Course “Efficient Boolean Reasoning”
TEST

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March 13^{th}, 2018

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[COPY WITH SOLUTIONS]

1

Let φ be a generic Boolean formula, and let $\varphi_1 \stackrel{\text{def}}{=} CNF(\varphi)$, s.c. $CNF()$ is the "classic" CNF conversion (i.e., that using DeMorgan's rules). Let $|\varphi|$ and $|\varphi_1|$ denote the size of φ and φ_1 respectively.

For each of the following sentences, say if it is true or false.

(a) $|\varphi_1|$ is in worst-case polynomial in size wrt. $|\varphi|$. [Solution: False.]

(b) φ_1 has the same number of distinct Boolean variables as φ has. [Solution: True.]

(c) A model for φ_1 (if any) is also a model for φ, and vice versa. [Solution: True.]

(d) φ_1 is valid if and only if φ is valid. [Solution: True.]

2

Consider the following CNF formula:

$$
\begin{array}{l}
(\neg A_1 \vee \neg A_2 \vee \neg A_3) \wedge \\
(\ A_4 \vee \neg A_7 \vee \neg A_5) \wedge \\
(\neg A_2 \vee \neg A_3 \vee \neg A_6) \wedge \\
(\neg A_8 \vee \neg A_1 \vee \neg A_3) \wedge \\
(\neg A_2 \vee \neg A_3 \vee \neg A_8) \wedge \\
(\neg A_6 \vee \neg A_5 \vee \neg A_8) \wedge \\
(\ A_8 \vee \neg A_3 \vee \neg A_7) \wedge \\
(\neg A_6 \vee \neg A_5 \vee \neg A_2) \wedge \\
(\ A_8 \vee \neg A_1 \vee \neg A_3) \wedge \\
(\neg A_3 \vee \neg A_4 \vee \neg A_3) \wedge \\
(\ A_7 \vee \neg A_2 \vee \neg A_1) \wedge \\
(\ A_1 \vee \neg A_2 \vee \neg A_3) \wedge \\
(\neg A_4 \vee \neg A_5 \vee \neg A_2) \wedge \\
(\neg A_8 \vee \neg A_7 \vee \neg A_1) \wedge \\
(\neg A_5 \vee \neg A_4 \vee \neg A_7) \wedge \\
(\neg A_4 \vee \neg A_2 \vee \neg A_5) \wedge \\
(\ A_3 \vee \neg A_6 \vee \neg A_7) \wedge \\
(\ A_3 \vee \neg A_4 \vee \neg A_2) \wedge \\
(\ A_6 \vee \neg A_2 \vee \neg A_8) \wedge \\
(\ A_1 \vee \neg A_2 \vee \neg A_6) \wedge \\
(\ A_2 \vee \neg A_3 \vee \neg A_4) \wedge \\
(\ A_5 \vee \neg A_6 \vee \neg A_2) \wedge \\
(\neg A_5 \vee \neg A_3 \vee \neg A_4)
\end{array}
$$

Decide *quickly* if it is satisfiable or not, and briefly explain why.

[Solution: It is a Horn formula with no positive unit clauses. Therefore, the assignment $\{\neg A_i\}_{i=1}^{8}$ is a model.]

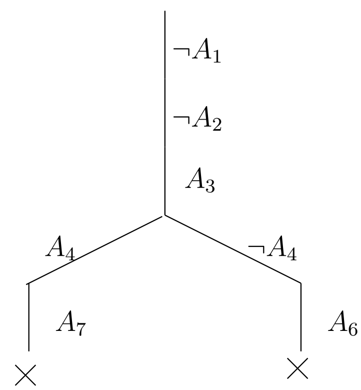
3

Using the basic DPLL algorithm, decide whether the following formula is satisfiable or not. (Write the search tree.)

$$\begin{array}{rcrcr} & & & & (\neg A_1) \wedge \\ & & (\ A_1 & \vee & \neg A_2) \wedge \\ (\ A_1 & \vee & A_2 & \vee & A_3) \wedge \\ (\ A_4 & \vee & \neg A_3 & \vee & A_6) \wedge \\ (\ A_4 & \vee & \neg A_3 & \vee & \neg A_6) \wedge \\ (\neg A_3 & \vee & \neg A_4 & \vee & A_7) \wedge \\ (\neg A_3 & \vee & \neg A_4 & \vee & \neg A_7) \end{array}$$

(Literal-selection criteria to your choice.)

