Mathematical Logics
Description Logic: Tbox and Abox

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Terminological axioms

- Two new logical symbols

  $\sqsubseteq$ (subsumption), with $\sigma \vDash C \sqsubseteq D$ iff $\sigma(C) \subseteq \sigma(D)$
  $\equiv$ (equivalence), with $\sigma \vDash C \equiv D$ iff $\sigma(C) = \sigma(D)$

  Inclusion axioms

  $C \sqsubseteq D$ has to be read "C is subsumed by (more specific than / less general than) D"

  Equality axioms

  $C \equiv D$ has to be read "C is equivalent to D"

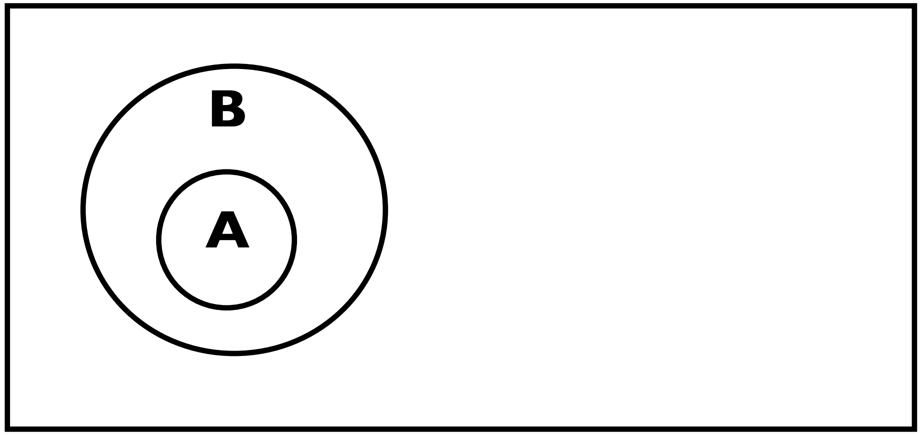
  NOTE: $C \equiv D$ holds iff both $C \sqsubseteq D$ and $D \sqsubseteq C$ hold

  NOTE: $\sqsubseteq$ and $\equiv$ in CLassL are the alter ego of the $\rightarrow$ and $\leftrightarrow$ in PL

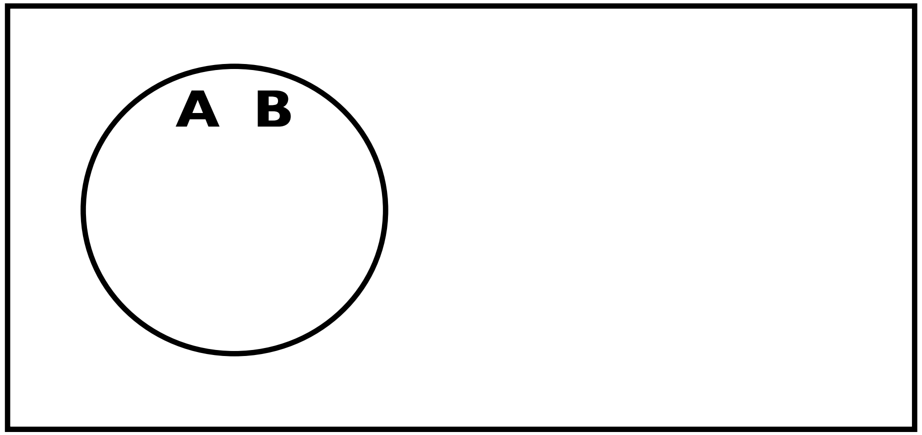
  NOTE: if $C \sqsubseteq D$, we can use the symbol $\sqsupseteq$ to say that $D \sqsupseteq C$ to be read "D subsumes (less specific than / more general than) C"

Semantics: Venn diagrams to represent axioms

- $\sigma(A \sqsubseteq B)$



- $\sigma(A \equiv B)$



Examples of terminological axioms

- **Inclusion Axioms (inclusions)**

  $\text{Master} \sqsubseteq \text{Student}$
  $\text{Woman} \sqsubseteq \text{Person}$
  $\text{Woman} \sqcup \text{Father} \sqsubseteq \text{Person}$

