Course “An Introduction to SAT and SMT”
Chapter 1: Propositional Satisfiability (SAT)

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Outline

1. Basics on SAT
2. Basic SAT-Solving techniques
3. Modern CDCL SAT Solvers
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
4. Tractable subclasses of SAT
5. Random k-SAT and Phase Transition
6. Advanced Functionalities: proofs, unsat cores, interpolants, optimization
7. Some Applications
   - Appl. #1: (Bounded) Planning
   - Appl. #2: Bounded Model Checking

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Boolean logic

Basic notation & definitions

- Boolean formula
  - $\top, \bot$ are formulas
  - A propositional atom $A_1, A_2, A_3, \ldots$ is a formula;
  - if φ_1 and φ_2 are formulas, then $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \varphi_1 \rightarrow \varphi_2, \varphi_1 \leftarrow \varphi_2, \varphi_1 \leftrightarrow \varphi_2$ are formulas.
- *Atoms*(φ): the set $\{A_1, \ldots, A_N\}$ of atoms occurring in φ.
- Literal: a propositional atom A_i (positive literal) or its negation $\neg A_i$ (negative literal)
  - Notation: if $l := \neg A_i$, then $\neg l := A_i$
- Clause: a disjunction of literals $\bigvee_j l_j$ (e.g., $(A_1 \vee \neg A_2 \vee A_3 \vee \ldots)$)
- Cube: a conjunction of literals $\bigwedge_j l_j$ (e.g., $(A_1 \wedge \neg A_2 \wedge A_3 \wedge \ldots)$)

Semantics of Boolean operators

- Truth table:

φ_1	φ_2	$\neg\varphi_1$	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

Note

- $\wedge$, $\vee$ and $\leftrightarrow$ are commutative:

$$(\varphi_1 \wedge \varphi_2) \iff (\varphi_2 \wedge \varphi_1)$$
$$(\varphi_1 \vee \varphi_2) \iff (\varphi_2 \vee \varphi_1)$$
$$(\varphi_1 \leftrightarrow \varphi_2) \iff (\varphi_2 \leftrightarrow \varphi_1)$$

- $\wedge$ and $\vee$ are associative:

$$((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \iff (\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)) \iff (\varphi_1 \wedge \varphi_2 \wedge \varphi_3)$$
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Syntactic Properties of Boolean Operators

$$
\begin{array}{rcl}
\neg\neg\varphi_1 & \Longleftrightarrow & \varphi_1 \\
(\varphi_1 \vee \varphi_2) & \Longleftrightarrow & \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\
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\neg(\varphi_1 \leftarrow \varphi_2) & \Longleftrightarrow & (\neg\varphi_1 \wedge \varphi_2) \\
(\varphi_1 \leftrightarrow \varphi_2) & \Longleftrightarrow & ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_1 \leftarrow \varphi_2)) \\
 & \Longleftrightarrow & ((\neg\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \neg\varphi_2)) \\
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 & \Longleftrightarrow & ((\varphi_1 \vee \varphi_2) \wedge (\neg\varphi_1 \vee \neg\varphi_2))
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Boolean logic can be expressed in terms of $\{\neg, \wedge\}$ (or $\{\neg, \vee\}$) only

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Tree and DAG representation of formulas: example

Formulas can be represented either as trees or as DAGS:

- DAG representation can be up to exponentially smaller

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

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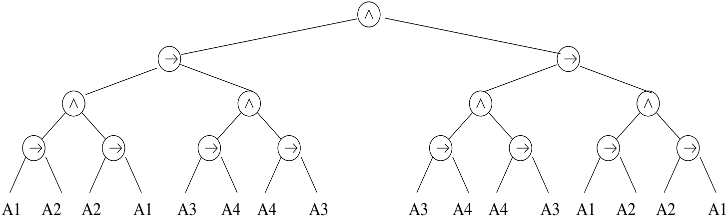
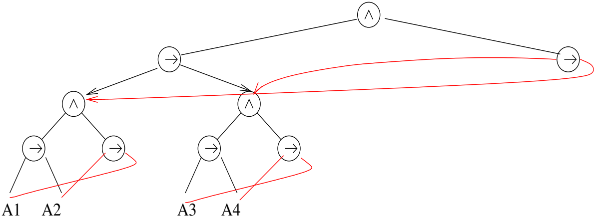
$$\begin{gathered}(((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)))\wedge \\ (((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1))))\end{gathered}$$

Tree and DAG repres. of formulas: example (cont)



Tree Representation



DAG Representation

Basic notation & definitions (cont)

- Total truth assignment μ for φ:
  $\mu : \mathit{Atoms}(\varphi) \longmapsto \{\top, \bot\}$.
- Partial Truth assignment μ for φ:
  $\mu : \mathcal{A} \longmapsto \{\top, \bot\}$, $\mathcal{A} \subset \mathit{Atoms}(\varphi)$.
- Set and formula representation of an assignment:
  - μ can be represented as a set of literals:
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies \{A_1, \neg A_2\}$
  - μ can be represented as a formula (cube):
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies (A_1 \wedge \neg A_2)$

Basic notation & definitions (cont)

- a total truth assignment μ satisfies φ ($\mu \models \varphi$):
  - $\mu \models A_i \iff \mu(A_i) = \top$
  - $\mu \models \neg\varphi \iff$ *not* $\mu \models \varphi$
  - $\mu \models \varphi_1 \wedge \varphi_2 \iff \mu \models \varphi_1$ *and* $\mu \models \varphi_2$
  - $\mu \models \varphi_1 \vee \varphi_2 \iff \mu \models \varphi_1$ *or* $\mu \models \varphi_2$
  - $\mu \models \varphi_1 \rightarrow \varphi_2 \iff$ *if* $\mu \models \varphi_1$, *then* $\mu \models \varphi_2$
  - $\mu \models \varphi_1 \leftrightarrow \varphi_2 \iff \mu \models \varphi_1$ *iff* $\mu \models \varphi_2$
- a partial truth assignment μ satisfies φ iff it makes φ evaluate to true (Ex: $\{A_1\} \models (A_1 \vee A_2)$)
  $\Longrightarrow$ if μ satisfies φ, then all its total extensions satisfy φ
  (Ex: $\{A_1, A_2\} \models (A_1 \vee A_2)$ and $\{A_1, \neg A_2\} \models (A_1 \vee A_2)$)
- φ is satisfiable iff $\mu \models \varphi$ for some μ
- φ_1 entails φ_2 ($\varphi_1 \models \varphi_2$): $\varphi_1 \models \varphi_2$ iff $\mu \models \varphi_1 \Longrightarrow \mu \models \varphi_2$ for every μ
- φ is valid ($\models \varphi$): $\models \varphi$ iff $\mu \models \varphi$ for every μ

Property

φ is valid $\iff \neg\varphi$ is not satisfiable

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Equivalence and equi-satisfiability

- φ_1 and φ_2 are equivalent iff, for every μ, $\mu \models \varphi_1$ iff $\mu \models \varphi_2$
- φ_1 and φ_2 are equi-satisfiable iff exists μ_1 s.t. $\mu_1 \models \varphi_1$ iff exists μ_2 s.t. $\mu_2 \models \varphi_2$
- φ_1, φ_2 equivalent
  $\Downarrow \quad \not\Uparrow$
  φ_1, φ_2 equi-satisfiable
- EX: $A_1 \vee A_2$ and $(A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$ are equi-satisfiable, not equivalent.
  $\{\neg A_1, A_2, A_3\} \models (A_1 \vee A_2)$, but
  $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when φ_2 is the result of applying some transformation T to φ_1: $\varphi_2 \stackrel{\text{def}}{=} T(\varphi_1)$:
  we say that T is validity-preserving [satisfiability-preserving] iff $T(\varphi_1)$ and φ_1 are equivalent [equi-satisfiable]

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  $\{\neg A_1, A_2, A_3\} \not\models (A_1 \vee \neg A_3) \wedge (A_3 \vee A_2)$
- Typically used when φ_2 is the result of applying some transformation T to φ_1: $\varphi_2 \stackrel{\text{def}}{=} T(\varphi_1)$:
  we say that T is validity-preserving [satisfiability-preserving] iff $T(\varphi_1)$ and φ_1 are equivalent [equi-satisfiable]

Complexity

- For N variables, there are up to 2^N truth assignments to be checked.
- The problem of deciding the satisfiability of a propositional formula is NP-complete
- The most important logical problems (validity, inference, entailment, equivalence, ...) can be straightforwardly reduced to satisfiability, and are thus (co)NP-complete.

$$\Downarrow$$

No existing worst-case-polynomial algorithm.

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
  - φ occurs positively in φ;
  - if $\neg\varphi_1$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ
  - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ, then φ_1 and φ_2 occur positively [negatively] in φ;
  - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
  - if $\varphi_1 \leftrightarrow \varphi_2$ occurs in φ, then φ_1 and φ_2 occur positively and negatively in φ;

Negative normal form (NNF)

- φ is in Negative normal form iff it is given only by the recursive applications of $\wedge, \vee$ to literals.
- every φ can be reduced into NNF:
  (i) substituting all $\rightarrow$'s and $\leftrightarrow$'s:

$$\begin{array}{lcl} \varphi_1 \rightarrow \varphi_2 & \Longrightarrow & \neg\varphi_1 \vee \varphi_2 \\ \varphi_1 \leftrightarrow \varphi_2 & \Longrightarrow & (\neg\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \neg\varphi_2) \end{array}$$

  (ii) pushing down negations recursively:

$$\begin{array}{lcl} \neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & \neg\varphi_1 \vee \neg\varphi_2 \\ \neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & \neg\varphi_1 \wedge \neg\varphi_2 \\ \neg\neg\varphi_1 & \Longrightarrow & \varphi_1 \end{array}$$

- The reduction is linear if a DAG representation is used.
- Preserves the equivalence of formulas.

NNF: example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

NNF: example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

$$\Downarrow$$

$$((((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))\wedge$$
$$(((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4))))$$

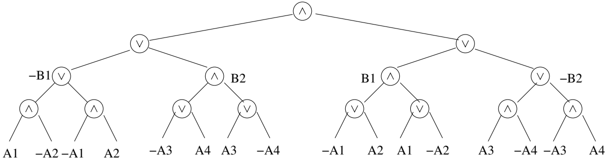
NNF: example

$$
\begin{gathered}
(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\
\Downarrow \\
((((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))\wedge \\
(((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))) \\
\Downarrow \\
((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))\wedge \\
(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))))
\end{gathered}
$$

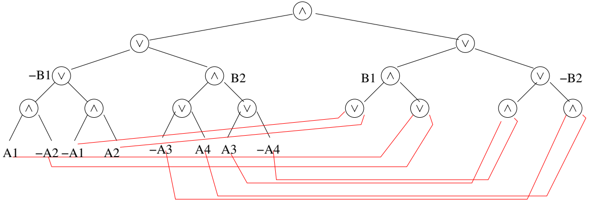
NNF: example

$$
\begin{array}{c}
(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4) \\
\Downarrow \\
((((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))\wedge \\
(((A_1 \rightarrow A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \rightarrow A_4) \wedge (A_3 \leftarrow A_4)))) \\
\Downarrow \\
((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))\wedge \\
(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))) \\
\Downarrow \\
((((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4)))\wedge \\
(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))))
\end{array}
$$

NNF: example (cont)



Tree Representation

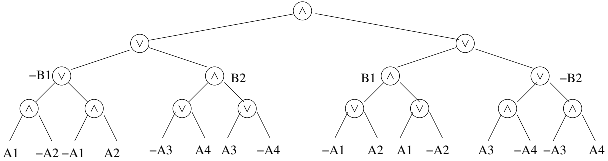


DAG Representation

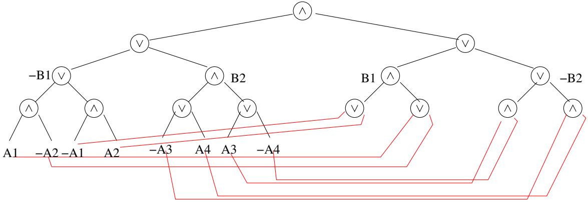
Note

For each non-literal subformula φ, φ and $\neg\varphi$ have different representations $\Longrightarrow$ they are not shared.

NNF: example (cont)



Tree Representation



DAG Representation

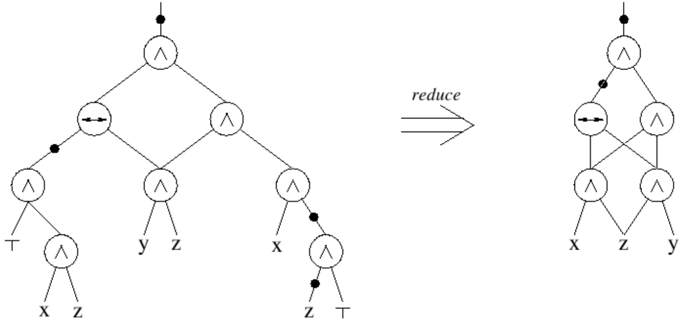
Note

For each non-literal subformula φ, φ and $\neg\varphi$ have different representations $\Longrightarrow$ they are not shared.

Optimized polynomial representations

And-Inverter Graphs, Reduced Boolean Circuits, Boolean Expression Diagrams

- Maximize the sharing in DAG representations: $\{\wedge, \leftrightarrow, \neg\}$-only, negations on arcs, sorting of subformulae, lifting of $\neg$'s over $\leftrightarrow$'s,...



Conjunctive Normal Form (CNF)

- φ is in Conjunctive normal form iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
- Easier to handle: list of lists of literals.
  $\Longrightarrow$ no reasoning on the recursive structure of the formula

Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,
  (i) converting it into NNF (not indispensible);
  (ii) applying recursively the DeMorgan's Rule:
  $$(\varphi_1 \wedge \varphi_2) \vee \varphi_3 \implies (\varphi_1 \vee \varphi_3) \wedge (\varphi_2 \vee \varphi_3)$$
- Worst-case exponential.
- $Atoms(CNF(\varphi)) = Atoms(\varphi)$.
- $CNF(\varphi)$ is equivalent to φ.
- Rarely used in practice.

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- Rarely used in practice.

Labeling CNF conversion $CNF_{label}(\varphi)$

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\begin{array}{lcl} \varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j)) \\ \varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j)) \\ \varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j)) \end{array}$$

  l_i, l_j being literals and B being a "new" variable.
- Worst-case linear.
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$.
- $CNF_{label}(\varphi)$ is equi-satisfiable w.r.t. φ.
- More used in practice.

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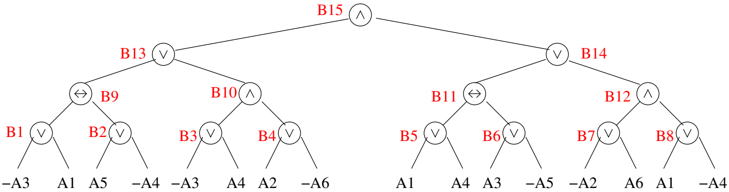
- $CNF_{label}(\varphi)$ is equi-satisfiable w.r.t. φ.
- More used in practice.

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \leftrightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \leftrightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j) \wedge$ $(B \vee \neg l_i \neg l_j)$
$CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j) \wedge$ $(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF conversion CNF_{label} – example



$$\begin{array}{ll}
CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) & \wedge \\
\ldots & \wedge \\
CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) & \wedge \\
CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) & \wedge \\
\ldots & \wedge \\
CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) & \wedge \\
CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) & \wedge \\
CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) & \wedge \\
CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) & \wedge \\
B_{15} &
\end{array}$$

Labeling CNF conversion CNF_{label} (improved)

- As in the previous case, applying instead the rules:

$$\begin{array}{llll}
\varphi \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\, CNF(B \rightarrow (l_i \vee l_j)) & \textit{if } (l_i \vee l_j) \textit{ pos.} \\
\varphi \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\, CNF((l_i \vee l_j) \rightarrow B) & \textit{if } (l_i \vee l_j) \textit{ neg.} \\
\varphi \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\, CNF(B \rightarrow (l_i \wedge l_j)) & \textit{if } (l_i \wedge l_j) \textit{ pos.} \\
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\varphi \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\, CNF(B \rightarrow (l_i \leftrightarrow l_j)) & \textit{if } (l_i \leftrightarrow l_j) \textit{ pos.} \\
\varphi \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\, CNF((l_i \leftrightarrow l_j) \rightarrow B) & \textit{if } (l_i \leftrightarrow l_j) \textit{ neg.}
\end{array}$$

- Smaller in size:

$$\begin{array}{ll}
CNF(B \rightarrow (l_i \vee l_j)) & = (\neg B \vee l_i \vee l_j) \\
CNF(((l_i \vee l_j) \rightarrow B)) & = (\neg l_i \vee B) \wedge (\neg l_j \vee B)
\end{array}$$

Labeling CNF conversion CNF_{label} (improved)

- As in the previous case, applying instead the rules:

$$\begin{array}{lllll}
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \vee l_j)) & \textit{if } (l_i \vee l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \vee l_j)|B] & \wedge\ CNF((l_i \vee l_j) \rightarrow B) & \textit{if } (l_i \vee l_j) \textit{ neg.} \\
\varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \wedge l_j)) & \textit{if } (l_i \wedge l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \wedge l_j)|B] & \wedge\ CNF((l_i \wedge l_j) \rightarrow B) & \textit{if } (l_i \wedge l_j) \textit{ neg.} \\
\varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\ CNF(B \rightarrow (l_i \leftrightarrow l_j)) & \textit{if } (l_i \leftrightarrow l_j) \textit{ pos.} \\
\varphi & \Longrightarrow & \varphi[(l_i \leftrightarrow l_j)|B] & \wedge\ CNF((l_i \leftrightarrow l_j) \rightarrow B) & \textit{if } (l_i \leftrightarrow l_j) \textit{ neg.}
\end{array}$$

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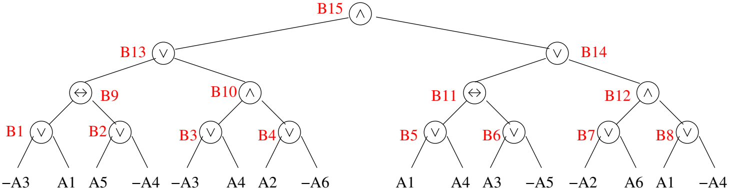
CNF(B \rightarrow (l_i \vee l_j)) & = (\neg B \vee l_i \vee l_j) \\
CNF(((l_i \vee l_j) \rightarrow B)) & = (\neg l_i \vee B) \wedge (\neg l_j \vee B)
\end{array}$$

Labeling CNF conversion $CNF_{label}(\varphi)$ (cont.)

