

Maximum-likelihood (ML) estimation

Setting

- A training data $\mathcal{D} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of i.i.d. examples for the target class y is available
- We assume the parameter vector θ has a fixed but unknown value
- We estimate such value maximizing its **likelihood** with respect to the training data:

$$\theta^* = \operatorname{argmax}_{\theta} p(\mathcal{D}|\theta) = \operatorname{argmax}_{\theta} \prod_{j=1}^n p(\mathbf{x}_j|\theta)$$

- The joint probability over \mathcal{D} decomposes into a product as examples are i.i.d (thus independent of each other given the distribution)

Maximum-likelihood estimation

Maximizing log-likelihood

- It is usually simpler to maximize the logarithm of the likelihood (monotonic):

$$\theta^* = \operatorname{argmax}_{\theta} \ln p(\mathcal{D}|\theta) = \operatorname{argmax}_{\theta} \sum_{j=1}^n \ln p(\mathbf{x}_j|\theta)$$

- Necessary conditions for the maximum can be obtained zeroing the gradient wrt to θ :

$$\nabla_{\theta} \sum_{j=1}^n \ln p(\mathbf{x}_j|\theta) = \mathbf{0}$$

- Points zeroing the gradient can be local or global maxima depending on the form of the distribution

Bayesian estimation

setting

- Assumes parameters θ_i are *random variables* with some known *prior* distribution
- Predictions for new examples are obtained *integrating* over all possible values for the parameters:

$$p(\mathbf{x}|y_i, \mathcal{D}_i) = \int_{\theta_i} p(\mathbf{x}, \theta_i|y_i, \mathcal{D}_i) d\theta_i$$

- probability of \mathbf{x} given each class y_i is independent of the other classes y_j , for simplicity we can again write:

$$p(\mathbf{x}|y_i, \mathcal{D}_i) \rightarrow p(\mathbf{x}|\mathcal{D}) = \int_{\theta} p(\mathbf{x}, \theta|\mathcal{D}) d\theta$$

- where \mathcal{D} is a dataset for a certain class y and θ the parameters of the distribution

Bayesian estimation

setting

$$p(\mathbf{x}|\mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}|\mathcal{D})d\boldsymbol{\theta} = \int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta}|\mathcal{D})d\boldsymbol{\theta}$$

- $p(\mathbf{x}|\boldsymbol{\theta})$ can be easily computed (we have both form and parameters of distribution, e.g. Gaussian)
- need to estimate the parameter posterior density given the training set:

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{D})}$$

Bayesian estimation

denominator

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{D})}$$

- $p(\mathcal{D})$ is a constant independent of $\boldsymbol{\theta}$ (i.e. it will no influence final Bayesian decision)
- if final *probability* (not only decision) is needed we can compute:

$$p(\mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}$$

Sufficient statistics

Definition

- Any function on a set of samples \mathcal{D} is a *statistic*
- A statistic $\mathbf{s} = \phi(\mathcal{D})$ is *sufficient* for some parameters $\boldsymbol{\theta}$ if:

$$P(\mathcal{D}|\mathbf{s}, \boldsymbol{\theta}) = P(\mathcal{D}|\mathbf{s})$$

- If $\boldsymbol{\theta}$ is a random variable, a sufficient statistic contains all relevant information \mathcal{D} has for estimating it:

$$p(\boldsymbol{\theta}|\mathcal{D}, \mathbf{s}) = \frac{p(\mathcal{D}|\boldsymbol{\theta}, \mathbf{s})p(\boldsymbol{\theta}|\mathbf{s})}{p(\mathcal{D}|\mathbf{s})} = p(\boldsymbol{\theta}|\mathbf{s})$$

Use

- A sufficient statistic allows to compress a sample \mathcal{D} into (possibly few) values
- Sample mean and covariance are sufficient statistics for true mean and covariance of the Gaussian distribution

Conjugate priors

Definition

- Given a likelihood function $p(x|\theta)$
- Given a prior distribution $p(\theta)$
- $p(\theta)$ is a *conjugate prior* for $p(x|\theta)$ if the posterior distribution $p(\theta|x)$ is in the same family as the prior $p(\theta)$

