Towards Defeasible Mappings for Tractable Description Logics (Technical Report)

Kunal Sengupta and Pascal Hitzler

Wright State University, Dayton OH 45435, USA
{sengupta.4,pascal.hitzler}@wright.edu

Abstract. We present a novel approach to denote mappings between $\mathcal{EL}$-based ontologies which are defeasible in the sense that such a mapping only applies to individuals if this does not cause an inconsistency. This provides the advantage of handling exceptions automatically and thereby avoiding logical inconsistencies that may be caused due to the traditional type of mappings. We consider the case where mappings from many possibly heterogeneous ontologies are one way links towards an overarching ontology. Questions can then be asked in terms of the concepts in the overarching ontology. We provide the formal semantics for the defeasible mappings and show that reasoning under such a setting is decidable even when the defeasible axioms apply to unknowns. Furthermore, we show that this semantics actually is strongly related to the idea of answer sets for logic programs.

1 Introduction

Description logic (DL) based knowledge representation is gaining in popularity and with that the number of domain ontologies is also on the rise. Especially in the medical domain, tractable fragments of DLs are heavily used. For example, SNOMED CT is a medical ontology which consists of more than 300,000 concepts, and which can be described in the description logic $\mathcal{EL}$ [1]. Smaller fragments of DLs are especially interesting for application scenarios where fast and efficient reasoning may be critical.

In this paper, we provide a formal framework for dealing with defeasible reasoning for smaller fragments of DLs, especially in the context of ontology alignment. In particular we consider a language $\mathcal{ER}_{\perp,\mathcal{O}}$ which allows for conjunction, existentials, role chains, disjointness of concepts and ABox statements and provide a semantics for one-way (defeasible) alignments from terms in several ontologies to one overarching ontology such that queries can be asked in terms of this overarching ontology, while answers may contain instances from several lower level ontologies. For defeasibility we take motivation from default logic [21] and define the semantics along similar lines. It turns out that combining DLs with default-like semantics is not very straightforward as unrestricted default applications may result in undecidability [2,22]. Previously, decidability was usually obtained for such logics by restricting defeasibility to known individuals, i.e. to a finite set of entities. In this paper, we show that the combination

$$Veg \sqcap NonVeg \sqsubseteq \bot \quad (1)$$
$$\exists consumes.EggFood \sqsubseteq NonVeg \quad (2)$$
$$contains \circ consumes \sqsubseteq consumes \quad (3)$$
$$\{juliet\} \sqsubseteq Veg \quad (4)$$
$$\{romeo\} \sqsubseteq Eggetarian \quad (5)$$
$$Eggetarian \sqsubseteq Vegetarian \quad (6)$$
$$Eggetarian \sqsubseteq \exists eats.Egg \quad (7)$$
$$Eggetarian \sqcap NonVegetarian \sqsubseteq \bot \quad (8)$$
$$\{caesar\} \sqsubseteq Vegetarian \quad (9)$$
$$\{caesar\} \sqsubseteq NotEggetarian \quad (10)$$
$$NotEggetarian \sqcap Eggetarian \sqsubseteq \bot \quad (11)$$
$$Vegetarian \equiv Veg \quad (12)$$
$$NonVeg \equiv NonVegetarian \quad (13)$$
$$EggFood \equiv Egg \quad (14)$$
$$eats \sqsubseteq consumes \quad (15)$$

Fig. 1. Example mapping with selected axioms.

of defeasible mappings with DLs presented here is decidable even without this type of restriction. Decidability in our setting results from our restriction to a tractable language in the $\mathcal{EL}$ family, together with the avoidance of recursion through the defeasible axioms resulting from our specific, but practically important application scenario, namely the one-way alignment of ontologies.

Indeed, similar concepts appear in several ontologies from heterogeneous domains, but these concepts may slightly differ semantically. The motivation of using defeasible axioms as alignments stems from the need to handle such heterogeneity among various data models. As we discuss in our previous work [22], DL axioms are semantically too rigid to be able to deal with alignments in such heterogeneous settings, in particular in the light of the fact that ontology alignment systems mostly rely on string similarity matching [7]. For example, the concept that represents those human beings who consume only vegetarian food may be part of two different domain ontologies but the notion of what vegetarian food means might slightly differ depending on the context, e.g. in some places eggs might be part of a typical vegetarian diet while in others this may not be so. Aligning these different world views appropriately cannot be done by simply mapping the respective concepts representing a "vegetarian person" in different ontologies, as claiming that they were equivalent may lead to inconsistencies.

For example consider the axioms in Figure 1 (see section 2 for explanations of the notation). Axioms 1–4 represent one ontology and axioms 5–11 another ontology. An alignment system may give alignments similar to axioms 12–15. Since *romeo* is an *Eggetarian* (axiom 5) he is also a *Vegetarian* (axiom 6). And since every *Vegetarian* is also a *Veg* as per the mapping axiom 12, *romeo* is a *Veg*. From axioms 5, 7, 14, 15 and 2 we obtain that *romeo* is also a *NonVeg*. But *Veg* and *NonVeg* are disjoint classes, so this results in an inconsistency. But applying the same rules to *caesar* does not cause an inconsistency. The usual process of repairing alignments like this is to remove mappings that cause the inconsistency [13]. But we would then lose the conclusion that *caesar* is also a *Veg*. If we replace the mapping axioms with defeasible axioms as introduced below, then we could achieve this outcome where we carry over the similarities while respecting the differences.

The paper is organized as follows. In section 2, we set the preliminaries by describing the language $\mathcal{ER}_{\perp,\mathcal{O}}$. The context of ontology mappings as well as the syntax and the semantics of defeasible mapping axioms along with the discussion on decidability is presented in section 3. Section 4 contains a description of the relation of the semantics of this approach with that of answer set programming for logic programs. Finally we discuss related work in section 5 and provide closing remarks in section 6.

2 The Description Logic $\mathcal{ER}_{\perp,\mathcal{O}}$

We consider the DL $\mathcal{ER}_{\perp,\mathcal{O}}$ (see [1] for further background). Let N_C be a set of *atomic concepts* (or *atomic classes*), let N_R be a set of *roles* and let N_I be a set of *individuals*, which contains an element $\iota_{R,D}$ for each pair $(R, D) \in N_R \times N_C$. These $\iota_{R,D}$ are called *auxilliary* individuals. *Complex class expressions* (short, *complex classes*) in $\mathcal{ER}_{\perp,\mathcal{O}}$ are defined using the grammar

$$C ::= A \mid \top \mid \perp \mid C_1 \sqcap C_2 \mid \exists R.C \mid \{a\},$$

where $A \in N_C$, $R \in N_R$ and C_1, C_2, C are complex class expressions. Furthermore, a *nominal class* (short, *nominal*) is represented as $\{a\}$, where $a \in N_I$. A *TBox* in $\mathcal{ER}_{\perp,\mathcal{O}}$ is a set of *general class inclusion* (*GCI*) axioms of the form $C \sqsubseteq D$, where C and D are complex classes. $C \equiv D$ abbreviates two GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$. An *RBox* in $\mathcal{ER}_{\perp,\mathcal{O}}$ is a set of *role inclusion* (*RI*) axioms of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$, where $R_1, \ldots, R_n, R \in N_R$. An *ABox* in $\mathcal{ER}_{\perp,\mathcal{O}}$ is a set of GCIs of the form $\{a\} \sqsubseteq C$ and $\{a\} \sqsubseteq \exists R.\{b\}$ where $\{a\}, \{b\}$ are nominals and C is a complex class.

An $\mathcal{ER}_{\perp,\mathcal{O}}$ *knowledge base* or *ontology* is a set of TBox, RBox and ABox statements which furthermore satisfy the condition that nominals occur only in ABox statements. This condition is a restriction of $\mathcal{ER}_{\perp,\mathcal{O}}$ as compared to, e.g., the allowed use of nominals in OWL 2 EL: While we allow for a full ABox, the TBox remains free of nominals. In particular, axioms such as $A \sqsubseteq \exists R.\{a\}$, with A an atomic or complex class other than a nominal, are not allowed.

An *initial* $\mathcal{ER}_{\perp,\mathcal{O}}$ knowledge base is an $\mathcal{ER}_{\perp,\mathcal{O}}$ knowledge base which does not contain any auxiliary individuals.

Example 1. The following is an example of an (initial) $\mathcal{ER}_{\perp,\mathcal{O}}$ knowledge base.

$$\mathit{Bird} \sqsubseteq \mathit{Fly} \qquad \mathit{Penguin} \sqcap \mathit{Fly} \sqsubseteq \perp$$

$$\mathit{Penguin} \sqsubseteq \mathit{Bird} \qquad \{tom\} \sqsubseteq \exists hasPet.Penguin$$

Next, we describe the semantics of the language $\mathcal{ER}_{\perp,\mathcal{O}}$ using the notion of interpretation. An *interpretation* $\mathcal{I}$ of an $\mathcal{ER}_{\perp,\mathcal{O}}$ knowledge base *KB* is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set of elements called the *domain of interpretation* and $\cdot^{\mathcal{I}}$ is the *interpretation function* that maps every individual

Axiom	Semantics
$\top$	$\Delta^{\mathcal{I}}$
$\bot$	$\emptyset$
$\{a\}$	$\{a^{\mathcal{I}}\}$
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
$C \sqsubseteq \exists R.D$	$C^{\mathcal{I}} = \{x \mid \text{there exists some } y \text{ with } (x,y) \in R^{\mathcal{I}} \text{ and } y \in D^{\mathcal{I}}\}$
$R_1 \circ R_2 \sqsubseteq R$	$R_1^{\mathcal{I}} \circ R_2^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\{a\} \sqsubseteq C$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
$\{a\} \sqsubseteq \exists R.D$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$

Table 1. Semantics of the language $\mathcal{ER}_{\bot,\mathcal{O}}$

in *KB* to an element of $\Delta^{\mathcal{I}}$, every concept in *KB* to a subset of $\Delta^{\mathcal{I}}$, and every role to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a *model* of an $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base *KB* if it satisfies all the TBox, RBox and ABox axioms in the sense of Table 1.

