Course “An Introduction to SAT and SMT”
Chapter 2: Satisfiability Modulo Theories

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Outline

1. Motivations and goals
2. Efficient SMT solving
   - Combining SAT with Theory Solvers
   - Theory Solvers for theories of interest
   - SMT for combinations of theories
3. Beyond Solving: advanced SMT functionalities
   - Proofs and unsatisfiable cores
   - Interpolants
   - All-SMT & Predicate Abstraction
   - SMT with cost optimization (Optimization Modulo Theories)
4. Conclusions & current research directions

Satisfiability Modulo Theories (SMT($\mathcal{T}$))

Satisfiability Modulo Theories (SMT($\mathcal{T}$))

The problem of deciding the satisfiability of (typically quantifier-free) formulas in some decidable first-order theory $\mathcal{T}$

- $\mathcal{T}$ can also be a combination of theories $\bigcup_i \mathcal{T}_i$.

SMT($\mathcal{T}$): theories of interest

Some theories of interest (e.g., for formal verification)

- Equality and Uninterpreted Functions ($\mathcal{EUF}$):
  $((x = y) \wedge (y = f(z))) \rightarrow (g(x) = g(f(z)))$
- Difference logic ($\mathcal{DL}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x - z \leq 6)$
- UTVPI ($\mathcal{UTVPI}$): $((x = y) \wedge (y - z \leq 4)) \rightarrow (x + z \leq 6)$
- Linear arithmetic over the rationals ($\mathcal{LRA}$):
  $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
- Linear arithmetic over the integers ($\mathcal{LIA}$):
  $(x := x_l + 2^{16} x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
- Arrays ($\mathcal{AR}$): $(i = j) \vee read(write(a, i, e), j) = read(a, j)$
- Bit vectors ($\mathcal{BV}$):
  $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$
- Non-Linear arithmetic over the reals ($\mathcal{NLA}(\mathbb{R})$) :
  $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = b + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
- ...

Satisfiability Modulo Theories (SMT($\mathcal{T}$)): Example

Example: SMT($\mathcal{LIA} \cup \mathcal{EUF} \cup \mathcal{AR}$)

$$\varphi \stackrel{\text{def}}{=} (d \geq 0) \wedge (d < 1) \wedge$$
$$((f(d) = f(0)) \rightarrow (read(write(V, i, x), i + d) = x + 1))$$

- involves arithmetical, arrays, and uninterpreted function/predicate symbols, plus Boolean operators
  - Is it consistent?
  - No:

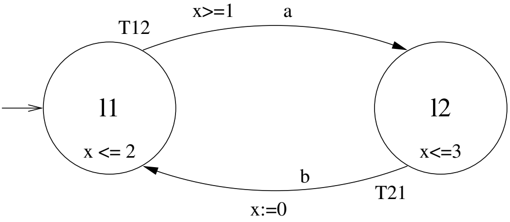
$$\begin{array}{ll} & \varphi \\ \Longrightarrow_{\mathcal{LIA}} & (d = 0) \\ \Longrightarrow_{\mathcal{EUF}} & (f(d) = f(0)) \\ \Longrightarrow_{Bool} & (read(write(V, i, x), i + d) = x + 1) \\ \Longrightarrow_{\mathcal{LIA}} & (read(write(V, i, x), i) = x + 1) \\ \Longrightarrow_{\mathcal{LIA}} & \neg(read(write(V, i, x), i) = x) \\ \Longrightarrow_{\mathcal{AR}} & \bot \end{array}$$

Some Motivating Applications

Interest in SMT triggered by some real-word applications

- Verification of Hybrid & Timed Systems
- Verification of RTL Circuit Designs & of Microcode
- SW Verification
- Planning with Resources
- Temporal reasoning
- Scheduling
- Compiler optimization
- ...

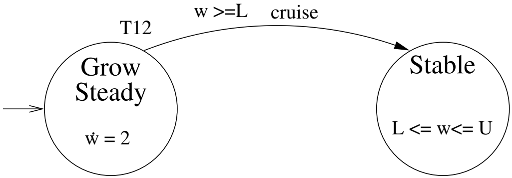
Verification of Timed Systems



- Bounded/inductive model checking of Timed Systems [6, 36, 58], ...
- Timed Automata encoded into $\mathcal{T}$-formulas:
  - discrete information (locations, transitions, events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \leq 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$ or $\mathcal{LRA}$ required

Verification of Hybrid Systems ...

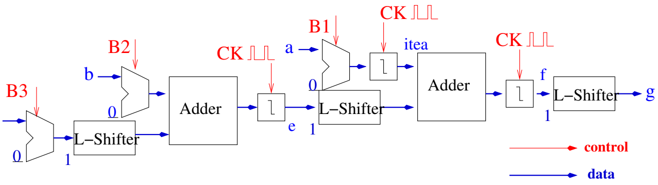


- Bounded model checking of Hybrid Systems [5],...
- Hybrid Automata encoded into $\mathcal{L}$-formulas:
  - discrete information (locs, trans., events) with Boolean vars.
  - timed information (clocks, elapsed time) with differences $(t_3 - x_3 \le 2)$, equalities $(x_4 = x_3)$ and linear constraints $(t_8 - x_8 = t_2 - x_2)$ on $\mathbb{Q}$
  - Evolution of Physical Variables (e.g., speed, pressure) with linear $(\omega_4 = 2\omega_3)$ and non-linear constraints $(P_1 V_1 = 4 T_1)$ on $\mathbb{Q}$
- Undecidable under simple hypotheses!

$\Longrightarrow$ SMT on $\mathcal{DL}(\mathbb{Q})$, $\mathcal{LRA}$ or $\mathcal{NLA}(\mathbb{R})$ required

Verification of HW circuit designs & microcode

g = 2*f
f' = itea+2*e
itea' = ITE(B1;a;0)
e' = ITE(B2;b;0)+2*ite(B3;c;0)



- SAT/SMT-based Model Checking & Equiv. Checking of RTL designs, symbolic simulation of μ-code [25, 22, 42]
- Control paths handled by Boolean reasoning
- Data paths information abstracted into theory-specific terms
  - words (bit-vectors, integers, $\mathcal{EUF}$ vars, ...): $\underline{a}[31:0]$, a
  - word operations: ($\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, $\mathcal{LIA}$, $\mathcal{NLA}(\mathbb{Z})$ operators) $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$, $(a = a_L + 2^{16} a_H)$, $(m_1 = store(m_0, l_0, v_0))$, ...
- Trades heavy Boolean reasoning ($\approx 2^{64}$ factors) with $\mathcal{T}$-solving

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, modulo-$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Verification of SW systems

```
...
i = 0;
acc = 0.0;
while (i<dim) {
   acc += V[i];
   i++;
}
...
```

$$
\begin{array}{ll}
\ldots. \wedge & \\
(i_0 = 0) \wedge & \\
(acc_0 = 0.0) \wedge & \\
((i_0 < dim) \rightarrow (& (acc_1 = acc_0 + read(V, i_0)) \wedge \\
 & (i_1 = i_0 + 1))) \wedge \\
(\neg(i_0 < dim) \rightarrow (& (acc_1 = acc_0) \wedge (i_1 = i_0))) \wedge \\
((i_1 < dim) \rightarrow (& (acc_2 = acc_1 + read(V, i_1)) \wedge \\
 & (i_2 = i_1 + 1))) \wedge \\
(\neg(i_1 < dim) \rightarrow (& (acc_2 = acc_1) \wedge (i_2 = i_1))) \wedge \\
\ldots &
\end{array}
$$

- Verification of SW code
  - BMC, K-induction, Check of proof obligations, interpolation-based model checking, symbolic simulation, concolic testing, ...

$\Longrightarrow$ SMT on $\mathcal{BV}$, $\mathcal{EUF}$, $\mathcal{AR}$, (modulo-)$\mathcal{LIA}$ [$\mathcal{NLA}(\mathbb{Z})$] required

Planning with Resources [82]

- SAT-bases planning augmented with numerical constraints
- Straightforward to encode into into SMT($\mathcal{LRA}$)

Example (sketch) [82]

$(Deliver)$ $\wedge$ // goal
$(MaxLoad)$ $\wedge$ // load constraint
$(MaxFuel)$ $\wedge$ // fuel constraint
$(Move \rightarrow MinFuel)$ $\wedge$ // move requires fuel
$(Move \rightarrow Deliver)$ $\wedge$ // move implies delivery
$(GoodTrip \rightarrow Deliver)$ $\wedge$ // a good trip requires
$(GoodTrip \rightarrow AllLoaded)$ $\wedge$ // a full delivery

---

$(MaxLoad \rightarrow (load \leq 30))$ $\wedge$ // load limit
$(MaxFuel \rightarrow (fuel \leq 15))$ $\wedge$ // fuel limit
$(MinFuel \rightarrow (fuel \geq 7 + 0.5 load))$ $\wedge$ // fuel constraint
$(AllLoaded \rightarrow (load = 45))$ //

(Disjunctive) Temporal Reasoning [79, 2]

- Temporal reasoning problems encoded as disjunctions of difference constraints

$$
\begin{array}{lll}
((x_1 - x_2 \leq 6) & \vee (x_3 - x_4 \leq -2)) & \wedge \\
((x_2 - x_3 \leq -2) & \vee (x_4 - x_5 \leq 5)) & \wedge \\
((x_2 - x_1 \leq 4) & \vee (x_3 - x_7 \leq -6)) & \wedge \\
\ldots & &
\end{array}
$$

- Straightforward to encode into into SMT($\mathcal{DL}$)

SMT and SMT solvers

Common fact about SMT problems from various applications

SMT requires capabilities for heavy Boolean reasoning combined with capabilities for reasoning in expressive decidable F.O. theories

- SAT alone not expressive enough
- standard automated theorem proving inadequate (e.g., arithmetic)
- may involve also numerical computation (e.g., simplex)

Modern SMT solvers

- combine SAT solvers with decision procedures (theory solvers)
  - contributions from SAT, Automated Theorem Proving (ATP), formal verification (FV) and operational research (OR)

Goal

Provide an overview of standard “lazy” SMT:

- foundations
- SMT-solving techniques
- beyond solving: advanced SMT functionalities
- ongoing research

We do not cover related approaches like:

- Eager SAT encodings
- Rewrite-based approaches

We refer to [71, 10] for an overview and references.

Notational remark (1): most/all examples in $\mathcal{LRA}$

For better readability, in most/all the examples of this presentation we will use the theory of linear arithmetic on rational numbers ($\mathcal{LRA}$) because of its intuitive semantics. E.g.:

$$(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$$

Nevertheless, analogous examples can be built with all other theories of interest.

Notational remark (2): “constants” vs. “variables”

- Consider, e.g., the formula:
  $(\neg A_1 \vee (3x_1 - 2x_2 - 3 \leq 5)) \wedge (A_2 \vee (-2x_1 + 4x_3 + 2 = 3))$
- How do we call A_1, A_2?:
  (*a*) Boolean/propositional variables?
  (*b*) uninterpreted 0-ary predicates?
- How do we call x_1, x_2, x_3?:
  (*a*) domain variables?
  (*b*) uninterpreted Skolem constants/0-ary uninterpreted functions?
- Hint:
  (*a*) typically used in SAT, CSP and OR communities
  (*b*) typically used in logic & ATP communities

Hereafter we call A_1, A_2 “Boolean/propositional variables” and x_1, x_2, x_3 "domain variables” (logic purists, please forgive me!)

Modern "lazy" SMT($\mathcal{T}$) solvers

A prominent "lazy" approach [45, 2, 82, 3, 8, 36] (aka "DPLL($\mathcal{T}$)")

- a CDCL SAT solver is used to enumerate truth assignments μ_i for (the Boolean abstraction of) the input formula φ
- a theory-specific solver *$\mathcal{T}$-solver* checks the $\mathcal{T}$-consistency of the set of $\mathcal{T}$-literals corresponding to each assignment

- Many techniques to maximize the benefits of integration [71, 10]
- Many lazy SMT tools available
(Barcelogic, CVC4, MathSAT, OpenSMT, Yices, Z3, ...)