Examples

Likelihood	Parameters	Conjugate prior
Binomial	p (probability)	Beta
Multinomial	\mathbf{p} (probability vector)	Dirichlet
Normal	μ (mean)	Normal
Multivariate normal	μ_i (mean vector)	Normal

Probability distributions

Bernoulli distribution

- Two possible values (outcomes): 1 (success), 0 (failure).
- Parameters: p probability of success.
- Probability mass function:

$$P(x; p) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

- $E[x] = p$
- $\text{Var}[x] = p(1 - p)$

Example: tossing a coin

- Head (success) and tail (failure) possible outcomes
- p is probability of head

Bernoulli distribution

Setting

- Boolean event: $x = 1$ for success, $x = 0$ for failure (e.g. tossing a coin)
- Parameters: θ = probability of success (e.g. head)
- Probability mass function

$$P(x|\theta) = \theta^x (1 - \theta)^{1-x}$$

- Beta conjugate prior:

$$P(\theta|\psi) = P(\theta|\alpha_h, \alpha_t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_h)\Gamma(\alpha_t)} \theta^{\alpha_h-1} (1 - \theta)^{\alpha_t-1}$$

Bernoulli distribution

Maximum likelihood estimation: example

- Dataset $\mathcal{D} = \{H, H, T, T, T, H, H\}$ of N realizations (e.g. head/tail coin toss results)

- Likelihood function:

$$p(\mathcal{D}|\theta) = \theta \cdot \theta \cdot (1 - \theta) \cdot (1 - \theta) \cdot (1 - \theta) \cdot \theta \cdot \theta = \theta^h (1 - \theta)^t$$

- Maximum likelihood parameter:

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln p(\mathcal{D}|\theta) = 0 &\quad \Rightarrow \quad \frac{\partial}{\partial \theta} h \ln \theta + t \ln (1 - \theta) = 0 \\ &\quad \quad \quad h \frac{1}{\theta} - t \frac{1}{1 - \theta} = 0 \\ &\quad \quad \quad h(1 - \theta) = t\theta \\ &\quad \quad \quad \theta = \frac{h}{h + t} \end{aligned}$$

- h, t are the sufficient statistics

Bernoulli distribution

Bayesian estimation: example

- Parameter posterior is proportional to:

$$P(\theta|\mathcal{D}, \psi) \propto P(\mathcal{D}|\theta)P(\theta|\psi) \propto \theta^h (1 - \theta)^t \theta^{\alpha_h - 1} (1 - \theta)^{\alpha_t - 1}$$

- i.e. the posterior has a beta distribution with parameters $h + \alpha_h, t + \alpha_t$:

$$P(\theta|\mathcal{D}, \psi) \propto \theta^{h + \alpha_h - 1} (1 - \theta)^{t + \alpha_t - 1}$$

- The prediction for a new event is the expected value of the posterior beta:

$$\begin{aligned} P(x|\mathcal{D}) &= \int P(x|\theta)P(\theta|\mathcal{D}, \psi)d\theta = \int \theta P(\theta|\mathcal{D}, \psi)d\theta \\ &= E_{P(\theta|\mathcal{D}, \psi)}[\theta] = \frac{h + \alpha_h}{h + t + \alpha_h + \alpha_t} \end{aligned}$$

Bernoulli distribution

Interpreting priors

- Our prior knowledge is encoded as a number $\alpha = \alpha_h + \alpha_t$ of imaginary experiments
- we assume α_h times we observed heads
- α is called *equivalent sample size*
- $\alpha \rightarrow 0$ reduces estimation to the classical ML approach (frequentist)