It is well-known that any such knowledge base can be cast into normal form, as follows.

Definition 1. *An initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base is in normal form if it contains axioms of only the following forms, where $C, C_1, C_2, D \in N_C$, $R, R_1, R_2 \in N_R$ and $a, b \in N_I$*

$$C \sqsubseteq D \qquad C_1 \sqcap C_2 \sqsubseteq D \qquad R_1 \circ R_2 \sqsubseteq R$$
$$\exists R.C \sqsubseteq D \qquad C_1 \sqcap C_2 \sqsubseteq \bot \qquad \{a\} \sqsubseteq C$$
$$C \sqsubseteq \exists R.D \qquad R_1 \sqsubseteq R \qquad \{a\} \sqsubseteq \exists R.\{b\}$$

Theorem 1. *For every initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base KB there exists a knowledge base KB' in normal form such that $KB \models \bar{A} \sqsubseteq B$ if and only if $KB' \models \bar{A} \sqsubseteq B$, where $\bar{A}$ is a class name or a nominal and B is a class name occurring in KB.*

Definition 2. *Given an initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base KB in normal form, we define the following:*

1. *Completion: the completion comp(KB) of KB is obtained from KB by exhaustively applying the completion rules from Figure 2.*
2. *Clash: a completion comp(KB) of KB contains a clash if $\{a\} \sqsubseteq \bot \in$ comp(KB), for some nominal class $\{a\}$.*

It is easily verified that repeated applications of completion rules on an initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base produces only axioms which are also in normal form, with one exception: Axioms of the form $\{a\} \sqsubseteq \exists R.D$, with $R \in N_R$ and $D \in N_C$, can also appear.

It is straightforward to show that *comp*(*KB*) is well-defined and that the completion process has a polynomial time complexity. This and the soundness

$$\bar{A} \sqsubseteq C, C \sqsubseteq D \mapsto \bar{A} \sqsubseteq D \quad (16)$$
$$\bar{A} \sqsubseteq C_1, \bar{A} \sqsubseteq C_2, C_1 \sqcap C_2 \sqsubseteq D \mapsto \bar{A} \sqsubseteq D \quad (17)$$
$$\bar{A} \sqsubseteq C, C \sqsubseteq \exists R.D \mapsto \bar{A} \sqsubseteq \exists R.D \quad (18)$$
$$\bar{A} \sqsubseteq \exists R.\bar{B}, \bar{B} \sqsubseteq C, \exists R.C \sqsubseteq D \mapsto \bar{A} \sqsubseteq D \quad (19)$$
$$\bar{C} \sqsubseteq \exists R.\bar{D}, \bar{D} \sqsubseteq \bot \mapsto \bar{C} \sqsubseteq \bot \quad (20)$$
$$\bar{A} \sqsubseteq \exists R.\bar{B}, R \sqsubseteq S \mapsto \bar{A} \sqsubseteq \exists S.\bar{B} \quad (21)$$
$$\bar{A} \sqsubseteq \exists R_1.\bar{B}, \bar{B} \sqsubseteq \exists R_2.\bar{C}, R_1 \circ R_2 \sqsubseteq R \mapsto \bar{A} \sqsubseteq \exists R.\bar{C} \quad (22)$$

Fig. 2. $\mathcal{ER}_{\bot,\mathcal{O}}$ completion rules. New axioms resulting from the rules are added to the existing axioms in KB. Symbols of the form $\bar{A}$ can be either a class name or a nominal class. We initialize comp(KB) with KB and $C \sqsubseteq C$, $\bot \sqsubseteq C$, $\bot \sqsubseteq \bot$ for all named classes $C \in N_C$.

and completeness results below are adapted from [1]. Since the proofs are relevant to understanding the discussions in this paper, we include them in the appendix.

Theorem 2. *(soundness and completeness) Let KB be an initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base in normal form. Then every model of KB is a model of comp(KB). Furthermore, if comp(KB) contains a clash then KB is inconsistent.*

Conversely, if A is a classname or a nominal and B is a classname such that $KB \models A \sqsubseteq B$, then $A \sqsubseteq B \in comp(KB)$. Furthermore, if KB is inconsistent then comp(KB) contains a clash.

Note now that the $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base given in Example 1 is inconsistent.

Central to the proof of Theorem 2 is the following construction, which we will also use later in this paper.

Given an $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base KB, let $\mathcal{I} = \mathcal{I}(KB)$ be defined as the following interpretation of $comp(KB)$.

$$\Delta^{\mathcal{I}} = \{a, x_C \mid C \text{ is a class name in } KB \text{ and } a \text{ is an individual in } KB\}$$

$$A^{\mathcal{I}} = \begin{cases} \emptyset, & \text{if } A \sqsubseteq \bot \in comp(KB) \\ \{x_C \mid C \sqsubseteq A \in comp(KB)\} \cup \{x_{\{a\}} \mid \{a\} \sqsubseteq A \in comp(KB)\}, & \\ & \text{if } A \sqsubseteq \bot \notin comp(KB) \end{cases}$$

$$\{a\}^{\mathcal{I}} = \begin{cases} \emptyset, & \text{if } \{a\} \sqsubseteq \bot \in comp(KB) \\ \{x_{\{a\}}\}, & \text{if } \{a\} \sqsubseteq \bot \notin comp(KB) \end{cases}$$

$$\begin{aligned} R^{\mathcal{I}} = \{&(x_C, x_D) \mid C \sqsubseteq \exists R.D \in comp(KB)\} \cup \\ &\{(x_{\{a\}}, x_D) \mid \{a\} \sqsubseteq \exists R.D \in comp(KB)\} \cup \\ &\{(x_{\{a\}}, x_{\{b\}}) \mid \{a\} \sqsubseteq \exists R.\{b\} \in comp(KB)\} \end{aligned}$$

The proof of Theorem 2 shows that $\mathcal{I}$ is a model of both $comp(KB)$ and KB.

3 Mapping Ontologies with $\mathcal{ER}_{\perp,\mathcal{O}}$-Defaults

We consider a rather specific but fundamentally important scenario, namely the integration of ontology-based information by means of an overarching ontology, as laid out and applied e.g. in [14,20] – see also the discussion of this in [22]. One of the central issues related to this type of information integration is how to obtain the mappings of the to-be-integrated ontologies to the overarching ontology, as the manual creation of these mappings is very costly for large ontologies.

However, methods for the automated creation of such mappings – commonly refered to as *ontology alignment* – are still rather crude [7,12], and are therefore prone to lead to inconsistencies of the integrated ontologies, as discussed in section 1. In order to deal with this, we introduce a defeasible mechanism to deal with such mappings. For simplicity of presentation we consider only two ontologies, with one taking the role of the overarching ontology. The other ontology can be considered the disjoint union of the ontologies which are to be integrated.

The following notion is going to be central.

Definition 3. *(defeasible axiom) A* defeasible axiom *is of the form* $C \sqsubseteq_d D$ *or* $R \sqsubseteq_d S$*, where* C, D *are class names and* R, S *are roles.*

Intuitively speaking, our intention with defeasible axioms is the following: It shall function just like a class inclusion axiom, unless it causes an inconsistency, in which case it should not apply to individuals causing this inconsistency. In a sense, such defeasible axioms act as a type of semantic debugging of mappings: The semantics itself encodes the removal of inconsistencies. More specifically speaking, given a defeasible axiom $C \sqsubseteq_d D$, instances of C will also be instances of D, except those instances of C which cause an inconsistency when also an instance of D. Such Cs are usually known as exceptions. Of course this intuitive understanding of defeasible axioms is not entirely straightforward to cast into a formal semantics.[1] We will give such a formal semantics in section 3.1 below.

Definition 4. *(mappings) Let* $\mathcal{O}_1, \mathcal{O}_2$ *be two consistent* $\mathcal{ER}_{\perp,\mathcal{O}}$ *knowledge bases. A (*defeasible*)* mapping *from* $\mathcal{O}1$ *to* $\mathcal{O}2$ *is a defeasible axioms with the left hand side of the axiom a concept or role from* $\mathcal{O}_1$*, and the right hand side a concept or role from* $\mathcal{O}_2$*.*

Note that here we restrict the mappings to axioms involving roles and atomic classes. However, we do so without loss of generality as $C \sqsubseteq_d D$, for complex classes C, D, can be replaced by adding the axiom $C \sqsubseteq A$ to $\mathcal{O}1$ and the axiom $B \sqsubseteq D$ to $\mathcal{O}2$, where A and B are new concept names, and replacing $C \sqsubseteq_d D$ in δ by $A \sqsubseteq_d B$. Similarly, our approach encompasses the specific case of*ontology population*, where $\mathcal{O}1$ is empty and all mappings are of the form $\{a\} \sqsubseteq_d C$.

Definition 5. *(mapped-tuple) Let* $\mathcal{O}_1, \mathcal{O}_2$ *be two ontologies in* $\mathcal{ER}_{\perp,\mathcal{O}}$ *with* δ *the set of defeasible mappings from* $\mathcal{O}_1$ *to* $\mathcal{O}_2$*. Then the tuple* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ *is called a* mapped-tuple*.*

[1] Different ways how to do this lead to different non-monotonic logics. This is a well-studied subfield of artificial intelligence, from which we take inspiration.