Basic schema: example

$\varphi =$

$c_1 : \neg(2v_2 - v_3 > 2) \vee A_1$

$c_2 : \neg A_2 \vee (v_1 - v_5 \leq 1)$

$c_3 : (3v_1 - 2v_2 \leq 3) \vee A_2$

$c_4 : \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

$c_5 : A_1 \vee (3v_1 - 2v_2 \leq 3)$

$c_6 : (v_2 - v_4 \leq 6) \vee (v_5 = 5 - 3v_4) \vee \neg A_1$

$c_7 : A_1 \vee (v_3 = 3v_5 + 4) \vee A_2$

true, *false*

$\varphi^p =$

$\neg B_1 \vee A_1$

$\neg A_2 \vee B_2$

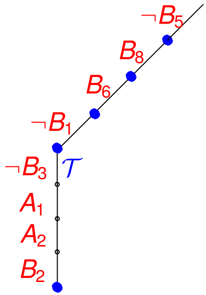
$B_3 \vee A_2$

$\neg B_4 \vee \neg B_5 \vee \neg A_1$

$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$



$$\mu^p = \{\neg B_5, B_8, B_6, \neg B_1, \neg B_3, A_1, A_2, B_2\}$$

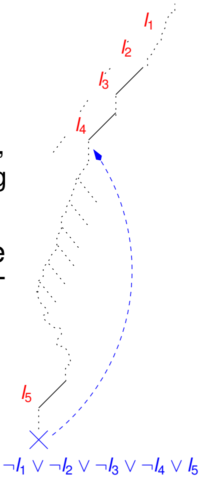
$$\mu = \{\underline{\neg(3v_1 - v_3 \leq 6)}, \underline{(v_3 = 3v_5 + 4)}, (v_2 - v_4 \leq 6), \neg(2v_2 - v_3 > 2), \neg(3v_1 - 2v_2 \leq 3), \underline{(v_1 - v_5 \leq 1)}\}$$

$\Longrightarrow$ inconsistent in $\mathcal{LRA}$ $\Longrightarrow$ backtrack

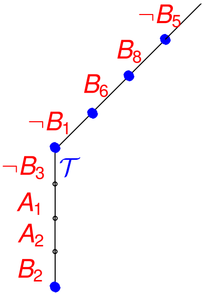
$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning [50, 82, 3, 8, 36]

- Similar to Boolean backjumping & learning
- important property of $\mathcal{T}$-solver:
  - extraction of $\mathcal{T}$-conflict sets: if μ is $\mathcal{T}$-unsatisfiable, then $\mathcal{T}$*-solver* (μ) returns the subset η of μ causing the $\mathcal{T}$-inconsistency of μ ($\mathcal{T}$-conflict set)
- If so, the $\mathcal{T}$-conflict clause $C := \neg\eta$ is used to drive the backjumping & learning mechanism of the SAT solver

  $\Longrightarrow$ lots of search saved
- the less redundant is η, the more search is saved



$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example



$\varphi =$

$c_1: \neg(2v_2 - v_3 > 2) \vee A_1$

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$c_4: \neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1$

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$\varphi^p =$

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$A_1 \vee B_3$

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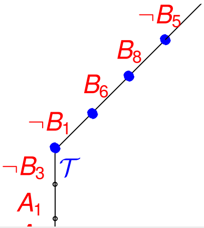
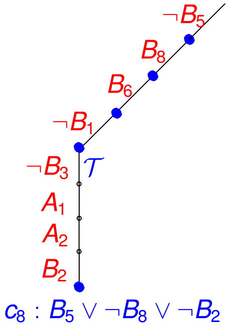


$\mathcal{T}$-Backjumping & $\mathcal{T}$-learning: example (2)



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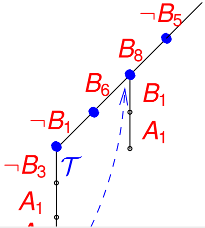
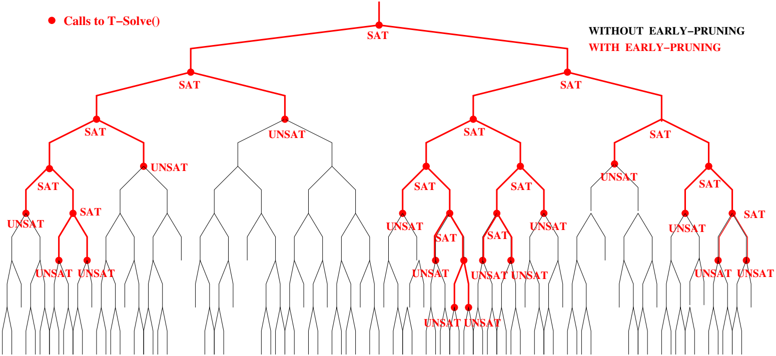


Early Pruning [45, 2, 82] I

- Introduce a $\mathcal{T}$-satisfiability test on intermediate assignments: if $\mathcal{T}$-*solver* returns UNSAT, the procedure backtracks.
  - benefit: prunes drastically the Boolean search
  - Drawback: possibly many useless calls to $\mathcal{T}$-*solver*



Early Pruning [45, 2, 82] II

- Different strategies for interleaving Boolean search steps and $\mathcal{T}$*-solver* calls
  - Eager E.P. [82, 11, 80, 44]): invoke $\mathcal{T}$*-solver* every time a new $\mathcal{T}$-atom is added to the assignment (unit propagations included)
  - Selective E.P.: Do not call $\mathcal{T}$*-solver* if the have been added only literals which hardly cause any $\mathcal{T}$-conflict with the previous assignment (e.g., Boolean literals, disequalities $(x - y \neq 3)$, $\mathcal{T}$-literals introducing new variables $(x - z = 3)$)
  - Weakened E.P.: for intermediate checks only, use weaker but faster versions of $\mathcal{T}$*-solver* (e.g., check μ on $\mathbb{R}$ rather than on $\mathbb{Z}$): $\{(x - y \leq 4), (z - x \leq -6), (z = y), (3x + 2y - 3z = 4)\}$

Early pruning: example

$$\varphi = \begin{array}{l} \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\ \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\ \{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge \\ \{\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1\} \wedge \\ \{A_1 \vee (3v_1 - 2v_2 \leq 3)\} \wedge \\ \{(v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge \\ \{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\}. \end{array}$$

$$\varphi^p = \begin{array}{l} \{\neg B_1 \vee A_1\} \wedge \\ \{\neg A_2 \vee B_2\} \wedge \\ \{B_3 \vee A_2\} \wedge \\ \{\neg B_4 \vee \neg B_5 \vee \neg A_1\} \wedge \\ \{A_1 \vee B_3\} \wedge \\ \{B_6 \vee B_7 \vee \neg A_1\} \wedge \\ \{A_1 \vee B_8 \vee A_2\}. \end{array}$$

- Suppose it is built the intermediate assignment:

$$\mu'^p = \neg B_1 \wedge \neg A_2 \wedge B_3 \wedge \neg B_5.$$

corresponding to the following set of $\mathcal{T}$-literals

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

- If $\mathcal{T}$*-solver* is invoked on μ', then it returns UNSAT, and DPLL backtracks without exploring any extension of μ'.

Early pruning: remark

Incrementality & Backtrackability of $\mathcal{T}$-*solvers*

- With early pruning, lots of incremental calls to $\mathcal{T}$-*solver*:

$\mathcal{T}$-*solver* (μ_1)	$\Rightarrow$ *Sat*	Undo μ_4, μ_3, μ_2		
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2)$	$\Rightarrow$ *Sat*	$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2')$	$\Rightarrow$ *Sat*
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2 \cup \mu_3)$	$\Rightarrow$ *Sat*	$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2' \cup \mu_3')$	$\Rightarrow$ *Sat*
$\mathcal{T}$-*solver* $(\mu_1 \cup \mu_2 \cup \mu_3 \cup \mu_4)$	$\Rightarrow$ *Unsat*	...	

$\Longrightarrow$ Desirable features of $\mathcal{T}$-*solvers*:

- incrementality: $\mathcal{T}$-*solver*$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{T}$-*solver*(μ_1) without restarting from scratch
- backtrackability (resettability): $\mathcal{T}$-*solver* can efficiently undo steps and return to a previous status on the stack

$\Longrightarrow$ $\mathcal{T}$-*solver* requires a stack-based interface

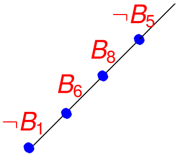
$\mathcal{T}$-Propagation [2, 3, 44]

- strictly related to early pruning
- important property of $\mathcal{T}$-*solver*:
  - $\mathcal{T}$-deduction: when a partial assignment μ is $\mathcal{T}$-satisfiable, $\mathcal{T}$-*solver* may be able to return also an assignment η to some unassigned atom occurring in φ s.t. $\mu \models_{\mathcal{T}} \eta$.
- If so:
  - the literal η is then unit-propagated;
  - optionally, a $\mathcal{T}$-deduction clause $C := \neg\mu' \vee \eta$ can be learned, μ' being the subset of μ which caused the deduction ($\mu' \models_{\mathcal{T}} \eta$)
  - lazy explanation: compute C only if needed for conflict analysis

$\Longrightarrow$ may prune drastically the search

Both $\mathcal{T}$-deduction clauses and $\mathcal{T}$-conflict clauses are called $\mathcal{T}$-lemmas since they are valid in $\mathcal{T}$

$\mathcal{T}$-propagation: example



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$\varphi^p =$

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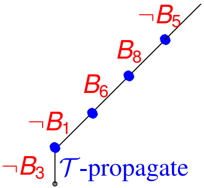
$A_1 \vee B_3$

$B_6 \vee B_7 \vee \neg A_1$

$A_1 \vee B_8 \vee A_2$

true, *false*



Pure-literal filtering [82, 3, 17]

Property

If we have non-Boolean $\mathcal{T}$-atoms occurring only positively [negatively] in the original formula φ (learned clauses are not considered), we can drop every negative [positive] occurrence of them from the assignment to be checked by $\mathcal{T}$*-solver* (and from the $\mathcal{T}$-deducible ones).

- increases the chances of finding a model
- reduces the effort for the $\mathcal{T}$*-solver*
- eliminates unnecessary "nasty" negated literals (e.g. negative equalities like $\neg(3v_1 - 9v_2 = 3)$ in $\mathcal{LIA}$ force splitting: $(3v_1 - 9v_2 > 3) \vee (3v_1 - 9v_2 < 3)$).
- may weaken the effect of early pruning.

Pure literal filtering: example

$$
\begin{aligned}
\varphi = \quad & \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\
& \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\
& \{(3v_1 - 2v_2 \le 3) \vee A_2\} \wedge \\
& \{\neg(2v_3 + v_4 \ge 5) \vee \neg(3v_1 - v_3 \le 6) \vee \neg A_1\} \wedge \\
& \{A_1 \vee (3v_1 - 2v_2 \le 3)\} \wedge \\
& \{(v_1 - v_5 \le 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge \\
& \{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\} \wedge \\
& \{(2v_2 - v_3 > 2) \vee \neg(3v_1 - 2v_2 \le 3) \vee (3v_1 - v_3 \le 6)\} \quad \textit{learned}
\end{aligned}
$$

$$\mu' = \{\neg(2v_2 - v_3 > 2), \neg A_2, (3v_1 - 2v_2 \le 3), \neg A_1, (v_3 = 3v_5 + 4), (3v_1 - v_3 \le 6)\}.$$

$\Longrightarrow$ Sat: $v_1 = v_2 = v_3 = 0, v_5 = -4/3$ is a solution

N.B. $(3v_1 - v_3 \le 6)$ "filtered out" from μ' because it occurs only negatively in the original formula φ

Preprocessing atoms [45, 50, 4]

Source of inefficiency: semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other]

$\Longrightarrow$ they may be assigned different [resp. identical] truth values.

$\Longrightarrow$ lots of redundant unsatisfiable assignment generated

Solution

Rewrite a priori trivially-equivalent atoms/literals into the same atom/literal.

Preprocessing atoms (cont.)