Multinomial distribution

Setting

- Categorical event with r states $x \in \{x^1, \dots, x^r\}$ (e.g. tossing a six-faced dice)
- One-hot encoding $\mathbf{z}(x) = [z_1(x), \dots, z_r(x)]$ with $z_k(x) = 1$ if $x = x^k$, 0 otherwise.
- Parameters: $\boldsymbol{\theta} = [\theta_1, \dots, \theta_r]$ probability of each state
- Probability mass function

$$P(x|\boldsymbol{\theta}) = \prod_{k=1}^r \theta_k^{z_k(x)}$$

- Dirichlet conjugate prior:

$$P(\boldsymbol{\theta}|\psi) = P(\boldsymbol{\theta}|\alpha_1, \dots, \alpha_r) = \frac{\Gamma(\alpha)}{\prod_{k=1}^r \Gamma(\alpha_k)} \prod_{k=1}^r \theta_k^{\alpha_k - 1}$$

Multinomial distribution

Maximum likelihood estimation: example

- Dataset \mathcal{D} of N realizations (e.g. results of tossing a dice)
- Likelihood function:

$$p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{j=1}^N \prod_{k=1}^r \theta_k^{z_k(x_j)} = \prod_{k=1}^r \theta_k^{N_k}$$

- Maximum likelihood parameter:

$$\theta_k = \frac{N_k}{N}$$

- N_1, \dots, N_r are the sufficient statistics

Multinomial distribution

Bayesian estimation: example

- Parameter posterior is proportional to:

$$P(\boldsymbol{\theta}|\mathcal{D}, \psi) \propto P(\mathcal{D}|\boldsymbol{\theta})P(\boldsymbol{\theta}|\psi) \propto \prod_{k=1}^r \theta_k^{N_k} \theta_k^{\alpha_k - 1}$$

- i.e. the posterior has a Dirichlet distribution with parameters $N_k + \alpha_k, k = 1, \dots, r$:

$$P(\boldsymbol{\theta}|\mathcal{D}, \psi) \propto \prod_{k=1}^r \theta_k^{N_k + \alpha_k - 1}$$

- The prediction for a new event is the expected value of the posterior Dirichlet:

$$P(x_k|\mathcal{D}) = \int \theta_k P(\boldsymbol{\theta}|\mathcal{D}, \psi) d\boldsymbol{\theta} = \mathbb{E}_{P(\boldsymbol{\theta}|\mathcal{D}, \psi)}[\theta_k] = \frac{N_k + \alpha_k}{N + \alpha}$$

APPENDIX

Appendix

Additional reference material

Maximum-likelihood estimation

Multivariate Gaussian case: proof (mean)

- The gradient wrt to the mean is:

$$\nabla_{\boldsymbol{\mu}} \sum_{j=1}^n -\frac{1}{2}(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) - \frac{1}{2} \ln (2\pi)^d |\Sigma| =$$
$$\sum_{j=1}^n \Sigma^{-1}(\mathbf{x}_j - \boldsymbol{\mu})$$

Note

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T A \mathbf{x} &= A^T \mathbf{x} + A \mathbf{x} \\ &= 2A \mathbf{x} \quad \text{for symmetric } A \end{aligned}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (mean)

- Setting the gradient to zero gives:

$$\begin{aligned} \sum_{j=1}^n \Sigma^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) &= \mathbf{0} \\ \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu}) &= \Sigma \mathbf{0} = \mathbf{0} \\ \sum_{j=1}^n \mathbf{x}_j &= \sum_{j=1}^n \boldsymbol{\mu} = n \boldsymbol{\mu} \\ \boldsymbol{\mu} &= \frac{1}{n} \sum_{j=1}^n \mathbf{x}_j \end{aligned}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

- The gradient wrt to the covariance is:

$$\begin{aligned} \frac{\partial}{\partial \Sigma} \sum_{j=1}^n -\frac{1}{2}(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) - \frac{1}{2} \ln (2\pi)^d |\Sigma| = \\ -\frac{1}{2} \left(\sum_{j=1}^n \frac{\partial}{\partial \Sigma} (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) + \sum_{j=1}^n \frac{\partial}{\partial \Sigma} \ln (2\pi)^d |\Sigma| \right) \end{aligned}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

$$\begin{aligned}\frac{\partial}{\partial \Sigma} (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) &= \\ (\mathbf{x}_j - \boldsymbol{\mu}) (\mathbf{x}_j - \boldsymbol{\mu})^t \frac{\partial}{\partial \Sigma} \Sigma^{-1} &= \\ -(\mathbf{x}_j - \boldsymbol{\mu}) (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2} &\end{aligned}$$

Note

Use matrix derivative rule:

$$\frac{\partial}{\partial B} \text{tr}(ABC) = CA$$

Where $A = (\mathbf{x}_j - \boldsymbol{\mu})^t$, $B = \Sigma^{-1}$, $C = (\mathbf{x}_j - \boldsymbol{\mu})$ and $\text{tr}(ABC) = ABC$ as ABC is a scalar.