[Solution:



$\Longrightarrow$ the formula is inconsistent.

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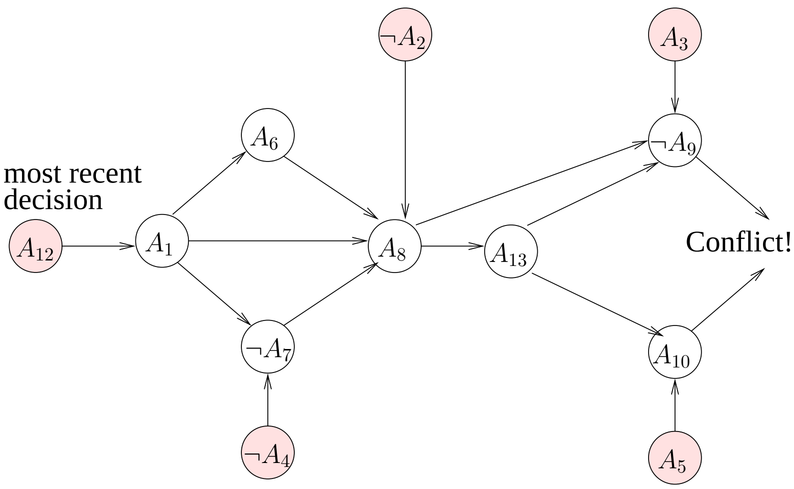
4

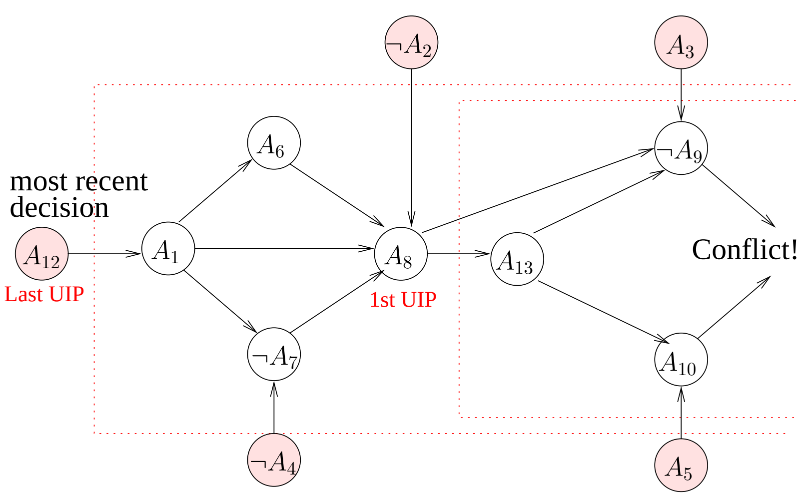
Consider the following implication graph:



A_{12} being the most recent decision literal. Write the conflict clauses generated by

1. the last UIP conflict analysis technique
2. the 1st UIP conflict analysis technique

[Solution:



1. Last UIP clause: $\neg A_{12} \vee A_2 \vee A_4 \vee \neg A_3 \vee \neg A_5$
2. 1st UIP clause: $\neg A_8 \vee \neg A_3 \vee \neg A_5$

]

5

Consider the following Boolean formulas:

$$\begin{aligned}\varphi_1 \stackrel{\text{def}}{=} \; & (\;\; A_1 \vee \;\; A_2) \;\; \wedge \\ & (\neg A_1 \vee \;\; A_2) \;\; \wedge \\ & (\;\; A_3 \vee \;\; A_4) \;\; \wedge \\ & (\neg A_3 \vee \;\; A_4)\end{aligned}$$

$$\begin{aligned}\varphi_2 \stackrel{\text{def}}{=} \; & (\neg A_2 \vee \;\; A_5) \;\; \wedge \\ & (\neg A_4 \vee \neg A_5)\end{aligned}$$

which are such that $\varphi_1 \wedge \varphi_2 \models \bot$. For each of the following formulas, say if it is a Craig interpolant for (φ_1, φ_2) or not.

[Solution: Recall that a Craig interpolant for (φ_1, φ_2) s.t. $\varphi_1 \wedge \varphi_2 \models \bot$ is a formula ψ s.t.