3.1 Semantics and Decidability

Given a mapped-tuple $(\mathcal{O}_1, \mathcal{O}_2, \delta)$, we define the formal semantics of the mappings following our intuitive reading as discussed above. Informally speaking, the semantics of $C \sqsubseteq_d D$ is similar to that of normal defaults as in Reiter's default logic [21]: if x is in C, then it can be assumed that x is also in D, unless it causes an inconsistency with respect to the current knowledge.

We define the semantics formally as follows. For each mapping axiom $C \sqsubseteq_d D$ in δ we define a set Cand that represents the set of axioms that could be possibly added to the completion of $\mathcal{O}_2$ as a result of the mapping axiom.

$$\mathsf{Cand}(C \sqsubseteq_d D) = \{\{a\} \sqsubseteq D \mid \{a\} \sqsubseteq C \in comp(\mathcal{O}_1)\} \quad (23)$$

Furthermore, we define the set $\mathsf{Cand}^{\mathsf{n}}$ as the power set of Cand for each mapping axiom.

$$\mathsf{Cand}^{\mathsf{n}}(C \sqsubseteq_d D) = \{X \mid X \subseteq \mathsf{Cand}(C \sqsubseteq_d D)\} \quad (24)$$

Similarly, we define the corresponding sets $\mathsf{Cand}_{\mathcal{R}}$ and $\mathsf{Cand}^{\mathsf{n}}_{\mathcal{R}}$ for mapping axioms involving roles.

$$\mathsf{Cand}_{\mathcal{R}}(R \sqsubseteq_d S) = \{\{a\} \sqsubseteq \exists S.\{b\} \mid \{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_1)\} \quad (25)$$
$$\mathsf{Cand}^{\mathsf{n}}_{\mathcal{R}}(R \sqsubseteq_d S) = \{X \mid X \subseteq \mathsf{Cand}_{\mathcal{R}}(R \sqsubseteq_d S)\} \quad (26)$$

Note that a and b may be auxiliary individuals.

Definition 6. *(mapped ontology) Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple. Define* selections *and the corresponding* mapped ontology *as follows:*

(i) For each mapping axiom of the form $C \sqsubseteq_d D \in \delta$, a selection *for $C \sqsubseteq_d D$ is any $\Sigma_{C \sqsubseteq_d D} \subseteq \mathsf{Cand}^{\mathsf{n}}(C \sqsubseteq_d D)$.*

(ii) For each mapping axiom of the form $R \sqsubseteq_d S \in \delta$, a selection *for $R \sqsubseteq_d S$ is any $\Sigma_{R \sqsubseteq_d S} \subseteq \mathsf{Cand}^{\mathsf{n}}_{\mathcal{R}}(R \sqsubseteq_d S)$*

(iii) Given selections for all mappings $\mu \in \delta$, we use Σ to denote their union $\Sigma = \bigcup_{\mu\in\delta} \Sigma_\mu$, and call Σ a selection *for the given mapped-tuple.*

(iv) $\mathcal{O}_2^{\Sigma} = comp(\mathcal{O}_2) \cup \bigcup_{X\in\Sigma} X$ is then called a mapped ontology.

Note that each mapped-tuple $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ can give rise to only a finite number of corresponding mapped ontologies, and the number is bounded by $|\mathsf{Cand}^{\mathsf{n}}(C \sqsubseteq_d D)|^{|\delta_1|} \times |\mathsf{Cand}^{\mathsf{n}}_{\mathcal{R}}(R \sqsubseteq_d S)|^{|\delta_2|}$, where δ_1 (respectively, δ_2) is the set of class (respectively, role) mappings contained in δ.

Definition 7. *(preferred mapping) Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple. Then for any two mapped ontologies $\mathcal{O}_2^{\Sigma^i}, \mathcal{O}_2^{\Sigma^j}$ we say $\mathcal{O}_2^{\Sigma^i} \succ \mathcal{O}_2^{\Sigma^j}$ or $\mathcal{O}_2^{\Sigma^i}$* is preferred over *$\mathcal{O}_2^{\Sigma^j}$, if all of the following hold.*
- $\Sigma^i_\mu \supseteq \Sigma^j_\mu$, for all $\mu \in \delta$
- $\Sigma^i_\mu \supset \Sigma^j_\mu$, for some $\mu \in \delta$
Note that μ can be of the form $C \sqsubseteq_d D$ or $R \sqsubseteq_d S$.

The notion of preferred mapping is used to identify the individuals to which the defeasible axioms maximally apply.

Definition 8. *(mapped completion and mapped entailment) Given a mapped-tuple* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$*, let* $\mathcal{O}_2^{\Sigma}$ *be a mapped ontology obtained from some selection* Σ*. Then the completion* $comp(\mathcal{O}_2^{\Sigma})$ *obtained by exhaustively applying the rules in Figure 2 is said to be a* mapped completion *of* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ *if* $\mathcal{O}_2^{\Sigma}$ *is consistent and there is no consistent mapped ontology* $\mathcal{O}_2^{\Sigma^i}$ *such that* $\mathcal{O}_2^{\Sigma^i} \succ \mathcal{O}_2^{\Sigma}$ *holds.*

Furthermore, let α *an axiom of the form* $\{a\} \sqsubseteq \{b\}$ *or* $\{a\} \sqsubseteq \exists R.\{b\}$*. Then* α *is* entailed *by* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$*, written* $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models_d \alpha$*, if* $\alpha \in comp(\mathcal{O}_2^{\Sigma})$ *for each mapped completion* $\mathcal{O}_2^{\Sigma}$ *of* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$*.*

Lemma 1. *A mapped-tuple* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ *always has a mapped completion.*

Proof. There are two conditions for obtaining a mapped completion $comp(\mathcal{O}_2^{\Sigma})$: (1) $\mathcal{O}_2^{\Sigma}$ is consistent, and (2) $\mathcal{O}_2^{\Sigma}$ is maximal with respect to $\succ$. It is clear that there is at least one Σ such that $\mathcal{O}_2^{\Sigma}$ is consistent, namely $\Sigma = \emptyset$. If this is the only Σ producing a consistent mapped ontology, then $comp(\mathcal{O}_2^{\Sigma})$ is the corresponding mapped completion. Now let $\mathcal{S}$ be the set of all selections which produce a consistent mapped ontology. We already know that $\mathcal{S}$ is finite, and so the set of corresponding consistent mapped ontologies is also finite, and therefore contains maximal elements with respect to the preference relation $\prec$. Each of these maximal elements is then a mapped completion of $(\mathcal{O}_1, \mathcal{O}_2, \delta)$.

Theorem 3. *The problem of entailment checking for a mapped-tuple* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ *is decidable.*

Proof. In order to check entailment, it suffices to obtain all the possible mapped completions as per definition 8. Since there is only a finite number of possible selections for $(\mathcal{O}_1, \mathcal{O}_2, \delta)$, then as argued in the proof of Lemma 1 there is only a finite number of corresponding mapped ontologies, and furthermore we know that exhaustive application of the completion rules terminates. Hence the task is decidable. □

3.2 Applying Defeasible Mappings to Unknowns

So far we have defined the semantics of defeasible mappings and a way to derive entailments. Using these mappings, queries can be asked in terms of concepts of the ontology which is being mapped to.

For instance let the ontology $\mathcal{O}_1$ have axioms

$$\{john\} \sqsubseteq USCitizen \qquad \{john\} \sqsubseteq Traveler$$
$$USCitizen \sqsubseteq \exists hasPassport.USPassport,$$

let the ontology $\mathcal{O}_2$ have axioms

$$Tourist \sqsubseteq \exists hasPP.Passport$$
$$\exists hasPP.AmericanPassport \sqsubseteq EuVisaNotRequired,$$

and let δ consist of the mappings

$$Traveler \sqsubseteq_d Tourist \qquad hasPassport \sqsubseteq_d hasPP$$
$$USPassport \sqsubseteq_d AmericanPassport.$$

We can then ask questions in terms of the concepts and roles of $\mathcal{O}_2$ like "list all the tourists," i.e., all instances that belong to the class *Tourist*, and we would get the answer *john*. But if we look carefully, we would also expect *john* as an instance of the class *EuVisaNotRequired*.

However, as per the semantics we have defined in the previous section, we would not be able to derive this conclusion. This is because the defeasible mappings do not apply to unknowns. In this case the unknown in question is *john*'s *USPassport*. We address this issue by modifying the semantics in order to apply the mappings to unknowns as well.

First of all, recall that the set N_I already contains the auxiliary individuals ι_{RC} for every $R \in N_R$ and $C \in N_C$ – we have not yet made use of them, but we will do so now. In fact, we now modify the completion rules in Figure 2 by adding two additional rules as follows, and where $a \in N_I$, i.e. a may also be an auxiliary individual.

$$\{a\} \sqsubseteq \exists R.D \mapsto \{a\} \sqsubseteq \exists R.\{\iota_{RD}\} \tag{27}$$
$$\{a\} \sqsubseteq \exists R.D \mapsto \{\iota_{RD}\} \sqsubseteq D \tag{28}$$

Furthermore, we retain all the definitions from section 3.1 starting from Cand, $\mathsf{Cand}_{\mathcal{R}}$ but using the completion $comp_u(\mathcal{O}_1)$ obtained by applying the completion rules in Figure 2 in conjunction with the new rules when producing selections. We still use *comp*, the previous version without the new rules, for all other steps.