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

$$\begin{aligned}\frac{\partial}{\partial \Sigma} \ln(2\pi)^d |\Sigma| &= \frac{1}{(2\pi)^d} |\Sigma|^{-1} \frac{\partial}{\partial \Sigma} (2\pi)^d |\Sigma| = \\ \frac{1}{(2\pi)^d} |\Sigma|^{-1} (2\pi)^d \frac{\partial}{\partial \Sigma} |\Sigma| &= |\Sigma|^{-1} \frac{\partial}{\partial \Sigma} |\Sigma| = \Sigma^{-1}\end{aligned}$$

Note

Use matrix derivative rule:

$$\frac{\partial}{\partial A} |A| = |A| A^{-1}$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

- Combining and putting equal to zero:

$$-\frac{1}{2} \left(\sum_{j=1}^n \underbrace{-(\mathbf{x}_j - \boldsymbol{\mu}) (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2}}_{\frac{\partial}{\partial \Sigma} (\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu})} + \sum_{j=1}^n \underbrace{\Sigma^{-1}}_{\frac{\partial}{\partial \Sigma} \ln(2\pi)^d |\Sigma|} \right) = 0$$

Maximum-likelihood estimation

Multivariate Gaussian case: proof (covariance)

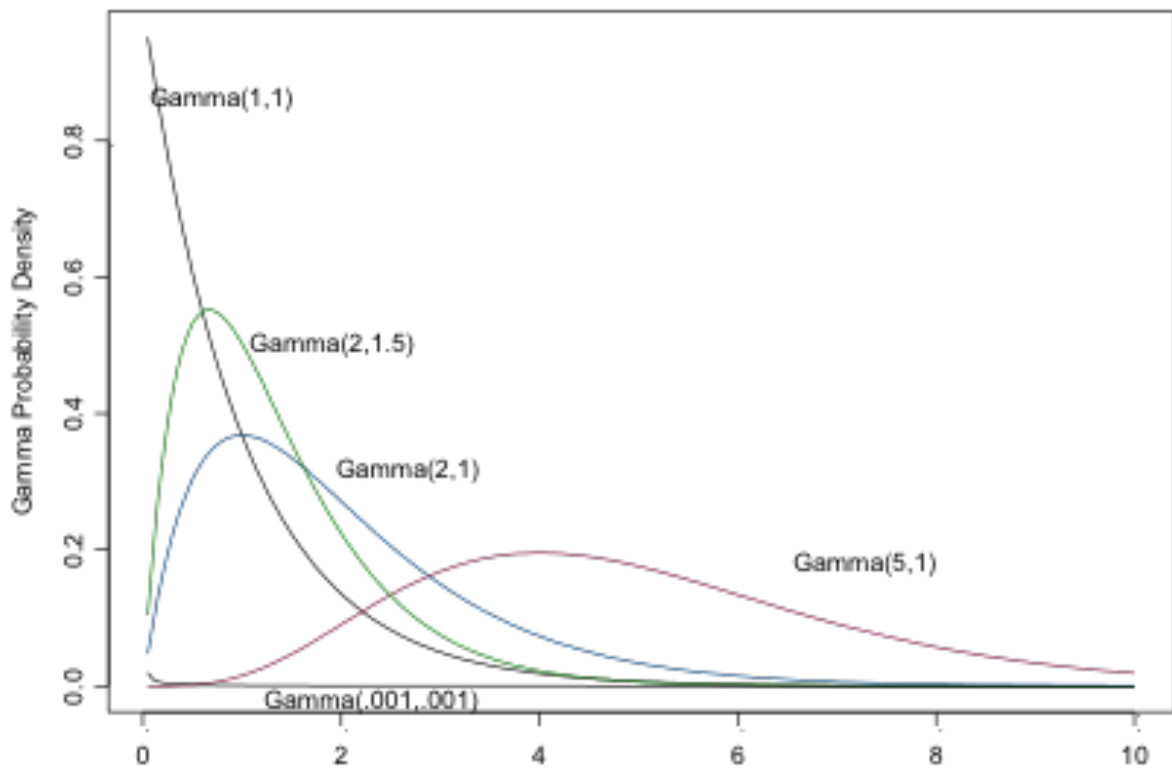
$$\begin{aligned}\sum_{j=1}^n \Sigma^{-1} &= \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2} \\ \Sigma^2 \sum_{j=1}^n \Sigma^{-1} &= \Sigma^2 \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \Sigma^{-2} \\ \sum_{j=1}^n \Sigma &= \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \\ n\Sigma &= \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t \\ \Sigma &= \frac{1}{n} \sum_{j=1}^n (\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})^t\end{aligned}$$