1. $\varphi_1 \models \psi$
2. $\psi \wedge \varphi_2 \models \bot$
3. all atoms in ψ occur in both φ_1 and φ_2.

]

(a)

$$\begin{aligned}& (\;\; A_1 \vee \;\; A_2) \;\; \wedge \\ & (\neg A_1 \vee \;\; A_2) \;\; \wedge \\ & (\;\; A_4)\end{aligned}$$

[Solution: No, it is not a solution because it does not verify condition 3.]

(b)

$$(\;\; A_2)$$

[Solution: No, it is not a solution because it does not verify condition 2.]

(c)

$$(\;\; A_2 \wedge \;\; A_4)$$

[Solution: Yes]

6

Consider the following formula in the theory $\mathcal{LRA}$ of linear arithmetic on the Rationals.

$$\begin{array}{rl} \varphi = & \{(v_1 - v_2 \leq 3) \ \vee A_2\} \quad \wedge \\ & \{\neg(2v_3 + v_4 \geq 5) \ \vee \underline{\neg(v_1 - v_3 \leq 6)} \ \vee \neg A_1\} \ \wedge \\ & \{A_1 \vee (v_1 - v_2 \leq 3)\} \quad \wedge \\ & \{\underline{(v_2 - v_4 \leq 6)} \ \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \ \wedge \\ & \{\underline{\neg(v_2 - v_3 > 2)} \ \vee A_1\} \wedge \\ & \{\underline{\neg A_2} \vee (v_1 - v_5 \leq 1)\} \wedge \\ & \{A_1 \vee \underline{(v_3 = v_5 + 6)} \ \vee A_2\} \end{array}$$

and consider the partial truth assignment μ given by the underlined literals above:

$$\{\neg(v_2 - v_3 > 2), \neg A_2, \neg(v_1 - v_3 \leq 6), (v_2 - v_4 \leq 6), (v_3 = v_5 + 6)\}.$$

1. Does (the Boolean abstraction of) μ propositionally satisfy (the Boolean abstraction of) φ?
2. Is μ satisfiable in $\mathcal{LRA}$?
   (a) If no, find a minimal conflict set for μ and the corresponding conflict clause C.
   (b) If yes, show one unassigned literal which can be deduced from μ, and show the corresponding deduction clause C.

[Solution:

1. No, since there are two clauses which are not satisfied.
2. Yes (there is no cycle among the constraints). $v_2 = v_3 = v_4 = 0.0$, $v_1 = 7.0$, $v_5 = -6.0$ is a solution.
   (a) ...
   (b) One possible deduction is:

   $$\{\neg(v_2 - v_3 > 2), \neg(v_1 - v_3 \leq 6)\} \models_{\mathcal{T}} \neg(v_1 - v_2 \leq 3).$$

   which corresponds to learning the deduction clause:

   $$(v_2 - v_3 > 2) \vee (v_1 - v_3 \leq 6) \vee \neg(v_1 - v_2 \leq 3).$$

   Another possible deduction is:

   $$\{(v_3 = v_5 + 6), \neg(v_1 - v_3 \leq 6)\} \models_{\mathcal{T}} \neg(v_1 - v_5 \leq 1).$$

   which corresponds to learning the deduction clause:

   $$\neg(v_3 = v_5 + 6) \vee (v_1 - v_3 \leq 6) \vee \neg(v_1 - v_5 \leq 1).$$

]

7

Consider the following $\mathcal{LRA}$ formula φ.

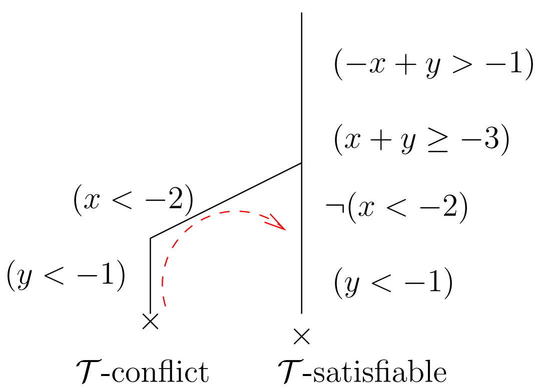
$$
\begin{array}{rcrcrc}
((-x+y>-1)) & & & & & \wedge \\
((x+y\geq -3) & \vee & \neg(-x+y>-1)) & & & \wedge \\
(\neg(x+y\geq -3) & \vee & \neg(x<-2) & \vee & (y<-1)) & \wedge \\
(\neg(-x+y>-1) & \vee & (x<-2) & \vee & (y<-1)) & \\
((x+y\geq -3) & \vee & \neg(5y-4z>1) & \vee & \neg(3v-5x>7)) & \wedge \\
((-x+y>-1) & \vee & (3v-5x>7) & \vee & \neg(5y-4z>1)) & \wedge
\end{array}
$$

