$ for k=3)
- easy problems ($L/N \leq\approx 3.8$) increase polynomially with N, hard problems increase exponentially with N
- Increasing L/N, satisfiable problems get harder, unsatisfiable problems get easier.

The phase transition phenomenon: CPU times/step # (cont.)



The phase transition phenomenon: CPU times/step # (cont.)



Outline

Many applications of SAT

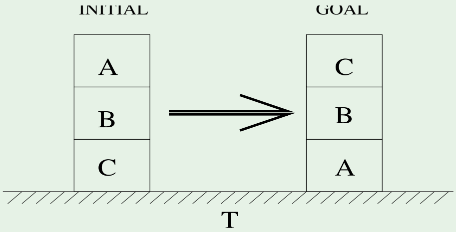
- Many successful applications of SAT:
  - Boolean circuits
  - (Bounded) Planning
  - (Bounded) Model Checking
  - Cryptography
  - Scheduling
  - ...
- All NP-complete problem can be (polynomially) converted to SAT.
- Key issue: find an efficient encoding.

Outline

The problem [29, 28, 46]

- Problem Given a set of action operators OP, (a representation of) an initial state I and goal state G, and a bound n, find a sequence of operator applications $o_1, .., o_n$, leading from the initial state to the goal state.
- Idea: Encode it into satisfiability problem of a Boolean formula φ

Example



$$
\begin{array}{ll}
\multicolumn{2}{l}{Move(b, s, d)} \\
\quad Precond: & Block(b) \wedge Clear(b) \wedge On(b, s) \wedge \\
& (Clear(d) \vee Table(d)) \wedge \\
& b \neq s \wedge b \neq d \wedge s \neq d \\
\quad Effect: & Clear(s) \wedge \neg On(b, s) \wedge \\
& On(b, d) \wedge \neg Clear(d)
\end{array}
$$

Encoding

- Initial states:

$$On_0(A, B), On_0(B, C), On_0(C, T), Clear_0(A).$$

- Goal states:

$$On_{2n}(C, B) \wedge On_{2n}(B, A) \wedge On_{2n}(A, T).$$

- Action preconditions and effects:

$$\begin{aligned} & Move_t(A, B, C) \rightarrow \\ & \quad Clear_{t-1}(A) \wedge On_{t-1}(A, B) \wedge Clear_{t-1}(C) \wedge \\ & \quad Clear_{t+1}(B) \wedge \neg On_{t+1}(A, B) \wedge \\ & \quad On_{t+1}(A, C) \wedge \neg Clear_{t+1}(C). \end{aligned}$$

Encoding: Frame axioms

- Classic

$$Move_t(A,B,T) \wedge Clear_{t-1}(C) \rightarrow Clear_{t+1}(C),$$
$$Move_t(A,B,T) \wedge \neg Clear_{t-1}(C) \rightarrow \neg Clear_{t+1}(C).$$

"At least one action" axiom:

$$\bigvee_{\substack{b,s,d \in \{A,B,C,T\} \\ b \neq s, b \neq d, s \neq d, b \neq T}} Move_t(b,s,d).$$

- Explanatory

$$\neg Clear_{t+1}(C) \wedge Clear_{t-1}(C) \rightarrow$$
$$Move_t(A,B,C) \vee Move_t(A,T,C) \vee Move_t(B,A,C) \vee Move_t(B,T,C).$$

Planning strategy

- Sequential for each pair of actions α and β, add axioms of the form $\neg\alpha_t \vee \neg\beta_t$ for each odd time step. For example, we will have:

$$\neg Move_t(A, B, C) \vee \neg Move_t(A, B, T).$$

- parallel for each pair of actions α and β, add axioms of the form $\neg\alpha_t \vee \neg\beta_t$ for each odd time step if α effects contradict β preconditions. For example, we will have

$$\neg Move_t(B, T, A) \vee \neg Move_t(A, B, C).$$

Encoding into SAT

- Assumption: the possible values of all the variables are bounded.
- Naive idea: Encode all possible ground predicates as Boolean variables.
  E.g.: $Move_1(B, T, A) \Longrightarrow$ `Move1_B_T_A`
- much more efficient encodings have been presented [28, 18]
- customizations of SAT solvers [22].