$CNF(B \rightarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i \vee l_j)$
$CNF(B \leftarrow (l_i \vee l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i) \wedge$ $(B \vee \neg l_j)$
$CNF(B \rightarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(\neg B \vee l_i) \wedge$ $(\neg B \vee l_j)$
$CNF(B \leftarrow (l_i \wedge l_j))$	$\Longleftrightarrow$	$(B \vee \neg l_i \neg l_j)$
$CNF(B \rightarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(\neg B \vee \neg l_i \vee l_j) \wedge$ $(\neg B \vee l_i \vee \neg l_j)$
$CNF(B \leftarrow (l_i \leftrightarrow l_j))$	$\Longleftrightarrow$	$(B \vee l_i \vee l_j) \wedge$ $(B \vee \neg l_i \vee \neg l_j)$

Labeling CNF conversion CNF_{label} – example



Basic

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4)) \quad \wedge$

$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8)) \quad \wedge$

$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10})) \quad \wedge$

$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12})) \quad \wedge$

$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14})) \quad \wedge$

B_{15}

Improved

$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_8 \rightarrow (A_1 \vee \neg A_4)) \quad \wedge$

$CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2)) \quad \wedge$

$\ldots \quad \wedge$

$CNF(B_{12} \rightarrow (B_7 \wedge B_8)) \quad \wedge$

$CNF(B_{13} \rightarrow (B_9 \vee B_{10})) \quad \wedge$

$CNF(B_{14} \rightarrow (B_{11} \vee B_{12})) \quad \wedge$

$CNF(B_{15} \rightarrow (B_{13} \wedge B_{14})) \quad \wedge$

B_{15}

Labeling CNF conversion CNF_{label} – further optimizations

- Do not apply CNF_{label} when not necessary:
  (e.g., $CNF_{label}(\varphi_1 \wedge \varphi_2) \Longrightarrow CNF_{label}(\varphi_1) \wedge \varphi_2$,
  if φ_2 already in CNF)
- Apply Demorgan's rules where it is more effective: (e.g.,
  $CNF_{label}(\varphi_1 \wedge (A \rightarrow (B \wedge C))) \Longrightarrow CNF_{label}(\varphi_1) \wedge (\neg A \vee B) \wedge (\neg A \vee C)$
- exploit the associativity of $\wedge$'s and $\vee$'s:
  $\ldots \underbrace{(A_1 \vee (A_2 \vee A_3))}_{B} \ldots \Longrightarrow \ldots CNF(B \leftrightarrow (A_1 \vee A_2 \vee A_3))\ldots$
- before applying CNF_{label}, rewrite the initial formula so that to maximize the sharing of subformulas (RBC, BED)
- ...

Outline

Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space (draw one line at a time).
- Requires analyzing $2^{|Atoms(\varphi)|}$ lines.
- Never used in practice.

Resolution [49, 15]

- Search for a refutation of φ
- φ is represented as a set of clauses
- Applies iteratively the resolution rule to pairs of clauses containing a conflicting literal, until a false clause is generated or the resolution rule is no more applicable
- Many different strategies

Resolution Rule

- Resolution of a pair of clauses with exactly one incompatible variable:

$$\frac{(\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{l}^{resolvent} \vee \overbrace{l'_{k+1} \vee \ldots \vee l'_m}^{C'}) \qquad (\overbrace{l_1 \vee \ldots \vee l_k}^{common} \vee \overbrace{\neg l}^{resolvent} \vee \overbrace{l''_{k+1} \vee \ldots \vee l''_n}^{C''})}{(\underbrace{l_1 \vee \ldots \vee l_k}_{common} \vee \underbrace{l'_{k+1} \vee \ldots \vee l'_m}_{C'} \vee \underbrace{l''_{k+1} \vee \ldots \vee l''_n}_{C''})}$$

- EXAMPLE:

$$\frac{(A \vee B \vee C \vee D \vee E) \qquad (A \vee B \vee \neg C \vee F)}{(A \vee B \vee D \vee E \vee F)}$$

- NOTE: many standard inference rules subcases of resolution:

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}\ (Transit.) \qquad \frac{A \quad A \rightarrow B}{B}\ (M.\ Ponens) \qquad \frac{\neg B \quad A \rightarrow B}{\neg A}$$

Resolution Rules [15, 14]: unit propagation

- Unit resolution:

$$\frac{\Gamma' \wedge (l) \wedge (\neg l \vee \bigvee_i l_i)}{\Gamma' \wedge (l) \wedge (\bigvee_i l_i)}$$

- Unit subsumption:

$$\frac{\Gamma' \wedge (l) \wedge (l \vee \bigvee_i l_i)}{\Gamma' \wedge (l)}$$

- Unit propagation = unit resolution + unit subsumption

"Deterministic" rule: applied before other "non-deterministic" rules!

Resolution: basic strategy [15]

```
function DP(Γ)
    if $\bot \in \Gamma$                                   /* unsat */
        then return False;
    if (Resolve() is no more applicable to Γ)   /* sat      */
        then return True;
    if {a unit clause (l) occurs in Γ}            /* unit     */
        then  $\Gamma := Unit\_Propagate(l, \Gamma)$);
        return   DP(Γ)
    A := select-variable(Γ);                     /* resolve  */
    $\Gamma = \Gamma \cup \bigcup_{A \in C', \neg A \in C''} \{Resolve(C', C'')\} \setminus \bigcup_{A \in C', \neg A \in C''} \{C', C''\}$;
    return   DP(Γ)
```

Hint: drops one variable $A \in Atoms(\Gamma)$ at a time

Resolution: Examples

$$(A_1 \vee A_2)\ \ (A_1 \vee \neg A_2)\ \ (\neg A_1 \vee A_2)\ \ (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2)\ \ (A_2 \vee \neg A_2)\ \ (\neg A_2 \vee A_2)\ \ (\neg A_2)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A_1 \vee A_2) \;\; (A_1 \vee \neg A_2) \;\; (\neg A_1 \vee A_2) \;\; (\neg A_1 \vee \neg A_2)$$

$$\Downarrow$$

$$(A_2) \;\; (A_2 \vee \neg A_2) \;\; (\neg A_2 \vee A_2) \;\; (\neg A_2)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A_1 \vee A_2)\ \ (A_1 \vee \neg A_2)\ \ (\neg A_1 \vee A_2)\ \ (\neg A_1 \vee \neg A_2)$$
$$\Downarrow$$
$$(A_2)\ \ (A_2 \vee \neg A_2)\ \ (\neg A_2 \vee A_2)\ \ (\neg A_2)$$
$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$
\begin{array}{c}
(A_1 \lor A_2)\ \ (A_1 \lor \neg A_2)\ \ (\neg A_1 \lor A_2)\ \ (\neg A_1 \lor \neg A_2) \\
\Downarrow \\
(A_2)\ \ (A_2 \lor \neg A_2)\ \ (\neg A_2 \lor A_2)\ \ (\neg A_2) \\
\Downarrow \\
\bot
\end{array}
$$

$\Longrightarrow$ UNSAT

Resolution: Examples (cont.)

$$(A \vee B \vee C) \;\; (B \vee \neg C \vee \neg F) \;\; (\neg B \vee E)$$

$$\Downarrow$$

$$(A \vee C \vee E) \;\; (\neg C \vee \neg F \vee E)$$

$$\Downarrow$$

$$(A \vee E \vee \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$(A \vee B \vee C) \;\; (B \vee \neg C \vee \neg F) \;\; (\neg B \vee E)$$

$$\Downarrow$$

$$(A \vee C \vee E) \;\; (\neg C \vee \neg F \vee E)$$

$$\Downarrow$$

$$(A \vee E \vee \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$
\begin{array}{c}
(A \vee B \vee C) \;\; (B \vee \neg C \vee \neg F) \;\; (\neg B \vee E) \\
\Downarrow \\
(A \vee C \vee E) \;\; (\neg C \vee \neg F \vee E) \\
\Downarrow \\
(A \vee E \vee \neg F)
\end{array}
$$

$\Longrightarrow$ SAT

Resolution: Examples (cont.)

$$(A \vee B \vee C)\ \ (B \vee \neg C \vee \neg F)\ \ (\neg B \vee E)$$
$$\Downarrow$$
$$(A \vee C \vee E)\ \ (\neg C \vee \neg F \vee E)$$
$$\Downarrow$$
$$(A \vee E \vee \neg F)$$

$\Longrightarrow$ SAT

Resolution: Examples

$$(A \lor B) \;\; (A \lor \neg B) \;\; (\neg A \lor C) \;\; (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(A) \;\; (\neg A \lor C) \;\; (\neg A \lor \neg C)$$
$$\Downarrow$$
$$(C) \;\; (\neg C)$$
$$\Downarrow$$
$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$(A \vee B)\ \ (A \vee \neg B)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(A)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C)$$

$$\Downarrow$$

$$(C)\ \ (\neg C)$$

$$\Downarrow$$

$$\bot$$

$\Longrightarrow$ UNSAT

Resolution: Examples

$$
\begin{array}{c}
(A \vee B)\ \ (A \vee \neg B)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C) \\
\Downarrow \\
(A)\ \ (\neg A \vee C)\ \ (\neg A \vee \neg C) \\
\Downarrow \\
(C)\ \ (\neg C) \\
\Downarrow \\
\bot
\end{array}
$$

$\Longrightarrow$ UNSAT

Resolution – summary

- Requires CNF
- Γ may blow up
  $\Longrightarrow$ May require exponential space
- Not very much used in Boolean reasoning (unless integrated with DPLL procedure in recent implementations)

Semantic tableaux [55]

- Search for an assignment satisfying φ
- applies recursively elimination rules to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_i$) for some i, the branch is closed, otherwise it is open.
- if no rule can be applied to an open branch μ, then $\mu \models \varphi$;
- if all branches are closed, the formula is not satisfiable;

Tableau elimination rules

$$\frac{\varphi_1 \wedge \varphi_2}{\begin{array}{c}\varphi_1\\ \varphi_2\end{array}} \quad \frac{\neg(\varphi_1 \vee \varphi_2)}{\begin{array}{c}\neg\varphi_1\\ \neg\varphi_2\end{array}} \quad \frac{\neg(\varphi_1 \to \varphi_2)}{\begin{array}{c}\varphi_1\\ \neg\varphi_2\end{array}} \qquad (\wedge\text{-elimination})$$

$$\frac{\neg\neg\varphi}{\varphi} \qquad (\neg\neg\text{-elimination})$$

$$\frac{\varphi_1 \vee \varphi_2}{\varphi_1 \qquad \varphi_2} \quad \frac{\neg(\varphi_1 \wedge \varphi_2)}{\neg\varphi_1 \qquad \neg\varphi_2} \quad \frac{\varphi_1 \to \varphi_2}{\neg\varphi_1 \qquad \varphi_2} \qquad (\vee\text{-elimination})$$

$$\frac{\varphi_1 \leftrightarrow \varphi_2}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \varphi_2 & \neg\varphi_2\end{array}} \quad \frac{\neg(\varphi_1 \leftrightarrow \varphi_2)}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \neg\varphi_2 & \varphi_2\end{array}} \qquad (\leftrightarrow\text{-elimination}).$$

Semantic Tableaux – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

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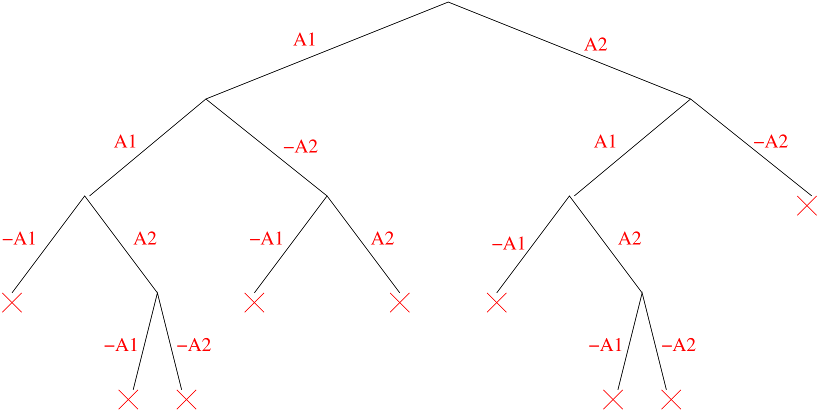


Tableau algorithm

```
function Tableau($\Gamma$)
    if $A_i \in \Gamma$ and $\neg A_i \in \Gamma$                          /* branch closed */
        then return False;
    if $(\varphi_1 \wedge \varphi_2) \in \Gamma$                                  /* $\wedge$-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1, \varphi_2\} \backslash \{(\varphi_1 \wedge \varphi_2)\}$);
    if $(\neg\neg\varphi_1) \in \Gamma$                                      /* $\neg\neg$-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1\} \backslash \{(\neg\neg\varphi_1)\}$);
    if $(\varphi_1 \vee \varphi_2) \in \Gamma$                                  /* $\vee$-elimination */
        then return   Tableau($\Gamma \cup \{\varphi_1\} \backslash \{(\varphi_1 \vee \varphi_2)\}$)   or
                      Tableau($\Gamma \cup \{\varphi_2\} \backslash \{(\varphi_1 \vee \varphi_2)\}$);
    ...
    return True;                                             /* branch expanded */
```

Semantic Tableaux – summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend
  $\Longrightarrow$ loved by logicians.
- Rather inefficient
  $\Longrightarrow$ avoided by computer scientists.
- Requires polynomial space

DPLL [15, 14]

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build an assignment μ satisfying φ;
- At each step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL rules

$$\frac{\varphi_1 \wedge (l)}{\varphi_1[l|\top]} \; (Unit)$$

$$\frac{\varphi}{\varphi[l|\top]} \; (l\ Pure)$$

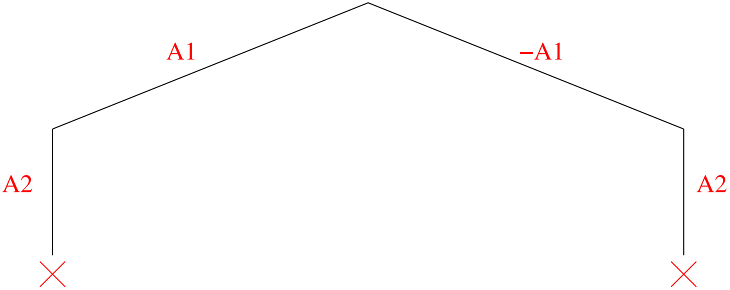
$$\frac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} \; (split)$$

(l is a pure literal in φ iff it occurs only positively).

- Split applied if and only if the others cannot be applied.
- Richer formalisms described in [57, 44, 45]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



DPLL Algorithm

```
function DPLL($\varphi, \mu$)
    if $\varphi = \top$                                    /* base      */
        then return True;
    if $\varphi = \bot$                                    /* backtrack */
        then return False;
    if {a unit clause ($l$) occurs in $\varphi$}           /* unit      */
        then return DPLL(assign($l, \varphi$), $\mu \wedge l$);
    if {a literal $l$ occurs pure in $\varphi$}            /* pure      */
        then return DPLL(assign($l, \varphi$), $\mu \wedge l$);
    $l$ := choose-literal($\varphi$);                      /* split     */
    return  DPLL(assign($l, \varphi$), $\mu \wedge l$)  or
            DPLL(assign($\neg l, \varphi$), $\mu \wedge \neg l$);
```

DPLL – summary

- Handles CNF formulas (non-CNF variant known [2, 25]).
- Branches on truth values
  $\Longrightarrow$ all instances of an atom assigned simultaneously
- Postpones branching as much as possible.
- Mostly ignored by logicians.
- (The grandfather of) the most efficient SAT algorithms
  $\Longrightarrow$ loved by computer scientists.
- Requires polynomial space
- Choose_literal() critical!
- Many very efficient implementations [61, 54, 4, 43].