- **Equality Axioms (equalities)**

  $\text{Student} \equiv \text{Pupil}$
  $\text{Parent} \equiv \text{Mother} \sqcup \text{Father}$
  $\text{Woman} \equiv \text{Person} \sqcap \text{Female}$

Tbox - Definitions and specializations

- A definition is an equality with an atomic concept on the left hand

$$\begin{aligned}
\mathit{Bachelor} &\equiv \mathit{Student} \sqcap \mathit{Undergraduate} \\
\mathit{Woman} &\equiv \mathit{Person} \sqcap \mathit{Female} \\
\mathit{Parent} &\equiv \mathit{Mother} \sqcup \mathit{Father}
\end{aligned}$$

- A specialization is an inclusion with an atomic concept on the left hand

$$\begin{aligned}
\mathit{Student} &\sqsubseteq \mathit{Person} \sqcap \mathit{Study} \\
\mathit{PhD} &\sqsubseteq \mathit{Student} \sqcap \mathit{Lecturer}
\end{aligned}$$

Tbox - Terminology

- A terminology (or TBox) is a set of definitions and specializations

Woman	$\equiv$ Person $\sqcap$ Female
Man	$\equiv$ Person $\sqcap$ $\neg$Woman
Student	$\sqsubseteq$ Person $\sqcap$ Study
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
PhD	$\sqsubseteq$ Student $\sqcap$ Lecturer

- Terminological axioms express constraints on the concepts of the language, i.e. they limit the possible models

- The TBox is the set of all the constraints on the possible models

Tbox - Normalization

- It is always possible to transform a specialization into a definition by introducing an auxiliary symbol as follows:

Woman $\sqsubseteq$ Person (the specialization)

Woman $\equiv$ Person $\sqcap$ Female (the normalized specialization)

- If from a TBox we transform all specializations into definitions we say we have normalized the TBox

- A TBox with definitions only is called a regular terminology

Tbox - Reasoning

Given two class-propositions P and Q, we want to reason about the following relations between them:

- Satisfiability w.r.t. T $\mathcal{T} \models P$?
- Subsumption $\mathcal{T} \models P \sqsubseteq Q$? $\mathcal{T} \models Q \sqsubseteq P$?
- Equivalence $\mathcal{T} \models P \sqsubseteq Q$ and $\mathcal{T} \models Q \sqsubseteq P$?
- Disjointness $\mathcal{T} \models P \sqcap Q \sqsubseteq \perp$?

Satisfiability with respect to a TBox T

- A concept P is satisfiable w.r.t. a terminology T, if there exists (or for all) an interpretation I with I ⊨ θ for all θ ∈ T, and such that I ⊨ P, namely I(P) is not empty

- In this case we also say that I is a model for P

- In other words, the interpretation I not only satisfies P, but also complies with all the constraints in T

NOTE: Instead of σ in DL literature the symbol I (interpretation) is preferred. Therefore, from now on we use it instead of σ.