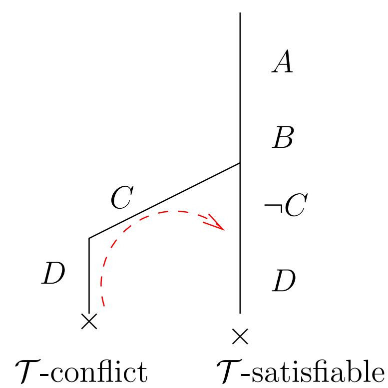
(a) Write the Boolean Abstraction of φ.

(b) Using the standard lazy SMT approach (literal-decision order, techniques and strategies to your choice), decide if φ is satisfiable in $\mathcal{LRA}$, plotting the corresponding search tree and producing the $\mathcal{T}$-lemmas involved.

[Solution: Using $\mathcal{T}$-backjumping, we show that the formula is $\mathcal{LRA}$-satisfiable, as shown in the following search tree:

$$
\begin{array}{rcrcrc}
((-x+y>-1)) & & & & & \wedge \\
((x+y\geq -3) & \vee & \neg(-x+y>-1)) & & & \wedge \\
(\neg(x+y\geq -3) & \vee & \neg(x<-2) & \vee & (y<-1)) & \wedge \\
(\neg(-x+y>-1) & \vee & (x<-2) & \vee & (y<-1)) & \\
((x+y\geq -3) & \vee & \neg(5y-4z>1) & \vee & \neg(3v-5x>7)) & \wedge \\
((-x+y>-1) & \vee & (3v-5x>7) & \vee & \neg(5y-4z>1)) & \wedge
\end{array}
\qquad
\begin{array}{rcrcrc}
(A) & & & & & \wedge \\
(B & \vee & \neg A) & & & \wedge \\
(\neg B & \vee & \neg C & \vee & D) & \wedge \\
(\neg A & \vee & C & \vee & D) & \\
(B & \vee & \neg E & \vee & \neg F) & \wedge \\
(A & \vee & F & \vee & \neg E) & \wedge
\end{array}
$$





because $\{(x+y\geq -3),(x<-2),(y<-1)\}$ is a conflict set for the first branch, but the second branch is satisfiable (e.g., $\mathcal{I} \stackrel{\text{def}}{=} \{x=-1.5, y=-1.5\}$ is a model for the second branch).

A faster result is obtained by deciding first $\neg C$ or D, from which the model $\mathcal{I}$ is found in one branch only.

]

8

Let $\mathcal{LA}(\mathbb{Q})$ be the logic of linear arithmetic over the rationals and $\mathcal{EUF}$ be the logic of equality and uninterpreted functions, and consider the following pure formula φ in the combined logic $\mathcal{LA}(\mathbb{Q}) \cup \mathcal{EUF}$:

$$(h = 3.0) \wedge (k = -2.0) \wedge (x = 1.0) \wedge (y = h + k) \wedge \quad (1)$$
$$(z = f(x)) \wedge (w = f(y)) \wedge \neg(g(z) = g(w)) \quad (2)$$

Say which variables are interface variables, list the interface equalities for this formula, and decide whether this formulas is $\mathcal{LA}(\mathbb{Q}) \cup \mathcal{EUF}$-satisfiable or not, using either Nelson-Oppen or Delayed Theory Combination.

[Solution: Only x and y occur both in $\mathcal{LA}(\mathbb{Q})$-atoms (1) and in $\mathcal{EUF}$-atoms (2). Thus x, y are the interface variables, and $x = y$ is the only interface equality.

Nelson-Oppen: From (1) in $\mathcal{LA}$ we infer the interface equality $(x = y)$. Adding the latter to (2), one gets a contradiction in $\mathcal{EUF}$.