- Sorting: $(v_1 + v_2 \leq v_3 + 1)$, $(v_2 + v_1 \leq v_3 + 1)$, $(v_1 + v_2 - 1 \leq v_3)$ $\Longrightarrow (v_1 + v_2 - v_3 \leq 1))$;
- Rewriting dual operators:
  $(v_1 < v_2)$, $(v_1 \geq v_2) \Longrightarrow (v_1 < v_2)$, $\neg(v_1 < v_2)$
- Exploiting associativity:
  $(v_1 + (v_2 + v_3) = 1)$, $((v_1 + v_2) + v_3) = 1) \Longrightarrow (v_1 + v_2 + v_3 = 1)$;
- Factoring $(v_1 + 2.0 v_2 \leq 4.0)$, $(-2.0 v_1 - 4.0 v_2 \geq -8.0)$, $\Longrightarrow$ $(0.25 v_1 + 0.5 v_2 \leq 1.0)$;
- Exploiting properties of $\mathcal{T}$:
  $(v_1 \leq 3)$, $(v_1 < 4) \Longrightarrow (v_1 \leq 3)$ if $v_1 \in \mathbb{Z}$;
- ...

Static Learning [2, 4]

- Often possible to quickly detect a priori short and "obviously inconsistent" pairs or triplets of literals occurring in φ.
  - mutual exclusion $\{x = 0, x = 1\}$,
  - congruence $\{(x_1 = y_1), (x_2 = y_2), \neg(f(x_1, x_2) = f(y_1, y_2))\}$,
  - transitivity $\{(x - y = 2), (y - z \leq 4), \neg(x - z \leq 7)\}$,
  - substitution $\{(x = y), (2x - 3z \leq 3), \neg(2y - 3z \leq 3)\}$
  - ...
- Preprocessing step: detect these literals and add blocking clauses to the input formula:
  (e.g., $\neg(x = 0) \vee \neg(x = 1)$)

$\Longrightarrow$ No assignment including one such group of literals is ever generated: as soon as all but one literals are assigned, the remaining one is immediately assigned false by unit-propagation.

Other optimization techniques

- $\mathcal{T}$-deduced-literal filtering
- Ghost-literal filtering
- $\mathcal{T}$-*solver* layering
- $\mathcal{T}$-*solver* clustering
- ...

(see [71, 10] for an overview)

Other SAT-solving techniques for SMT?

Frequently-asked question:

Are CDCL SAT solvers the only suitable Boolean Engines for SMT?

Some previous attempts:

- Ordered Binary Decision Diagrams (OBDDs) [83, 60, 1]
- Stochastic Local Search [49]

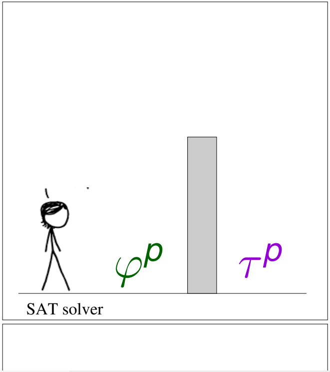
CDCL based currently much more efficient.

SMT formulas = “partially-invisible” SAT formulas

An SMT problem φ from the perspective of a SAT solver:

- a “partially-invisible” Boolean CNF formula $\varphi^p \wedge \tau^p$:
  - φ^p: the Boolean abstraction of the input formula φ
  - τ^p: (the B. abst. of) the set τ of all $\mathcal{T}$-lemmas on atoms in φ.

φ $\mathcal{T}$-satisfiable iff $\varphi^p \wedge \tau^p$ satisfiable.



Example

φ:		φ^p:	
c_1:	$\{A_1\}$	c_1:	$\{A_1\}$
c_2:	$\{\neg A_1 \vee (x - z > 4)\}$	c_2:	$\{\neg A_1 \vee B_1\}$
c_3:	$\{\neg A_3 \vee A_1 \vee (y \geq 1)\}$	c_3:	$\{\neg A_3 \vee A_1 \vee B_2\}$
c_4:	$\{\neg A_2 \vee \neg(x - z > 4) \vee \neg A_1\}$	c_4:	$\{\neg A_2 \vee \neg B_1 \vee \neg A_1\}$
c_5:	$\{(x - y \leq 3) \vee \neg A_4 \vee A_5\}$	c_5:	$\{B_3 \vee \neg A_4 \vee A_5\}$
c_6:	$\{\neg(y - z \leq 1) \vee (x + y = 1) \vee \neg A_5\}$	c_6:	$\{\neg B_4 \vee B_5 \vee \neg A_5\}$
c_7:	$\{A_3 \vee \neg(x + y = 0) \vee A_2\}$	c_7:	$\{A_3 \vee \neg B_6 \vee A_2\}$
c_8:	$\{\neg A_3 \vee (z + y = 2)\}$	c_8:	$\{\neg A_3 \vee B_7\}$
$\mathcal{T}$:	(all possible $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of φ)	τ^p:	
c_9:	$\{\neg(x + y = 0) \vee \neg(x + y = 1)\}$	c_9:	$\{\neg B_6 \vee \neg B_5\}$
c_{10}:	$\{\neg(x - z > 4) \vee \neg(x - y \leq 3) \vee \neg(y - z \leq 1)\}$	c_{10}:	$\{\neg B_1 \vee \neg B_3 \vee \neg B_4\}$
c_{11}:	$\{(x - z > 4) \vee (x - y \leq 3) \vee (y - z \leq 1)\}$	c_{11}:	$\{B_1 \vee B_3 \vee B_4\}$
c_{12}:	$\{\neg(x - z > 4) \vee \neg(x + y = 1) \vee \neg(z + y = 2)\}$	c_{12}:	$\{\neg B_1 \vee \neg B_5 \vee \neg B_7\}$
c_{13}:	$\{\neg(x - z > 4) \vee \neg(x + y = 0) \vee \neg(z + y = 2)\}$	c_{13}:	$\{\neg B_1 \vee \neg B_6 \vee \neg B_7\}$
...	...	...	...

$\mu_1^p : \{A_1, B_1, \neg A_2, A_3, \neg A_4, \neg A_5, \neg B_6, B_5, B_3, B_4, B_7, \neg B_2\}$

$\mu_1 : \{(x - z > 4), \neg(x + y = 0), (x + y = 1), (x - y \leq 3), (y - z \leq 1), (z + y = 2), \neg(y \geq 1$

satisfies φ^p, but violates both c_{10} and c_{12} in τ^p.

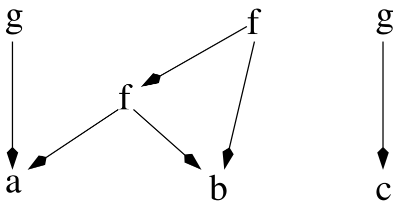
$\mathcal{T}$-solvers for Equality and Uninterpreted Functions ($\mathcal{EUF}$)

- Typically used as a "core" $\mathcal{T}$*-solver*
- $\mathcal{EUF}$ polynomial: $O(n \cdot log(n))$
- Fully incremental and backtrackable (stack-based)
- use a congruence closure data structures (E-Graphs) [40, 64, 35], based on the Union-Find data-structure for equivalence classes
- Supports efficient $\mathcal{T}$-propagation
  - Exhaustive for positive equalities
  - Incomplete for disequalities
- Supports Lazy explanations and conflict generation
  - However, minimality not guaranteed
- Supports efficient extensions
  (e.g., Integer offsets, Bit-vector slicing and concatenation)

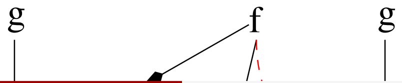
$\mathcal{T}$-solvers for $\mathcal{EUF}$: Example

Idea (sketch): given the set of terms occurring in the formula represented as nodes in a DAG (aka term bank),

- if $(t = s)$, then merge the eq. classes of t and s
- if $\forall i \in 1...k$, t_i and s_i pairwise belong to the same eq. classes, then merge the the eq. classes of $f(t_1, ..., t_k)$ and $f(s_1, ..., s_k)$
- if $(t \neq s)$ and t and s belong to the same eq. class, then conflict



$$f(a, b) = a$$
$$f(f(a, b), b) = c$$
$$g(a) \neq g(c)$$

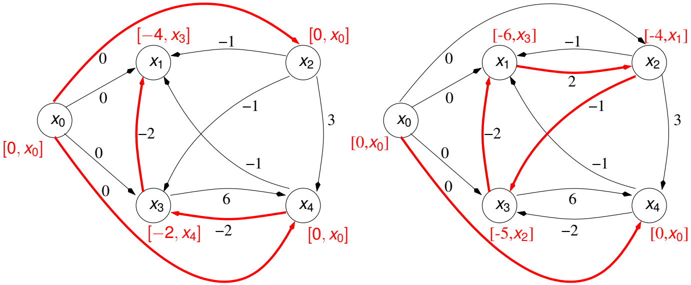


$$f(a, b) = a$$
$$f(f(a, b), b) = c$$
$$g(a) \neq g(c)$$

$\mathcal{T}$-solvers for Difference logic ($\mathcal{DL}$)

- $\mathcal{DL}$ polynomial: $O(\#vars \cdot \#constraints)$
- variants of the Bellman-Ford shortest-path algorithm: a negative cycle reveals a conflict [65, 34]
- Ex:

$$\{(x_1 - x_2 \le -1), (x_1 - x_4 \le -1), (x_1 - x_3 \le -2), (x_2 - x_1 \le 2), (x_3 - x_4 \le -2), (x_3 - x_2 \le -1), (x_4 - x_2 \le 3), (x_4 - x_3 \le 6)\}$$



$\Longrightarrow$ Sat $\qquad$ $\Longrightarrow$ Unsat

$\mathcal{T}$-solvers for Linear arithmetic over the rationals ($\mathcal{LRA}$)

- EX: $\{(s_1 - s_2 \leq 5.2), (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0), \neg(s_1 = s_0)\}$
- $\mathcal{LRA}$ polynomial
- variants of the simplex LP algorithm [41]
- [41] allows for detecting conflict sets & performing $\mathcal{T}$-propagation
- strict inequalities $t < 0$ rewritten as $t + \epsilon \leq 0$, ϵ treated symbolically

$$\overset{\mathcal{B}}{\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix}} = \begin{bmatrix} \dots A_{1j} \dots \\ \vdots \\ A_{i1} \dots A_{ij} \dots A_{iM} \\ \vdots \\ \dots A_{Nj} \dots \end{bmatrix} \overset{\mathcal{N}}{\begin{bmatrix} x_{N+1} \\ \vdots \\ x_j \\ \vdots \\ x_{N+M} \end{bmatrix}};$$

Invariant: $\beta(x_j) \in [l_j, u_j]\ \forall x_j \in \mathcal{N}$

Remark: infinite precision arithmetic

In order to avoid incorrect results due to numerical errors and to overflows, all $\mathcal{T}$*-solvers* for $\mathcal{LRA}$, $\mathcal{LIA}$ and their subtheories which are based on numerical algorithms must be implemented on top of infinite-precision-arithmetic software packages.