Ordered Binary Decision Diagrams (OBDDs) [12]]

Canonical representation of Boolean formulas

- "If-then-else" binary direct acyclic graphs (DAGs) with one root and two leaves: 1, 0 (or $\top$,$\bot$; or T, F)
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priori.
- Paths leading to 1 represent models
  Paths leading to 0 represent counter-models

Note

Some authors call them Reduced Ordered Binary Decision Diagrams (ROBDDs)

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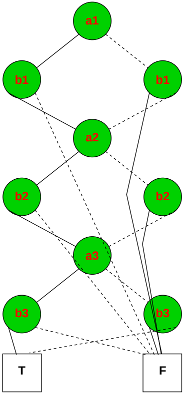
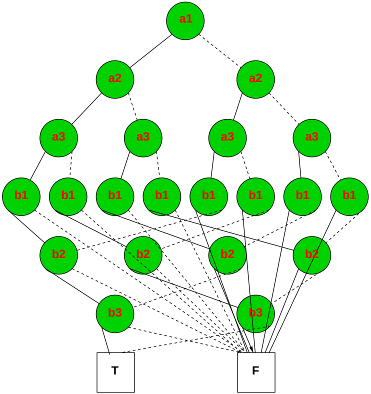
OBDD - Examples

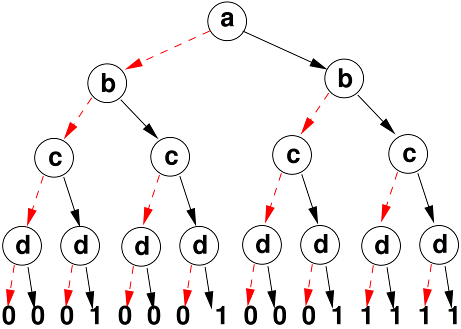




OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Ordered Decision Trees

- Ordered Decision Tree: from root to leaves, variables are encountered always in the same order
- Example: Ordered Decision tree for $\varphi = (a \land b) \lor (c \land d)$



From Ordered Decision Trees to OBDD’s: reductions

- Recursive applications of the following reductions:
  - share subnodes: point to the same occurrence of a subtree (via hash consing)
  - remove redundancies: nodes with same left and right children can be eliminated (“if A then B else B” $\Longrightarrow$ “B”)

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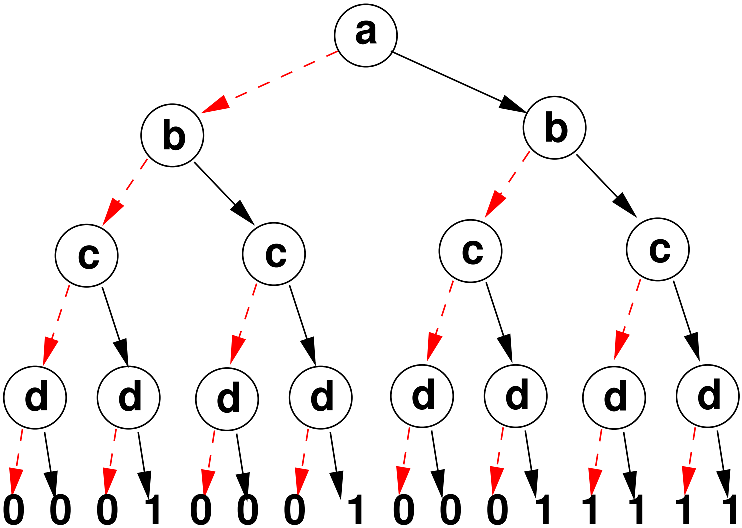
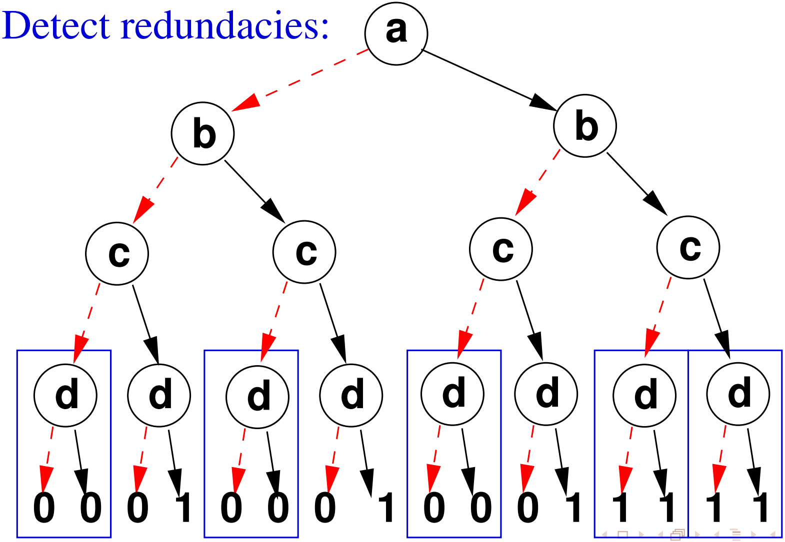
Reduction: example



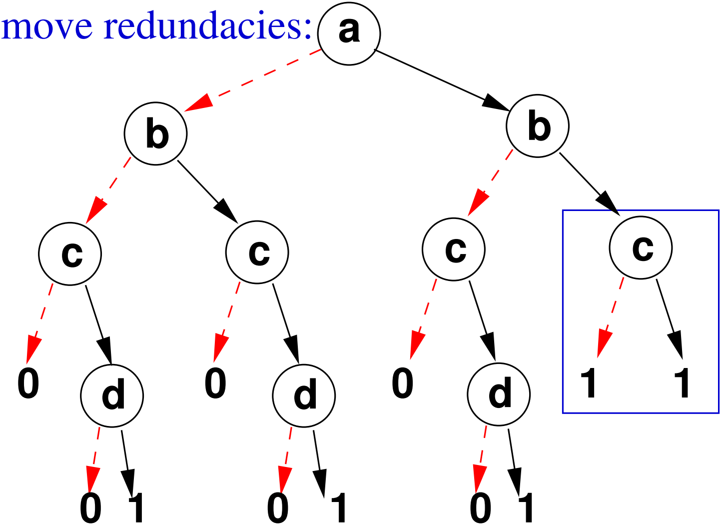
Reduction: example

Detect redundacies:



Reduction: example

Remove redundacies:



Reduction: example

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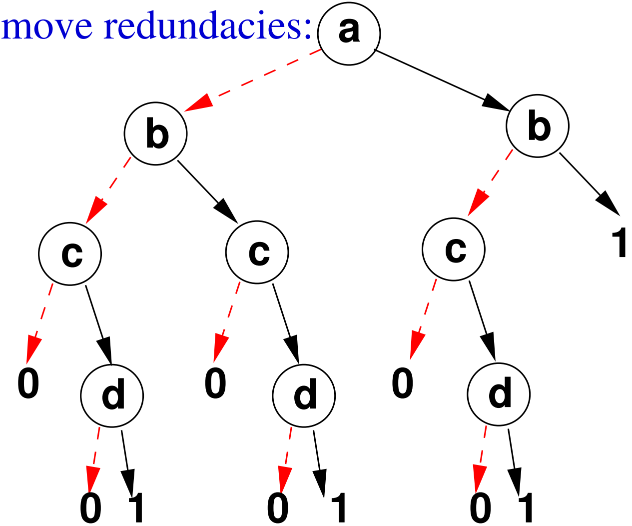
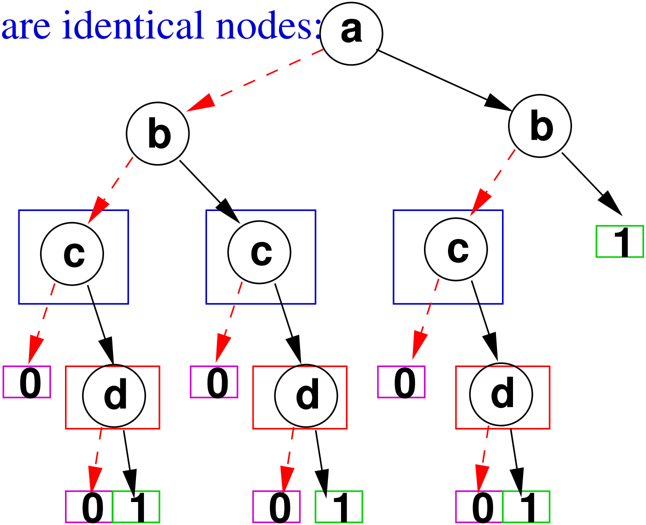


Reduction: example

Share identical nodes:



Reduction: example

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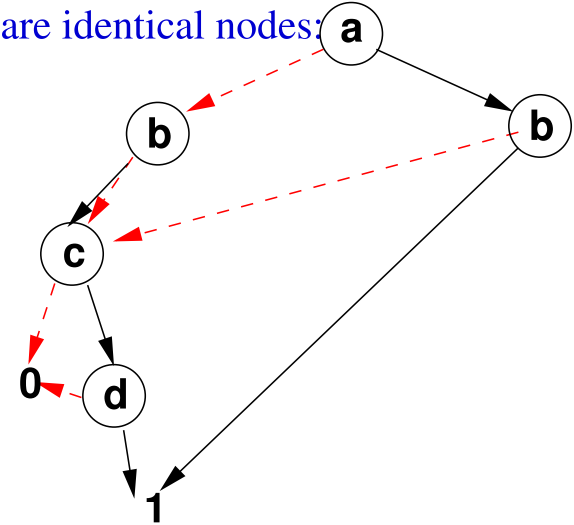

Reduction: example

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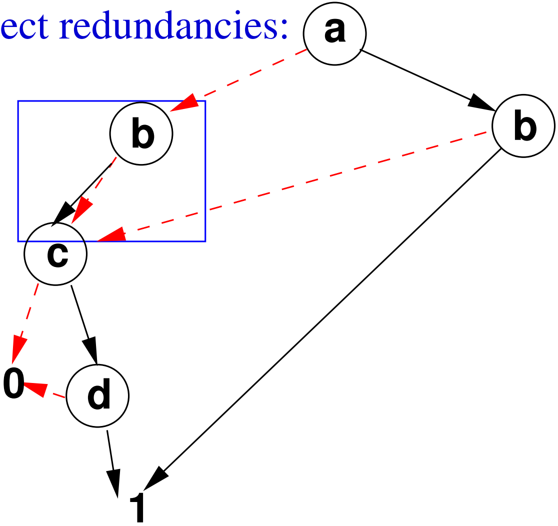
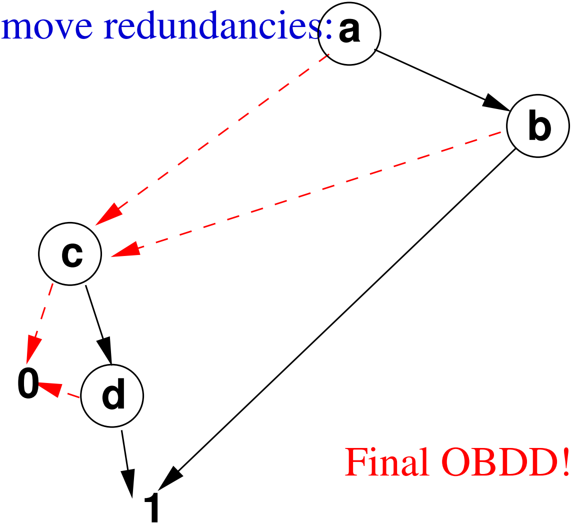


Reduction: example

Remove redundancies:



Final OBDD!

Recursive structure of an OBDD

Assume the variable ordering $A_1, A_2, ..., A_n$:

$$\begin{array}{lcl} OBDD(\top, \{A_1, A_2, ..., A_n\}) & = & 1 \\ OBDD(\bot, \{A_1, A_2, ..., A_n\}) & = & 0 \\ OBDD(\varphi, \{A_1, A_2, ..., A_n\}) & = & \textit{if } A_1 \\ & & \textit{then } OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\}) \\ & & \textit{else } OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\}) \end{array}$$

Incrementally building an OBDD

- $obdd\_build(\top, \{...\}) := 1$,
- $obdd\_build(\bot, \{...\}) := 0$,
- $obdd\_build(A_i, \{...\}) := ite(A_i, 1, 0)$,
- $obdd\_build((\neg\varphi), \{A_1, ..., A_n\}) :=$
  $apply(\neg, obdd\_build(\varphi, \{A_1, ..., A_n\}))$
- $obdd\_build((\varphi_1\ op\ \varphi_2), \{A_1, ..., A_n\}) :=$
  $reduce($
  $apply(\quad op,$
  $obdd\_build(\varphi_1, \{A_1, ..., A_n\}),\quad op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
  $obdd\_build(\varphi_2, \{A_1, ..., A_n\})$
  $))$

"$ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot})$" is "If A_i Then $\varphi_i^{\top}$ Else $\varphi_i^{\bot}$"

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- $obdd\_build((\varphi_1\ op\ \varphi_2), \{A_1, ..., A_n\}) :=$

```
reduce(
  apply(   op,
           obdd_build(φ1, {A1, ..., An}),   op ∈ {∧, ∨, →, ↔}
           obdd_build(φ2, {A1, ..., An})
))
```

"$ite(A_i, \varphi_i^\top, \varphi_i^\bot)$" is "If A_i Then $\varphi_i^\top$ Else $\varphi_i^\bot$"

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$$\begin{array}{lll} reduce(& & \\ \quad apply(& op, & \\ & obdd\_build(\varphi_1, \{A_1, ..., A_n\}), & op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ & obdd\_build(\varphi_2, \{A_1, ..., A_n\}) & \\ )\,) & & \end{array}$$

"$ite(A_i, \varphi_i^\top, \varphi_i^\bot)$" is "If A_i Then $\varphi_i^\top$ Else $\varphi_i^\bot$"

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```
reduce(
  apply(   op,
           $obdd\_build(\varphi_1, \{A_1, ..., A_n\})$,   $op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
           $obdd\_build(\varphi_2, \{A_1, ..., A_n\})$
) )
```

"$ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot})$" is "If A_i Then $\varphi_i^{\top}$ Else $\varphi_i^{\bot}$"

Incrementally building an OBDD (cont.)

- $apply\ (op,\ O_i, O_j) := (O_i\ op\ O_j)$ **if** $(O_i, O_j \in \{1, 0\})$
- $apply\ (\neg,\ ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot})) :=$
  $ite(A_i, apply(\neg, \varphi_i^{\top}), apply(\neg, \varphi_i^{\bot}))$
- $apply\ (op,\ ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot}),\ ite(A_j, \varphi_j^{\top}, \varphi_j^{\bot})) :=$
  if $(A_i = A_j)$ **then** $ite(A_i,\ apply\ (op, \varphi_i^{\top}, \varphi_j^{\top}),$
  $apply\ (op, \varphi_i^{\bot}, \varphi_j^{\bot}))$
  if $(A_i < A_j)$ **then** $ite(A_i,\ apply\ (op, \varphi_i^{\top}, ite(A_j, \varphi_j^{\top}, \varphi_j^{\bot})),$
  $apply\ (op, \varphi_i^{\bot}, ite(A_j, \varphi_j^{\top}, \varphi_j^{\bot})))$
  if $(A_i > A_j)$ **then** $ite(A_j,\ apply\ (op, ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot}), \varphi_j^{\top}),$
  $apply\ (op, ite(A_i, \varphi_i^{\top}, \varphi_i^{\bot}), \varphi_j^{\bot}))$

$op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Incrementally building an OBDD (cont.)

- $apply\ (op,\ O_i, O_j) := (O_i\ op\ O_j)$ **if** $(O_i, O_j \in \{1, 0\})$
- $apply\ (\neg,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot)) :=$
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- $apply\ (op,\ ite(A_i, \varphi_i^\top, \varphi_i^\bot),\ ite(A_j, \varphi_j^\top, \varphi_j^\bot)) :=$
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Incrementally building an OBDD (cont.)

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  if $(A_i > A_j)$ **then** $ite(A_j,\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\top),\ apply\ (op, ite(A_i, \varphi_i^\top, \varphi_i^\bot), \varphi_j^\bot))$

$op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Incrementally building an OBDD (cont.)

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$\begin{aligned}
& apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\
= \quad & ite(A_1,\ apply(\vee, \top, ite(A_1, \top, \bot)),\ apply(\vee, \bot, ite(A_2, \top, \bot))\ ) \\
= \quad & ite(A_1,\ \top,\ ite(A_2, \top, \bot)\ )
\end{aligned}$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2)$, $(A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$\begin{aligned}
& apply(\wedge,\ \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)},\ \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)}, \\
= \quad & ite(A_1,\ apply(\wedge, \top, \top),\ apply(\wedge,\ ite(A_2, \top, \bot),\ ite(A_2, \bot, \top)) \\
= \quad & ite(A_1,\ \top,\ ite(A_2,\ apply(\wedge, \top, \bot),\ apply(\wedge, \bot, \top))) \\
= \quad & ite(A_1,\ \top,\ ite(A_2,\ \bot,\ \bot)) \\
= \quad & ite(A_1,\ \top,\ \bot)
\end{aligned}$$

Incrementally building an OBDD (cont.)

- Ex: build the obdd for $A_1 \vee A_2$ from those of A_1, A_2 (order: A_1, A_2):

$$\begin{aligned} & apply(\vee, \overbrace{ite(A_1, \top, \bot)}^{A_1}, \overbrace{ite(A_2, \top, \bot)}^{A_2}) \\ = \quad & ite(A_1,\ apply(\vee, \top, ite(A_1, \top, \bot)),\ apply(\vee, \bot, ite(A_2, \top, \bot))\ ) \\ = \quad & ite(A_1,\ \top,\ ite(A_2, \top, \bot)\ ) \end{aligned}$$

- Ex: build the obdd for $(A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)$ from those of $(A_1 \vee A_2),\ (A_1 \vee \neg A_2)$ (order: A_1, A_2):

$$\begin{aligned} & apply(\wedge,\ \overbrace{ite(A_1, \top, ite(A_2, \top, \bot))}^{(A_1 \vee A_2)},\ \overbrace{ite(A_1, \top, ite(A_2, \bot, \top))}^{(A_1 \vee \neg A_2)}, \\ = \quad & ite(A_1,\ apply(\wedge, \top, \top),\ apply(\wedge,\ ite(A_2, \top, \bot),\ ite(A_2, \bot, \top)) \\ = \quad & ite(A_1,\ \top,\ ite(A_2,\ apply(\wedge, \top, \bot),\ apply(\wedge, \bot, \top))) \\ = \quad & ite(A_1,\ \top,\ ite(A_2,\ \bot,\ \bot)) \\ = \quad & ite(A_1,\ \top,\ \bot) \end{aligned}$$

OBBD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

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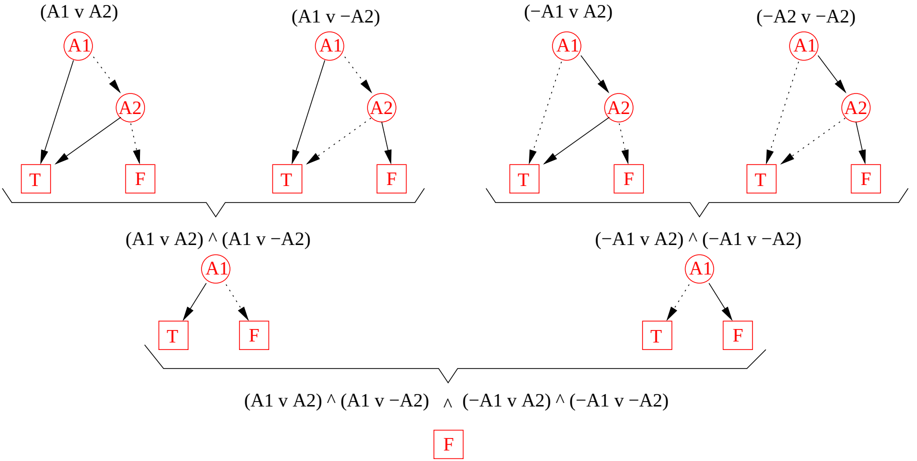

Critical choice of variable Orderings in OBDD's

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$$

Linear size

Exponential size

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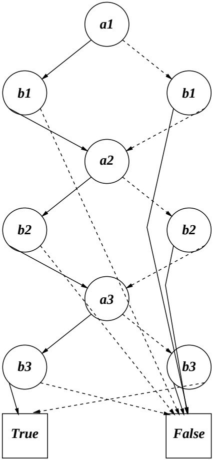

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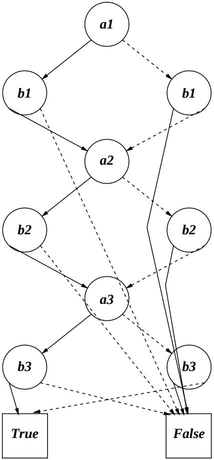

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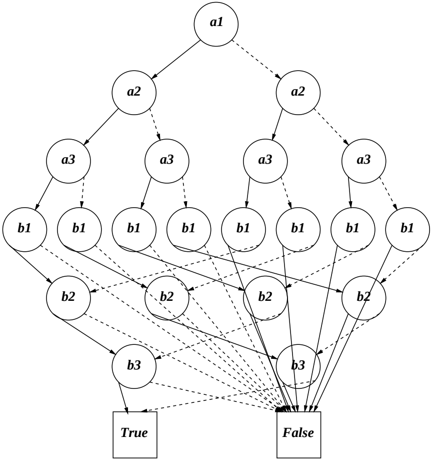

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OBDD's as canonical representation of Boolean formulas

- An OBDD is a canonical representation of a Boolean formula: once the variable ordering is established, equivalent formulas are represented by the same OBDD:

$$\varphi_1 \leftrightarrow \varphi_2 \iff OBDD(\varphi_1) = OBDD(\varphi_2)$$

- equivalence check requires constant time!
  $\Longrightarrow$validity check requires constant time! ($\varphi \leftrightarrow \top$)
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Exponentiality of OBDD's

- The size of OBDD's may grow exponentially wrt. the number of variables in worst-case
- Consequence of the canonicity of OBDD's (unless P = co-NP)
- Example: there exist no polynomial-size OBDD representing the electronic circuit of a bitwise multiplier

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- the equivalence check between two OBDDs is simple
    - are they the same OBDD? (⟹ constant time)
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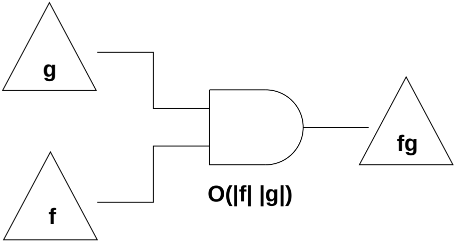

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Boolean quantification

Shannon's expansion:

- If v is a Boolean variable and f is a Boolean formula, then
  $$\begin{aligned}\exists v.f &:= f|_{v=0} \vee f|_{v=1}\\ \forall v.f &:= f|_{v=0} \wedge f|_{v=1}\end{aligned}$$
- v does no more occur in $\exists v.f$ and $\forall v.f$!!
- Multi-variable quantification: $\exists(w_1,\ldots,w_n).f \;:=\; \exists w_1 \ldots \exists w_n.f$

- Intuition:
  - $\mu \models \exists v.f$ iff exists $tvalue \in \{\top,\bot\}$ s.t. $\mu \cup \{v := tvalue\} \models f$
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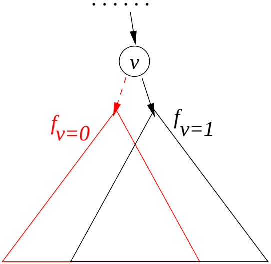
OBDD's and Boolean quantification

- OBDD's handle quantification operations quite efficiently
  - if f is a sub-OBDD labeled by variable v, then $f|_{v=1}$ and $f|_{v=0}$ are the "then" and "else" branches of f



$\Longrightarrow$ lots of sharing of subformulae!

OBDD – summary

- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
- Canonical representation of a Boolean formula.
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Represents all models and counter-models of the formula.
- Require exponential space in worst-case
- Very efficient for some practical problems (circuits, symbolic model checking).

Incomplete SAT techniques: GSAT, WSAT [53, 52]

- Hill-Climbing techniques: GSAT, WSAT
- looks for a complete assignment;
- starts from a random assignment;
- Greedy search: looks for a better “neighbor” assignment
- Avoid local minima: restart & random walk

The GSAT algorithm [53]