Bayesian estimation

Gamma distribution

- Defined in the interval $[0, \infty]$
- Parameters: $\alpha > 0$ (shape) $\beta > 0$ (rate)
- Probability density function:

$$p(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$



- $E[x] = \frac{\alpha}{\beta}$
- $\text{Var}[x] = \frac{\alpha}{\beta^2}$

Note

Used to model the prior distribution of the *precision* (inverse variance, i.e. $\lambda = 1/\sigma^2$).

Bayesian estimation

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

- Examples are drawn from:

$$p(x|\mu, \lambda) \sim N(\mu, 1/\lambda)$$

- The Prior of mean and precision is the NormalGamma distribution:

$$\begin{aligned} p(\mu, \lambda) &= p(\mu|\lambda)p(\lambda) = N\left(\mu|\mu_0, \frac{1}{\kappa_0\lambda}\right)Ga(\lambda|\alpha_0, \beta_0) \\ &= NG(\mu, \lambda|\mu_0, \kappa_0, \alpha_0, \beta_0) \end{aligned}$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$
a posteriori parameter density

$$\begin{aligned}
 p(\mu, \lambda | \mathcal{D}) &= \frac{1}{\mathcal{D}} \underbrace{\prod_{j=1}^n \frac{\lambda^{1/2}}{\sqrt{2\pi}} \exp\left[-\frac{\lambda}{2}(x_j - \mu)^2\right]}_{p(x_j | \mu, \lambda)} \underbrace{\frac{(\kappa_0 \lambda)^{1/2}}{\sqrt{2\pi}} \exp\left[-\frac{\kappa_0 \lambda}{2}(\mu - \mu_0)^2\right]}_{p(\mu | \lambda)} \\
 &\quad \underbrace{\frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \lambda^{\alpha_0 - 1} \exp(-\beta_0 \lambda)}_{p(\lambda)} \\
 &\propto \lambda^{\alpha_0 + n/2 - 1} \exp(-\beta_0 \lambda) \lambda^{1/2} \exp - \frac{\lambda}{2} \left[\sum_{j=1}^n (x_j - \mu)^2 - \kappa_0 (\mu - \mu_0)^2 \right]
 \end{aligned}$$

a posteriori parameter density is still NormalGamma

$$p(\mu, \lambda | \mathcal{D}) = NG(\mu, \lambda | \mu_n, \kappa_n, \alpha_n, \beta_n)$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

a posteriori parameter density is still NormalGamma

$$p(\mu, \lambda | \mathcal{D}) = NG(\mu, \lambda | \mu_n, \kappa_n, \alpha_n, \beta_n)$$

where

$$\begin{aligned}
 \mu_n &= \frac{\kappa_0 \mu_0 + n \hat{\mu}_n}{k_0 + n} \\
 \kappa_n &= k_0 + n \\
 \alpha_n &= \alpha_0 + n/2 \\
 \beta_n &= \beta_0 + \frac{1}{2} \sum_{j=1}^n (x_j - \hat{\mu}_n)^2 + \frac{\kappa_0 n (\hat{\mu}_n - \mu_0)^2}{2(\kappa_0 + n)}
 \end{aligned}$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