$\mathcal{T}$-solvers for Linear arithmetic over the integers ($\mathcal{LIA}$)

- EX: $\{(x := x_l + 2^{16}x_h), (x \geq 0), (x \leq 2^{16} - 1)\}$
- $\mathcal{LIA}$ NP-complete
- combination of many techniques: simplex, branch&bound, cutting planes, ... [41, 47]

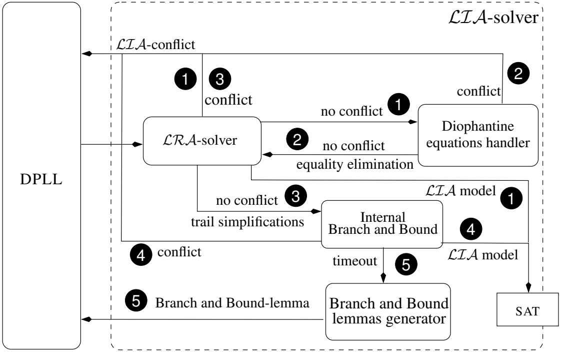


Figure courtesy of A. Griggio [47]

$\mathcal{T}$-solvers for Arrays ($\mathcal{AR}$)

- EX: $(write(A, i, v) = write(B, i, w)) \wedge \neg(v = w)$
- NP-complete
- congruence closure ($\mathcal{EUF}$) plus on-the-fly instantiation of array's axioms:

$$\forall a.\forall i.\forall e.\ (read(write(a, i, e), i) = e), \tag{1}$$

$$\forall a.\forall i.\forall j.\forall e.\ ((i \neq j) \rightarrow read(write(a, i, e), j) = read(a, j)), \tag{2}$$

$$\forall a.\forall b.\ (\forall i.(read(a, i) = read(b, i)) \rightarrow (a = b)). \tag{3}$$

- EX:

$$\begin{array}{ll} Input: & (write(A, i, v) = write(B, i, w)) \wedge \neg(v = w) \\ inst.(1): & (read(write(A, i, v), i) = v) \\ & (read(write(B, i, w), i) = w) \\ \models_{\mathcal{EUF}} & (v = w) \\ \models_{Bool} & \bot \end{array}$$

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$)

Bit vectors ($\mathcal{BV}$)

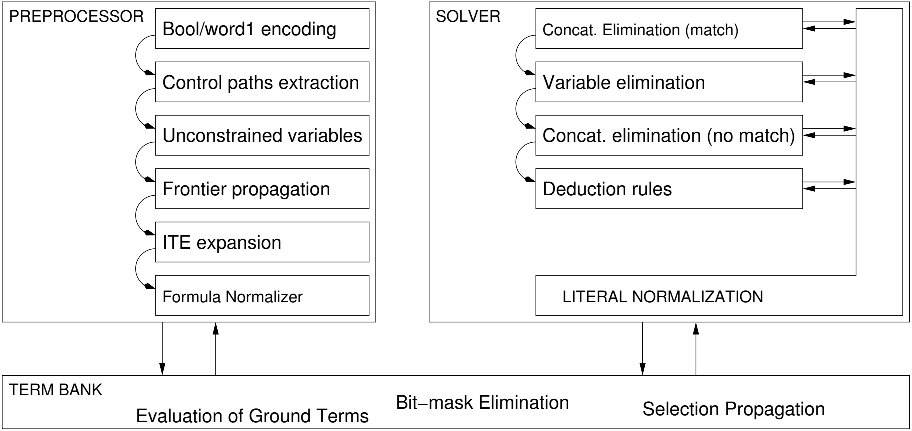
- EX: $\{(x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[16]}[3:0]), ...\}$
- NP-hard
- involve complex word-level operations: word partition/concat, modulo-2^N arithmetic, shifts, bitwise-operations, multiplexers, ...
- $\mathcal{T}$-solving: combination of rewriting & simplification techniques with either:
  - final encoding into $\mathcal{LIA}$ [19, 22]
  - final encoding into SAT (lazy bit-blasting) [25, 43, 21, 42]

Eager approach

Most solvers use an eager approach for $\mathcal{BV}$ (e.g., [21]):

- Heavy preprocessing, based on rewriting rules
- bit-blasting

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$) [cont.]



Example borrowed from [22]

$\mathcal{T}$-solvers for Bit vectors ($\mathcal{BV}$) [cont.]

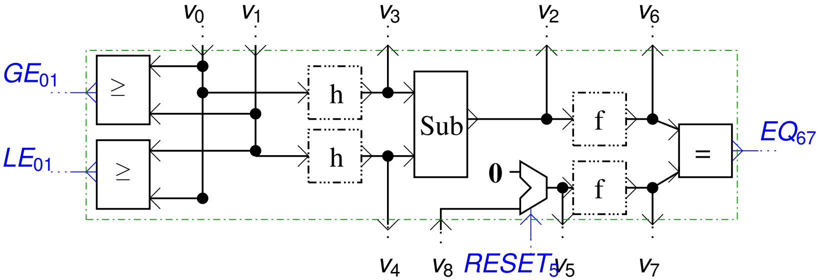
Lazy bit-blasting

- Two nested SAT solvers
- bit-blast each $\mathcal{BV}$ atom ψ_i
  $\Longrightarrow \Phi \stackrel{\text{def}}{=} \bigwedge_i (A_i \leftrightarrow BB(\psi_i))$,
  A_i fresh variables labeling $\mathcal{BV}$-atoms ψ_i in φ
  $\Longrightarrow \varphi$ $\mathcal{BV}$-satisfiable iff $\varphi^p \wedge \Phi$ satisfiable
- Exploit SAT under assumptions
  - let μ^p an assignment for φ^p, s.t. $\mu^p \stackrel{\text{def}}{=} \{[\neg]A_1, ..., [\neg]A_n\}$
  - $\mathcal{T}$-solver for $\mathcal{BV}$: $SAT_{assumption}(\Phi, \mu^p)$
  - If UNSAT, generate the unsat core $\eta^p \subseteq \mu^p$

  $\Longrightarrow$ $\neg\eta^p$ used as blocking clause

SMT for combined theories: $SMT(\bigcup_i \mathcal{T}_i)$

Problem: Many problems can be expressed as SMT problems only in combination of theories $\bigcup_i \mathcal{T}_i$ — $SMT(\bigcup_i \mathcal{T}_i)$



$$
\begin{array}{ll}
\mathcal{LIA}: & (GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge \\
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge \\
\mathcal{LIA}: & (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
\mathcal{EUF} \text{ or } \mathcal{LIA}: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \\
\mathcal{EUF}: & (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{EUF} \text{ or } \mathcal{LIA}: & (EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \ldots.
\end{array}
$$

SMT for combined theories: SMT($\mathcal{T}_1 \cup \mathcal{T}_2$)

- Standard approach for combining $\mathcal{T}_i$-*solver*'s:
  (deterministic) Nelson-Oppen/Shostak (N.O.) [61, 63, 77]
  - based on deduction and exchange of equalities on shared variables
  - combined $\mathcal{T}_i$-*solver*'s integrated with a SAT tool
- More-recent alternative approaches: Delayed Theory Combination [15, 14] and Model-Based Theory Combination [37]
  - based on Boolean search on equalities on shared variables
  - $\mathcal{T}_i$-*solver*'s integrated directly with a SAT tool

Problem:

N.O. approaches have some drawbacks and limitations when used within a SMT framework

Background: Pure Formulas

Consider two theories T_1, T_2 with equality and disjoint signatures Σ_1, Σ_2

- W.l.o.g. we assume all input formulas $\phi \in T_1 \cup T_2$ are pure.
  - A formula ϕ is pure iff every atom in ϕ is i-pure for some $i \in \{1, 2\}$.
  - An atom/literal in ϕ is *i*-pure if only $=$, variables and symbols from Σ_i can occur in ϕ

Purification:

maps a formula into an equisatisfiable pure formula by labeling terms with fresh variables

$$(f(\underbrace{x + 3y}_{w}) = g(\underbrace{2x - y}_{t})) \qquad [\textit{not pure}]$$

$$\Downarrow$$

$$(w = x + 3y) \wedge (t = 2x - y) \wedge (f(w) = g(t)) \quad [\textit{pure}]$$

Background: Interface equalities

Interface variables & equalities

- A variable v occurring in a pure formula ϕ is an interface variable iff it occurs in both 1-pure and 2-pure atoms of ϕ.
- An equality $(v_i = v_j)$ is an interface equality for ϕ iff v_i, v_j are interface variables for ϕ.
- We denote the interface equality $v_i = v_j$ by "e_{ij}"

Example:

$$
\begin{array}{ll}
\mathcal{LIA}: & (GE_{01} \leftrightarrow (v_0 \geq v_1)) \wedge (LE_{01} \leftrightarrow (v_0 \leq v_1)) \wedge \\
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge \\
\mathcal{LIA}: & (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
\mathcal{EUF} \text{ or } \mathcal{LIA}: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \\
\mathcal{EUF}: & (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{EUF} \text{ or } \mathcal{LIA}: & (EQ_{67} \leftrightarrow (v_6 = v_7)) \wedge \ldots.
\end{array}
$$

v_0, v_1, v_2, v_3, v_4, v_5 are interface variables, v_6, v_7, v_8 are not
$\Longrightarrow (v_0 = v_1)$ is an interface equality, $(v_0 = v_6)$ is not.

Background: Stably-infinite & Convex Theories

Stably-infinite Theories

A theory T is stably-infinite iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T.

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$, $\mathcal{LIA}$ are stably-infinite
- bit-vector theories typically are not stably-infinite

Convex Theories

A theory T is convex iff, for every collection $l_1, ..., l_k, l', l''$ of literals in T s.t. l', l'' are in the form $(x = y)$, x, y being variables, we have that:
$\{l_1, ..., l_k\} \models_T (l' \vee l'') \iff \{l_1, ..., l_k\} \models_T l'$ *or* $\{l_1, ..., l_k\} \models_T l''$

- $\mathcal{EUF}$, $\mathcal{DL}$, $\mathcal{LRA}$ are convex
- $\mathcal{LIA}$ is not convex:
  $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \models ((v = v_0) \vee (v = v_1)),$
  $\{(v_0 = 0), (v_1 = 1), (v \geq v_0), (v \leq v_1)\} \not\models (v = v_0)$
  $\{(v_0 = 0), (v_1 = 1), (v \geq 0), (v \leq v_1)\} \not\models (v = v_1)$

$SMT(\bigcup_i \mathcal{T}_i)$ via "classic" Nelson-Oppen

Main idea

Combine two or more $\mathcal{T}_i$-*solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 78]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
- important improvements and evolutions [69, 7, 40]

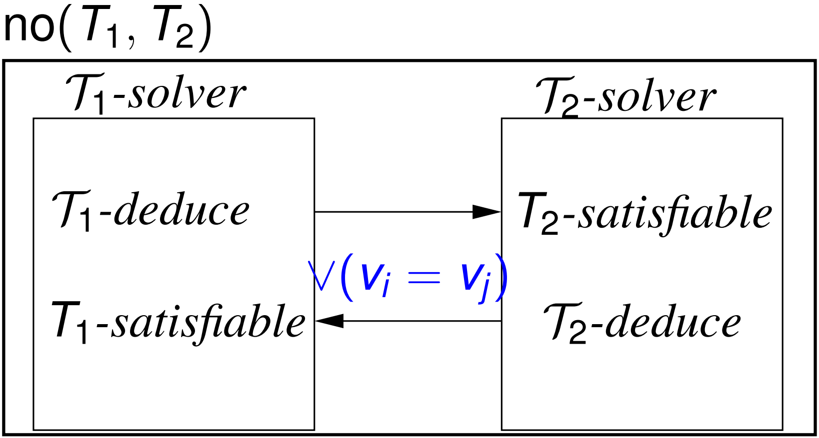
Schema of N.O. combination of T-solvers: no(T_1, T_2)

For $i \in \{1, 2\}$, let T_i be a stably infinite theory admitting a satisfiability T_i-solver, and μ_i a set of i-pure literals.
We want to to decide the $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiability of $\mu_1 \cup \mu_2$

- each T_i-solver, in turn
  - checks the T_i-satisfiability of μ_i,
  - deduces all the (disjunctions of) interface equalities which derive from μ_i
  - passes them to T_j-solve, $j \neq i$, which adds them to μ_j

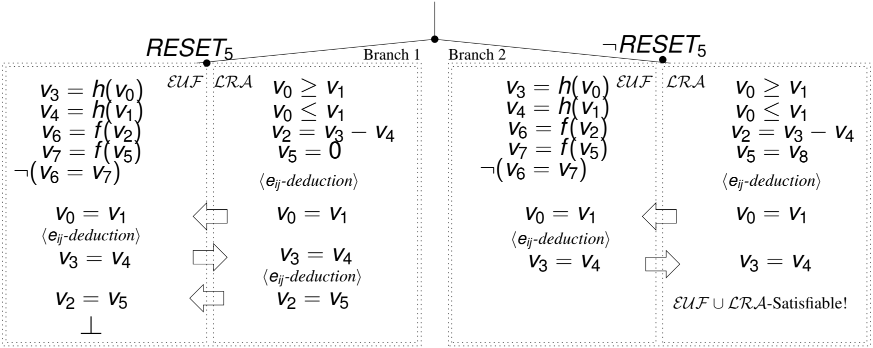
  until either:
  - one T_i-solver detects inconsistency ($\mu_1 \cup \mu_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$-unsat)
  - no more deductions are possible ($\mu_1 \cup \mu_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$-sat)
- disjunctions of literals (due to non-convexity) force case-splitting

Schema of N.O. combination of T-solvers: no(T_1, T_2)

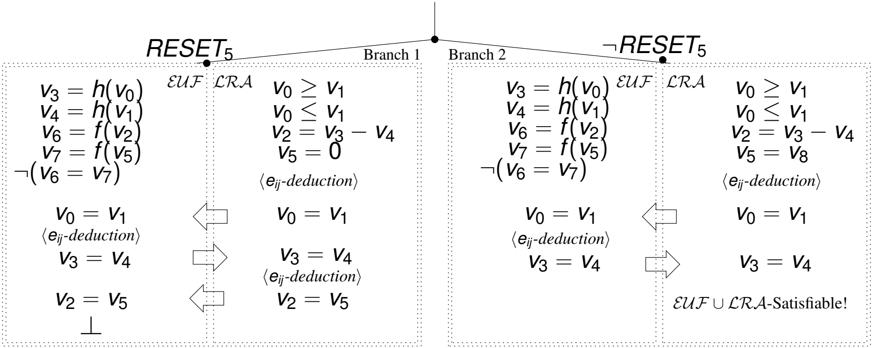


N.O.: example (convex theory)

$$\begin{array}{ll} \mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\ \mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\ Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7). \end{array}$$



N.O.: example (convex theory) [cont.]