```
function GSAT($\varphi$)
    for $i := 1$ to Max-tries do
        $\mu$ := rand-assign($\varphi$);
        for $j := 1$ to Max-flips do
            if ($score(\varphi, \mu) = 0$)
                then return True;
                else Best-flips := hill-climb($\varphi, \mu$);
                    $A_i$ := rand-pick(Best-flips);
                    $\mu$ := flip($A_i, \mu$);
        end
    end
    return "no satisfying assignment found".
```

The WalkSAT algorithm(s) [52]

```
function WalkSAT($\varphi$)
    for $i := 1$ to Max-tries do
        $\mu$ := rand-assign($\varphi$);
        for $j := 1$ to Max-flips do
            if ($score(\varphi, \mu) = 0$)
                then return True;
                else C := randomly-pick-clause(unsat-clauses($\varphi, \mu$));
                    $A_i$ := heuristically-select-variable(C);
                    $\mu$ := flip($A_i, \mu$);
        end
    end
    return "no satisfying assignment found".
```

- many variants available [27, 58, 5]

SLS SAT solvers – summary

- Handle only CNF formulas.
- Incomplete
- Extremely efficient for some (satisfiable) problems.
- Require polynomial space
- Used in Artificial Intelligence (e.g., planning)
- Lots of variants (see e.g. [31])
- Non-CNF Variants: [50, 51, 6]

Outline

1. Basics on SAT
2. Basic SAT-Solving techniques
3. Modern CDCL SAT Solvers
    - Conflict-Driven Clause-Learning SAT solvers
    - Further Improvements
4. Tractable subclasses of SAT
5. Random k-SAT and Phase Transition
6. Advanced Functionalities: proofs, unsat cores, interpolants, optimization
7. Some Applications
    - Appl. #1: (Bounded) Planning
    - Appl. #2: Bounded Model Checking

Variants of DPLL

DPLL is a family of algorithms.

- backjumping & learning
- preprocessing: (subsumption, 2-simplification, resolution)
- different branching heuristics
- restarts
- (horn relaxation)
- ...

“Classic” chronological backtracking

DPLL implements “classic” chronological backtracking:

- variable assignments (literals) stored in a stack
- each variable assignments labeled as “unit”, “open”, “closed”
- when a conflict is encountered, the stack is popped up to the most recent open assignment l
- l is toggled, is labeled as “closed”, and the search proceeds.

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
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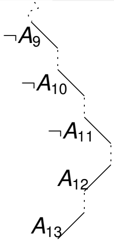

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(initial assignment)

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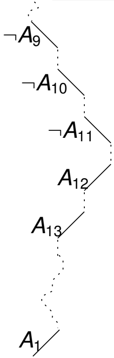

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (branch on A_1)

Classic chronological backtracking – example

$c_1 : \neg A_1 \lor A_2$ ✓

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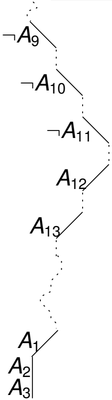

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$

(unit A_2, A_3)

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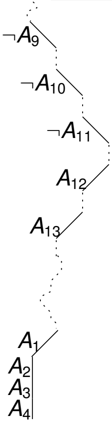

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4\}$
(unit A_4)

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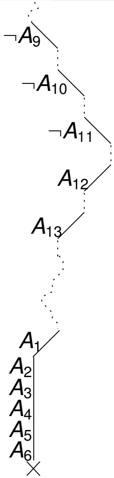

$\{..., \neg A_9, \neg A_{10}, \neg A_{1\neg A_4 1}, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4, A_5, A_6\}$

(unit A_5, A_6)$\Longrightarrow$ conflict

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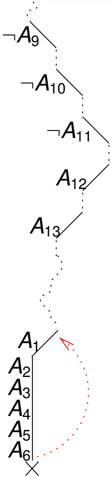

$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

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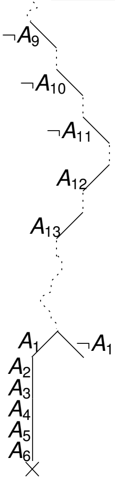
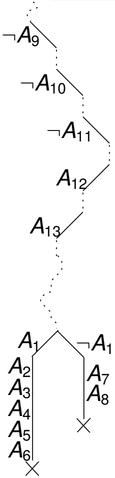


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1\}$
(unit $\neg A_1$)

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$ ✓
$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
$c_8 : A_1 \vee A_8$ ✓
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×
$\ldots$



$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1, A_7, A_8\}$
(unit A_7, A_8) $\Longrightarrow$ conflict

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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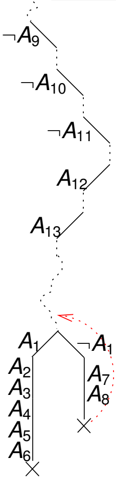
...



$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots\}$
$\Longrightarrow$ backtrack to the most recent open branching point

Classic chronological backtracking – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
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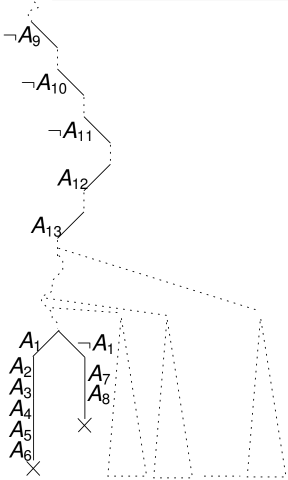
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ lots of useless search before backtracking up to A_{13}!

Classic chronological backtracking: drawbacks

- often the branch heuristic delays the “right” choice
- chronological backtracking always backtracks to the most recent branching point, even though a higher backtrack could be possible $\Longrightarrow$ lots of useless search!

Outline

1. Basics on SAT
2. Basic SAT-Solving techniques
3. Modern CDCL SAT Solvers
   - Conflict-Driven Clause-Learning SAT solvers
   - Further Improvements
4. Tractable subclasses of SAT
5. Random k-SAT and Phase Transition
6. Advanced Functionalities: proofs, unsat cores, interpolants, optimization
7. Some Applications
   - Appl. #1: (Bounded) Planning
   - Appl. #2: Bounded Model Checking

Conflict-Driven Clause-Learning (CDCL) SAT solvers

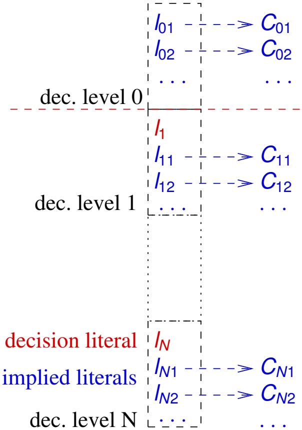
Conflict-Driven Clause-Learning (CDCL) SAT solvers [54, 43, 18, 37]

- Evolution of Davis-Putnam-Longeman-Loveland (DPLL) [15, 14]
- non-recursive: stack-based representation of data structures
- Perform conflict-directed backtracking (backjumping) and learning
- efficient data structures for doing and undoing assignments (e.g., two-watched-literal scheme)
- perform search restarts
- ...

Dramatically efficient: solve industrial-derived problems with $\approx 10^7$ Boolean variables and $\approx 10^7 - 10^8$ clauses!

Stack-based representation of a truth assignment μ

- assign one truth-value at a time (add one literal to a stack representing μ)
- stack partitioned into decision levels:
  - one decision literal
  - its implied literals
  - each implied literal tagged with the clause causing its unit-propagation (antecedent clause)
- equivalent to an implication graph



l_{01} → C_{01}
l_{02} → C_{02}
...   ...
dec. level 0

l_1
l_{11} → C_{11}
l_{12} → C_{12}
dec. level 1 ...   ...

decision literal l_N
implied literals l_{N1} → C_{N1}
l_{N2} → C_{N2}
dec. level N ...   ...

Implication graph

- An implication graph is a DAG s.t.:
    - each node represents a variable assignment (literal)
    - each edge $l_i \xmapsto{c} l$ is labeled with a clause
    - the node of a decision literal has no incoming edges
    - all edges incoming into a node l are labeled with the same clause c, s.t. $l_1 \xmapsto{c} l, ..., l_n \xmapsto{c} l$ iff $c = \neg l_1 \vee ... \vee \neg l_n \vee l$
      (c is said to be the antecedent clause of l)
    - when both l and $\neg l$ occur in the graph, we have a conflict.
- Intuition:
    - representation of the dependencies between literals in μ
    - the graph contains $l_1 \xmapsto{c} l, ..., l_n \xmapsto{c} l$ iff l has been obtained from $l_1, ..., l_n$ by unit propagation on c
    - a partition of the graph with all decision literals on one side and the conflict on the other represents a conflict set

Implication graph - example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

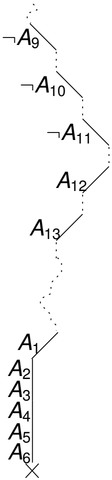
$c_6 : \neg A_5 \vee \neg A_6$ ×

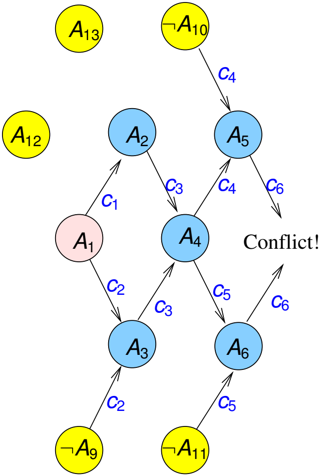
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





Unique implication point - UIP [63]

- A node I in an implication graph is an unique implication point (UIP) for the last decision level iff every path from the last decision node to both the conflict nodes passes through I.
    - the most recent decision node is an UIP (last UIP)
    - all other UIP's have been assigned after the most recent decision

Unique implication point - UIP - example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

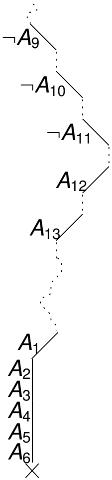
$c_6 : \neg A_5 \lor \neg A_6$ ×

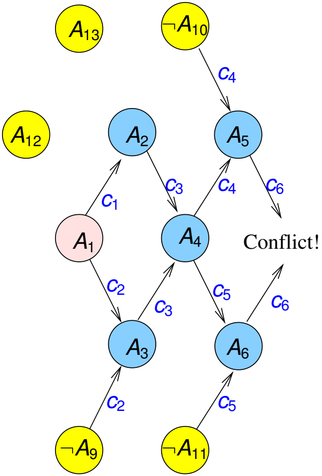
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...





- A_1 is the last UIP
- A_4 is the 1^{st} UIP

Schema of a CDCL DPLL solver [54, 64]