Returning to the example above, $comp_u(\mathcal{O}_1)$ now becomes

$$\{john\} \sqsubseteq USCitizen \qquad \{john\} \sqsubseteq Traveler$$
$$USCitizen \sqsubseteq \exists hasPassport.USPassport \qquad \{\iota_{hpp,usp}\} \sqsubseteq USPassport$$
$$\{john\} \sqsubseteq \exists hasPassport.\{\iota_{hpp,usp}\},$$

and from the mappings we obtain

$$\mathcal{O}_2^{\Sigma} = comp(\mathcal{O}_2) \cup \{ \quad \{john\} \sqsubseteq Tourist, \quad \{john\} \sqsubseteq \exists hasPP.\{\iota_{hpp,usp}\}, \quad \{\iota_{hpp,usp}\} \sqsubseteq AmericanPassport\} \quad \}.$$

Note, that this $\mathcal{O}_2^{\Sigma}$ is the only maximal mapped ontology. When we apply the completion rules of Figure 2 on $\mathcal{O}_2^{\Sigma}$, rule 19 will produce the axiom $\{john\} \sqsubseteq$ *EuVisaNotRequired*.

We now show that, under this new version, default mappings behave just as ordinary mappings provided no inconsistencies arise. This is of course exactly what we would like to obtain, i.e., the new semantics is conservative in this respect and "kicks in" only if needed due to inconsistencies.

Theorem 4. *Let* $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ *be a mapped-tuple such that for any selection* Σ, $\mathcal{O}_2^{\Sigma}$ *is consistent. Let* α *be an* $\mathcal{ER}_{\perp,\mathcal{O}}$ *axiom of the form* $\{a\} \sqsubseteq C$ *or* $\{a\} \sqsubseteq$

$\exists R.\{b\}$*, where* a, b *are named individuals from* $\mathcal{O}_1$ *and* C, R *are class names, respectively role names, from* $\mathcal{O}_2$*. Then* $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models \alpha$ *if and only if* $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta} \models \alpha$*, where* $\bar{\delta}$ *is exactly the same as* δ *but with all* $\sqsubseteq_d$ *replaced by* $\sqsubseteq$*.*

Proof. In this case there is only one relevant selection Σ, namely the full selection, since for every possible Σ, $\mathcal{O}_2^{\Sigma}$ is consistent.

Consider an interpretation $\mathcal{I} = \mathcal{I}(\mathcal{O}_2^{\Sigma})$ of $\mathcal{O}_2^{\Sigma}$, defined as at the end of Section 2, and recall that $\mathcal{I} \models \mathcal{O}_2^{\Sigma}$.

Let $\mathcal{I}'$ be an interpretation of $\mathcal{O}_1 \cup \mathcal{O}_2^{\Sigma}$ which extends $\mathcal{I}$ such that $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \cup \{x_C \mid C \in N_C^{\mathcal{O}_1}\}$, and for all $C \in N_C^{\mathcal{O}_1}$ and $R \in N_R^{\mathcal{O}_1}$, $C^{\mathcal{I}'}$ and $D^{\mathcal{I}'}$ are constructed from $comp_u(\mathcal{O}_1)$ exactly as it is done for $\mathcal{I}$ from $comp(\mathcal{O}_2^{\Sigma})$. Then clearly $\mathcal{I}' \models \mathcal{O}_2^{\Sigma} \cup \mathcal{O}_1$. Furthermore, axioms of the form $\{a\} \sqsubseteq C$, $\{a\} \sqsubseteq \exists R.\{b\}$ where $a, b \in N_I^{\mathcal{O}_1}, C \in N_C^{\mathcal{O}_2}$ and $R \in N_R^{\mathcal{O}_2}$ are only produced from the axioms of $\mathcal{O}_2^{\Sigma}$.

Moreover, $\mathcal{I}' \models \mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta}$ holds. To prove this it suffices to show that $\mathcal{I}'$ satisfies all axioms $C \sqsubseteq D \in \bar{\delta}$ and $R \sqsubseteq S \in \bar{\delta}$ since we already know that $\mathcal{I}' \models \mathcal{O}_1 \cup \mathcal{O}_2$. And indeed, for every axiom $C \sqsubseteq D \in \bar{\delta}$ (which also means $C \sqsubseteq_d D \in \delta$), we know that if $\{a\} \sqsubseteq C \in comp_u(\mathcal{O}_1)$ then $\{a\} \sqsubseteq D \in \mathcal{O}_2^{\Sigma}$. Hence, by definition of $\mathcal{I}'$, $a \in C^{\mathcal{I}'} \cap D^{\mathcal{I}'}$. Similarly, for every axiom $R \sqsubseteq S \in \bar{\delta}$, we know that whenever $\{a\} \sqsubseteq \exists R.\{b\} \in comp_u(\mathcal{O}_1)$, we have $\{a\} \sqsubseteq \exists S.\{b\} \in \mathcal{O}_2^{\Sigma}$, and by definition of $\mathcal{I}'$, we obtain $(a, b) \in R^{\mathcal{I}'}, S^{\mathcal{I}'}$.

So now, in particular, if $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta} \models \alpha$ then $\mathcal{I}' \models \alpha$, and therefore $\mathcal{I} \models \alpha$, since α does not contain any class or role names from $\mathcal{O}_1$. By definition of $\mathcal{I}$, we then obtain $\alpha \in comp(\mathcal{O}_2^{\Sigma})$ and consequently $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models \alpha$ as required.

Conversely, consider an interpretation $\mathcal{I} = \mathcal{I}(\overline{\mathcal{O}})$ of $\overline{\mathcal{O}} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta}$ obtained as defined at the end of Section 2, and recall that $\mathcal{I} \models \overline{\mathcal{O}}$.

Now consider $\mathcal{O}' = comp_u(\mathcal{O}_1) \cup comp(\mathcal{O}_2) \cup \delta \cup \Sigma$ and note that $\mathcal{O}_2^{\Sigma} = comp(\mathcal{O}_2) \cup \Sigma \subseteq \mathcal{O}'$ and also that $\overline{\mathcal{O}} \subseteq \mathcal{O}'$. Let $\mathcal{I}' = \mathcal{I}(\mathcal{O}')$ be obtained as defined at the end of Section 2, and recall that $\mathcal{I}' \models \mathcal{O}'$. By construction, we also obtain $\mathcal{I}' \models \mathcal{O}_2^{\Sigma}$ and also that $\mathcal{I}'$ and $\mathcal{I}$ coincide on the signature of $\overline{\mathcal{O}}$.

So now, in particular, if $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models \alpha$, for α as in the statement of the theorem, then $\mathcal{I}' \models \alpha$, and therefore $\mathcal{I} \models \alpha$, and by definition of $\mathcal{I}$ we obtain $\alpha \in comp(\mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta})$. Consequently, $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \bar{\delta} \models \alpha$ as required. □

4 Relationship with Answer sets

The above semantics is inspired by Reiter's default logic, as already mentioned. Formally, we show that it is very closely related with the prominent answer set semantics from logic programming, which in turn has a well-established relationship to Reiter's default logic. We first recall the definition of answer sets from [10], see [11] for exhaustive background reading.

Definition 9. *(answer sets) An* extended program *is a logic program that contains rules of the form*

$$L_1, \ldots, L_m, not\ L_{m+1}, \ldots, not\ L_n \rightarrow L_0,$$

	Axiom	Rule
1	$C \sqsubseteq D$	$C(x) \rightarrow D(x)$
2	$C \sqsubseteq \bot$	$C(x) \rightarrow \neg C(x)$
3	$\exists R.C \sqsubseteq D$	$R(x,y) \wedge C(y) \rightarrow D(x)$
4	$C_1 \sqcap C_2 \sqsubseteq_d D$	$C_1(x) \wedge C_2(x) \rightarrow D(x)$
5	$C_1 \sqcap C_2 \sqsubseteq_d \bot$	$C_1(x) \rightarrow \neg C_2(x), C_2(x) \rightarrow \neg C_1(x)$
6	$R_1 \sqcap R$	$R_1(x,y) \rightarrow R(x,y)$
7	$R_1 \circ R_2 \sqsubseteq R$	$R_1(x,y) \wedge R_2(y,z) \rightarrow R(x,z)$
8	$\{a\} \sqsubseteq C$	$\rightarrow C(a)$
9	$\{a\} \sqsubseteq \exists R.\{b\}$	$\rightarrow R(a,b)$

Table 2. Rewriting of axioms to rules

where $0 \leq m \leq n$ *and each* L_i *is a literal* A *or* $\neg A$. $\neg$ *denotes so-called* classical negation, *as opposed to not which denotes* default negation.

For Π *an extended program that contains no variables and does not contain not , let Lit be the set of ground literals in the language of* Π. *The* answer set $\alpha(\Pi)$ *of* Π *is the smallest subset* S *of Lit such that*

1. *for any rule* $L_1, \ldots, L_m \rightarrow L_0 \in \Pi$, *if* $L_1, \ldots L_2 \in S$, *then* $L_0 \in S$, *and*
2. *if* S *contains a pair of complementary literals, then* $S = Lit$.

ForΠ *a (general) extended program and Lit the set of all literals in the language of* Π, *define* Π^S, *for a set* $S \subseteq Lit$, *as the extended program obtained by deleting, from* Π,

1. *each rule that has some not* L *in its body with* $L \in S$, *and*
2. *all expressions of the form not* L *in the bodies of the remaining rules.*

Finally, S *is an* answer set *of* Π *if* $S = \alpha(\Pi^S)$.

Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple. We now define an extended program $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$ as follows. For every axiom of the form $C \sqsubseteq_d D \in \delta$ and for all $\{a\} \sqsubseteq C \in comp(\mathcal{O}_1)$, we add rules of the following form to $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$.

$$\rightarrow C(a) \tag{29}$$

$$C(a), not\ \neg D(a) \rightarrow D(a) \tag{30}$$

For mapping axioms of the form $R \sqsubseteq_d S \in \delta$, we add the following rules.

$$\rightarrow R(a,b) \tag{31}$$

$$R(a,b), not\ \neg S(a,b) \rightarrow S(a,b) \tag{32}$$

Furthermore, we add to $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$ all possible groundings of the rules obtained by rewriting $comp(\mathcal{O}_2)$ as per the rules in Table 2, using all the individuals that occur in $\mathcal{O}_1, \mathcal{O}_2$.

It should be noted that we do not provide a transformation for axioms of the form $C \sqsubseteq \exists R.D$ in Table 2. This is because for representing defeasible axioms in logic programs we need the classical negation [10] and to represent axioms with existentials on the right hand side we require existential rules. Although a stable model semantics for existential rules has been defined in [17], it is not

$$\{a\} \sqsubseteq C \quad (33) \qquad C \sqsubseteq_d D \quad (35) \qquad D \sqcap E \sqsubseteq \bot \quad (37)$$
$$\{a\} \sqsubseteq B \quad (34) \qquad B \sqsubseteq_d E \quad (36) \qquad D \sqsubseteq F \quad (38)$$
$$E \sqsubseteq F \quad (39)$$

Fig. 3. Example mapping

defined for extended programs with classical negation. Furthermore, it is not straightforward to extend the approach from [17] to extended programs. So we restrict ourselves to showing that our reduction works for the case when axioms of the form $C \sqsubseteq \exists R.D$ are not present. This is sufficient to show that our approach aligns well with the answer set programming semantics.

Example 2. Consider the axioms listed in Figure 3 where axioms 33, and 34 are from $\mathcal{O}_1$, axioms 37, 38, and 39 are from $\mathcal{O}_2$ and the axioms 35, and 36 represent the set δ of defeasible mappings. The corresponding extended program $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$ is as follows.

$$\rightarrow C(a) \qquad C(a) \wedge \text{not } \neg D(a) \rightarrow D(a) \qquad D(a) \rightarrow \neg E(a) \qquad D(a) \rightarrow F(a)$$
$$\rightarrow B(a) \qquad B(a) \wedge \text{not } \neg E(a) \rightarrow E(a) \qquad E(a) \rightarrow \neg D(a) \qquad E(a) \rightarrow F(a)$$

Note there are two answer sets, $S_1 = \{C(a), B(a), D(a), \neg E(a), F(a)\}$ and $S_2 = \{C(a), B(a), E(a), \neg B(a), F(a)\}$, for $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$.

Definition 10. *Let $\mathcal{O}_2^\Sigma$ be a mapped ontology for $(\mathcal{O}_1, \mathcal{O}_2, \delta)$, and let $comp(\mathcal{O}_2^\Sigma)$ be a corresponding mapped completion. Then we define the* mapped answer set *$S(\mathcal{O}_2^\Sigma)$ to be the following set.*

$$\{C(a) \mid C \sqsubseteq_d D \in \delta \text{ and } \{a\} \sqsubseteq C \in comp(\mathcal{O}_1)\} \cup$$
$$\{R(a,b) \mid R \sqsubseteq_d S \in \delta \text{ and } \{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_1) \cup$$
$$\{C(a) \mid \{a\} \sqsubseteq C \in comp(\mathcal{O}_2^\Sigma)\} \cup$$
$$\{\neg D(a) \mid C \sqsubseteq_d D \in \delta, \{a\} \sqsubseteq C \in comp(\mathcal{O}_1) \text{ and } \{a\} \sqsubseteq D \notin comp(\mathcal{O}_2^\Sigma)\} \cup$$
$$\{R(a,b) \mid \{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_2^\Sigma)\} \cup$$
$$\{\neg S(a,b) \mid R \sqsubseteq_d S \in \delta, \{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_1) \text{ and } \{a\} \sqsubseteq \exists S.\{b\} \notin comp(\mathcal{O}_2^\Sigma)\}$$

Lemma 2. *Let $\mathcal{O}_2^\Sigma$ be a mapped ontology for $(\mathcal{O}_1, \mathcal{O}_2, \delta)$, and let $comp(\mathcal{O}_2^\Sigma)$ be a corresponding mapped completion. Then the mapped answer set $S(\mathcal{O}_2^\Sigma)$ is an answer set of $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$.*

The proofs of this lemma and the next can be found in the appendix.

Lemma 3. *Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple and let S be an answer set of $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta) = \overline{\Pi}$. Then $S = S(\mathcal{O}_2^\Sigma)$ for some mapped ontology $\mathcal{O}_2^\Sigma$ of $(\mathcal{O}_1, \mathcal{O}_2, \delta)$.*

The following theorem is a now direct consequence of Lemmas 2 and 3.

Theorem 5. *Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple. Then $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models_d \{a\} \sqsubseteq C$ if, and only if, $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta) \models_S C(a)$, where $\models_S$ represents stable model entailment.*

5 Related Work

This work is relevant to at least two areas of work, (1) advancing the use of non-monotonic logics in description logics, and especially in the $\mathcal{EL}$ family, and (2) providing a robust mapping language.

We introduced the use of defeasible semantics to denote mappings in [22], but therein we had to impose a rather significant restriction that exceptions to the default rules may occur only in the known individuals, a restriction which we could completely lift with the approach and setting described in the earlier sections of this work.

With respect to repairing ontology alignments there are approaches like [1,16,19]. The work in [19] is specifically close in spirit to our approach, though we provide a much more detailed semantic treatment which is closely related to Reiter's defaults and answer set programming. Furthermore, we also include defeasible axioms for roles and obtain a mild tractability result. Our approach also forms a basis for a mapping language rather than focusing on the repairing of ontology alignments.

In terms of integration of non-monotonic logics with DLs, recent work [6,4,5] has been proposed in integrating the semantics of rational closure and KLM style semantics to DLs. These are alternative semantics to defaults and thus give a different perspective for apply defeasible logic to DLs. A plethora of other proposals have been made for the integration of non-monotonic logics with DLs, and we refer the reader to [15] which provides pointers to most of the prominent relevant work.

Similar in spirit to to our approach, though on a different logic, is also [3].

6 Conclusion

In this paper we provide an extension for the description logic $\mathcal{ER}_{\perp,\mathcal{O}}$ with the ability to have defeasible mappings between ontologies. This work should be easily extendable to other logics in the $\mathcal{EL}$ family,provided soundness and completeness proofs can be obtained for the base logic along similar lines. We show a reduction from our semantics of defeasible mappings to that of answer set programming. This shows that the approach outlined here is very close to the original notion of defaults. Furthermore, the application of defaults is not limited to named individuals but also applies to unknowns that are implicitly referred to in the knowledge base due to existentials.

Of course, our resulting logic appears to be no longer tractable. However, it should be remarked that the application of a monotonic semantics is completely impossible in the context of inconsistencies coming from the mappings, and repair approaches currently require human intervention and are generally employed at the level of axioms, rather than individuals. Some form of paraconsistent reasoning [18] may be a more efficient contender, but then paraconsistent approaches such as [18] tend to miss many desired consequences.

As a part of future work we consider a smart algorithmization for entailment checking that would perform with reasonable efficiency. One such approach would be to find a method to generate rules that act as templates which could be used to check which selections used to create the mapped ontologies would lead to inconsistencies without actually running the completion algorithm on the mapped ontologies. We also plan to implement the algorithm and perform a detailed evaluation of its performance with respect to time when compared to the monotonic extensions and also with respect to the quality of entailments obtained by defeasible mappings compared to traditional alignments produced by automatic alignment systems. We could make use of data made available by the ontology alignment evaluation initiative [8,9]. Good results would lead to a solid framework towards a robust mapping language for tractable ontology languages.

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Appendix

A Proof of Theorem 2

Theorem 2. *(soundness and completeness) Let KB be an initial $\mathcal{ER}_{\bot,\mathcal{O}}$ knowledge base in normal form. Then every model of KB is a model of comp(KB). Furthermore, if comp(KB) contains a clash then KB is inconsistent.*

Conversely, if A is a classname or a nominal and B is a classname such that $KB \models A \sqsubseteq B$, then $A \sqsubseteq B \in comp(KB)$. Furthermore, if KB is inconsistent then comp(KB) contains a clash.

Proof. There are two cases to consider one in which $comp(KB)$ does not contain a clash and other in which $comp(KB)$ contains a clash. We first consider the case when there is no clash. Let $\mathcal{I}$ be a model of KB. Hence, $\mathcal{I}$ satisfies all the axioms in KB. It suffices to show that if the Left hand side (LHS) of a completion rule holds under $\mathcal{I}$ then so does the right hand side then by induction on the rule application $\mathcal{I}$ is also a model of $comp(KB)$.

For rule 16, we have from the LHS $\mathcal{I} \models C \sqsubseteq D$, $\bar{A} \sqsubseteq C$. $\bar{A}$ can be either a concept or a nominal. If $\bar{A}$ is a concept then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ from which we get $A^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Therefore, $\mathcal{I} \models A \sqsubseteq D$. If $\bar{A}$ is a nominal $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $a^{\mathcal{I}} \in C^{\mathcal{I}}$ from which we get $a^{\mathcal{I}} \in C^{\mathcal{I}}$. Therefore, $\mathcal{I} \models \{a\} \sqsubseteq D$.