$\mathcal{EUF}$-conflict : $$((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_5)) \rightarrow \bot$$

$\mathcal{LRA}$-deduction : $$((v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$$

$\mathcal{EUF}$-deduction : $$((v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$$

$\mathcal{LRA}$-deduction : $$((v_0 \geq v_1) \wedge (v_0 \leq v_1)) \rightarrow (v_0 = v_1)$$

$\Longrightarrow$

$\mathcal{EUF} \cup \mathcal{LRA}$-conflict : $$((v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \neg(v_6 = v_7) \wedge (v_2 = v_3 - v_4) \wedge (v_5 = 0) \wedge (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_0 \geq v_1)) \rightarrow \bot.$$

N.O.: example (non-convex theory)

$\mu_{\mathcal{LIA}}$

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$

$v_1 \leq 1$ $\quad v_3 = 0$

$v_2 \geq v_6$ $\quad v_4 = 1$

$v_2 \leq v_6 + 1$

$\mu_{\mathcal{EUF}}$

$\neg(f(v_1) = f(v_2))$

$\neg(f(v_2) = f(v_4))$

$f(v_3) = v_5$

$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$

$v_1 \geq 0$

$v_1 \leq 1$

$v_2 \geq v_6$

$v_2 \leq v_6$

v_1

$SMT(\bigcup_i \mathcal{T}_i)$ via "classic" Nelson-Oppen

Main idea

Combine two or more $\mathcal{T}_i$-*solvers* into one $(\bigcup_i \mathcal{T}_i)$-solver via Nelson-Oppen/Shostak (N.O.) combination procedure [62, 78]

- based on the deduction and exchange of equalities between shared variables/terms (interface equalities, e_{ij}s)
- important improvements and evolutions [69, 7, 40]
- drawbacks [23, 24]:
  - require (possibly expensive) deduction capabilities from $\mathcal{T}_i$-*solvers*
  - [with non-convex theories] case-splits forced by the deduction of disjunctions of e_{ij}'s
  - generate (typically long) $(\bigcup_i \mathcal{T}_i)$-lemmas, without interface equalities $\Longrightarrow$ no backjumping & learning from e_{ij}-reasoning

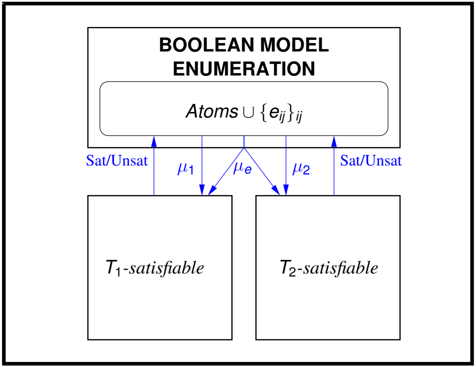
$SMT(\bigcup_i \mathcal{T}_i)$ via Delayed Theory Combination (DTC)

Main idea

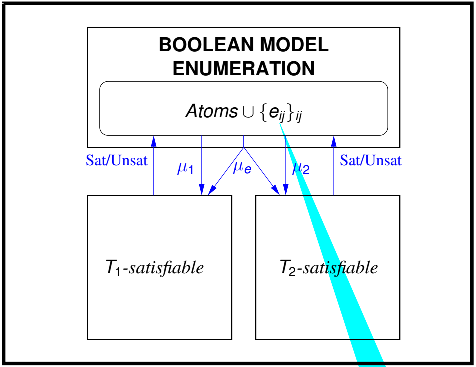
Delegate to the CDCL SAT solver part/most of the (possibly very expensive) reasoning effort on interface equalities previously due to the $\mathcal{T}_i$*-solvers* (e_{ij}-deduction, case-split). [15, 16, 24]

- based on Boolean reasoning on interface equalities via CDCL (plus $\mathcal{T}$-propagation)
- important improvements and evolutions [37, 9]
- feature wrt N.O. [23, 24]
  - do not require (possibly expensive) deduction capabilities from $\mathcal{T}_i$*-solvers*
  - with non-convex theories, case-splits on e_{ij}'s handled by SAT
  - generate $\mathcal{T}_i$-lemmas with interface equalities $\Longrightarrow$ backjumping & learning from e_{ij}-reasoning

DTC: Basic schema

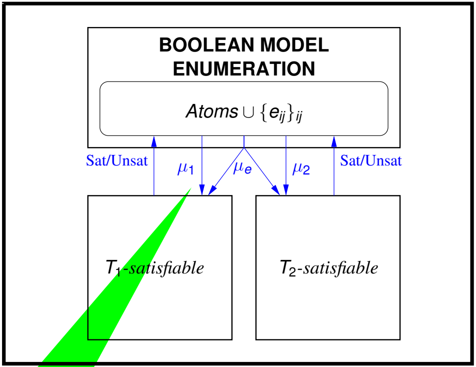


DTC: Basic schema



The boolean solver assigns values not only to atoms in $Atoms(\phi)$, but also to interface equalities $\{(v_i = v_j)\}_{ij}$:
$\mu = \mu_1 \cup \mu_2 \cup \mu_e, \quad \mu_e := \{[\neg](v_i = v_j) | v_i, v_j \in \mu_1 \cup \mu_2\}$

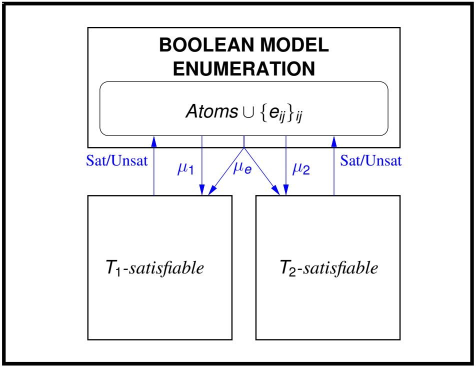
DTC: Basic schema



Each $\mathcal{T}_i$-*solver* interacts only with the boolean solver

- receives $\mu_i' := \mu_i \cup \mu_e$ from Bool
- checks the T_i-satisfiability of μ_i'

DTC: Basic schema



...until either:

- some μ propositionally satisfies ϕ and both $\mu_i' := \mu_i \cup \mu_e$ are T_i-consistent
  $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-sat)
- no more assignment μ are available
  $\Longrightarrow$ (ϕ is $\mathcal{T}_1 \cup \mathcal{T}_2$-unsat)

DTC: enhanced schema

- DPLL-based assignment enumeration on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$,
  $\Longrightarrow$ benefits of state-of-the-art SAT techniques
- Early pruning: invoke the $\mathcal{T}_i$-*solver*'s before every Boolean decision
  $\Longrightarrow$ total assignments generated only when strictly necessary
- Branching: branching on e_{ij}'s postponed
  $\Longrightarrow$ Boolean search on e_{ij}'s performed only when strictly necessary
- Theory-Backjumping & Learning: e_{ij}'s are involved in conflicts
  $\Longrightarrow$ e_{ij}'s can be assigned by unit propagation
- [Theory-deduction & learning: $\mathcal{T}_i$-*solver* deduces unassigned literals l on $Atoms(\phi) \cup \{e_{ij}\}_{ij}$
  - l is passed back to the Boolean solver, which unit-propagates it
  - the deduction $\mu' \models l$ is learned as a clause $\mu' \rightarrow l$ (deduction clause)]
- ...

DTC: example w.out $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:	$\mu_{\mathcal{LIA}}$:	
$\neg(f(v_1) = f(v_2))$	$v_1 \geq 0$	$v_5 = v_4 - 1$
$\neg(f(v_2) = f(v_4))$	$v_1 \leq 1$	$v_3 = 0$
$f(v_3) = v_5$	$v_2 \geq v_6$	$v_4 = 1$
$f(v_1) = v_6$	$v_2 \leq v_6 + 1$	

DTC: example with $\mathcal{T}$-prop. (non-convex theory)

$\mu_{\mathcal{EUF}}$:

$v_1) = f(v_2))$
$v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

$v_1 \geq 0$ $\quad v_5 = v_4 - 1$
$v_1 \leq 1$ $\quad v_3 = 0$
$v_2 \geq v_6$ $\quad v_4 = 1$
$v_2 \leq v_6 + 1$

$\mu_{\mathcal{EUF}}$:

$\neg(f(v_1) = f(v_2))$
$\neg(f(v_2) = f(v_4))$
$f(v_3) = v_5$
$f(v_1) = v_6$

$\mu_{\mathcal{LIA}}$:

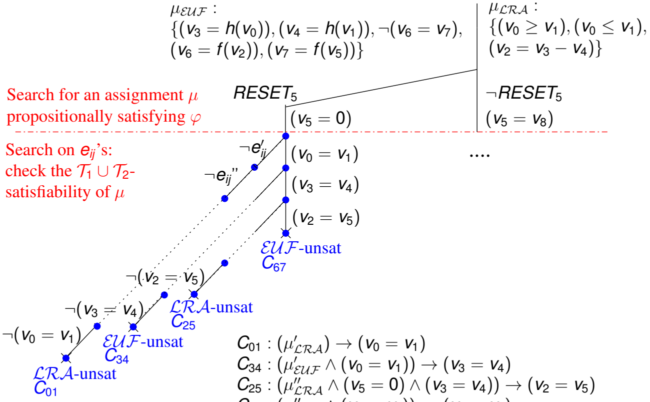
$v_1 \geq 0$ $\quad v$
$v_1 \leq 1$ $\quad v$
$v_2 \geq v_6$ $\quad v$
$v_2 \leq v_6 + 1$

$\mathcal{LIA}$-deduce $(v$

$C_{13} : (\mu'_{\mathcal{LIA}}) \rightarrow ((v_1 = v$

DTC: example without $\mathcal{T}$-propagation (convex theory)

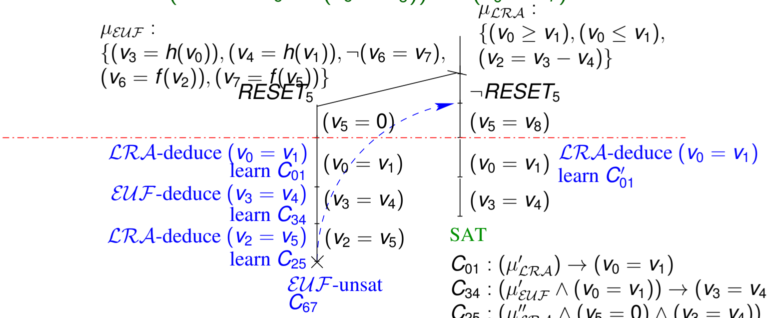
$$\begin{array}{ll}
\mathcal{EUF}: & (v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge \\
\mathcal{LRA}: & (v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge \\
Both: & (\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7).
\end{array}$$



$C_{01}: (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1)$
$C_{34}: (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$
$C_{25}: (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$
$C_{67}: (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)$

DTC: example with $\mathcal{T}$-propagation (convex theory)

$\mathcal{EUF}$: $(v_3 = h(v_0)) \wedge (v_4 = h(v_1)) \wedge (v_6 = f(v_2)) \wedge (v_7 = f(v_5)) \wedge$
$\mathcal{LRA}$: $(v_0 \geq v_1) \wedge (v_0 \leq v_1) \wedge (v_2 = v_3 - v_4) \wedge (RESET_5 \rightarrow (v_5 = 0)) \wedge$
Both : $(\neg RESET_5 \rightarrow (v_5 = v_8)) \wedge \neg(v_6 = v_7)$.