```
Function CDCL-SAT (formula: $\varphi$, assignment & $\mu$) {
        status := preprocess($\varphi,\mu$);
        while (1) {
            while (1) {
                status := deduce($\varphi,\mu$);
                if (status == Sat)
                    return Sat;
                if (status == Conflict) {
                    $\langle$blevel,$\eta\rangle$ := analyze_conflict($\varphi,\mu$);
                    //$\eta$ is a conflict set
                    if (blevel == 0)
                        return Unsat;
                    else backtrack(blevel,$\varphi,\mu$);
                }
                else break;
            }
            decide_next_branch($\varphi,\mu$);
}       }
```

Schema of a CDCL DPLL solver [54, 64]

- `preprocess(`φ, μ`)` simplifies φ into an easier equisatisfiable formula (and updates μ if it is the case)
- `decide_next_branch(`φ, μ`)` chooses a new decision literal from φ according to some heuristic, and adds it to μ
- `deduce(`φ, μ`)` performs all deterministic assignments (unit), and updates φ, μ accordingly.
- `analyze_conflict(`φ, μ`)` Computes the subset η of μ causing the conflict (conflict set), and returns the “wrong-decision” level suggested by η (“0” means that η is entirely assigned at level 0, i.e., a conflict exists even without branching);
- `backtrack(blevel,`φ, μ`)` undoes the branches up to blevel, and updates φ, μ accordingly

Example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...

Example

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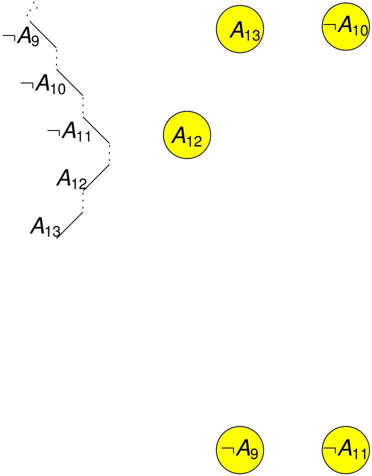
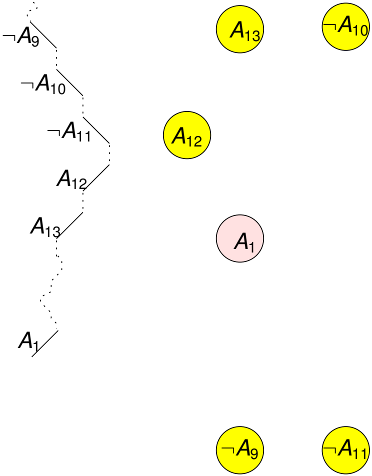


$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
(Initial assignment. Note: $c_1, ..., c_9$ inconsistent.)

Example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
$c_6 : \neg A_5 \vee \neg A_6$
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
$c_8 : A_1 \vee A_8$ ✓
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1\}$
... (decide A_1)

Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

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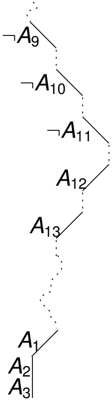
$c_6 : \neg A_5 \vee \neg A_6$

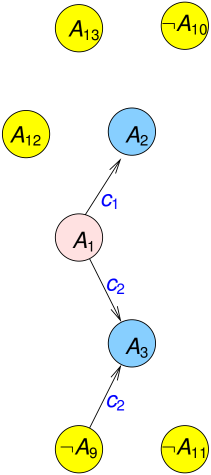
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., A_1, A_2, A_3\}$
(unit A_2, A_3)

Example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$

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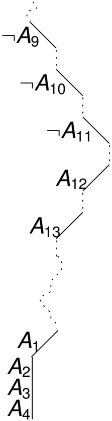
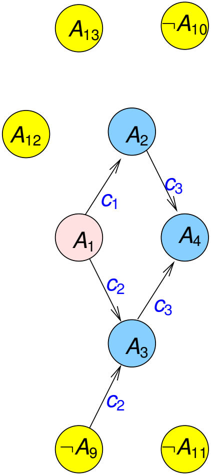
$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...





$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4\}$

(unit A_4)

Example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

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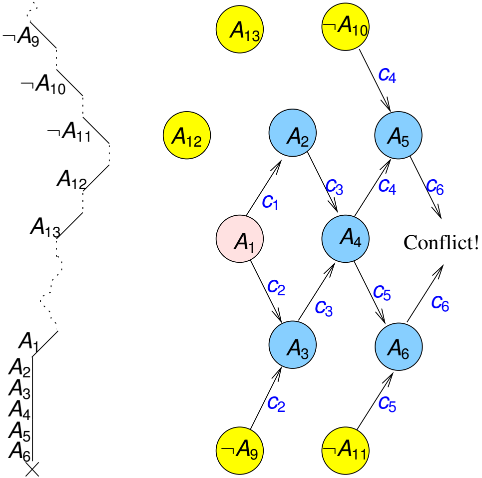
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{1}{}_{\neg A_4}1, A_{12}, A_{13}, ..., A_1, A_2, A_3, A_4, A_5, A_6\}$

(unit A_5, A_6) $\Longrightarrow$ conflict

Backjumping and learning: general ideas [4, 54]

- When a branch μ fails:
  (i) conflict analysis: reveal the sub-assignment $\eta \subseteq \mu$ causing the failure (conflict set η)
  (ii) learning: add the conflict clause $C \overset{\text{def}}{=} \neg\eta$ to the clause set
  (iii) backjumping: use η to decide the point where to backtrack
- may jump back up much more than one decision level in the stack $\Longrightarrow$ may avoid lots of redundant search!!.
- we illustrate two main backjumping & learning strategies:
  - the original strategy presented in [54]
  - the state-of-the-art 1^{st}UIP strategy of [63]

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

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criterium: decision

...until C contains only decision literals

$$\dfrac{\neg A_1 \vee A_2 \qquad \dfrac{\neg A_1 \vee A_3 \vee A_9 \qquad \dfrac{\neg A_2 \vee \neg A_3 \vee A_4 \qquad \dfrac{\neg A_4 \vee A_5 \vee A_{10} \qquad \dfrac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{\textit{Conflicting cl.}}}{\neg A_4 \vee \neg A_5 \vee A_{11}}\ (A_6)}{\neg A_4 \vee A_{10} \vee A_{11}}\ (A_5)}{\neg A_2 \vee \neg A_3 \vee A_{10} \vee A_{11}}\ (A_4)}{\neg A_2 \vee \neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}\ (A_3)}{\neg A_1 \vee A_9 \vee A_{10} \vee A_{11}}\ (A_2)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterium: last UIP

... until C contains only one literal assigned at current decision level: the decision literal (last UIP)

$$\dfrac{\neg A_1 \lor A_2 \qquad \dfrac{\neg A_1 \lor A_3 \lor A_9 \qquad \dfrac{\neg A_2 \lor \neg A_3 \lor A_4 \qquad \dfrac{\neg A_4 \lor A_5 \lor A_{10} \qquad \dfrac{\neg A_4 \lor A_6 \lor A_{11} \qquad \overbrace{\neg A_5 \lor \neg A_6}^{\mathit{Conflicting\ cl.}}}{\neg A_4 \lor \neg A_5 \lor A_{11}}\ (A_6)}{\neg A_4 \lor A_{10} \lor A_{11}}\ (A_5)}{\neg A_2 \lor \neg A_3 \lor A_{10} \lor A_{11}}\ (A_4)}{\neg A_2 \lor \neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\ (A_3)}{\neg A_1 \lor A_9 \lor A_{10} \lor A_{11}}\ (A_2)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

criterium: 1st UIP

... until C contains only one literal assigned at current decision level (1st UIP)

$$\dfrac{\neg A_4 \vee A_5 \vee A_{10} \qquad \dfrac{\neg A_4 \vee A_6 \vee A_{11} \qquad \overbrace{\neg A_5 \vee \neg A_6}^{Conflicting\ cl.}}{\neg A_4 \vee \neg A_5 \vee A_{11}}\ (A_6)}{\underbrace{\neg A_4}_{1st\ UIP} \vee A_{10} \vee A_{11}}\ (A_5)$$

Conflict analysis

1. $C :=$ falsified clause (conflicting clause)
2. repeat
   (i) resolve the current clause C with the antecedent clause of the last unit-propagated literal l in C

   until C verifies some given termination criteria

Note:

$\varphi \models C$, so that C can be safely added to C.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis

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$\varphi \models C$, so that C can be safely added to C.

Note:

Equivalent to finding a partition in the implication graph of μ with all decision literals on one side and the conflict on the other.

Conflict analysis and implication graph - example

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

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$c_6 : \neg A_5 \lor \neg A_6$ ×

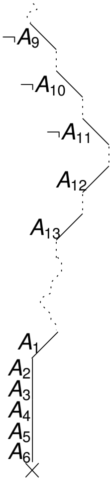
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

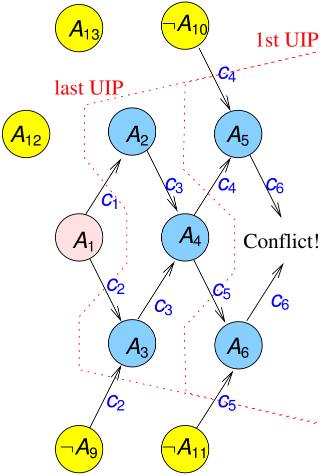
$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...

Note: in this case decision and last-UIP criteria produce the same partition



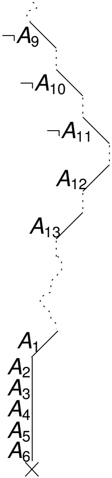


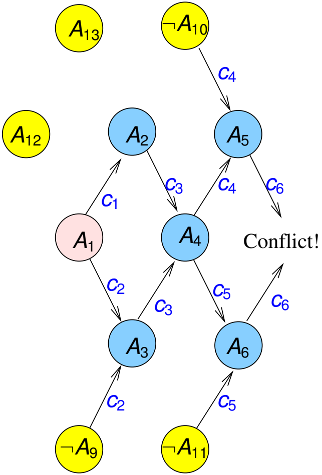
The original backjumping and learning strategy of [54]

- Idea: when a branch μ fails,
  - (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \overset{\text{def}}{=} \neg\eta$ via resolution from the falsified clause (conflicting clause) using the "Decision" criterion;
  - (ii) learning: add the conflict clause C to the clause set
  - (iii) backjumping: backtrack to the most recent branching point s.t. the stack does not fully contain η, and then unit-propagate the unassigned literal on C

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$ ✓
$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓
$c_6 : \neg A_5 \vee \neg A_6$ ×
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓
$c_8 : A_1 \vee A_8$ ✓
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
...





$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{1_{\neg A_4}1}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6) $\Longrightarrow$ conflict

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

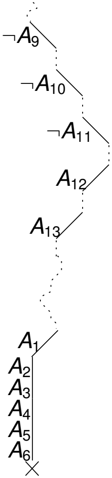
$c_6 : \neg A_5 \vee \neg A_6$ ×

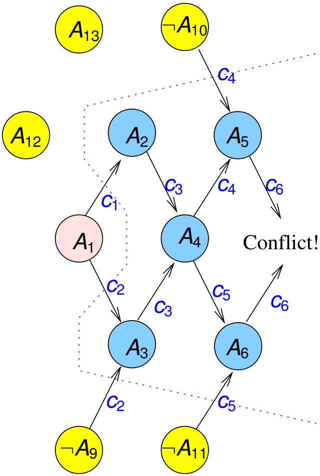
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





$\Longrightarrow$ Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ (last-UIP schema)

$\Longrightarrow$ learn the conflict clause $c_{10} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
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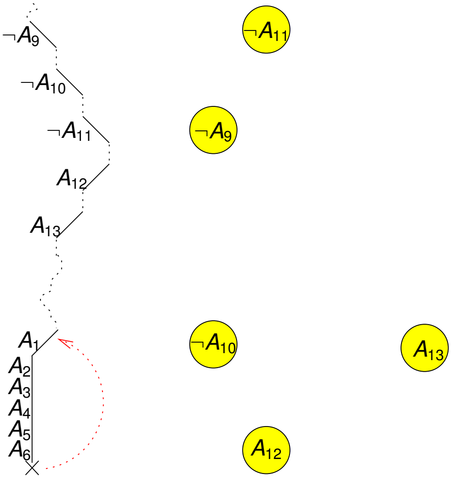
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$
$c_8 : A_1 \vee A_8$
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$
...



$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ...\}$
$\Longrightarrow$ backtrack up to A_1

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$ √

$c_2 : \neg A_1 \vee A_3 \vee A_9$ √

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

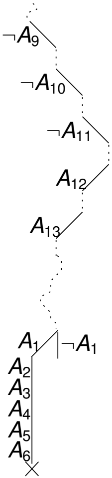
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

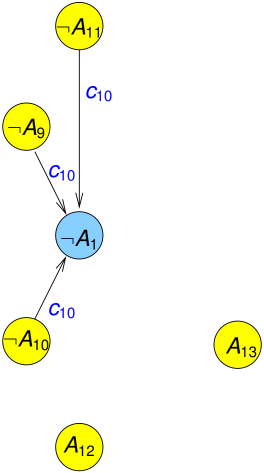
$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ √

...

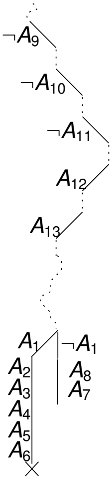


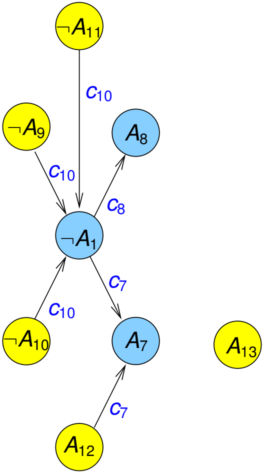


$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1\}$
(unit $\neg A_1$)

The original backjumping strategy – example

$c_1 : \neg A_1 \lor A_2$ ✓
$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$
$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓
$c_8 : A_1 \lor A_8$ ✓
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$ ✓
...





$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1, A_7, A_8\}$
(unit A_7, A_8)

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

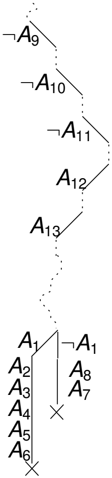
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

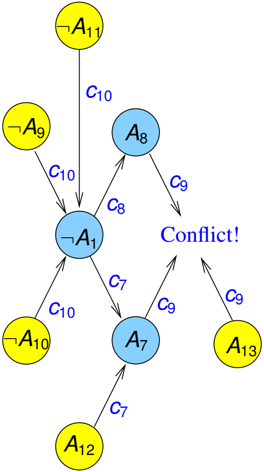
$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓

...





$\{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, ..., \neg A_1, A_7, A_8\}$

Conflict!

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

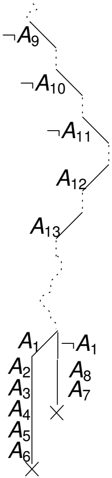
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

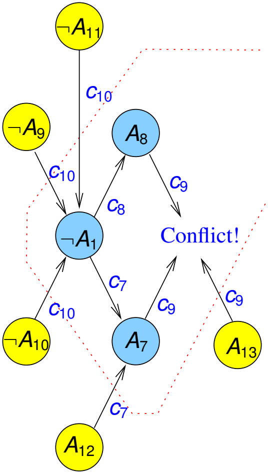
$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ×

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓

...





$\Longrightarrow$ conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\}$.

$\Longrightarrow$ learn $C_{11} := A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

The original backjumping strategy – example

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$

$c_6 : \neg A_5 \vee \neg A_6$

$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

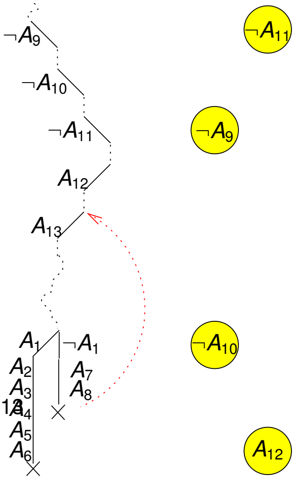
$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$

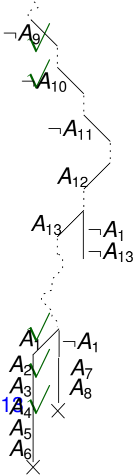
...

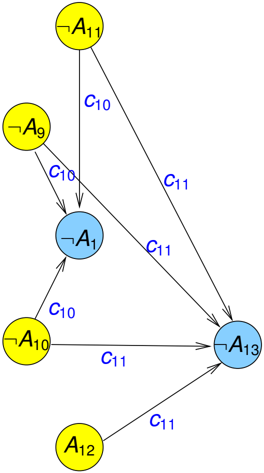


$\Longrightarrow$ backtrack to A_{13} $\Longrightarrow$ Lots of search saved!

The original backjumping strategy – example

$c_1 : \neg A_1 \lor A_2$
$c_2 : \neg A_1 \lor A_3 \lor A_9$
$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$
$c_4 : \neg A_4 \lor A_5 \lor A_{10}$
$c_5 : \neg A_4 \lor A_6 \lor A_{11}$
$c_6 : \neg A_5 \lor \neg A_6$
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$
$c_8 : A_1 \lor A_8$
$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$
$c_{10} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_1$
$c_{11} : A_9 \lor A_{10} \lor A_{11} \lor \neg A_{12} \lor \neg A_{13}$
...





$\Longrightarrow$ backtrack to A_{13}, set A_{13} and A_1 to $\bot$,...

State-of-the-art backjumping and learning [63]

- Idea: when a branch μ fails,
  (i) conflict analysis: find the conflict set $\eta \subseteq \mu$ by generating the conflict clause $C \stackrel{\text{def}}{=} \neg\eta$ via resolution from the falsified clause, according to the 1^{st}UIP strategy
  (ii) learning: add the conflict clause C to the clause set
  (iii) backjumping: backtrack to the highest branching point s.t. the stack contains all-but-one literals in η, and then unit-propagate the unassigned literal on C

1st UIP strategy – example (7)

$c_1 : \neg A_1 \vee A_2$ ✓

$c_2 : \neg A_1 \vee A_3 \vee A_9$ ✓

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$ ✓

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

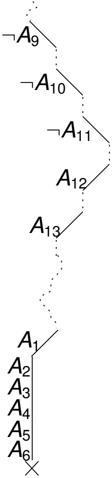
$c_6 : \neg A_5 \vee \neg A_6$ ×

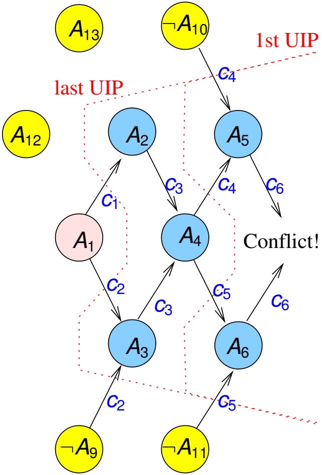
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$ ✓

$c_8 : A_1 \vee A_8$ ✓

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

...





$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$

1st UIP strategy and backjumping [63]

- The added conflict clause states the reason for the conflict
- The procedure backtracks to the most recent decision level of the variables in the conflict clause which are not the UIP.
- then the conflict clause forces the negation of the UIP by unit propagation.

E.g.: $c_{10} := A_{10} \vee A_{11} \vee \neg A_4$
$\Longrightarrow$ backtrack to A_{11}, then assign $\neg A_4$

1st UIP strategy – example (7)

$c_1 : \neg A_1 \lor A_2$ ✓

$c_2 : \neg A_1 \lor A_3 \lor A_9$ ✓

$c_3 : \neg A_2 \lor \neg A_3 \lor A_4$ ✓

$c_4 : \neg A_4 \lor A_5 \lor A_{10}$ ✓

$c_5 : \neg A_4 \lor A_6 \lor A_{11}$ ✓

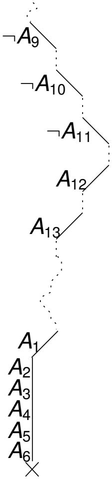
$c_6 : \neg A_5 \lor \neg A_6$ ×

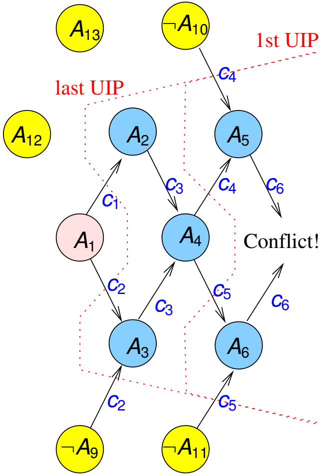
$c_7 : A_1 \lor A_7 \lor \neg A_{12}$ ✓

$c_8 : A_1 \lor A_8$ ✓

$c_9 : \neg A_7 \lor \neg A_8 \lor \neg A_{13}$

...





$\Longrightarrow$ Conflict set: $\{\neg A_{10}, \neg A_{11}, A_4\}$, learn $c_{10} := A_{10} \lor A_{11} \lor \neg A_4$

1st UIP strategy – example (8)

$c_1 : \neg A_1 \vee A_2$
$c_2 : \neg A_1 \vee A_3 \vee A_9$
$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$
$c_4 : \neg A_4 \vee A_5 \vee A_{10}$
$c_5 : \neg A_4 \vee A_6 \vee A_{11}$
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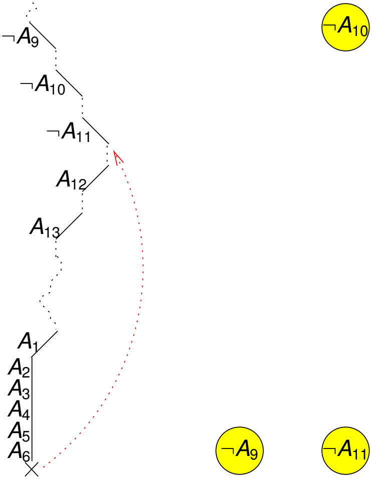
...



$\Longrightarrow$ backtrack up to A_{11} $\Longrightarrow \{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}\}$

1st UIP strategy – example (9)

$c_1 : \neg A_1 \vee A_2$

$c_2 : \neg A_1 \vee A_3 \vee A_9$

$c_3 : \neg A_2 \vee \neg A_3 \vee A_4$

$c_4 : \neg A_4 \vee A_5 \vee A_{10}$ ✓

$c_5 : \neg A_4 \vee A_6 \vee A_{11}$ ✓

$c_6 : \neg A_5 \vee \neg A_6$

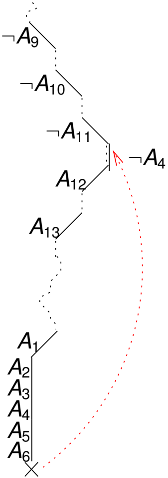
$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

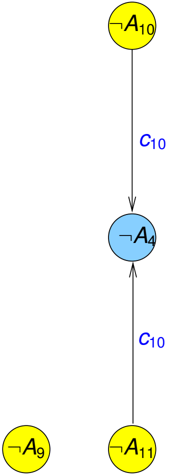
$c_8 : A_1 \vee A_8$

$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$c_{10} : A_{10} \vee A_{11} \vee \neg A_4$ ✓

...





$\Longrightarrow$ unit propagate $\neg A_4 \Longrightarrow \{..., \neg A_9, \neg A_{10}, \neg A_{11}, A_4\}$...

1st UIP strategy and backjumping – intuition

- An UIP is a single reason implying the conflict at the current level
- substituting the 1st UIP for the last UIP
    - does not enlarge the conflict
    - requires less resolution steps to compute C
    - may require involving less decision literals from other levels
- by backtracking to the most recent decision level of the variables in the conflict clause which are not the UIP:
    - jump higher
    - allows for assigning (the negation of) the UIP as high as possible in the search tree.

Learning [4, 54]

Idea: When a conflict set η is revealed, then $C \overset{\text{def}}{=} \neg\eta$ added to φ
$\Longrightarrow$ the solver will no more generate an assignment containing η: when $|\eta| - 1$ literals in η are assigned, the other is set $\perp$ by unit-propagation on C
$\Longrightarrow$ Drastic pruning of the search!

Learning – example

$c_1 : \neg A_1 \vee A_2$

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$c_7 : A_1 \vee A_7 \vee \neg A_{12}$

$c_8 : A_1 \vee A_8$

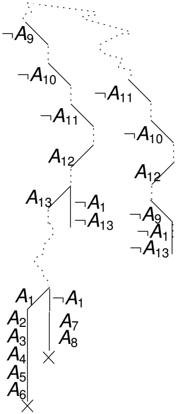
$c_9 : \neg A_7 \vee \neg A_8 \vee \neg A_{13}$ ✓

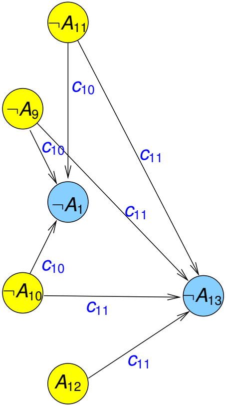
$c_{10} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$ ✓

$c_{11} : A_9 \vee A_{10} \vee A_{11} \vee \neg A_{12} \vee \neg A_{13}$ ✓

...

$\Longrightarrow$ Unit: $\{\neg A_1, \neg A_{13}\}$





Drawbacks of Learning & Clause discharging

Problem with Learning

Learning can generate exponentially-many clauses

- may cause a blowup in space
- may drastically slow down BCP

A solution: clause discharging

Techniques to drop learned clauses when necessary

- according to their size
- according to their activity.

A clause is currently active if it occurs in the current implication graph (i.e., it is the antecedent clause of a literal in the current assignment).

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Drawbacks of Learning & Clause discharging

- Is clause-discharging safe?
- Yes, if done properly.

Property (see, e.g., [45])

In order to guarantee correctness, completeness & termination of a CDCL solver, it suffices to keep each clause until it is active.
$\Longrightarrow$ CDCL solvers require polynomial space

“Lazy” Strategy

- when a clause is involved in conflict analisis, increase its activity
- when needed, drop the least-active clauses

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State-of-the-art backjumping and learning: intuitions

- **Backjumping:** allows for climbing up to many decision levels in the stack
  - intuition: " go back to the oldest decision where you'd have done something different if only you had known C"

  $\Longrightarrow$ may avoid lots of redundant search
- **Learning:** in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
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- Learning: in future branches, when all-but-one literals in η are assigned, the remaining literal is assigned to false by unit-propagation:
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State-of-the-art backjumping and learning: intuitions

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Remark: the “quality” of conflict sets

- Different ideas of “good” conflict set
    - Backjumping: if causes the highest backjump (“local” role)
    - Learning: if causes the maximum pruning (“global” role)
- Many different strategies implemented (see, e.g., [4, 54, 63])

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Preprocessing: (sorting plus) subsumption

- Detect and remove subsumed clauses:

$$\begin{array}{c}\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee l_3 \vee l_1) \wedge \varphi_3 \\ \Downarrow \\ \varphi_1 \wedge (l_1 \vee l_2) \wedge \varphi_2 \wedge \varphi_3\end{array}$$

Preprocessing: detect & collapse equivalent literals [11]

Repeat:

(i) build the implication graph induced by binary clauses

(ii) detect strongly connected cycles $\Longrightarrow$ equivalence classes of literals

(iii) perform substitutions

(iv) perform unit and pure literal.

Until (no more simplification is possible).

- Ex:

$$\varphi_1 \wedge (\neg l_2 \vee l_1) \wedge \varphi_2 \wedge (\neg l_3 \vee l_2) \wedge \varphi_3 \wedge (\neg l_1 \vee l_3) \wedge \varphi_4$$
$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

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- Ex:

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$$\Downarrow_{l_1 \leftrightarrow l_2 \leftrightarrow l_3}$$
$$(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)[l_2 \leftarrow l_1; l_3 \leftarrow l_1;]$$

- Very effective in many application domains.

Preprocessing: resolution (and subsumption) [3]

- Apply some basic steps of resolution (and simplify):

$$\begin{gathered}
\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee \neg l_1) \wedge \varphi_3 \\
\Downarrow_{resolve} \\
\varphi_1 \wedge (l_2) \wedge \varphi_2 \wedge \varphi_3 \\
\Downarrow_{unit-propagate} \\
(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[l_2 \leftarrow \top]
\end{gathered}$$

Branching heuristics

- Branch is the source of non-determinism for DPLL
  $\Longrightarrow$ critical for efficiency
- many branch heuristics conceived in literature.

Some example heuristics

- MOMS heuristics: pick the literal occurring most often in the minimal size clauses
  $\Longrightarrow$ fast and simple, many variants
- Jeroslow-Wang: choose the literal with maximum

$$score(l) := \Sigma_{l \in c \ \& \ c \in \varphi} 2^{-|c|}$$

  $\Longrightarrow$ estimates l's contribution to the satisfiability of φ
- Satz [33]: selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set
  $\Longrightarrow$ maximizes the effects of unit propagation
- VSIDS [43]: variable state independent decaying sum
  - “static”: scores updated only at the end of a branch
  - “local”: privileges variable in recently learned clauses

Restarts [26]

(according to some strategy) restart DPLL

- abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- may significantly reduce the overall search space

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Tractable subclasses of SAT

- SAT in general is an NP-complete problem
- Some subclasses of SAT are tractable
- Two noteworthy tractable subclasses of SAT:
  - Horn Formulas (Horn-SAT)
  - 2-CNF formulas (2-SAT)

Horn Formulas

- A Horn formula is a CNF Boolean formula s.t. each clause contains at most one positive literal.

  $A_1 \vee \neg A_2$
  $A_2 \vee \neg A_3 \vee \neg A_4$
  $\neg A_5 \vee \neg A_3 \vee \neg A_4$
  A_3
- Intuition: implications between positive Boolean variables:

$$\begin{array}{rl} A_2 \rightarrow & A_1 \\ (A_3 \wedge A_4) \rightarrow & A_2 \\ (A_5 \wedge A_3 \wedge A_4) \rightarrow & \bot \\ & A_3 \end{array}$$

Formulas reducible to Horn

- Remark: Some non-Horn formulas can be reduced to Horn by simply renaming literals

$$\begin{array}{l} A_1 \vee A_2 \\ \neg A_2 \vee \neg A_3 \vee \neg A_4 \\ \neg A_5 \vee \neg A_3 \vee \neg A_4 \\ A_3 \end{array} \Longrightarrow_{B:=\neg A_2} \begin{array}{l} A_1 \vee \neg B \\ B \vee \neg A_3 \vee \neg A_4 \\ \neg A_5 \vee \neg A_3 \vee \neg A_4 \\ A_3 \end{array}$$

Tractability of Horn Formulas

Property

Checking the satisfiability of Horn formulas requires polynomial time

Hint:

(i) Eliminate unit clauses by propagating their value;
$\Longrightarrow$ Every clause contains at least one negative literal.

(ii) Assign all variables to $\bot$;

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Checking the satisfiability of Horn formulas requires polynomial time

Hint:

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$\Longrightarrow$ Every clause contains at least one negative literal.

(ii) Assign all variables to $\bot$;

A simple polynomial procedure for Horn-SAT