Interpreting the posterior

- Posterior mean is weighted average of prior (μ_0) and sample (μ_n) means, weighted by κ_0 and n respectively

$$\mu_n = \frac{\kappa_0 \mu_0 + n \hat{\mu}_n}{k_0 + n}$$

- Posterior κ_n is increased by the number of samples n

$$\kappa_n = k_0 + n$$

- Posterior α_n is increased by half the number of samples n

$$\alpha_n = \alpha_0 + n/2$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

Interpreting the posterior

- Posterior sum of squares (β_n) is sum of prior sum of squares (β_0) and sample sum of squares $\frac{1}{2} \sum_{j=1}^n (x_j - \hat{\mu}_n)^2$ and a term due to the discrepancy between the sample mean and the prior mean.

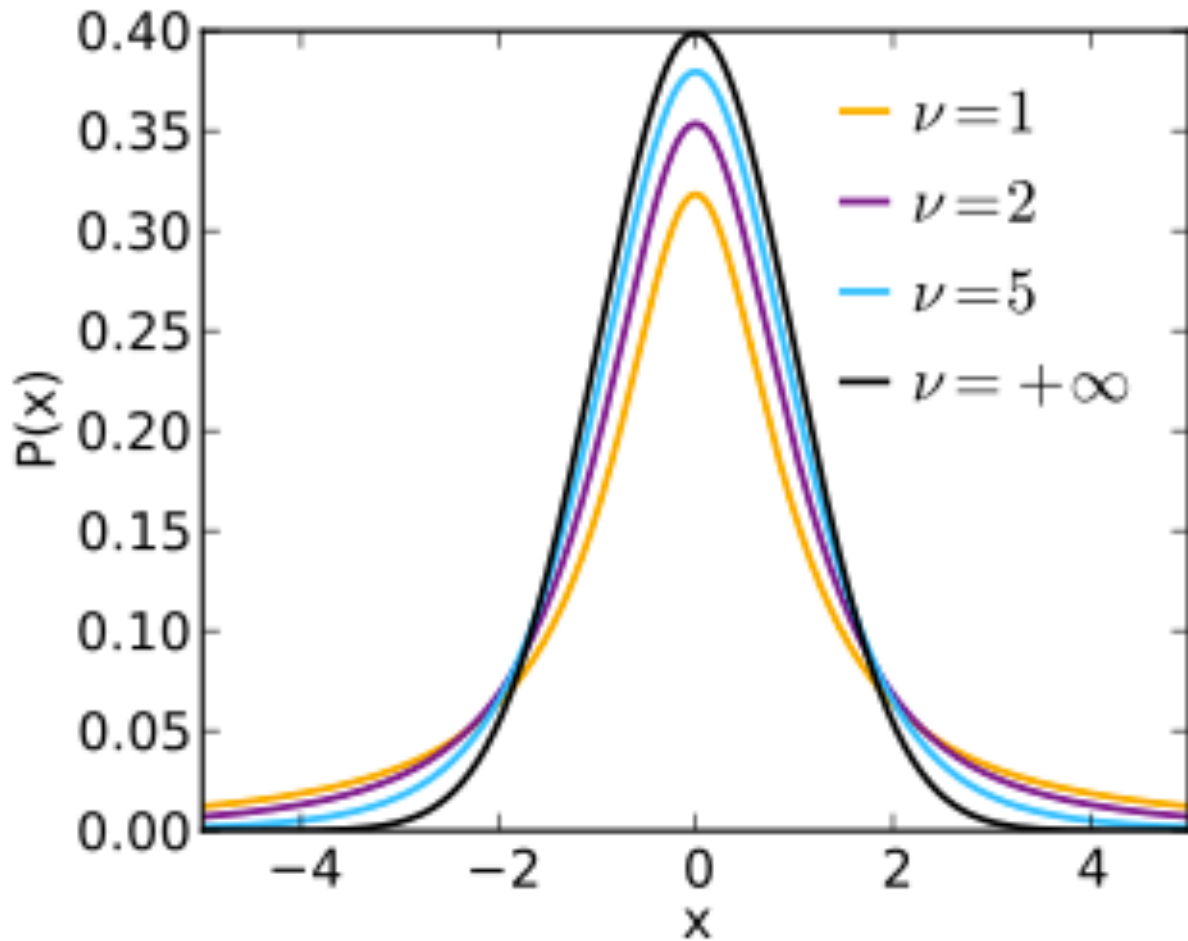
$$\beta_n = \beta_0 + \frac{1}{2} \sum_{j=1}^n (x_j - \hat{\mu}_n)^2 + \frac{\kappa_0 n (\hat{\mu}_n - \mu_0)^2}{2(\kappa_0 + n)}$$