For rule 17, we have from the LHS $\mathcal{I} \models C_1 \sqcap C_2 \sqsubseteq D, \bar{A} \sqsubseteq C_1, \bar{A} \sqsubseteq C_2$, if $\bar{A}$ is a concept then $C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, $A^{\mathcal{I}} \subseteq C_1^{\mathcal{I}}$, and $A^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$. Therefore, we have $A^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and consequently $\mathcal{I} \models A \sqsubseteq D$. If $\bar{A}$ is a nominal then $C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, $a^{\mathcal{I}} \in C_1^{\mathcal{I}}$, and $a^{\mathcal{I}} \in C_2^{\mathcal{I}}$. Therefore, we have $a^{\mathcal{I}} \in D^{\mathcal{I}}$ and consequently $\mathcal{I} \models \{a\} \sqsubseteq D$.

For rule 18, we have from the LHS $\mathcal{I} \models C \sqsubseteq \exists R.D, \bar{A} \sqsubseteq C$. If $\bar{A}$ is a concept then, $C^{\mathcal{I}} \subseteq (\exists R.D)^{\mathcal{I}}, A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$. Therefore, we have $A^{\mathcal{I}} \subseteq (\exists R.D)^{\mathcal{I}}$ and consequently $\mathcal{I} \models A \sqsubseteq \exists R.D$. However, if $\bar{A}$ is a nominal then $C^{\mathcal{I}} \subseteq (\exists R.D)^{\mathcal{I}}$, $a^{\mathcal{I}} \in C^{\mathcal{I}}$. Therefore, we have $a^{\mathcal{I}} \in (\exists R.D)^{\mathcal{I}}$ and consequently $\mathcal{I} \models \{a\} \sqsubseteq \exists R.D$.

For rule 19, from the LHS we have $\mathcal{I} \models \exists R.C \sqsubseteq D, \bar{A} \sqsubseteq \exists R.\bar{B}, \bar{B} \sqsubseteq C$, then we have three possible cases. Both $\bar{A}, \bar{B}$ are concepts A, B respectively, then $B^{\mathcal{I}} \subseteq C^{\mathcal{I}}, A^{\mathcal{I}} \subseteq (\exists R.B)^{\mathcal{I}}, (\exists R.C)^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Consider, $A^{\mathcal{I}} \subseteq (\exists R.B)^{\mathcal{I}}$, which means $A^{\mathcal{I}} \subseteq \{x \in \Delta^{\mathcal{I}} \mid \text{there exists a } y \in \Delta^{\mathcal{I}} \text{ with } (x,y) \in R^{\mathcal{I}} \text{ and } y \in B^{\mathcal{I}}\}$, and since $B^{\mathcal{I}} \subseteq C^{\mathcal{I}}$, we get, $A^{\mathcal{I}} \subseteq \{x \in \Delta^{\mathcal{I}} \mid \text{there exists a } y \in \Delta^{\mathcal{I}} \text{ with } (x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$, which means $A^{\mathcal{I}} \subseteq (\exists R.C)^{\mathcal{I}}$. And since, $(\exists R.C)^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, we get $A^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Therefore, $\mathcal{I} \models A \sqsubseteq D$. If $\bar{A}$ is a nominal $\{a\}$ and $\bar{B}$ is a concept B. The proof of this case is similar to that of the first case and can be obtained by replacing all occurances of $A^{\mathcal{I}} \subseteq$ with $a^{\mathcal{I}} \in$ and $A \sqsubseteq$ with $\{a\} \sqsubseteq$. Finally if, $\bar{A}, \bar{B}$ are nominals $\{a\}, \{b\}$ respectively. Again, we can reuse the arguments of the first case, we replace all occurances of $A^{\mathcal{I}} \subseteq$ with $a^{\mathcal{I}} \in$, $A \sqsubseteq$ with $\{a\} \sqsubseteq$, and all the occurances of B with $\{b\}$ and $B^{\mathcal{I}}$ with $\{b^{\mathcal{I}}\}$.

For rule 20, from LHS we have $\bar{C} \sqsubseteq \exists R.\bar{D}, \bar{D} \sqsubseteq \bot$. If $\bar{C}, \bar{D}$ are concepts C, D respectively, we get $C^{\mathcal{I}} = \{x \mid \text{ there exists a } y \in D^{\mathcal{I}} \text{ such that } (x,y) \in R^{\mathcal{I}}\}$, but $D^{\mathcal{I}} = \emptyset$. Hence, $C^{\mathcal{I}} = \emptyset$. The other two cases can be proved in a similar manner.

For rule 21, from LHS we have $\mathcal{I} \models R \sqsubseteq S$ and $\bar{A} \sqsubseteq \exists R.\bar{B}$, then again we have three cases. $\bar{A}, \bar{B}$ are concepts A, B respectively. We have, $A^{\mathcal{I}} \subseteq (\exists R.B)^{\mathcal{I}}$, which means $A^{\mathcal{I}} \subseteq \{x \in \Delta^{\mathcal{I}} \mid$ there exists a $y \in \Delta^{\mathcal{I}}$ with $(x, y) \in R^{\mathcal{I}}$ and $y \in B^{\mathcal{I}}\}$, and since $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$, we get, $A^{\mathcal{I}} \subseteq \{x \in \Delta^{\mathcal{I}} \mid$ there exists a $y \in \Delta^{\mathcal{I}}$ with $(x, y) \in S^{\mathcal{I}}$ and $y \in B^{\mathcal{I}}\}$, subsequently, $A^{\mathcal{I}} \subseteq (\exists S.B)^{\mathcal{I}}$. Therefore, $\mathcal{I} \models A \sqsubseteq \exists S.C$. The other two cases are similar to that of Rule - 19 and the proofs can be obtained in the similar manner as done for Rule - 19.

For rule 22, from LHS we have $\bar{A} \sqsubseteq \exists R_1.\bar{B}, \bar{B} \sqsubseteq \exists R_2\bar{C}, R_1 \circ R_2 \sqsubseteq R$. $\bar{A}, \bar{B}, \bar{C}$ are concepts A, B, C respectively, then $B^{\mathcal{I}} \subseteq \{v \in \Delta^{\mathcal{I}} \mid$ there exists a $w \in \Delta^{\mathcal{I}}$ such that $(v, w) \in R_2^{\mathcal{I}}$ and $w \in C^{\mathcal{I}}\}$ and $A^{\mathcal{I}} \subseteq \{u \in \Delta^{\mathcal{I}} \mid$ there exists a $v \in \Delta^{\mathcal{I}}$ such that $(u, v) \in R_1^{\mathcal{I}}$ and $v \in B^{\mathcal{I}}\}$. Therefore, $A^{\mathcal{I}} \subseteq \{u \in \Delta^{\mathcal{I}} \mid$ there exists a $v \in \Delta^{\mathcal{I}}$ such that $(u, v) \in R_1^{\mathcal{I}}$ and $v \in \{v \in \Delta^{\mathcal{I}} \mid$ there exists a $w \in \Delta^{\mathcal{I}}$ such that $(v, w) \in R_2^{\mathcal{I}}$ and $w \in C^{\mathcal{I}}\}\}$. And since, $R_1^{\mathcal{I}} \circ R_2^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, we have, $A^{\mathcal{I}} \subseteq \{u \in \Delta^{\mathcal{I}} \mid$ there exists a $w \in \Delta^{\mathcal{I}}$ with $(u, w) \in R^{\mathcal{I}}$ and $w \in C^{\mathcal{I}}\}$, which means $A^{\mathcal{I}} \subseteq (\exists R.C)^{\mathcal{I}}$, therefore $\mathcal{I} \models A \sqsubseteq \exists R.C$.

Note in the rest of the cases we can have a similar proof with adjustments similar to that in the proofs of the previous rules. We simply list down all the possibilities. $\bar{A}$ is a nominal $\{a\}$, and $\bar{B}, \bar{C}$ are concepts B, C, respectively. $\bar{A}, \bar{B}$ are nominals $\{a\}, \{b\}$, respectively and $\bar{C}$ is a concept C. $\bar{A}, \bar{B}, \bar{C}$ are nominals $\{a\}, \{b\}, \{c\}$, respectively.

Now, if *comp*(*KB*) contains a clash of the form $\{a\} \sqsubseteq \bot$, then as shown above any model of *KB* should also satisfy all the axioms of *comp*(*KB*), therefore *KB* $\models \{a\} \sqsubseteq \bot$. *KB* is inconsistent.

This shows the first part of the theorem.

For the converse, consider an interpretation $\mathcal{I}$ of *comp*(*KB*) as follows:

$$\Delta^{\mathcal{I}} = \{x_C \mid C \in N_C\} \cup \{x_{\{a\}} \mid \{a\} \in \mathit{KB}\}$$

$$A^{\mathcal{I}} = \begin{cases} \emptyset, & \text{if } A \sqsubseteq \bot \in \mathit{comp}(\mathit{KB}) \\ \{x_C \mid C \sqsubseteq A \in \mathit{comp}(\mathit{KB})\} \cup \{x_{\{a\}} \mid \{a\} \sqsubseteq A \in \mathit{comp}(\mathit{KB})\}, & \\ & \text{if } A \sqsubseteq \bot \notin \mathit{comp}(\mathit{KB}) \end{cases}$$

$$\{a\}^{\mathcal{I}} = \begin{cases} \emptyset, & \text{if } \{a\} \sqsubseteq \bot \in \mathit{comp}(\mathit{KB}) \\ \{x_{\{a\}}\}, & \text{if } \{a\} \sqsubseteq \bot \notin \mathit{comp}(\mathit{KB}) \end{cases}$$

$$\begin{aligned} R^{\mathcal{I}} = \{&(x_C, x_D) \mid C \sqsubseteq \exists R.D \in \mathit{comp}(\mathit{KB})\} \cup \\ &\{(x_{\{a\}}, x_D) \mid \{a\} \sqsubseteq \exists R.D \in \mathit{comp}(\mathit{KB})\} \cup \\ &\{(x_{\{a\}}, x_{\{b\}}) \mid \{a\} \sqsubseteq \exists R.\{b\} \in \mathit{comp}(\mathit{KB})\} \end{aligned}$$

We now show that $\mathcal{I}$ is also a model of *comp*(*KB*) whenever *comp*(*KB*) is clash free, and thereby, also a model of *KB* $\subseteq$ *comp*(*KB*). Let $\alpha \in$ *comp*(*KB*). If α is of the form $C \sqsubseteq D$, $C \sqsubseteq \bot$, $\{a\} \sqsubseteq D$, $C \sqsubseteq \exists R.D$, $\{a\} \sqsubseteq \exists R.D$, $\{a\} \sqsubseteq \exists R.\{b\}$, there is nothing to show.