$C_{01} : (\mu'_{\mathcal{LRA}}) \rightarrow (v_0 = v_1)$
$C_{34} : (\mu'_{\mathcal{EUF}} \wedge (v_0 = v_1)) \rightarrow (v_3 = v_4)$
$C_{25} : (\mu''_{\mathcal{LRA}} \wedge (v_5 = 0) \wedge (v_3 = v_4)) \rightarrow (v_2 = v_5)$
$C_{67} : (\mu''_{\mathcal{EUF}} \wedge (v_2 = v_5)) \rightarrow (v_6 = v_7)$

DTC + Model-based heuristic (aka Model-Based Theory Combination) [37]

- Initially, no interface equalities generated
- When a model is found, check against all the possible interface equalities
  - If $\mathcal{T}_1$ and $\mathcal{T}_2$ agree on the implied equalities, then return SAT
  - Otherwise, branch on equalities implied by $\mathcal{T}_1$-model but not by $\mathcal{T}_2$-model
- "Optimistic" approach, similar to axiom instantiation

Beyond Solving: advanced SAT & SMT functionalities

Advanced SMT functionalities (very important in FV):

- Building proofs of $\mathcal{T}$-unsatisfiability
- Extracting $\mathcal{T}$-unsatisfiable Cores
- Computing Craig interpolants
- Performing All-SMT and Predicate Abstraction
- Deciding/optimizing SMT problems with costs

Building (Resolution) Proofs of $\mathcal{T}$-Unsatisfiability

Resolution proof of $\mathcal{T}$-unsatisfiability

Very similar to building proofs with plain SAT:

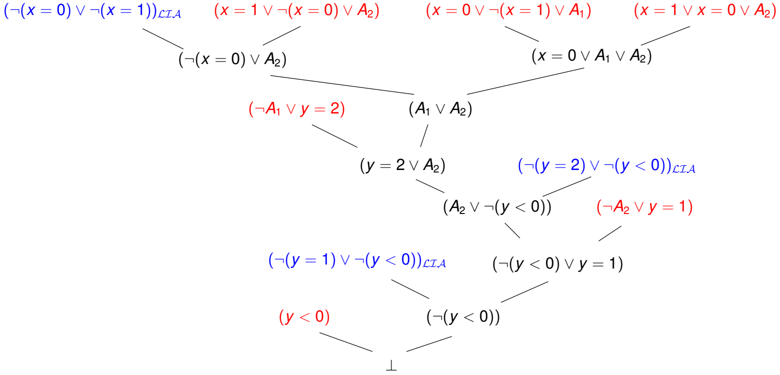
- resolution proofs whose leaves are original clauses and $\mathcal{T}$-lemmas returned by the $\mathcal{T}$-*solver* (i.e., $\mathcal{T}$-conflict and $\mathcal{T}$-deduction clauses)
- built by backward traversal of implication graphs, as in CDCL SAT
- Sub-proofs of $\mathcal{T}$-lemmas can be built in some $\mathcal{T}$-specific deduction framework if requested

Important for:

- certifying $\mathcal{T}$-unsatisfiability results
- computing unsatisfiable cores
- computing interpolants

Building Proofs of $\mathcal{T}$-Unsatisfiability: example

$$(x = 0 \lor \neg(x = 1) \lor A_1) \land (x = 0 \lor x = 1 \lor A_2) \land (\neg(x = 0) \lor x = 1 \lor A_2) \land$$
$$(\neg A_2 \lor y = 1) \land (\neg A_1 \lor x + y > 3) \land (y < 0) \land (A_2 \lor x - y = 4) \land (y = 2 \lor \neg A_1) \land (x \geq 0),$$



relevant original clauses, irrelevant original clauses, $\mathcal{T}$-lemmas

Example: proof on non-strict $\mathcal{LRA}$ inequalities

- A proof of unsatisfiability for a set of non-strict $\mathcal{LRA}$ inequalities can be obtained by building a linear combination of such inequalities, each time eliminating one or more variables, until you get a contradictory inequality on constant values.
- Example:

$$\varphi \stackrel{\text{def}}{=} (0 \le x_1 - 3x_2 + 1), (0 \le x_1 + x_2), (0 \le x_3 - 2x_1 - 3), (0 \le 1 - 2x_3).$$

A proof of unsatisfiability P for φ is the following:

$$\dfrac{\dfrac{(0 \le x_1 - 3x_2 + 1) \quad (0 \le x_1 + x_2)}{\text{COMB } (0 \le 4x_1 + 1) \textit{ with coeffs 1 and 3}} \qquad \dfrac{(0 \le x_3 - 2x_1 - 3) \quad (0 \le 1 - 2x_3)}{\text{COMB } (0 \le -4x_1 - 5) \textit{ with coeffs 2 and 1}}}{\text{COMB } (0 \le -4) \textit{ with coeffs 1 and 1}}$$

- It is possible to produce such proof from an inconsistent tableau in Simplex procedure for $\mathcal{LRA}$ [30, 32]
- It is straightforward to produce such proof from a negative cycle in the graph-based procedure for $\mathcal{DL}$ [30, 32]

Extraction of $\mathcal{T}$-unsatisfiable cores

The problem

Given a $\mathcal{T}$-unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) $\mathcal{T}$-unsatisfiable subset ($\mathcal{T}$-unsatisfiable core)

- wide literature in SAT
- Some implementations, very few literature for SMT [29, 56]
- We recognize three approaches:
  - Proof-based approach (CVClite, MathSAT): byproduct of finding a resolution proof
  - Assumption-based approach (Yices): use extra variables labeling clauses, as in the plain Boolean case
  - Lemma-Lifting approach [29] : use an external (possibly-optimized) Boolean unsat-core extractor

The proof-based approach to $\mathcal{T}$-unsat cores

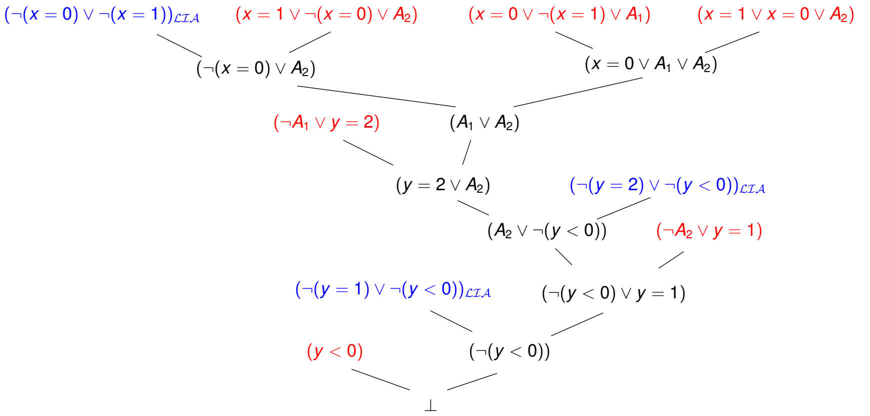
Idea (adapted from [84])

Unsatisfiable core of φ:

- in SAT: the set of leaf clauses of a resolution proof of unsatisfiability of φ
- in SMT($\mathcal{T}$): the set of leaf clauses of a resolution proof of $\mathcal{T}$-unsatisfiability of φ, minus the $\mathcal{T}$-lemmas

The proof-based approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \lor \neg(x = 1) \lor A_1) \land (x = 0 \lor x = 1 \lor A_2) \land (\neg(x = 0) \lor x = 1 \lor A_2)\land$$
$$(\neg A_2 \lor y = 1)\land(\neg A_1 \lor x+y > 3)\land(y < 0)\land(A_2 \lor x-y = 4)\land(y = 2 \lor \neg A_1)\land(x \geq 0),$$



The assumption-based approach to $\mathcal{T}$-unsat cores

Let φ be $\bigwedge_{i=1}^{n} C_i$ s.t. φ inconsistent.

Idea (adapted from [57])

1. each clause C_i in φ is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh "selector" variable
2. the resulting formula is checked for satisfiability under the assumption of all S_i's
3. final conflict clause at dec. level 0: $\bigvee_j \neg S_j$ $\Longrightarrow \{C_j\}_j$ is the unsat core

- extends straightforwardly to SMT($\mathcal{T}$).

The assumption-based approach to $\mathcal{T}$-unsat cores: Example

$$(S_1 \rightarrow (x = 0 \vee \neg(x = 1) \vee A_1)) \wedge (S_2 \rightarrow (x = 0 \vee x = 1 \vee A_2)) \wedge$$
$$(S_3 \rightarrow (\neg(x = 0) \vee x = 1 \vee A_2)) \wedge (S_4 \rightarrow (\neg A_2 \vee y = 1)) \wedge$$
$$(S_5 \rightarrow (\neg A_1 \vee x + y > 3)) \wedge (S_6 \rightarrow y < 0) \wedge$$
$$(S_7 \rightarrow (A_2 \vee x - y = 4)) \wedge (S_8 \rightarrow (y = 2 \vee \neg A_1)) \wedge (S_9 \rightarrow x \geq 0)$$

Conflict analysis (Yices 1.0.6) returns:

$$\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_6 \vee \neg S_7 \vee \neg S_8,$$

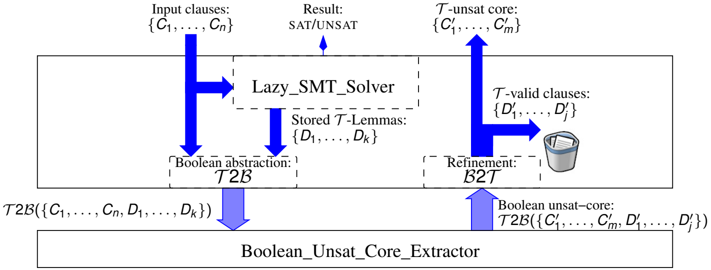
corresponding to the unsat core in red.

The lemma-lifting approach to $\mathcal{T}$-unsat cores

Idea [29, 33]

(i) The $\mathcal{T}$-lemmas D_i are valid in $\mathcal{T}$

(ii) The conjunction of φ with all the $\mathcal{T}$-lemmas $D_1, \ldots, D_k$ is propositionally unsatisfiable: $\mathcal{T}2\mathcal{B}(\varphi \wedge \bigwedge_{i=1}^{n} D_i) \models \bot$.



- interfaces with an external Boolean Unsat-core Extractor

$\Longrightarrow$ benefits for free of all state-of-the-art size-reduction techniques

The lemma-lifting approach to $\mathcal{T}$-unsat cores: example

$$(x = 0 \vee \neg(x = 1) \vee A_1) \wedge (x = 0 \vee x = 1 \vee A_2) \wedge (\neg(x = 0) \vee x = 1 \vee A_2) \wedge$$
$$(\neg A_2 \vee y = 1) \wedge (\neg A_1 \vee x + y > 3) \wedge (y < 0) \wedge (A_2 \vee x - y = 4) \wedge (y = 2 \vee \neg A_1) \wedge (x \geq 0),$$

1. The SMT solver generates the following set of $\mathcal{LIA}$-lemmas:

$$\{(\neg(x = 1) \vee \neg(x = 0)),\ (\neg(y = 2) \vee \neg(y < 0)),\ (\neg(y = 1) \vee \neg(y < 0))\}.$$

2. The following formula is passed to the external Boolean core extractor

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (B_6 \vee \neg A_1) \wedge B_7 \wedge$$
$$(\neg B_1 \vee \neg B_0) \wedge (\neg B_6 \vee \neg B_4) \wedge (\neg B_2 \vee \neg B_4)$$

which returns the unsat core in red.

3. The unsat-core is mapped back, the three $\mathcal{T}$-lemmas are removed $\Longrightarrow$ the final $\mathcal{T}$-unsat core (in red above).

Computing (Craig) Interpolants in SMT

Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models_{\mathcal{T}} \perp$, a *Craig interpolant* is a formula I s.t.:

a) $A \models_{\mathcal{T}} I$,

b) $I \wedge B \models_{\mathcal{T}} \perp$,

c) $I \preceq A$ and $I \preceq B$.

"$I \preceq A$" meaning that all uninterpreted (in $\mathcal{T}$) symbols in I occur in A.

- Very important in many FV applications
- A few works presented for various theories:
  - $\mathcal{EUF}$ [59, 70], $\mathcal{DL}$ [30, 32], $\mathcal{UTVPI}$ [31, 32], $\mathcal{LRA}$ [59, 70, 30, 32], $\mathcal{LIA}$ [51, 18, 48], $\mathcal{BV}$ [52], ...