Outline

The problem [7, 6]

Ingredients:

- A system written as a Kripke structure $M := \langle S, I, T, \mathcal{L} \rangle$
  - S: set of states
  - I: set of initial states
  - T: transition relation
  - $\mathcal{L}$: labeling function
- A property f written as a LTL formula:
  - a propositional literal p
  - $h \wedge g$, $h \vee g$, $\mathbf{X}g$, $\mathbf{G}g$, $\mathbf{F}g$,$h\mathbf{U}g$ and $h\mathbf{R}g$,
    X, **G**, **F**, **U**, **R** "next", "globally", "eventually", "until" and "releases"
- an integer k (bound)

Problem:

Is there an execution path of M of length k satisfying the temporal property f?:

$$M \models_k \mathbf{f}$$

The problem [7, 6]

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Problem:

Is there an execution path of M of length k satisfying the temporal property f?:

$$M \models_k \mathbf{f}$$

The encoding

Equivalent to the satisfiability problem of a Boolean formula $[[M, f]]_k$ defined as follows:

$$[[M, f]]_k \quad := \quad [[M]]_k \wedge [[f]]_k \tag{1}$$

$$[[M]]_k \quad := \quad I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}), \tag{2}$$

$$[[f]]_k \quad := \quad (\neg \bigvee_{l=0}^{k} T(s_k, s_l) \wedge [[f]]_k^0) \vee \bigvee_{l=0}^{k} (T(s_k, s_l) \wedge {}_l[[f]]_k^0), \tag{3}$$

The encoding of $[[f]]_k^i$ and ${}_l[[f]]_k^i$

f	$[[f]]_k^i$	${}_l[[f]]_k^i$
p	p_i	p_i
$\neg p$	$\neg p_i$	$\neg p_i$
$h \wedge g$	$[[h]]_k^i \wedge [[g]]_k^i$	${}_l[[h]]_k^i \wedge {}_l[[g]]_k^i$
$h \vee g$	$[[h]]_k^i \vee [[g]]_k^i$	${}_l[[h]]_k^i \vee {}_l[[g]]_k^i$
$\mathbf{X}g$	$[[g]]_k^{i+1}$ *if* $i < k$ $\bot$ *otherwise.*	${}_l[[g]]_k^{i+1}$ *if* $i < k$ ${}_l[[g]]_k^l$ *otherwise.*
$\mathbf{G}g$	$\bot$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$\mathbf{F}g$	$\bigvee_{j=i}^{k} [[g]]_k^j$	$\bigvee_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$h\mathbf{U}g$	$\bigvee_{j=i}^{k} \left([[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} [[h]]_k^n \right)$	$\bigvee_{j=i}^{k} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_l[[h]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[h]]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_l[[h]]_k^n \right)$
$h\mathbf{R}g$	$\bigvee_{j=i}^{k} \left([[h]]_k^j \wedge \bigwedge_{n=i}^{j} [[g]]_k^n \right)$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j \vee$ $\bigvee_{j=i}^{k} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{j} {}_l[[g]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[g]]_k^n \wedge \bigwedge_{n=l}^{j} {}_l[[g]]_k^n \right)$

Example: **F***p* (reachability)

- $f := \mathbf{F}p$: is there a reachable state in which p holds?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{j=0}^{k} p_j$$

Example: $\mathbf{G}p$

- $f := \mathbf{G}p$: is there a path where p holds forever?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{l=0}^{k} T(s_k, s_l) \wedge \bigwedge_{j=0}^{k} p_j$$

Example: $\mathbf{GF}q \wedge \mathbf{F}p$ (fair reachability)

- $f := \mathbf{GF}q \wedge \mathbf{F}p$: is there a reachable state in which p holds provided that q holds infinitely often?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{j=0}^{k} p_j \wedge \bigvee_{l=0}^{k} \left(T(s_k, s_l) \wedge \bigvee_{j=l}^{k} q \right)$$

Bounded Model Checking

- very efficient for some problems
- lots of enhancements [7, 1, 54, 58, 12]

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