For α of the form $C_1 \sqcap C_2 \sqsubseteq D$, let $x_C \in C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$, then $C \sqsubseteq C_1 \in$ *comp*(*KB*), and $C \sqsubseteq C_2 \in$ *comp*(*KB*), and thus $C \sqsubseteq D \in$ *comp*(*KB*) by rule - 16. Hence, $x_C \in D$ as required. For α of the form $\exists R.A \sqsubseteq B$, let $(x_C, x_D) \in R^{\mathcal{I}}$ and

$x_D \in A^{\mathcal{I}}$, then $D \sqsubseteq A \in comp(KB)$ and $C \sqsubseteq \exists R.D \in comp(KB)$ and by rule - 19 we have $C \sqsubseteq B$ and thus $x_C \in B^{\mathcal{I}}$ as required.

For α of the form $R \sqsubseteq S$, let $(x_C, x_D) \in R^{\mathcal{I}}$, $C \sqsubseteq \exists R.D \in comp(KB)$, and from rule - 21, we have $C \sqsubseteq \exists S.D$, and thus $(x_C, x_D) \in S^{\mathcal{I}}$ as required. For α of the form $R_1 \circ R_2 \sqsubseteq R$, let $(x_C, x_D) \in R_1^{\mathcal{I}}, (x_D, x_E) \in R_2^{\mathcal{I}}$. Then $C \sqsubseteq \exists R_1.D \in comp(KB)$ and $D \sqsubseteq \exists R_2.E \in comp(KB)$, then by rule - 22, we get $C \sqsubseteq \exists R.E$, and thus $(x_C, x_E) \in R^{\mathcal{I}}$.

For α of the form $C_1 \sqcap C_2 \sqsubseteq \bot$, we prove by contradiction. Let $x_C \in C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$, then $C \sqsubseteq C_1, C \sqsubseteq C_2 \in comp(KB)$, Then from rule 17, we get $C \sqsubseteq \bot$ but as per definition of $\mathcal{I}$ we have $C^{\mathcal{I}} = \emptyset$.

Let $A \sqsubseteq B \notin comp(KB)$, then we know $x_A \in A^{\mathcal{I}}$ and $x_A \notin B^{\mathcal{I}}$ since $A \sqsubseteq B \notin comp(KB)$. Then $x_A \in B^{\mathcal{I}} \setminus A^{\mathcal{I}}$ and $\mathcal{I} \not\models A \sqsubseteq B$, therefore $KB \not\models A \sqsubseteq B$.

Let $\{a\} \sqsubseteq B \notin comp(KB)$, then we know $x_{\{a\}} \in \{a\}^{\mathcal{I}}$ and clearly $x_{\{a\}} \notin B^{\mathcal{I}}$ as $\{a\} \sqsubseteq B \notin comp(KB)$, thus $\mathcal{I} \not\models \{a\} \sqsubseteq B$ and $KB \not\models \{a\} \sqsubseteq B$.

Let $comp(KB)$ be clash free then there is no axiom of the form $\{a\} \sqsubseteq \bot$, then as shown above $\mathcal{I}$ is a model of $comp(KB)$, KB, thus KB is consistent.

The claim is proven by contraposition.

B Another worked example

In the following we show another example of the working of the semantics of the defeasible mappings.

Example 3. Consider $\mathcal{O}_1 = \{\{a\} \sqsubseteq A, A \sqsubseteq \exists R.B, B \sqsubseteq \exists R.D\}$, $\mathcal{O}_2 = \{\exists R'.D' \sqsubseteq E', \exists R'.E' \sqsubseteq F'\}$ and $\delta = \{R \sqsubseteq_d R', D \sqsubseteq_d D'\}$

In that case,

$$\begin{aligned} comp_u(\mathcal{O}_1) = &\mathcal{O}_1 \cup \\ &\{\{a\} \sqsubseteq \exists R.B\} \cup \\ &\{\{a\} \sqsubseteq \exists R.\{\iota_{RB}\}, \{\iota_{RB}\} \sqsubseteq B\} \cup \\ &\{\{\iota_{RB}\} \sqsubseteq \exists R.\{\iota_{RD}\}, \{\iota_{RD} \sqsubseteq D\}\} \end{aligned} \tag{40}$$

$$\mathsf{Cand}(D \sqsubseteq_d D') = \{\{\iota_{RD}\} \sqsubseteq D'\} \tag{41}$$

$$\mathsf{Cand}_{\mathcal{R}}(R \sqsubseteq_d R') = \{\{a\} \sqsubseteq \exists R'.\{\iota_{RB}\}, \{\iota_{RB}\} \sqsubseteq \exists R'.\{\iota_{RD}\}\} \tag{42}$$

Note that there will be only one $\mathcal{O}_2^{\Sigma}$ for which we get mapped completion, it is the one corresponding to the maximum selection for each mapping which does not result in an inconsistency. In this case $\Sigma = \mathsf{Cand}(D \sqsubseteq D') \cup \mathsf{Cand}_{\mathcal{R}}(R \sqsubseteq R')$ is the only maximal selection that leads to a consistent mapped completion. Therefore, we have just one $comp(\mathcal{O}_2^{\Sigma})$,

$$comp(\mathcal{O}_2^{\Sigma}) = comp(\mathcal{O}_2) \cup \Sigma \cup \{\{\iota_{RB}\} \sqsubseteq E', \{a\} \sqsubseteq F'\}. \tag{43}$$

Therefore, $(\mathcal{O}_1, \mathcal{O}_2, \delta) \models \{a\} \sqsubseteq F'$, which is what we would expect when defeasible mappings also apply to unknowns. If we did not have the auxiliary individuals in $comp_u(\mathcal{O}_1)$, then this would not have been possible.

C Proofs of Lemmas from Section 4

Lemma 2. *Let $\mathcal{O}_2^\Sigma$ be a mapped ontology for $(\mathcal{O}_1, \mathcal{O}_2, \delta)$, and let $comp(\mathcal{O}_2^\Sigma)$ be a corresponding mapped completion. Then the mapped answer set $S(\mathcal{O}_2^\Sigma)$ is an answer set of $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$.*

Proof. We use $\overline{\Pi}$ as a short notation for $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta)$. We need to show that (1) $S(\mathcal{O}_2^\Sigma) \subseteq Lit$ (the set of all literals in $\overline{\Pi}$), (2) $S(\mathcal{O}_2^\Sigma) = \alpha(\overline{\Pi}^{S(\mathcal{O}_2^\Sigma)})$. Condition (1) trivially holds. For condition (2), we first assume that $\delta = \emptyset$ then $\overline{\Pi}^{S(\mathcal{O}_2^\Sigma)} = \overline{\Pi}$, since $\overline{\Pi}$ does not contain any *not* in the rules. For this case all the rules of $\overline{\Pi}$ are constructed using the rules of Table 2 followed by grounding. We show that $S(\mathcal{O}_2^\Sigma) = \alpha(\overline{\Pi})$ by showing that $S(\mathcal{O}_2^\Sigma)$ satisfies the conditions of definition 9. We only need to show that $S(\mathcal{O}_2^\Sigma)$ satisfies condition (1), since $\mathcal{O}_2$ is consistent. For all rules in $\overline{\Pi}$ of the form $C(a) \rightarrow D(a)$ generated using the transformation 1 of table 2, $S(\mathcal{O}_2^\Sigma)$ satisfies the condition: if $C(a) \in S(\mathcal{O}_2^\Sigma)$ then $D(a) \in S(\mathcal{O}_2^\Sigma)$. $C(a) \in S(\mathcal{O}_2^\Sigma)$ whenever $\{a\} \sqsubseteq C \in comp(\mathcal{O}_2^\Sigma)$ and if $C \sqsubseteq D \in comp(\mathcal{O}_2^\Sigma)$ then $\{a\} \sqsubseteq D \in comp(\mathcal{O}_2^\Sigma)$, therefore, $D(a) \in S(\mathcal{O}_2^\Sigma)$ by definition of $S(\mathcal{O}_2^\Sigma)$.

For rules of the form $C(a) \rightarrow \neg C(a)$ in $\overline{\Pi}$ generated using transformation 2, $S(\mathcal{O}_2^\Sigma)$ trivially satisfies the condition: if $C(a) \in S(\mathcal{O}_2^\Sigma)$ then $\neg C(a) \in S(\mathcal{O}_2^\Sigma)$ as $\mathcal{O}_2^\Sigma$ is consistent, therefore, no axiom of the form $\{a\} \sqsubseteq \bot \in comp(\mathcal{O}_2^\Sigma)$.