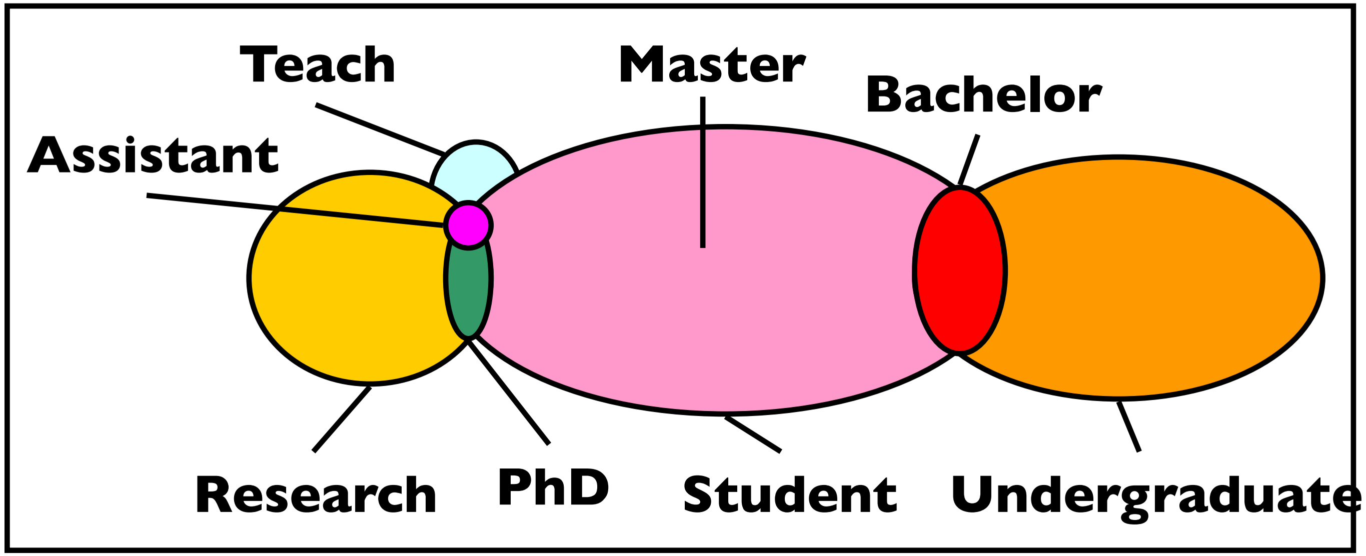
Satisfiability with respect to a TBox T

- Suppose we describe the students/listeners in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

- The TBox is satisfiable. A possible model is:



In this model the two concepts **Bachelor** and **Assistant** are satisfiable w.r.t. T, while the concept **Assistant** $\sqcap$ **Bachelor** is not.

TBox reasoning: subsumption

- Let T be a TBox. Subsumption (with respect to T):

$$T \models P \sqsubseteq Q \ (P \sqsubseteq_T Q)$$

A concept P is subsumed by a concept Q with respect to T if $I(P) \subseteq I(Q)$ for every model I of T

NOTE: subsumption is a property of <u>all models</u>. Used to implement entailment and validity (when T empty)

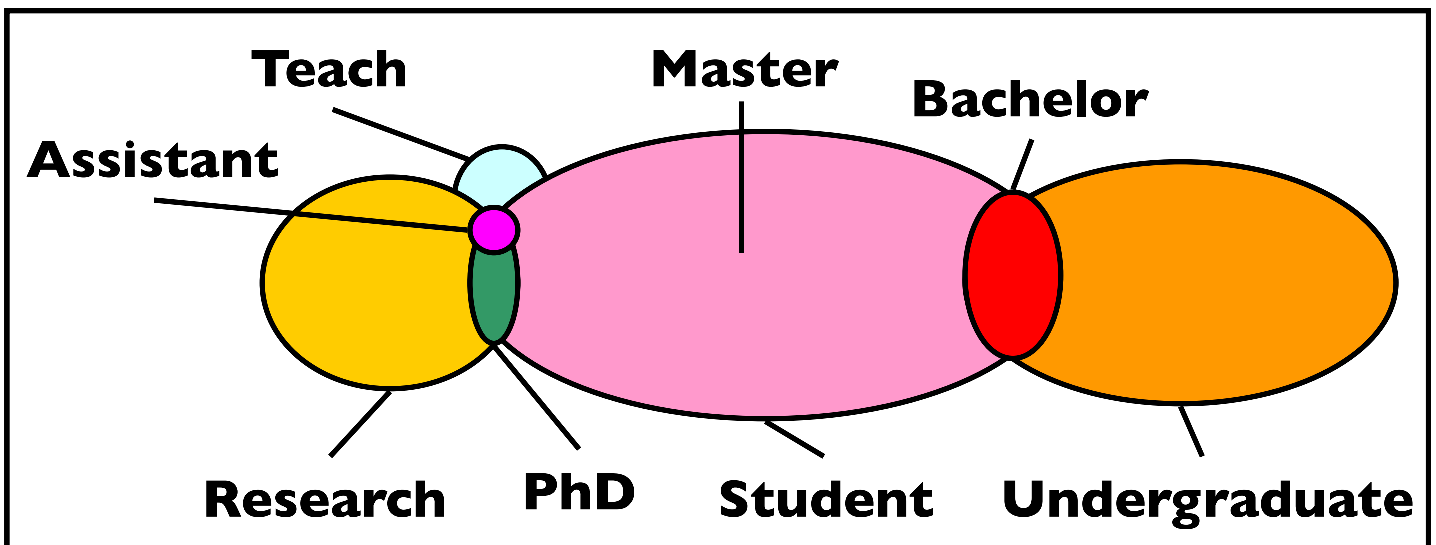
Subsumption with respect to a TBox T (1)