```
function Horn_SAT(formula $\varphi$, assignment & $\mu$) {
    Unit_Propagate($\varphi$, $\mu$);
    if ($\varphi == \bot$)
        then return UNSAT;
    else {
        $\mu := \mu \cup \bigcup_{A_i \not\in \mu} \{\neg A_i\}$;
        return SAT;
} }

function Unit_Propagate(formula & $\varphi$, assignment & $\mu$)
    while ($\varphi \neq \top$ and $\varphi \neq \bot$ and {a unit clause ($l$) occurs in $\varphi$}) do {
        $\varphi = assign(\varphi, l)$;
        $\mu := \mu \cup \{l\}$;
} }
```

Example

$$\begin{array}{llll} \neg A_1 & \vee\ A_2 & \vee \neg A_3 \\ A_1 & \vee \neg A_3 & \vee \neg A_4 \\ \neg A_2 & \vee \neg A_4 & \\ A_3 & \vee \neg A_4 & \\ A_4 & & \end{array}$$

Example

$$\begin{array}{lll} \neg A_1 & \vee\ A_2 & \vee\neg A_3 \\ \ \ A_1 & \vee\neg A_3 & \vee\neg A_4 \\ \neg A_2 & \vee\neg A_4 & \\ \ \ A_3 & \vee\neg A_4 & \\ \ \ A_4 & & \end{array}$$

$\mu := \{A_4 := \top\}$

Example

$$
\begin{array}{rlll}
\neg A_1 & \vee & A_2 & \vee \neg A_3 \\
A_1 & \vee \neg A_3 & & \vee \neg A_4 \\
\neg A_2 & \vee \neg A_4 & & \\
A_3 & \vee \neg A_4 & & \\
A_4 & & &
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example

$$\begin{array}{lll} \neg A_1 & \vee\ A_2 & \vee \neg A_3 \\ A_1 & \vee \neg A_3 & \vee \neg A_4 \\ \neg A_2 & \vee \neg A_4 & \\ A_3 & \vee \neg A_4 & \\ A_4 & & \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example

$$
\begin{array}{rlll}
\neg A_1 & \vee \; A_2 & \vee \neg A_3 & \times \\
A_1 & \vee \neg A_3 & \vee \neg A_4 & \\
\neg A_2 & \vee \neg A_4 & & \\
A_3 & \vee \neg A_4 & & \\
A_4 & & &
\end{array}
$$

$$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_1 := \top\} \Longrightarrow \text{UNSAT}$$

Example 2

$$\begin{array}{lll} A_1 & \vee \neg A_2 & \\ A_2 & \vee \neg A_5 & \vee \neg A_4 \\ A_4 & \vee \neg A_3 & \\ A_3 & & \end{array}$$

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$$\begin{array}{lll} A_1 & \vee \neg A_2 & \\ A_2 & \vee \neg A_5 & \vee \neg A_4 \\ A_4 & \vee \neg A_3 & \\ A_3 & & \end{array}$$

$\mu := \{A_3 := \top, A_4 := \top\} \Longrightarrow$ SAT

2-CNF Formulas

- A 2-CNF formula is a CNF formula in which each clause has (at most) two literals.

  $A_1 \vee \neg A_2$
  $A_2 \vee \neg A_3$
  $\neg A_5 \vee \neg A_3$
  $A_3 \vee \neg A_1$
  A_5
- Checking the satisfiability of 2-CNF formulas requires polynomial time

Tractability of 2-CNF Formulas

Graph-based approach:

(i) Build the implication graph of the formula

(ii) check if it has a cycle containing both A_i and $\neg A_i$ for some i (e.g., by Tarjan's algorithm)
$\Longrightarrow$ the formula is unsatisfiable iff such cycle exists

- requires linear time

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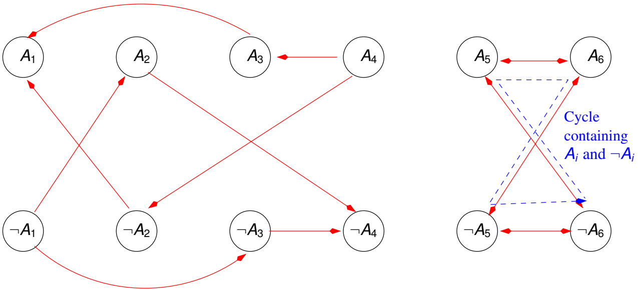
Example:

$A_1 \vee A_2$
$A_1 \vee \neg A_3$
$\neg A_2 \vee \neg A_4$
$A_3 \vee \neg A_4$
A_4
$\neg A_5 \vee A_6$
$A_5 \vee A_6$
$A_5 \vee \neg A_6$
$\neg A_5 \vee \neg A_6$



Cycle containing A_i and $\neg A_i$

Tractability of 2-CNF Formulas

Idea

Let φ, l s.t. $var(l) \in \varphi$ and $(\varphi \wedge l) \not\models_{BCP} \bot$.

- φ': clauses remained after BCP
- φ'': clauses removed by BCP

Suppose φ' is UNSAT. Can we conclude anything about φ?

- Case φ is $>$2-CNF: No!
  - there may be (non-unit) clauses $C \in \varphi'$ s.t. $(\neg l \vee C) \in \varphi$

  $\Longrightarrow$ $\varphi \neq \varphi' \wedge \varphi''$ and $\varphi' \models \bot \not\Longrightarrow \varphi \models \bot$

  $\Longrightarrow$ we must check also $\varphi \wedge \neg l$
- Case φ is 2-CNF: Yes!
  - there cannot be clause $C \in \varphi'$ s.t. $(\neg l \vee C) \in \varphi$

  $\Longrightarrow$ $\varphi = \varphi' \wedge \varphi''$ and $\varphi' \models \bot \Longrightarrow \varphi \models \bot$

  $\Longrightarrow$ φ is UNSAT

Note: we need to check first that $(\varphi \wedge l) \not\models_{BCP} \bot$:
If $(\varphi \wedge l) \models_{BCP} \bot$, then $\varphi' \models \bot \not\Longrightarrow \varphi \models \bot$ (see later Example 2).

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Note: we need to check first that $(\varphi \wedge l) \not\models_{BCP} \bot$:
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A simple polynomial procedure for 2-SAT