For rules of the form $R(x, y) \wedge C(y) \rightarrow D(x)$ in $\overline{\Pi}$ generated using transformation 3, $S(\mathcal{O}_2^\Sigma)$ satisfies the condition: if $R(a, b), C(b) \in S(\mathcal{O}_2^\Sigma)$ then $D(a) \in S(\mathcal{O}_2^\Sigma)$. We have $\exists R.C \sqsubseteq D \in comp(\mathcal{O}_2^\Sigma)$ by definition of the transformation rule. $R(a, b), C(b) \in S(\mathcal{O}_2^\Sigma)$ whenever $\{a\} \sqsubseteq \exists R.\{b\}, \{b\} \sqsubseteq C \in comp(\mathcal{O}_2^\Sigma)$ respectively by definition of $S(\mathcal{O}_2^\Sigma)$. Then $\{a\} \sqsubseteq D \in comp(\mathcal{O}_2^\Sigma)$ should hold by rule 19 of the completion rules of $\mathcal{ER}_{\perp,\mathcal{O}}$. Hence, $D(a) \in S(\mathcal{O}_2^\Sigma)$.

For rules of the form $C_1(a) \wedge C_2(a) \rightarrow D(a)$ in $\overline{\Pi}$ generated using transformation 4, $S(\mathcal{O}_2^\Sigma)$ satisfies the condition: if $C_1(a), C_2(a) \in S(\mathcal{O}_2^\Sigma)$ then $D(a) \in S(\mathcal{O}_2^\Sigma)$. This is because $C_1(a), C_2(a) \in S(\mathcal{O}_2^\Sigma)$ whenever $\{a\} \sqsubseteq C_1, \{a\} \sqsubseteq C_2 \in comp(\mathcal{O}_2^\Sigma)$, since $C_1 \sqcap C_2 \sqsubseteq D \in comp(\mathcal{O}_2^\Sigma)$, $\{a\} \in D$ should be in $comp(\mathcal{O}_2^\Sigma)$, therefore, $D(a) \in S(\mathcal{O}_2^\Sigma)$.

Again for rules in $\overline{\Pi}$ generated using transformation 5 are trivially satisfied by $S(\mathcal{O}_2^\Sigma)$ since $\mathcal{O}_2^\Sigma$ is consistent.

For rules of the form $R_1(a, b) \rightarrow R(a, b)$ in $\overline{\Pi}$ generated using transformation 6, $S(\mathcal{O}_2^\Sigma)$ satisfies the condition: if $R_1(a, b) \in S(\mathcal{O}_2^\Sigma)$ then $R(a, b) \in S(\mathcal{O}_2^\Sigma)$. $R_1(a, b) \in S(\mathcal{O}_2^\Sigma)$ whenever $\{a\} \sqsubseteq \exists R1.\{b\} \in comp(\mathcal{O}_2^\Sigma)$ and since, $R_1 \sqsubseteq R \in comp(\mathcal{O}_2^\Sigma)$, $\{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_2^\Sigma)$ should hold, therefore, $R(a, b) \in S(\mathcal{O}_2^\Sigma)$.

For rules of the form $R_1(a, b) \wedge R_2(b, c) \rightarrow R(a, c)$ in $\overline{\Pi}$ generated using transformation 7, $S(\mathcal{O}_2^\Sigma)$ satisfies the condition: if $R_1(a, b), R_2(b, c) \in S(\mathcal{O}_2^\Sigma)$ then $R(a, c) \in S(\mathcal{O}_2^\Sigma)$. $R_1(a, b), R_2(b, c) \in S(\mathcal{O}_2^\Sigma)$ whenever $\{a\} \sqsubseteq \exists R_1.\{b\}, \{b\} \sqsubseteq \exists R_2.\{c\} \in comp(\mathcal{O}_2^\Sigma)$ and since $R_1 \circ R_2 \sqsubseteq R \in comp(\mathcal{O}_2^\Sigma)$, we have $\{a\} \sqsubseteq \exists R.C \in comp(\mathcal{O}_2^\Sigma)$. Hence, $R(a, c) \in S(\mathcal{O}_2^\Sigma)$.

For rules in $\overline{\Pi}$ generated using transformations 8 and 9, it is easy to see that $S(\mathcal{O}_2^\Sigma)$ trivially satisfy these rules since it follows directly from the definition of $S(\mathcal{O}_2^\Sigma)$.

Now consider the case when $\delta \neq \emptyset$. The mappings in δ gives rise to rules in $\overline{\Pi}$ of the form of rules in 29, 30, 31 and 32. For rules of type 29, 31 there is nothing to show and $S(\mathcal{O}_2^{\Sigma})$ satisfies such rules (facts) by definition of $S(\mathcal{O}_2^{\Sigma})$. Now rules of type 30, 32 are the only rules that contain *not*. Now consider $\overline{\Pi}^{S(\mathcal{O}_2^{\Sigma})}$ as per the transformations in definition 9. Rules of type 30 are either deleted or they transformed to the form $C(a) \rightarrow D(a)$. $S(\mathcal{O}_2^{\Sigma})$ satisfies rules of this form since $C(a) \in S(\mathcal{O}_2^{\Sigma})$ if $C \sqsubseteq_d D \in \delta$ and $\{a\} \sqsubseteq C \in comp(\mathcal{O}_1)$. Note that for such cases there are only two possibilities $D(a) \in S(\mathcal{O}_2^{\Sigma})$ or $\neg D(a) \in S(\mathcal{O}_2^{\Sigma})$. Clearly, $\neg D(a) \notin S(\mathcal{O}_2^{\Sigma})$, otherwise the rule would have been deleted. Hence, $D(a) \in S(\mathcal{O}_2^{\Sigma})$. Similarly, rules of the form 32 are either deleted or transformed into $R(a,b) \rightarrow S(a,b)$. $R(a,b) \in S(\mathcal{O}_2^{\Sigma})$ holds if $R \sqsubseteq_d S \in \delta$ and $\{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_1)$. Clearly $S(a,b) \in S(\mathcal{O}_2^{\Sigma})$ otherwise the rule would have been deleted. Note the arguments made above for the case of $\delta = \emptyset$ still hold and thereby $S(\mathcal{O}_2^{\Sigma}) = \alpha(\overline{\Pi}^{S(\mathcal{O}_2^{\Sigma})})$. It can also be seen from the definition of $S(\mathcal{O}_2^{\Sigma})$, it is in fact the smallest subset of *Lit* satisfying these conditions. □

Lemma 3. *Let $(\mathcal{O}_1, \mathcal{O}_2, \delta)$ be a mapped-tuple and let S be an answer set of $\Pi(\mathcal{O}_1, \mathcal{O}_2, \delta) = \overline{\Pi}$. Then $S = S(\mathcal{O}_2^{\Sigma})$ for some mapped ontology $\mathcal{O}_2^{\Sigma}$ of $(\mathcal{O}_1, \mathcal{O}_2, \delta)$.*

Proof. First we construct $\mathcal{O}_2^{\Sigma}$ from S. Let $\Sigma_{C \sqsubseteq_d D} = \{\{a\} \sqsubseteq D \mid D(a) \in S\}$ for each mapping axiom $C \sqsubseteq_d D \in \delta$ and $\Sigma_{R \sqsubseteq_d S} = \{\{a\} \sqsubseteq \exists S.\{b\} \mid R(a,b) \in S\}$ for each mapping axiom $R \sqsubseteq_d S \in \delta$. Let Σ be the collection of all such $\Sigma_{C \sqsubseteq_d D}$ and $\Sigma_{R \sqsubseteq_d S}$. Then $\mathcal{O}_2^{\Sigma} = comp(\mathcal{O}_2) \cup \{X \mid X \in \Sigma\}$ is a mapped ontology as per definition 6. It remains to be shown that $comp(\mathcal{O}_2^{\Sigma})$ is a mapped completion. We assume $\mathcal{O}_2^{\Sigma}$ is not a mapped completion which means there is some $\mathcal{O}_2^{\Sigma^i}$ with $comp(\mathcal{O}_2^{\Sigma^i})$ a mapped completion and $\mathcal{O}_2^{\Sigma^i} \succ \mathcal{O}_2^{\Sigma}$. Therefore, for some mapping axiom $C \sqsubseteq_d D \in \delta$ or $R \sqsubseteq_d S \in \delta$ we have $\Sigma^i_{C \sqsubseteq_d D} \succ \Sigma_{C \sqsubseteq_d D}$ or $\Sigma^i_{R \sqsubseteq_d S} \succ \Sigma_{R \sqsubseteq_d S}$. For the case when $\Sigma^i_{C \sqsubseteq_d D} \succ \Sigma_{C \sqsubseteq_d D}$ then there is an axiom $\{a\} \sqsubseteq D$ with $\{a\} \sqsubseteq C \in comp(\mathcal{O}_1)$ and $\{a\} \sqsubseteq D \in \mathcal{O}_2^{\Sigma^i}$, $\{a\} \sqsubseteq D \notin \mathcal{O}_2^{\Sigma}$. For the case when $\Sigma^i_{R \sqsubseteq_d S} \succ \Sigma_{R \sqsubseteq_d S}$ there is an axiom $\{a\} \sqsubseteq \exists S.\{b\} \in \mathcal{O}_2^{\Sigma^i}$ with $\{a\} \sqsubseteq \exists R.\{b\} \in comp(\mathcal{O}_1)$ and $\{a\} \sqsubseteq \exists S.\{b\} \notin \mathcal{O}_2^{\Sigma}$. Also from lemma 2, $S(\mathcal{O}_2^{\Sigma^i})$ is an answer set of $\overline{\Pi}$ with $D(a) \in S(\mathcal{O}_2^{\Sigma^i})$ or $S(a,b) \in S(\mathcal{O}_2^{\Sigma^i})$. But that could not be the case since $S \subset S(\mathcal{O}_2^{\Sigma^i})$ is an answer set of $\overline{\Pi}$ and an answer set should be the minimal subset of *Lit* satisfying the rules of $\overline{\Pi}$ as per definition 9. □