A General Algorithm

Algorithm: Interpolant generation for SMT($\mathcal{T}$) [68, 59]

(i) Generate a resolution proof of $\mathcal{T}$-unsatisfiability $\mathcal{P}$ for $A \wedge B$.

(ii) ...

(iii) Foreach $\mathcal{T}$-lemma $\neg\eta$ in $\mathcal{P}$, generate an interpolant I_η for $(\eta \setminus B, \eta \downarrow B)$.

(iv) For every original leaf clause C in $\mathcal{P}$, set $I_C \stackrel{\text{def}}{=} C \downarrow B$ if $C \in A$, and $I_C \stackrel{\text{def}}{=} \top$ if $C \in B$.

(v) For every inner node C of $\mathcal{P}$ obtained by resolution from $C_1 \stackrel{\text{def}}{=} p \vee \phi_1$ and $C_2 \stackrel{\text{def}}{=} \neg p \vee \phi_2$, set $I_C \stackrel{\text{def}}{=} I_{C_1} \vee I_{C_2}$ if p does not occur in B, and $I_C \stackrel{\text{def}}{=} I_{C_1} \wedge I_{C_2}$ otherwise.

(vi) Output $I_\perp$ as an interpolant for (A, B).

"$\eta \setminus B$" [resp. "$\eta \downarrow B$"] is the set of literals in η whose atoms do not [resp. do] occur in B.

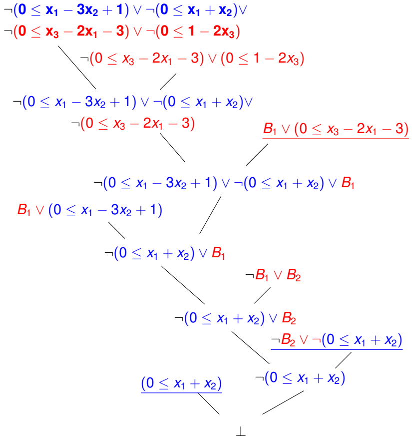
- row 2. only takes place where $\mathcal{T}$ comes in to play

$\Longrightarrow$ Reduced to the problem of finding an interpolant for two sets of $\mathcal{T}$-literals (Boolean and $\mathcal{T}$-specific component decoupled)

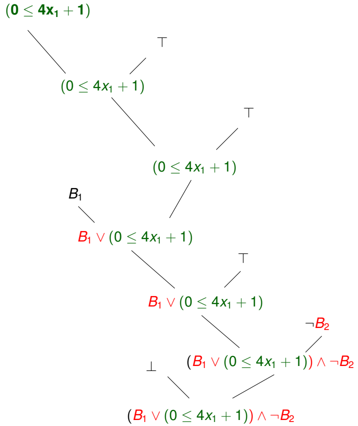
Computing Craig Interpolants in SMT: example

$$A \stackrel{\text{def}}{=} (B_1 \vee (0 \le x_1 - 3x_2 + 1)) \wedge (0 \le x_1 + x_2) \wedge (\neg B_2 \vee \neg(0 \le x_1 + x_2))$$

$$B \stackrel{\text{def}}{=} (\neg(0 \le x_3 - 2x_1 - 3) \vee (0 \le 1 - 2x_3)) \wedge (\neg B_1 \vee B_2) \wedge (B_1 \vee (0 \le x_3 - 2x_1 - 3))$$



original proof



interpolant proof

McMillan's algorithm for non-strict $\mathcal{LRA}$ inequalities

$$A \stackrel{\text{def}}{=} \{(0 \leq x_1 - 3x_2 + 1), (0 \leq x_1 + x_2\}$$
$$B \stackrel{\text{def}}{=} \{(0 \leq x_3 - 2x_1 - 3), (0 \leq 1 - 2x_3)\}.$$

A proof of unsatisfiability P for $A \wedge B$ is the following:

$$\dfrac{\dfrac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\textsc{Comb}\ (0 \leq 4x_1 + 1)\ \textit{with coeffs 1 and 3}} \qquad \dfrac{(0 \leq x_3 - 2x_1 - 3) \quad (0 \leq 1 - 2x_3)}{\textsc{Comb}\ (0 \leq -4x_1 - 5)\ \textit{with coeffs 2 and 1}}}{\textsc{Comb}\ (0 \leq -4)\ \textit{with coeffs 1 and 1}}$$

By replacing inequalities in B with $(0 \leq 0)$, we obtain the proof P':

$$\dfrac{\dfrac{(0 \leq x_1 - 3x_2 + 1) \quad (0 \leq x_1 + x_2)}{\textsc{Comb}\ (0 \leq 4x_1 + 1)} \qquad \dfrac{(0 \leq 0) \quad (0 \leq 0)}{\textsc{Comb}\ (0 \leq 0)}}{\textsc{Comb}\ (0 \leq 4x_1 + 1)}$$

Thus, the interpolant obtained is $(0 \leq 4x_1 + 1)$.

Example: interpolation algorithms for difference logic

- An inference-based algorithm [59]

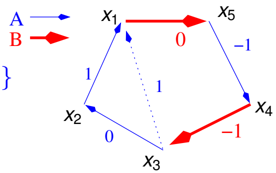
$$
\dfrac{\dfrac{\dfrac{\dfrac{(0 \le x_1 - x_2 + 1) \quad (0 \le x_2 - x_3)}{\text{COMB} \quad (0 \le x_1 - x_3 + 1)} \quad (0 \le x_4 - x_5 - 1)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)} \quad (0 \le 0)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)} \quad (0 \le 0)}{\text{COMB} \quad (0 \le x_1 - x_3 + x_4 - x_5)}
$$

$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + x_4 - x_5)$ (not in $\mathcal{DL}$, and weaker).

- A graph-based algorithm [30, 32]

$$
A \stackrel{\text{def}}{=} \{\overbrace{(0 \le x_1 - x_2 + 1), (0 \le x_2 - x_3)}^{\textit{Chord}:\ (0 \le x_1 - x_3 + 1)}, (0 \le x_4 - x_5 - 1)\}
$$

$$
B \stackrel{\text{def}}{=} \{(0 \le x_5 - x_1), (0 \le x_3 - x_4 - 1)\}.
$$



$\Longrightarrow$ Interpolant: $(0 \le x_1 - x_3 + 1) \land (0 \le x_4 - x_5 - 1)$ (still in $\mathcal{DL}$)

All-SAT/All-SMT

- All-SAT: enumerate all truth assignments satisfying φ
- All-SMT: enumerate all $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ
- All-SMT over an "important" subset of atoms $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$: enumerate all assignments over **P** which can be extended to $\mathcal{T}$-satisfiable truth assignments propositionally satisfying φ $\Longrightarrow$can compute predicate abstraction
- Algorithms:
  - BCLT [53]
    each time a $\mathcal{T}$-satisfiable assignment $\{l_1, ..., l_n\}$ is found, perform conflict-driven backjumping as if the restricted clause $(\bigvee_i \neg l_i) \downarrow \mathbf{P}$ belonged to the clause set
  - MathSAT/NuSMV [26]
    As above, plus the Boolean search of the SMT solver is driven by an OBDD.

Predicate Abstraction

Predicate abstraction

if $\varphi(\mathbf{v})$ is a SMT formula over the domain variables $\mathbf{v} \stackrel{\text{def}}{=} \{v_j\}_j$, $\{\gamma_i\}_i$ is a set of "relevant" predicates over $\mathbf{v}$, and $\mathbf{P} \stackrel{\text{def}}{=} \{P_i\}_i$ a set of Boolean labels, then:

$$
\begin{aligned}
& \mathit{PredAbs}_{\mathbf{P}}(\varphi) \\
\stackrel{\text{def}}{=} \; & \exists \mathbf{v}.(\; \varphi(\mathbf{v}) \wedge \bigwedge_i P_i \leftrightarrow \gamma_i(\mathbf{v}) \;) \\
= \; & \bigvee \left\{ \mu \;\middle|\; \begin{array}{c} \mu \text{ truth assignment on } \mathbf{P} \\ \text{s.t. } \mu \wedge \varphi \wedge \bigwedge_i (P_i \leftrightarrow \gamma_i) \text{ is } \mathcal{T}\text{-satisfiable} \end{array} \right\}
\end{aligned}
$$

- projection of φ over (the Boolean abstraction of) the set $\{\gamma_i\}_i$.
- essential step in FV: extracts finite-state abstractions from a infinite state space

Predicate Abstraction: example

$$\begin{array}{rcl} \varphi & \stackrel{\text{def}}{=} & (v_1 + v_2 > 12) \\ \gamma_1 & \stackrel{\text{def}}{=} & (v_1 + v_2 = 2) \\ \gamma_2 & \stackrel{\text{def}}{=} & (v_1 - v_2 < 10) \end{array}$$

$$\Downarrow$$

$$\begin{array}{rcl} PreAbs(\varphi)_{\{P_1, P_2\}} & \stackrel{\text{def}}{=} & \exists\, v_1\ v_2\,.\left(\begin{array}{ll} (v_1 + v_2 > 12) & \wedge \\ (P_1 \leftrightarrow (v_1 + v_2 = 2)) & \wedge \\ (P_2 \leftrightarrow (v_1 - v_2 < 10)) & \end{array}\right) \\ & = & (\neg P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \\ & = & \neg P_1. \end{array}$$

SMT with Pseudo-Boolean (PB) cost-minimization

The problem

SMT($\mathcal{T}$) problem φ for some $\mathcal{T}$, augmented with cost functions:
$cost^i = \sum_{j=1}^{N^i} ite(P^{ij}, c_1^{ij}, c_2^{ij})$, s.t. $cost^i \in (l^i, u^i]$, $c_{\{1,2\}}^{ij} > 0$

- Decision problem: is there a model complying with cost ranges?
- Optimization problem: find model minimizing some $cost^i$.

- allows for encoding MaxSAT/MaxSMT and PseudoBoolean

Proposed solution: [66, 27]

- SMT($\mathcal{T} \cup \mathcal{C}$), $\mathcal{C}$ is an ad-hoc “theory of costs”
- a specialized very-fast theory-solver for $\mathcal{C}$ added to MathSAT
  - very fast & aggressive search pruning and theory-propagation
- cost minimization handled by linear or binary search

SMT($\mathcal{T} \cup \mathcal{C}$): main ideas

- A "theory of costs" $\mathcal{C}$:
  - Cost variables $cost^i$
  - "bound cost" $BC(cost^i, k)$: "$cost^i \le k$"
  - "incur cost" $IC(cost^i, j, k^i_j)$: "the jth addend of $cost^i := k^i_j$
  - "$cost^i = \sum_{j=1}^{N^i} ite(P^i_j, k^i_j, 0), \; s.t. \; cost^i \in (l^i, u^i]$" encoded as
    $\neg BC(cost^i, l^i) \wedge BC(cost^i, u^i) \wedge \bigwedge_{j=1}^{N^i}(P^i_j \leftrightarrow IC(cost^i, j, k^i_j))$
- very-fast theory solver: $\mathcal{C}$*-solver*
  1. $IC(cost^i, j, k^i_j) = \top \Longrightarrow cost^i = cost^i + k^i_j$
  2. $cost^i > ub^i \Longrightarrow$ conflict
  3. $cost^i + \{total\ cost\ of\ all\ unassigned\ IC's\} \le lb^i \Longrightarrow$ conflict
  4. $IC(cost^i, j, k^i_j) = \top$ causes 2. $\Longrightarrow \mathcal{C}$-propagate $\neg IC(cost^i, j, k^i_j)$
  5. $IC(cost^i, j, k^i_j) = \bot$ causes 3. $\Longrightarrow \mathcal{C}$-propagate $IC(cost^i, j, k^i_j)$
- no symbol shared with $\mathcal{T}$
  $\Longrightarrow$ independent theory solvers for $\mathcal{T}$ and $\mathcal{C}$

Optimization Modulo Theories with $\mathcal{LA}$ $\mathcal{LRA}$ costs I

Ingredients

- an SMT formula φ on $\mathcal{LA} \cup \mathcal{T}$ $\mathcal{LRA} \cup \mathcal{T}$
  - $\mathcal{LA}$ can be $\mathcal{LRA}$, $\mathcal{LIA}$ or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
  - $\mathcal{LA}\mathcal{LRA}$ and $\mathcal{T}_i$ disjoint Nelson-Oppen theories
- a $\mathcal{LA}$ $\mathcal{LRA}$ variable [term] "*cost*" occurring in φ
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t. lb $\leq$ *cost* $<$ ub (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{LA}$ $\mathcal{LRA}$ costs (OMT($\mathcal{LA} \cup \mathcal{T}$) OMT($\mathcal{LRA} \cup \mathcal{T}$))

Find a model for φ whose value of *cost* is minimum.