Univariate normal case: unknown μ and $\lambda = 1/\sigma^2$

Computing the posterior predictive

$$\begin{aligned} p(x|\mathcal{D}) &= \int_{\mu} \int_{\lambda} p(x|\mu, \lambda) p(\mu, \lambda|\mathcal{D}) d\mu d\lambda \\ &= \frac{P(x, \mathcal{D})}{P(\mathcal{D})} = t_{2\alpha_n} \left(x | \mu_n, \frac{\beta_n(\kappa_n + 1)}{\alpha_n \kappa_n} \right) \end{aligned}$$

- It is a T-distribution with mean μ_n and precision $\frac{\beta_n(\kappa_n + 1)}{\alpha_n \kappa_n}$ (proof omitted)



Bayesian estimation

Wishart distribution

- Defined over $d \times d$ positive semi-definite matrix
- Parameters: $\nu > d - 1$ (degree of freedom) $T > 0$ ($d \times d$ scale matrix)
- Probability density function:

$$p(X; \nu, T) = \frac{1}{2^{\nu d/2} |T|^{\nu/2} \Gamma_d(\nu/2)} |X|^{\frac{\nu-d-1}{2}} \exp -\frac{1}{2} \text{tr}(T^{-1}X)$$

- $E[X] = \nu T$
- $\text{Var}[X_{ij}] = \nu(T_i i^2 + T_{ii} T_{jj})$

Note

Used to model the prior distribution of the *precision* matrix (inverse covariance matrix, i.e. $\Lambda = \Sigma^{-1}$). T is the prior covariance

Bayesian estimation

Multivariate normal case: unknown μ and Σ

- Examples are drawn from:

$$p(x|\mu, \Lambda) \sim N(\mu, \Lambda^{-1})$$

- The Prior of mean and precision is the NormalWishart distribution:

$$p(\mu, \Lambda) = p(\mu|\Lambda)p(\Lambda) = N(\mu|\mu_0, (\kappa_0\Lambda)^{-1})Wi(\Lambda|\nu, T)$$

Multivariate normal case: unknown μ and Σ

a posteriori parameter density

$$p(\mu, \Lambda|\mathcal{D}) = N(\mu|\mu_n, (\kappa_n\Lambda)^{-1})Wi(\Lambda|\nu_n, T_n)$$

where

$$\begin{aligned}\mu_n &= \frac{\kappa_0\mu_0 + n\hat{\mu}_n}{\kappa_0 + n} \\ T_n &= T + \sum_{i=1}^n (x_i - \hat{\mu}_n)(x_i - \hat{\mu}_n)^T + \frac{\kappa_0 n}{\kappa_0 + n} (\mu_0 - \hat{\mu}_n)(\mu_0 - \hat{\mu}_n)^T \\ \nu_n &= \nu + n \quad \kappa_n = \kappa_0 + n\end{aligned}$$

Computing the posterior predictive

$$p(x|\mathcal{D}) = t_{\nu_n-d+1} \left(x \mid \mu_n, \frac{T_n(\kappa_n + 1)}{\kappa_n(\nu_n - d + 1)} \right)$$