- Suppose we describe the students/listeners in a course:

Undergraduate	$\sqsubseteq$	$\neg$ Teach
Bachelor	$\equiv$	Student $\sqcap$ Undergraduate
Master	$\equiv$	Student $\sqcap$ $\neg$ Undergraduate
PhD	$\equiv$	Master $\sqcap$ Research
Assistant	$\equiv$	PhD $\sqcap$ Teach

TBox T

- **T $\models$ PhD $\sqsubseteq$ Student**



Subsumption with respect to a TBox T (2)

PhD $\sqsubseteq$ Student

<u>Proof</u>:

$$\begin{aligned}
&\text{PhD} \\
&\equiv \text{Master} \sqcap \text{Research} \\
&\equiv (\text{Student} \sqcap \neg\, \text{Undergraduate}) \sqcap \text{Research} \\
&\sqsubseteq \text{Student}
\end{aligned}$$

TBox reasoning: equivalence

- Let T be a TBox. Equivalence (with respect to T):

$$(T \vDash P \equiv Q)\ (P \equiv_T Q)$$

Two concepts P and Q are equivalent with respect to T if I(P) = I(Q) for every model I of T.

NOTE: equivalence is a property of all models.

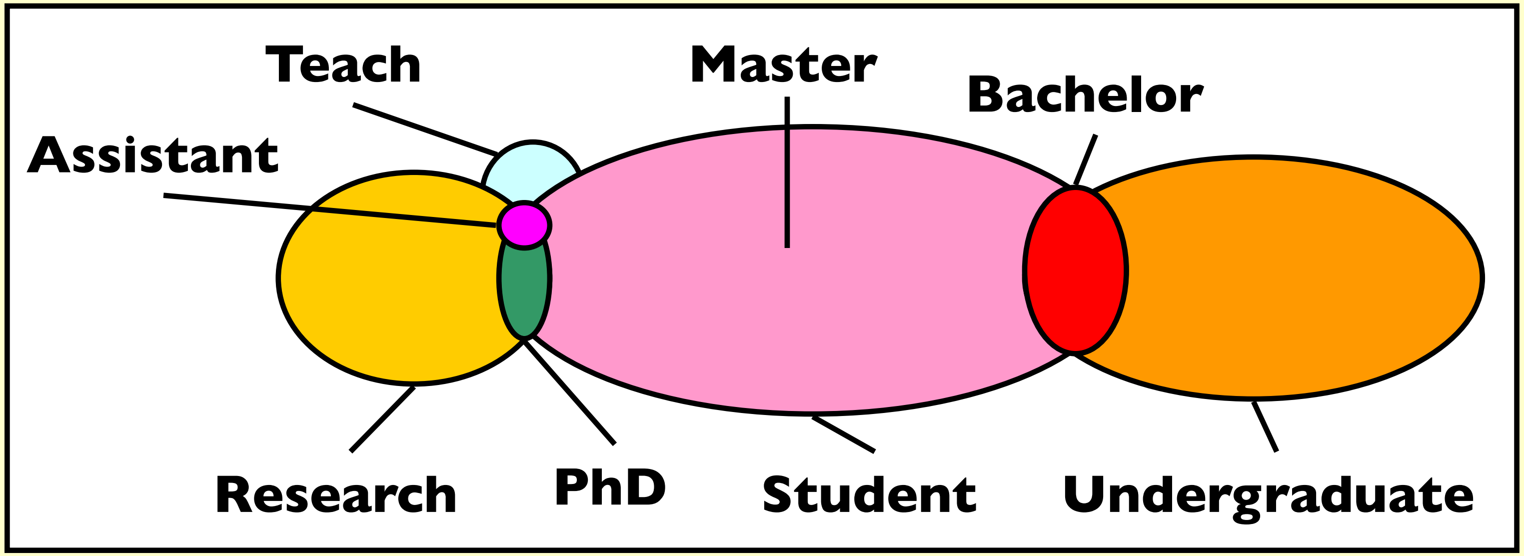
Equivalence with respect to a TBox T (I)