- maximization dual

Optimization Modulo Theories with $\mathcal{LA}$ $\mathcal{LRA}$ costs II

We restrict to the case $\mathcal{LA} = \mathcal{LRA}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{LRA}$)).

Basic idea [72]:

SMT($\mathcal{LRA}$) augmented with a LP optimization routine:

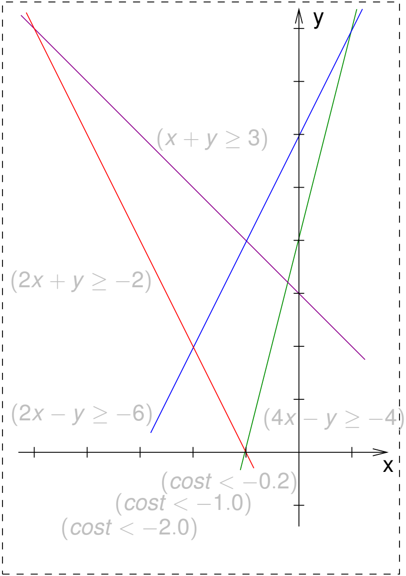
- once each assignment μ is found $\mathcal{LRA}$-satisfiable, an LP optimization is invoked, finding the minimum *min*
- ($cost < min$) is learned
- the search proceeds, until UNSAT
  $\Longrightarrow$ the latest value of *min* is returned

Optimization Modulo Theories with $\mathcal{LA}$ $\mathcal{LRA}$ costs III

Extensions

- both linear and binary search, and combination [72, 73]
- cost minimization embedded inside the CDCL search [72, 73]
- combination with other theories: OMT($\mathcal{LRA} \cup \mathcal{T}$) via DTC [73]
- extension to integers via ILP techniques: OMT($\mathcal{LIA} \cup \mathcal{T}$) [13, 76, 54]
- extension to multiple independent objectives [55, 13, 76]
- incremental OMT [13, 76]
- other combinations of objectives (min-max, lexicograpohic) [13, 76]
- OMT with Pareto fronts [13].

A toy example (linear search)



OMT with Independent Objectives (aka Boxed OMT) [55, 76]

The problem: $\langle \varphi, \{cost_1, ..., cost_k\} \rangle$ [55]

Given $\langle \varphi, \mathcal{C} \rangle$ s.t.:

- φ is the input formula
- $\mathcal{C} \stackrel{\text{def}}{=} \{cost_1, ..., cost_k\}$ is a set of $\mathcal{LA}$-terms on variables in φ,

$\langle \varphi, \mathcal{C} \rangle$ is the problem of finding a set of independent $\mathcal{LA}$-models $\mathcal{M}_1, ..., \mathcal{M}_k$ s.t. s.t. each $\mathcal{M}_i$ makes $cost_i$ minimum.

Notes

- derives from SW verification problems [55]
- equivalent to k independent problems $\langle \varphi, cost_1 \rangle, ..., \langle \varphi, cost_k \rangle$
- intuition: share search effort for the different objectives
- generalizes to OMT($\mathcal{LA} \cup \mathcal{T}$) straightforwardly

OMT with Multiple Objectives [55, 13, 76]

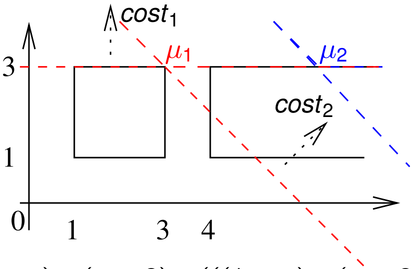
Solution

- Intuition: when a $\mathcal{T}$-consistent satisfying assignment μ is found,

```
foreach cost_i
    min_i := min{min_i, Tsolver.minimize(μ, cost_i)};
learn ⋁_i(cost_i < min_i);    // (cost_i < −∞) ≡ ⊥
```

  proceed until UNSAT;
- Notice:
  - for each μ, guaranteed improvement of at least one min_i
  - in practice, for each μ, multiple $cost_i$ minima are improved
- Implemented improvements:
  - (a) drop previous clauses $\bigvee_i(cost_i < min_i)$
  - (b) $(cost_i < min_i)$ pushed in μ first: if $\mathcal{T}$-inconsistent, skip minimization
  - (c) learn $\neg(cost_i < min_i) \vee (cost_i < min_i^{old})$, s.t. min_i^{old} previous min_i
    $\Longrightarrow$ reuse previously-learned clauses like $\neg(cost_i < min_i^{old}) \vee C$

Boxed OMT: Example [55, 76]



$$\varphi \;=\; (1 \le y) \wedge (y \le 3) \wedge (((1 \le x) \wedge (x \le 3)) \vee (x \ge 4))$$
$$\wedge\; (cost_1 = -y) \wedge (cost_2 = -x - y)$$

$\mu_1 \;=\; \{(1 \le y), (y \le 3), (1 \le x), (x \le 3)\} \Longrightarrow$ SAT $\Longrightarrow [-3, -6]$

$\Longrightarrow$ *learn* $\{(cost_1 < -3) \vee (cost_2 < -6)\}$

$\mu_2 \;=\; \{(1 \le y), (y \le 3), (x \ge 4)\} \Longrightarrow$ SAT $\Longrightarrow [-3, -\infty]$

$\Longrightarrow$ *learn* $\{(cost_1 < -3)\}$

$\Longrightarrow$ UNSAT

OMT with Lexicographic Combination of Objectives [13]

The problem

Find one optimal model $\mathcal{M}$ minimizing $\underline{c} \stackrel{\text{def}}{=} cost_1, cost_2, ..., cost_k$ lexicographically.

Solution

- Intuition:

  $\{$*minimize* $cost_1\}$
  when UNSAT
  $\{$*substitute unit clause* $(cost_1 < min_1)$ *with* $(cost_1 = min_1)\}$
  $\{$*minimize* $cost_2\}$
  ...
- improvement:
  - each time UNSAT is found, add $\bigwedge_i (cost_i \leq \mathcal{M}_i(cost_i))$ to φ

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) I

SMT with Pseudo-Boolean Constraints & Weighted MaxSMT

$$OMT + PB: \quad \sum_j w_j \cdot A_j,\ w_i > 0 \quad //(\sum_j ite(A_j, w_j, 0))$$

$$\Downarrow$$

$$\sum_j x_j,\ x_j\ fresh$$

$$\text{s.t.} \quad \ldots \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

$$MaxSMT: \quad \langle \varphi_h, \bigwedge_j \psi_j \rangle \quad s.t.\ \psi_j\ soft,\ w_j = weight(\psi_j),\ w_i > 0$$

$$\Downarrow$$

$$minimize\ \sum_j x_j,\ x_j, A_j\ fresh$$

$$\varphi_h \wedge \bigwedge_j (A_j \vee \psi_j) \wedge \bigwedge_j (\neg A_j \vee (x_j = w_j)) \wedge (A_j \vee (x_j = 0))$$

$$\wedge (x_j \geq 0) \wedge (x_j \leq w_j)$$

Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$
\begin{array}{ll}
OMT+PB: & \sum_j w_j \cdot A_j,\ w_i > 0\ \ //(\sum_j ite(A_j, w_j, 0)) \\
 & \Downarrow \\
 & \sum_j x_j,\ x_j\ fresh \\
\text{s.t.} & ... \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\
 & \wedge (x_j \geq 0) \wedge (x_j \leq w_j)
\end{array}
$$

Range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$" logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i's are assigned :
  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \ \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
  $\Longrightarrow$ drastic pruning of the search
- same for weighted MaxSMT

Optimization problems encoded into OMT($\mathcal{LA} \cup \mathcal{T}$) II

OMT with Min-Max [Max-Min] optimization

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$, find a solution which minimizes the maximum value among $\{cost_1, ..., cost_k\}$. (Max-Min dual.)

- Frequent in some applications (e.g. [74, 81])

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge \bigwedge_i (cost_i \le cost), cost\}$ s.t. *cost* fresh.

OMT with linear combinations of costs

Given $\langle \varphi, \{cost_1, ..., cost_k\}\rangle$ and a set of weights $\{w_1, ..., w_k\}$, find a solution which minimizes $\sum_i w_i \cdot cost_i$.

$\Longrightarrow$ encode into OMT($\mathcal{LA} \cup \mathcal{T}$) problem $\{\varphi \wedge (cost = \sum_i w_i \cdot cost_i), cost\}$ s.t. *cost* fresh.

These objectives can be composed with other OMT($\mathcal{LA}$) objectives.

Other OMT Functionalities [hints]

Incremental interface [13, 76]

Allows for pushing/popping sub-formulas into a stack, and then run OMT incrementally over them, reusing previous search.

- useful in some applications (e.g., BMC with parametric systems)
- straightforward variant of incremental SAT and SMT solvers

Pareto Fronts [13, 12]

- Given $cost_1, cost_2$, compute $\mathcal{M}_1, ..., \mathcal{M}_i, ..., \mathcal{M}_j, ...$ s.t.:
  - either $\mathcal{M}_i(cost_1) > \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) > \mathcal{M}_j(cost_2)$ and $\mathcal{M}_i(cost_1) < \mathcal{M}_j(cost_1)$ or $\mathcal{M}_i(cost_2) < \mathcal{M}_j(cost_2)$
  - for each $\mathcal{M}_i$, no $\mathcal{M}'$ dominates $\mathcal{M}_i$
- no objective can be improved without degrading some other one

Some OMT tools

- BCLT [66, 54]
  http://www.cs.upc.edu/~oliveras/bclt-main.html
- OPTIMATHSAT [72, 74, 76, 75], on top of MATHSAT [28]
  http://optimathsat.disi.unitn.it
- SYMBA [55], on top of Z3 [38]
  https://bitbucket.org/arieg/symba/src
- νZ [13, 12], on top of Z3 [38]
  http://z3.codeplex.com

Conclusions

- SMT very popular, due to successful application in many domains
- Combines techniques from SAT, ATP and operational research
- Not only satisfiability, but also advanced functionalities

Open/ongoing research directions

- Solving:
  - improve efficiency (e.g. $\mathcal{BV}$, $\mathcal{AR}$, $\mathcal{LIA}$ & their combinations)
    "a never-ending fight against the search-space explosion problem [E. Clarke, Turing-award winner 2007]"
  - develop efficient solvers for other theories ($\mathcal{NLA}(\mathbb{R})$, $\mathcal{NLA}(\mathbb{Z})$)
  - develop new theories & solvers (e.g., floating-point arithmetic)
  - ...
- Functionalities
  - Interpolation in some theories ($\mathcal{LIA}$, $\mathcal{BV}$) still very challenging
  - Predicate abstraction (AllSMT) still a bottleneck in SMT-based FV
  - SMT with costs/optimization still in very early stage
  - ...
- Combination of SMT solvers and ATP (SMT with quantifiers)
- Integration & customization of SMT solvers with (FV) tools
- See also [67]

Links I

- survey papers:
  - Roberto Sebastiani: "Lazy Satisfiability Modulo Theories". Journal on Satisfiability, Boolean Modeling and Computation, JSAT. Vol. 3, 2007. Pag 141–224, ©IOS Press.
  - Clark Barrett, Roberto Sebastiani, Sanjit Seshia, Cesare Tinelli "Satisfiability Modulo Theories". Part II, Chapter 26, The Handbook of Satisfiability. 2009. ©IOS press.
  - Leonardo de Moura and Nikolaj Bjørner. "Satisfiability modulo theories: introduction and applications". Communications of the ACM, 54 (9), 2011. ©ACM press.
- web links:
  - The SMT library SMT-LIB: `http://goedel.cs.uiowa.edu/smtlib/`
  - The SMT Competition SMT-COMP: `http://www.smtcomp.org/`
  - The SAT/SMT Schools `http://satassociation.org/sat-smt-school.html`

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The list of references above is by no means intended to be all-inclusive. I apologize both with the authors and with the readers for all the relevant works which are not cited here.


"That's all Folks!"