Delayed Theory Combination: By unit-propagation, φ causes only one branch containing all its literals. Then the SAT solver assigns first a negative value to the interface equality, adding $\neg(x = y)$ to the assignment, which is found inconsistent in $\mathcal{LA}(\mathbb{Q})$:

$$(h = 3.0) \wedge (k = -2.0) \wedge (x = 1.0) \wedge (y = h + k) \wedge \neg(x = y). \quad (3)$$

Then the SAT solver backtracks, adding $(x = y)$ to the assignment, which is found inconsistent in $\mathcal{EUF}$:

$$(z = f(x)) \wedge (w = f(y)) \wedge \neg(g(z) = g(w)) \wedge (x = y). \quad (4)$$

Thus, with either technique, we can conclude that φ is $\mathcal{LA}(\mathbb{Q}) \cup \mathcal{EUF}$-unsatisfiable.]

9

Consider the following set of clauses φ in the theory of linear arithmetic on the Integers $\mathcal{LIA}$.

$$\left\{\begin{array}{l}(\neg(x=0)\vee\neg(x=1)),\\(\neg(x=0)\vee\quad(x=1)),\\(\quad(x=0)\vee\neg(x=1)),\\(\quad(x=0)\vee\quad(x=1))\end{array}\right\}$$

Say which of the following sets is a $\mathcal{LIA}$-unsatisfiable core of φ and which is not. For each one, explain why.

(a)

$$\left\{\begin{array}{l}(\neg(x=0)\vee\quad(x=1)),\\(\quad(x=0)\vee\neg(x=1)),\\(\quad(x=0)\vee\quad(x=1))\end{array}\right\}$$

[Solution: yes, because it is a subset of φ and it is inconsistent in $\mathcal{LIA}$.]

(b)

$$\left\{\begin{array}{l}(\neg(x=0)\vee\neg(x=1)),\\(\quad(x=0)\vee\neg(x=1)),\\(\quad(x=0)\vee\quad(x=1))\end{array}\right\}$$

[Solution: no, because is not inconsistent in $\mathcal{LIA}$ (e.g., $(x=0)$ is a solution).]

(c)

$$\left\{\begin{array}{l}(\neg(x=0)\vee\quad(x=1)),\\(\quad(x=0)\vee\neg(x=1)),\\(\quad(x=0)\vee\quad(x=1)),\\((x=y))\end{array}\right\}$$

[Solution: no, because it is not a subset of φ.]

10

Consider the following formulas in difference logic ($\mathcal{DL}$):

$$\begin{array}{rcll} \varphi_1 & \stackrel{\text{def}}{=} & (x_4 - x_5 \leq -2) & \wedge \\ & & (x_5 - x_6 \leq -4) & \wedge \\ & & (x_1 - x_2 \leq 3) & \wedge \\ & & (x_2 - x_3 \leq 1) & \\ & & & \\ \varphi_2 & \stackrel{\text{def}}{=} & (x_6 - x_1 \leq 0) & \wedge \\ & & (x_3 - x_4 \leq 1) & \end{array}$$

which are such that $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \bot$. For each of the following formulas, say if it is a Craig interpolant in $\mathcal{DL}$ for (φ_1, φ_2), and explain why.

[Solution: Recall that a Craig interpolant for (φ_1, φ_2) s.t. $\varphi_1 \wedge \varphi_2 \models_{\mathcal{DL}} \bot$ is a formula ψ s.t.

1. $\varphi_1 \models_{\mathcal{DL}} \psi$
2. $\psi \wedge \varphi_2 \models_{\mathcal{DL}} \bot$
3. all symbols in ψ occur in both φ_1 and φ_2.

]

(a) $(x_1 - x_2 + x_4 - x_6 \leq -3)$

[Solution: no, because x_2 is not a symbol occurring in φ_2. Moreover, it is not a $\mathcal{DL}$ formula.]

(b) $(x_1 - x_3 \leq 4)$

[Solution: No, because it violates condition 2.]

(c) $(x_1 - x_3 \leq 4) \wedge$
$(x_4 - x_6 \leq -6)$

[Solution: yes, because it is a $\mathcal{DL}$ formula and it verifies all conditions 1., 2., 3.]