- Suppose we describe the students/listeners in a course:

$$
\begin{array}{lll}
\text{Undergraduate} & \sqsubseteq & \neg\,\text{Teach} \\
\text{Bachelor} & \equiv & \text{Student} \sqcap \text{Undergraduate} \\
\text{Master} & \equiv & \text{Student} \sqcap \neg\,\text{Undergraduate} \\
\text{PhD} & \equiv & \text{Master} \sqcap \text{Research} \\
\text{Assistant} & \equiv & \text{PhD} \sqcap \text{Teach}
\end{array}
$$

TBox T

- **T $\models$ Student $\equiv$ Bachelor $\sqcup$ Master**



Equivalence with respect to a TBox T (2)

Student $\equiv$ Bachelor $\sqcup$ Master

Proof:

Bachelor $\sqcup$ Master

$\equiv$ (Student $\sqcap$ Undergraduate) $\sqcup$ Master

$\equiv$ (Student $\sqcap$ Undergraduate) $\sqcup$ (Student $\sqcap \neg$ Undergraduate)

$\equiv$ Student $\sqcap$ (Undergraduate $\sqcup \neg$Undergraduate)

$\equiv$ Student $\sqcap \top$

$\equiv$ Student

Tbox reasoning: disjointness

- Let T be a TBox. Disjointness (with respect to T):

$$T \models P \sqcap Q \sqsubseteq \perp \quad (P \sqcap Q \sqsubseteq_T \perp)$$

Two concepts P and Q are disjoint with respect to T if their intersection is empty, $I(P) \cap I(Q) = \varnothing$, for every model I of T.

NOTE: disjointness is a property of <u>all models</u>

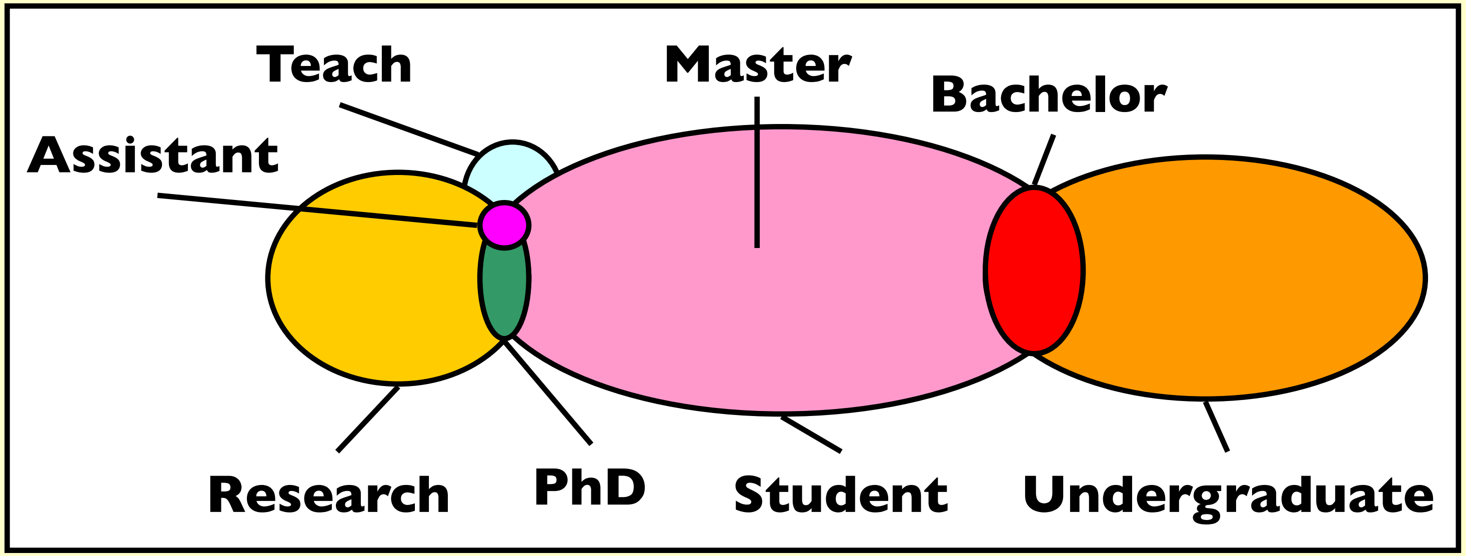
Disjointness with respect to a TBox T (1)