```
function 2_SAT(formula $\varphi$, assignment & $\mu$) {
    Unit_Propagate($\varphi, \mu$);
    if ($\varphi == \bot$) then return UNSAT;
    if ($\varphi == \top$) then return SAT;
    while True do {
        {choose some literal l occurring in $\varphi$};
        save($\varphi, \mu$);
        $\varphi := \varphi \wedge l$;
        Unit_Propagate($\varphi, \mu$);
        if ($\varphi == \bot$) then {
            retrieve($\varphi, \mu$);
            $\varphi = \varphi \wedge \neg l$;
            Unit_Propagate($\varphi, \mu$); }
        if ($\varphi == \bot$) then return UNSAT;
        if ($\varphi == \top$) then return SAT;
} };
```

Example

$$\begin{array}{rl}
A_1 & \vee\ \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee\ \ A_6 \\
A_5 & \vee\ \ A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}$$

Example

$$\begin{array}{rcl}
A_1 & \vee & A_2 \\
A_1 & \vee & \neg A_3 \\
\neg A_2 & \vee & \neg A_4 \\
A_3 & \vee & \neg A_4 \\
A_4 & & \\
\neg A_5 & \vee & A_6 \\
A_5 & \vee & A_6 \\
A_5 & \vee & \neg A_6 \\
\neg A_5 & \vee & \neg A_6
\end{array}$$

$\mu := \{A_4 := \top\}$

Example

$$\begin{array}{rcl} A_1 & \vee & A_2 \\ A_1 & \vee & \neg A_3 \\ \neg A_2 & \vee & \neg A_4 \\ A_3 & \vee & \neg A_4 \\ A_4 & & \\ \neg A_5 & \vee & A_6 \\ A_5 & \vee & A_6 \\ A_5 & \vee & \neg A_6 \\ \neg A_5 & \vee & \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example

$$
\begin{array}{rl}
A_1 & \vee\ \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee\ \ A_6 \\
A_5 & \vee\ \ A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot\}$ (Select $\neg A_6$)

Example

$$\begin{array}{rl} A_1 & \vee\ A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee\ A_6 \\ A_5 & \vee\ A_6 \quad \times \\ A_5 & \vee \neg A_6 \\ \neg A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot, A_5 := \bot\} \Longrightarrow$ backtrack

Example

$$
\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6 \\
\neg A_5 & \vee \neg A_6
\end{array}
$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top\}$ (Select A_6)

Example

$$\begin{array}{rcll}
A_1 & \vee & A_2 & \\
A_1 & \vee & \neg A_3 & \\
\neg A_2 & \vee & \neg A_4 & \\
A_3 & \vee & \neg A_4 & \\
A_4 & & & \\
\neg A_5 & \vee & A_6 & \\
A_5 & \vee & A_6 & \\
A_5 & \vee & \neg A_6 & \times \\
\neg A_5 & \vee & \neg A_6 &
\end{array}$$

$$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top, A_5 := \top\} \Longrightarrow \text{UNSAT}$$

Example 2

$$\begin{array}{rl}
A_1 & \vee A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee A_6 \\
A_5 & \vee A_6 \\
A_5 & \vee \neg A_6
\end{array}$$

Example 2

$$\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6
\end{array}$$

$\mu := \{A_4 := \top\}$

Example 2

$$\begin{array}{rl} A_1 & \vee \ A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee \ A_6 \\ A_5 & \vee \ A_6 \\ A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top\}$

Example 2

$$\begin{array}{rl} A_1 & \vee \quad A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee \quad A_6 \\ A_5 & \vee \quad A_6 \\ A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot\}$

Example 2

$$\begin{array}{rl} A_1 & \vee\ A_2 \\ A_1 & \vee \neg A_3 \\ \neg A_2 & \vee \neg A_4 \\ A_3 & \vee \neg A_4 \\ A_4 & \\ \neg A_5 & \vee\ A_6 \\ A_5 & \vee\ A_6 \\ A_5 & \vee \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot\}$ (Select $\neg A_6$)

Example 2

$$\begin{array}{rcl} A_1 & \vee & A_2 \\ A_1 & \vee & \neg A_3 \\ \neg A_2 & \vee & \neg A_4 \\ A_3 & \vee & \neg A_4 \\ A_4 & & \\ \neg A_5 & \vee & A_6 \\ A_5 & \vee & A_6 \quad \times \\ A_5 & \vee & \neg A_6 \end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \bot, A_5 := \bot\} \Longrightarrow$ backtrack

Example 2

$$\begin{array}{rl}
A_1 & \vee \ A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee \ A_6 \\
A_5 & \vee \ A_6 \\
A_5 & \vee \neg A_6
\end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top\}$ (Select A_6)

Example 2

$$\begin{array}{rl}
A_1 & \vee A_2 \\
A_1 & \vee \neg A_3 \\
\neg A_2 & \vee \neg A_4 \\
A_3 & \vee \neg A_4 \\
A_4 & \\
\neg A_5 & \vee A_6 \\
A_5 & \vee A_6 \\
A_5 & \vee \neg A_6
\end{array}$$

$\mu := \{A_4 := \top, A_3 := \top, A_2 := \bot, A_6 := \top, A_5 := \top\} \Longrightarrow$ SAT

Outline

The satisfiability of k-CNF (k-SAT) [20]

- k-CNF: CNF s.t. all clauses have *k* literals
- the satisfiability of 2-CNF is polynomial
- the satisfiability of k-CNF is NP-complete for $k \geq 3$
- every k-CNF formula can be converted into 3-CNF:

$$l_1 \lor l_2 \lor \ldots \lor l_{k-1} \lor l_k$$

$$\Downarrow$$

$$\begin{gathered}(l_1 \lor l_2 \lor B_1) \land \\ (\neg B_1 \lor l_3 \lor B_2) \land \\ \ldots \\ (\neg B_{k-4} \lor l_{k-2} \lor B_{k-3}) \land \\ (\neg B_{k-3} \lor l_{k-1} \lor l_k)\end{gathered}$$

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Random K-CNF formulas generation

Random k-CNF formulas with N variables and L clauses:

```
DO
    (i) pick with uniform probability a set of $k$ atoms over $N$
    (ii) randomly negate each atom with probability 0.5
    (iii) create a disjunction of the resulting literals
UNTIL $L$ different clauses have been generated;
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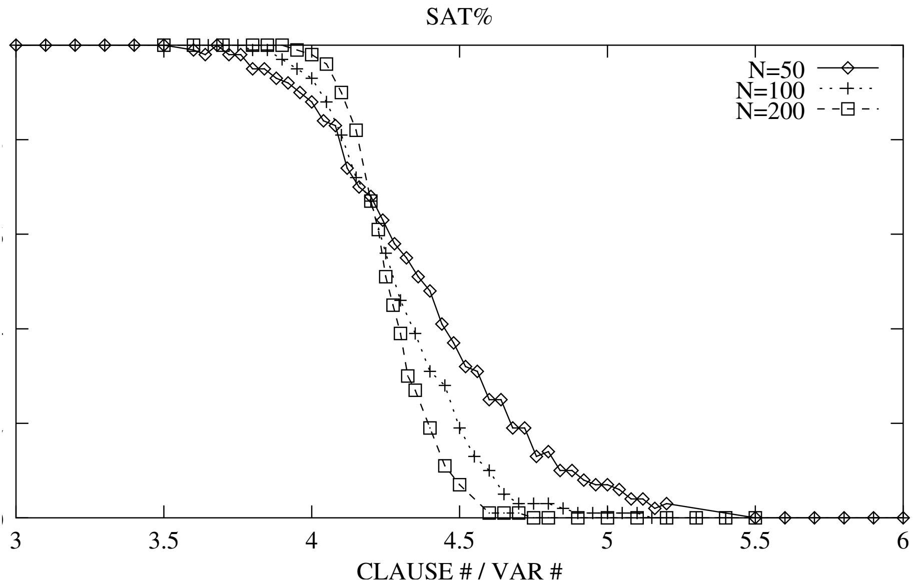
Random k-SAT plots

- fix k and N
- for increasing L, randomly generate and solve (500,1000,10000,...) problems with k, L, N
- plot
  - satisfiability percentages
  - median/geometrical mean CPU time/# of steps

  against L/N

The phase transition phenomenon: SAT % Plots [41, 32]

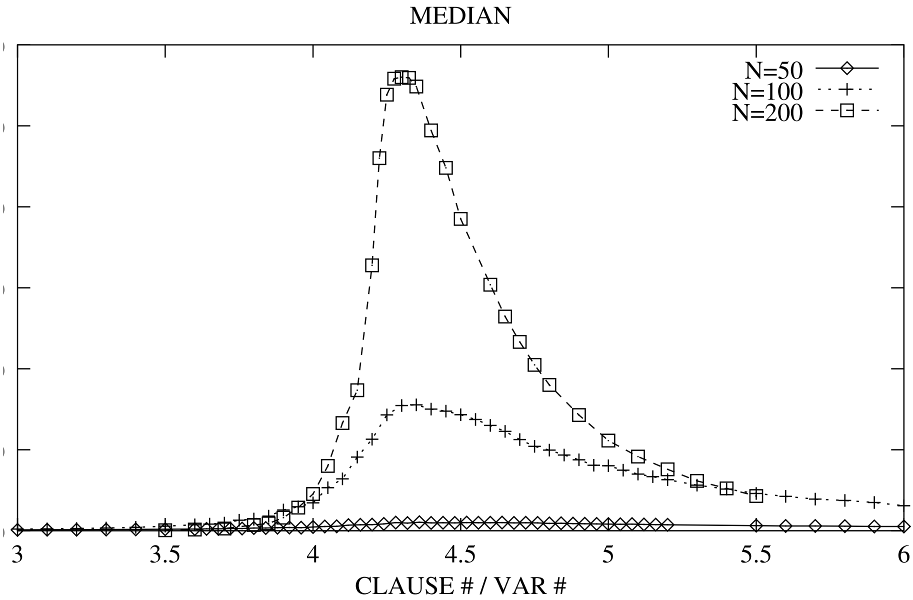
- Increasing L/N we pass from 100% satisfiable to 100% unsatisfiable formulas
- the decay becomes steeper with N
- for $N \to \infty$, the plot converges to a step in the cross-over point ($L/N \approx 4.28$ for k=3)
- Revealed for many other NP-complete problems
- Many theoretical models [59, 21, 32, 16, 42]
- Strong relation with Thermodynamics

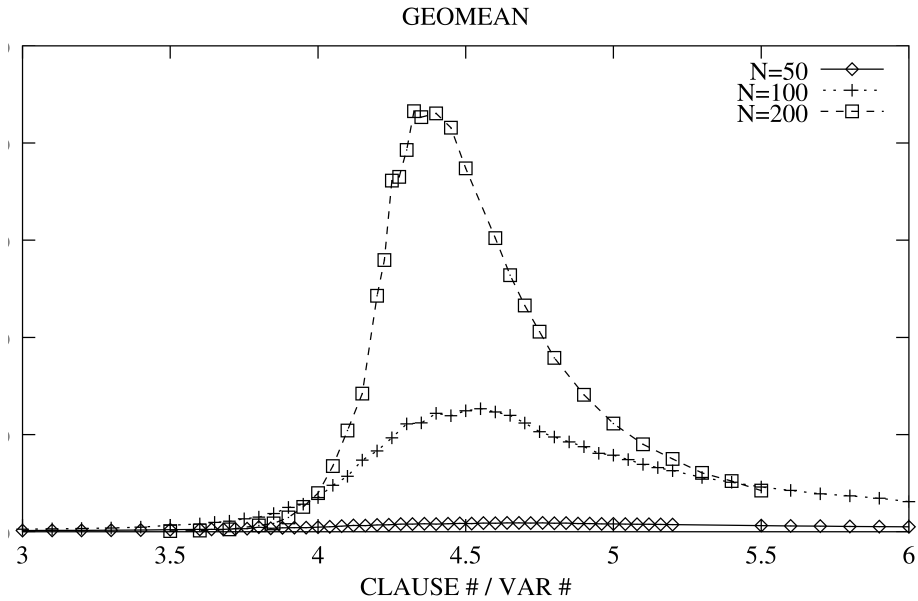

SAT%
N=50
N=100
N=200
3
3.5
4
4.5
5
5.5
6
CLAUSE # / VAR #

The phase transition phenomenon: CPU times/step

Using search algorithms (DPLL):

- Increasing L/N we pass from easy problems, to very hard problems down to hard problems
- the peak is centered in the 50% satisfiable point
- the decay becomes steeper with N
- for $N \to \infty$, the plot converges to an impulse in the cross-over point ($L/N \approx 4.28$ for k=3)
- easy problems ($L/N \leq\approx 3.8$) increase polynomially with N, hard problems increase exponentially with N
- Increasing L/N, satisfiable problems get harder, unsatisfiable problems get easier.


MEDIAN
N=50
N=100
N=200
3
3.5
4
4.5
5
5.5
6
CLAUSE # / VAR #


GEOMEAN
N=50
N=100
N=200
3
3.5
4
4.5
5
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6
CLAUSE # / VAR #

Outline

1. Basics on SAT
2. Basic SAT-Solving techniques
3. Modern CDCL SAT Solvers
    - Conflict-Driven Clause-Learning SAT solvers
    - Further Improvements
4. Tractable subclasses of SAT
5. Random k-SAT and Phase Transition
6. Advanced Functionalities: proofs, unsat cores, interpolants, optimization
7. Some Applications
    - Appl. #1: (Bounded) Planning
    - Appl. #2: Bounded Model Checking

Advanced functionalities

Advanced SAT functionalities (very important in formal verification):

- Computing SAT under assumptions & Incremental SAT solving
- Building proofs of unsatisfiability
- Extracting unsatisfiable Cores
- Computing Craig Interpolants

SAT under assumptions: $SAT(\varphi, \{l_1, ..., l_n\})$ [18]

- Many SAT solvers allow for solving a CNF formula φ under a set of assumption literals $\mathcal{A} \stackrel{\text{def}}{=} \{l_1, ..., l_n\}$: $SAT(\varphi, \{l_1, ..., l_n\})$
  - $SAT(\varphi, \{l_1, ..., l_n\})$: same result as $SAT(\varphi \wedge \bigwedge_{i=1}^{n} l_i)$
  - often useful to call the same formula with different assumption lists: $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2), ...$
- Idea:
  - $l_1, ..., l_n$ "decided" at decision level 0 before starting the search
  - if backjump to level 0 on $C \stackrel{\text{def}}{=} \neg\eta$ s.t. $\eta \subseteq \mathcal{A}$, then return UNSAT
  - if the "decision" strategy for conflict analysis is used, then η is the subset of assumptions causing the inconsistency

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Selection of sub-formulas

Let φ be $\bigwedge_{i=1}^{n} C_i$.

Idea [18, 35]

- let $S_1 ... S_n$ be fresh Boolean atoms (selection variables).
- let $\mathcal{A} \overset{\text{def}}{=} \{S_{i_1}, ..., S_{i_K}\} \subseteq \{S_1, ..., S_n\}$
- SAT($\bigwedge_{i=1}^{n} (\neg S_i \vee C_i), \mathcal{A}$): same as SAT($\bigwedge_{i=i_1}^{i_k} (C_i)$)

$\Longrightarrow$ allows for "selecting" (activating) only a subset of the clauses in φ at each call.

Incremental SAT solving [18, 17]

- Many CDCL solvers provide a stack-based incremental interface
  - it is possible to push/pop ϕ_i into a stack of formulas $\Phi \stackrel{\text{def}}{=} \{\phi_1, ..., \phi_k\}$
  - check incrementally the satisfiability of $\bigwedge_{i=1}^{k} \phi_i$.
- Maintains the status of the search from one call to the other
  - in particular it records the learned clauses (plus other information)

$\Longrightarrow$ reuses search from one call to another

- Very useful in many applications (in particular in FV)
- Simple idea [18, 17]: incremental calls $SAT(\varphi, \mathcal{A}_1), SAT(\varphi, \mathcal{A}_2)$,...
  - $\varphi \stackrel{\text{def}}{=} \bigwedge_i (\neg A_i \vee \phi_i)$, $\mathcal{A}_i \subseteq \{A_1, ..., A_k\}\ \forall\, i$,
  - stack-based interface for $\mathcal{A} \stackrel{\text{def}}{=} \{A_1, A_2, ...\}$

learned clauses safely reused from call to call even if assumptions have been removed

- learned clauses C_j s.t. $\varphi \models C_j$
- C_j may be in the form $\neg A_j \vee C_j'$ s.t. $A_i \not\in \mathcal{A}_i \Longrightarrow C_j$ not reused

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Building Proofs of Unsatisfiability

- When φ is unsat, it is very important to build a (resolution) proof of unsatisfiability:
  - to verify the result of the solver
  - to understand a “reason” for unsatisfiability
  - to build unsatisfiable cores and interpolants
- can be built by keeping track of the resolution steps performed when constructing the conflict clauses.

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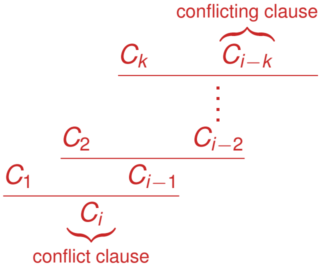
- recall: each conflict clause C_i learned is computed from the conflicting clause C_{i-k} by backward resolving with the antecedent clause of one literal



- $C_1, ..., C_k$, and C_{i-k} can be original or learned clauses
- each resolution (sub)proof can be easily tracked:

```
k i-k -> i-k-1

...

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1 i-1 -> i
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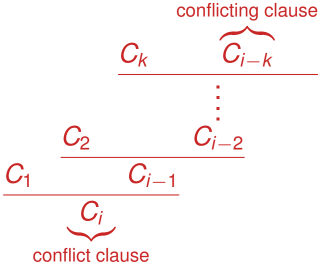

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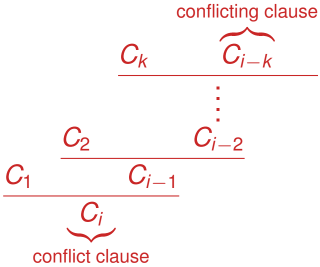

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```
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...
2 i-2 -> i-1
1 i-1 -> i
```
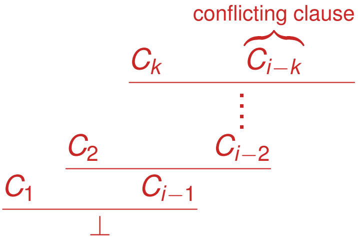

Building Proofs of Unsatisfiability

- ... in particular, if φ is unsatisfiable, the last step produces “false” as conflict clause:



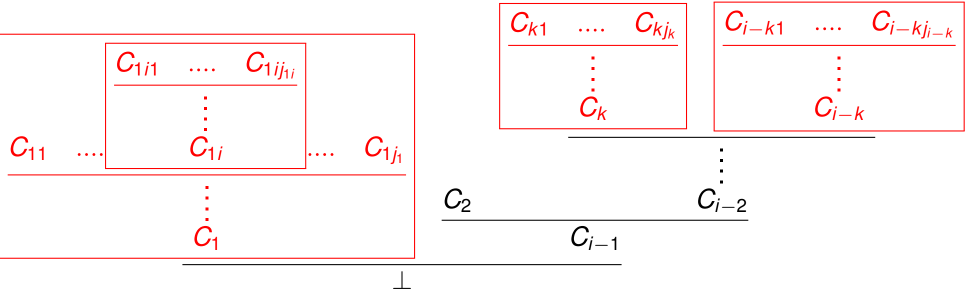
- note: $C_1 = l$, $C_{i-1} = \neg l$ for some literal l
- $C_1, ..., C_k$, and C_{i-k} can be original or learned clauses...