- Suppose we describe the students/listeners in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

- **T $\models$ Undergraduate $\sqcap$ Assistant $\sqsubseteq \perp$**



Disjointness with respect to a TBox T (2)

It can be proved showing that:

$T \models \text{Undergraduate} \sqcap \text{Assistant} \sqsubseteq \bot$

<u>Proof</u>:

$$
\begin{aligned}
\text{Undergraduate} \sqcap \text{Assistant} & \\
& \sqsubseteq \neg\text{Teach} \sqcap \text{Assistant} \\
& \equiv \neg\text{Teach} \sqcap \text{PhD} \sqcap \text{Teach} \\
& \equiv \bot \sqcap \text{PhD} \\
& \equiv \bot
\end{aligned}
$$

Exercise

Suspect we describe the students/attendees in a course:

Undergraduate	$\sqsubseteq \neg$ Teach
Bachelor	$\equiv$ Student $\sqcap$ Undergraduate
Master	$\equiv$ Student $\sqcap \neg$ Undergraduate
PhD	$\equiv$ Master $\sqcap$ Research
Assistant	$\equiv$ PhD $\sqcap$ Teach

TBox T

- Is Bachelor $\sqcap$ PhD satisfiable?

  NO!

- Consider the following propositions:

  Assistant, Student, Bachelor, Teach, PhD, Master $\sqcap$ Teach

  1. Which pairs are subsumed/supersumed?
  2. Which pairs are disjoint?

Abox - Syntax

- The second component of the knowledge base is the world description, the ABox.
- In an ABox one introduces individuals, by giving them names, and one *asserts* properties about them.
- We denote individual names as a, b, c,…
- An assertion with concept C is called concept assertion (or simply assertion) in the form:

C(a), C(b), C(c), …

Student(paul)
Professor(fausto)

To be read:
paul belongs to (is in) Student
fausto belongs to (is in) Professor

Abox - Semantics

- We give semantics to ABoxes by extending interpretations to individual names

- An interpretation $I: L \rightarrow \Delta^I$ not only maps atomic concepts to sets, but in addition it maps each individual name a to an element $a^I \in \Delta^I$, namely

$$I(a) = a^I \in \Delta^I$$

- Unique name assumption (UNA). We assume that distinct individual names denote distinct objects in the domain

NOTE: Δ^I denotes the domain of interpretation, a denotes the symbol used for the individual (the name), while a^I is the actual individual of the domain.

Abox - Semantics

Δ^I = {Fausto, Jack, Paul, Mary}

A

Student(paul)
Professor(fausto)

We mean that:

I(paul) $\in$ I(Student) I(fausto) $\in$ I(Professor)

I(paul) = Paul
I(fausto) = Fausto

I (Professor) = {Fausto}
I (Student) = {Jack, Paul, Mary}

Individuals in the TBox

- Sometimes, it is convenient to allow individual names (also called nominals) not only in the ABox, but also in the TBox
- They are used to construct concepts. The most basic one is the "**set**" constructor, written:

$$\{a_1,\ldots,a_n\}$$

It defines a concept, without giving it a name, by enumerating its elements, with the semantics:

$$\{a_1,\ldots,a_n\}^I = \{a_1^I,\ldots,a_n^I\}$$

Student ≡ {Jack, Paul, …, Mary} (the name is optional)

Abox - Reasoning Services

Given an ABox A, we can reason (w.r.t. a TBox T) about the following:

- Satisfiability/Consistency: An ABox A is consistent with respect to T if there is an interpretation I which is a model of both A and T.