Building Proofs of Unsatisfiability

Starting from the previous proof of unsatisfiability, repeat recursively:

- for every learned leaf clause C_i, substitute C_i with the resolution proof generating it

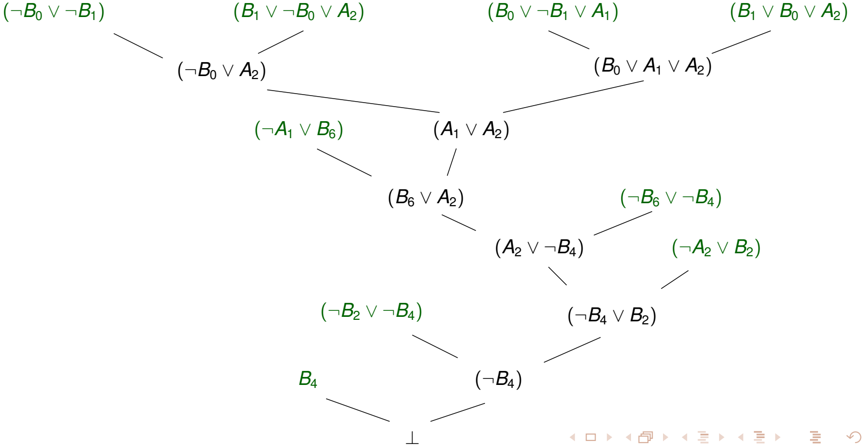
until all leaf clauses are original clauses



$\Longrightarrow$ we obtain a resolution proof of unsatisfiability for (a subset of) the clauses in φ

Building Proofs of Unsatisfiability: example

$$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$$
$$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$$



$(\neg B_0 \vee \neg B_1)$ $(B_1 \vee \neg B_0 \vee A_2)$ $(B_0 \vee \neg B_1 \vee A_1)$ $(B_1 \vee B_0 \vee A_2)$

$(\neg B_0 \vee A_2)$ $(B_0 \vee A_1 \vee A_2)$

$(\neg A_1 \vee B_6)$ $(A_1 \vee A_2)$

$(B_6 \vee A_2)$ $(\neg B_6 \vee \neg B_4)$

$(A_2 \vee \neg B_4)$ $(\neg A_2 \vee B_2)$

$(\neg B_2 \vee \neg B_4)$ $(\neg B_4 \vee B_2)$

B_4 $(\neg B_4)$

$\perp$

Extraction of unsatisfiable cores

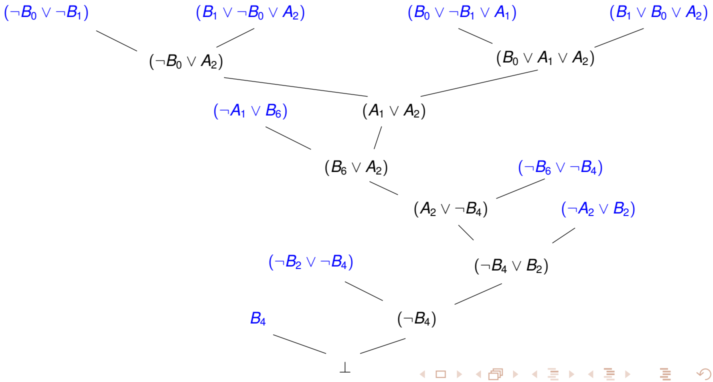
- Problem: given an unsatisfiable set of clauses, extract from it a (possibly small/minimal/minimum) unsatisfiable subset

  $\Longrightarrow$ unsatisfiable cores (aka (Minimal) Unsatisfiable Subsets, (M)US)
- Lots of literature on the topic [65, 36, 39, 46, 62, 28, 22, 10]
- We recognize two main approaches:
  - Proof-based approach [65]: byproduct of finding a resolution proof
  - Assumption-based approach [36]: use extra variables labeling clauses
- many optimizations for further reducing the size of the core:
  - repeat the process up to fixpoit
  - remove clauses one-by one, until satisfiability is obtained
  - combinations of the two processed above
  - ...

The proof-based approach to unsat-core extraction [65]

Unsat core: the set of leaf clauses of a resolution proof

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge$
$(\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$



The assumption-based approach to unsat-core extraction [36]

Based on the following process:

(i) each clause C_i is substituted by $\neg S_i \vee C_i$, s.t. S_i fresh "selector" variable

(ii) before starting the search each S_i is forced to true.

(iii) final conflict clause at dec. level 0: $\bigvee_j \neg S_j$

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$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg B_0 \vee \neg B_1) \wedge (\neg B_2 \vee \neg B_4) \wedge (\neg A_2 \vee B_2) \wedge (\neg A_1 \vee B_3) \wedge$
$B_4 \wedge (A_2 \vee B_5) \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1) \wedge B_7$

(i) add selector variables:
$(\neg S_1 \vee B_0 \vee \neg B_1 \vee A_1) \wedge (\neg S_2 \vee B_0 \vee B_1 \vee A_2) \wedge (\neg S_3 \vee \neg B_0 \vee B_1 \vee A_2) \wedge$
$(\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge$
$(\neg S_7 \vee \neg A_1 \vee B_3) \wedge (\neg S_8 \vee B_4) \wedge (\neg S_9 \vee A_2 \vee B_5) \wedge (\neg S_{10} \vee \neg B_6 \vee \neg B_4) \wedge$
$(\neg S_{11} \vee B_6 \vee \neg A_1) \wedge (\neg S_{12} \vee B_7)$

(ii) The conflict analysis returns:
$\neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \neg S_4 \vee \neg S_5 \vee \neg S_6 \vee \neg S_8 \vee \neg S_{10} \vee \neg S_{11},$

(iii) corresponding to the unsat core:
$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
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$B_4 \wedge (\neg B_6 \vee \neg B_4) \wedge (B_6 \vee \neg A_1)$

The assumption-based approach to unsat-core extraction

$(B_0 \vee \neg B_1 \vee A_1) \wedge (B_0 \vee B_1 \vee A_2) \wedge (\neg B_0 \vee B_1 \vee A_2) \wedge$
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(i) add selector variables:
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$(\neg S_4 \vee \neg B_0 \vee \neg B_1) \wedge (\neg S_5 \vee \neg B_2 \vee \neg B_4) \wedge (\neg S_6 \vee \neg A_2 \vee B_2) \wedge$
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Computing Craig Interpolants in SAT

Let "$X \preceq Y$", X, Y being Boolean formulas, denote the fact that all Boolean atoms in X occur also in Y.

Definition: Craig Interpolant

Given an ordered pair (A, B) of formulas such that $A \wedge B \models \perp$, a *Craig interpolant* is a formula I s.t.:

a) $A \models I$,

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- Very important in many Formal Verification applications
- A few works presented [47, 38, 40]

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(i) Generate a resolution proof of unsatisfiability $\mathcal{P}$ for $A \wedge B$.

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Computing Craig Interpolants in SAT: example

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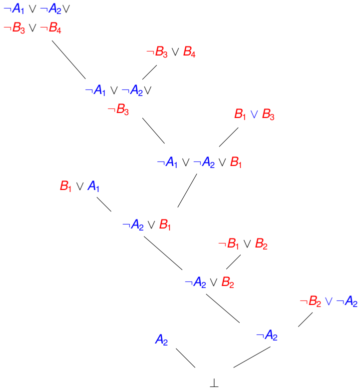
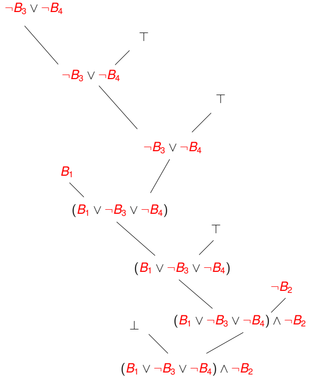


original proof



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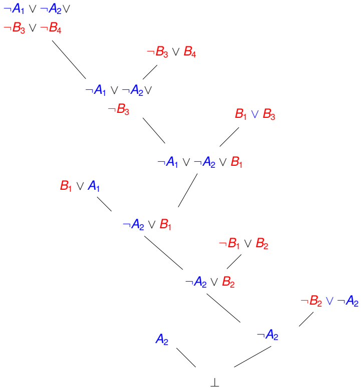

$\neg A_1 \vee \neg A_2 \vee \neg B_3 \vee \neg B_4$

$\neg B_3 \vee B_4$

$\neg A_1 \vee \neg A_2 \vee \neg B_3$

$B_1 \vee B_3$

$\neg A_1 \vee \neg A_2 \vee B_1$

$B_1 \vee A_1$

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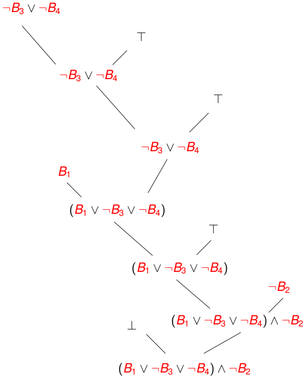
$\neg B_2 \vee \neg A_2$

A_2

$\neg A_2$

$\perp$

original proof



$\neg B_3 \vee \neg B_4$

$\top$

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B_1

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$\neg B_2$

$\perp$

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- **MaxSAT:** given a pair of CNF formulas $\langle \varphi_h, \varphi_s \rangle$ s.t. $\varphi_h \wedge \varphi_s \models \bot$, $\varphi_s \overset{\text{def}}{=} \{C_1, ..., C_k\}$, find a truth assignment μ satisfying φ_h and maximizing the amount of the satisfied clauses in φ_s.
- **Weighted MaxSAT:** given also the positive integer penalties $\{w_1, ..., w_k\}$, μ must satisfy φ_h and maximize the sum of penalties of the satisfied clauses in φ_s
- Generalization of SAT to optimization
  $\Longrightarrow$ much harder than SAT
- Many different approaches (see e.g. [34])
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$$\varphi_h \overset{\text{def}}{=} (A_1 \vee A_2) \qquad \varphi_s \overset{\text{def}}{=} \left(\begin{array}{ccc} (\ \ A_1 \vee \neg A_2) & \wedge & [4] \\ (\neg A_1 \vee \ \ A_2) & \wedge & [3] \\ (\neg A_1 \vee \neg A_2) & \wedge & [2] \end{array} \right)$$

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- Many different approaches (see e.g. [34])
- EX:

$$\varphi_h \overset{\text{def}}{=} (A_1 \vee A_2) \qquad \varphi_s \overset{\text{def}}{=} \left(\begin{array}{lll} (\ \ A_1 \vee \neg A_2) & \wedge & [4] \\ (\neg A_1 \vee \ \ A_2) & \wedge & [3] \\ (\neg A_1 \vee \neg A_2) & \wedge & [2] \end{array}\right)$$

$\Longrightarrow \mu = \{A_1, A_2\}$ (penalty = 2)

Outline

1. Basics on SAT
2. Basic SAT-Solving techniques
3. Modern CDCL SAT Solvers
    - Conflict-Driven Clause-Learning SAT solvers
    - Further Improvements
4. Tractable subclasses of SAT
5. Random k-SAT and Phase Transition
6. Advanced Functionalities: proofs, unsat cores, interpolants, optimization
7. **Some Applications**
    - Appl. #1: (Bounded) Planning
    - Appl. #2: Bounded Model Checking

Many applications of SAT

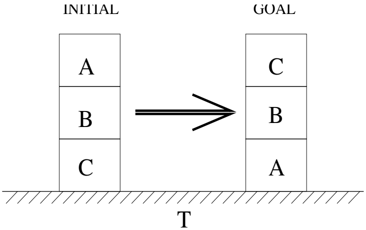
- Many successful applications of SAT:
    - Boolean circuits
    - (Bounded) Planning
    - (Bounded) Model Checking
    - Cryptography
    - Scheduling
    - ...
- All NP-complete problem can be (polynomially) converted to SAT.
- Key issue: find an efficient encoding.

Outline

The problem [30, 29, 48]

- Problem Given a set of action operators OP, (a representation of) an initial state I and goal state G, and a bound n, find a sequence of operator applications $o_1, .., o_n$, leading from the initial state to the goal state.
- Idea: Encode it into satisfiability problem of a Boolean formula φ

Example



$$\begin{array}{lll} \mathit{Move}(b,s,d) & & \\ & \mathit{Precond}: & \mathit{Block}(b) \wedge \mathit{Clear}(b) \wedge \mathit{On}(b,s) \wedge \\ & & (\mathit{Clear}(d) \vee \mathit{Table}(d)) \wedge \\ & & b \neq s \wedge b \neq d \wedge s \neq d \\ & \mathit{Effect}: & \mathit{Clear}(s) \wedge \neg \mathit{On}(b,s) \wedge \\ & & \mathit{On}(b,d) \wedge \neg \mathit{Clear}(d) \end{array}$$

Encoding

- Initial states:

$$On_0(A,B), On_0(B,C), On_0(C,T), Clear_0(A).$$

- Goal states:

$$On_{2n}(C,B) \wedge On_{2n}(B,A) \wedge On_{2n}(A,T).$$

- Action preconditions and effects:

$$\begin{array}{l} Move_t(A,B,C) \rightarrow \\ \qquad Clear_{t-1}(A) \wedge On_{t-1}(A,B) \wedge Clear_{t-1}(C) \wedge \\ \qquad Clear_{t+1}(B) \wedge \neg On_{t+1}(A,B) \wedge \\ \qquad On_{t+1}(A,C) \wedge \neg Clear_{t+1}(C). \end{array}$$

Encoding: Frame axioms

- Classic

$$Move_t(A,B,T) \wedge Clear_{t-1}(C) \rightarrow Clear_{t+1}(C),$$
$$Move_t(A,B,T) \wedge \neg Clear_{t-1}(C) \rightarrow \neg Clear_{t+1}(C).$$

“At least one action” axiom:

$$\bigvee_{\substack{b,s,d \in \{A,B,C,T\} \\ b \neq s, b \neq d, s \neq d, b \neq T}} Move_t(b,s,d).$$

- Explanatory

$$\neg Clear_{t+1}(C) \wedge Clear_{t-1}(C) \rightarrow$$
$$Move_t(A,B,C) \vee Move_t(A,T,C) \vee Move_t(B,A,C) \vee Move_t(B,$$

Planning strategy

- Sequential for each pair of actions α and β, add axioms of the form $\neg\alpha_t \vee \neg\beta_t$ for each odd time step. For example, we will have:

$$\neg Move_t(A, B, C) \vee \neg Move_t(A, B, T).$$

- parallel for each pair of actions α and β, add axioms of the form $\neg\alpha_t \vee \neg\beta_t$ for each odd time step if α effects contradict β preconditions. For example, we will have

$$\neg Move_t(B, T, A) \vee \neg Move_t(A, B, C).$$

Encoding into SAT

- Assumption: the possible values of all the variables are bounded.
- Naive idea: Encode all possible ground predicates as Boolean variables.
  E.g.: $Move_1(B, T, A) \Longrightarrow$ `Move1_B_T_A`
- much more efficient encodings have been presented [29, 19]
- customizations of SAT solvers [23].

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The problem [8, 7]

Ingredients:

- A system written as a Kripke structure $M := \langle S, I, T, \mathcal{L} \rangle$
  - S: set of states
  - I: set of initial states
  - T: transition relation
  - $\mathcal{L}$: labeling function
- A property f written as a LTL formula:
  - a propositional literal p
  - $h \wedge g$, $h \vee g$, $\mathbf{X}g$, $\mathbf{G}g$, $\mathbf{F}g$,$h\mathbf{U}g$ and $h\mathbf{R}g$,
    X, **G**, **F**, **U**, **R** “next”, “globally”, “eventually”, “until” and “releases”
- an integer k (bound)

The problem (cont.)

Problem:
Is there an execution path of M of length k satisfying the temporal property f?:

$$M \models_k \mathbf{f}$$

The encoding

Equivalent to the satisfiability problem of a Boolean formula $[[M,f]]_k$ defined as follows:

$$
\begin{aligned}
[[M,f]]_k &:= [[M]]_k \wedge [[f]]_k \quad (1)\\
[[M]]_k &:= I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}), \quad (2)\\
[[f]]_k &:= (\neg \bigvee_{l=0}^{k} T(s_k, s_l) \wedge [[f]]_k^0) \vee \bigvee_{l=0}^{k} (T(s_k, s_l) \wedge {}_l[[f]]_k^0), \quad (3)
\end{aligned}
$$

The encoding of $[[f]]_k^i$ and ${}_l[[f]]_k^i$

f	$[[f]]_k^i$	${}_l[[f]]_k^i$
p	p_i	p_i
$\neg p$	$\neg p_i$	$\neg p_i$
$h \wedge g$	$[[h]]_k^i \wedge [[g]]_k^i$	${}_l[[h]]_k^i \wedge {}_l[[g]]_k^i$
$h \vee g$	$[[h]]_k^i \vee [[g]]_k^i$	${}_l[[h]]_k^i \vee {}_l[[g]]_k^i$
$\mathbf{X}g$	$[[g]]_k^{i+1}$ $\quad$ $if\ i < k$ $\bot$ $\quad$ $otherwise.$	${}_l[[g]]_k^{i+1}$ $\quad$ $if\ i < k$ ${}_l[[g]]_k^{l}$ $\quad$ $otherwise.$
$\mathbf{G}g$	$\bot$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$\mathbf{F}g$	$\bigvee_{j=i}^{k} [[g]]_k^j$	$\bigvee_{j=min(i,l)}^{k} {}_l[[g]]_k^j$
$h\mathbf{U}g$	$\bigvee_{j=i}^{k} \left([[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} [[h]]_k^n \right)$	$\bigvee_{j=i}^{k} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{j-1} {}_l[[h]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[g]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[h]]_k^n \wedge \bigwedge_{n=l}^{j-1} {}_l[[h]]_k^n \right)$
$h\mathbf{R}g$	$\bigvee_{j=i}^{k} \left([[h]]_k^j \wedge \bigwedge_{n=i}^{j} [[g]]_k^n \right)$	$\bigwedge_{j=min(i,l)}^{k} {}_l[[g]]_k^j \vee$ $\bigvee_{j=i}^{k} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{j} {}_l[[g]]_k^n \right) \vee$ $\bigvee_{j=l}^{i-1} \left({}_l[[h]]_k^j \wedge \bigwedge_{n=i}^{k} {}_l[[g]]_k^n \wedge \bigwedge_{n=l}^{j} {}_l[[g]]_k^n \right)$

Example: **F**p (reachability)

- $f := \mathbf{F}p$: is there a reachable state in which p holds?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{j=0}^{k} p_j$$

Example: **G**p

- $f := \mathbf{G}p$: is there a path where p holds forever?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{l=0}^{k} T(s_k, s_l) \wedge \bigwedge_{j=0}^{k} p_j$$

Example: **GF**$q \wedge$ **F**p (fair reachability)

- $f :=$ **GF**$q \wedge$ **F**p: is there a reachable state in which p holds provided that q holds infinitely often?
- $[[M, f]]_k$ is:

$$I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1}) \wedge \bigvee_{j=0}^{k} p_j \wedge \bigvee_{l=0}^{k} \left(T(s_k, s_l) \wedge \bigvee_{j=l}^{k} q \right)$$

Bounded Model Checking

- very efficient for some problems
- lots of enhancements [8, 1, 56, 60, 13]

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- The SAT Association
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  `http://www.satlive.org/index.jsp`
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