- Instance checking: checking whether an assertion C(*a*) is entailed by an ABox, i.e. checking whether *a* belongs to C.

  A ⊨ C(*a*) if every interpretation that satisfies A also satisfies C(*a*).

- Instance retrieval: given a concept C, retrieve all the instances *a* which satisfy C.

- Concept realization: given *a* set of concepts and an individual *a* find the most specific concept(s) C (w.r.t. subsumption ordering) such that A ⊨ C(*a*).

Reasoning via expansion of the ABox

- Reasoning services over an ABox w.r.t. an acyclic TBox can be reduced to checking an expanded ABox.
- We define the expansion of an ABox A with respect to T as the ABox A' that is obtained from A by replacing each concept assertion C(*a*) with the assertion C'(*a*), with C' the expansion of C with respect to T.
- A is consistent with respect to T iff its expansion A' is consistent
- A is consistent iff A is satisfiable (*), i.e. non contradictory.
- (*) in PL, under the usual translation, with C(a) considered as a proposition different from C(b)

Abox - Reasoning via Expansion

T

$Undergraduate \sqsubseteq \neg Teach$

$Bachelor \equiv Student \sqcap Undergraduate$

$Master \equiv Student \sqcap \neg Undergraduate$

$PhD \equiv Master \sqcap Research$

$Assistant \equiv PhD \sqcap Teach$

A

Master(Chen)

PhD(Enzo)

Assistant(Rui)

- The expansion of A w.r.t. T:

Master(Chen)	PhD(Enzo)	Assistant(Rui)
Student(Chen)	Master(Enzo)	PhD(Rui)
¬Undergraduate(Chen)	Research(Enzo)	Teach(Rui)
	Student(Enzo)	Master(Rui)
	¬Undergraduate(Enzo)	Research(Rui)
		Student(Rui)
		¬Undergraduate(Rui)

Abox - Consistency

- Satisfiability/Consistency: An ABox A is consistent with respect to T if there is an interpretation I which is a model of both A and T.
- We simply say that A is consistent if it is consistent with respect to the empty TBox

T

Parent $\equiv$ Mother $\sqcup$ Father
Father $\equiv$ Male $\sqcap$ hasChild
Mother $\equiv$ Female $\sqcap$ hasChild
Male $\equiv$ Person $\sqcap$ $\neg$ Female

A

Mother(Mary)
Father(Mary)

A is <u>not</u> consistent w.r.t. T: In fact, from the expansion of T we get that Mother and Father are disjoint.

A is consistent (w.r.t. the empty TBox, no constraints)

Abox - Instance checking

- Instance checking: checking whether an assertion C(*a*) is entailed by an ABox, i.e. checking whether a belongs to C.
- A $\vDash$ C(a) if every interpretation that satisfies A also satisfies C(a).
- A $\vDash$ C(a) iff A $\cup$ {$\neg$ C(a)} is inconsistent

Consider T and A from the previous example.

Is Phd(Rui) entailed?
YES! The assertion is in the expansion of A.

Abox - Instance retrieval

- Instance retrieval: given a concept C, retrieve all the instances *a* which satisfy C.
- Implementation: A trivial, but not optimixed implementation consists in doing instance checking for all instances.

Consider T and A from the previous example.

Find all the instances of $\neg$Undergraduate
Looking at the expansion of A we have {Chen, Enzo, Rui}

Abox - Concept realization

- Concept realization: given *a* set of concepts and an individual a find the most specific concept(s) C (w.r.t. subsumption ordering) such that A $\models$ C(a).

- Dual problem of Instance retrieval

- Implementation: A trivial, but not optimixed implementation consists in doing instance checking for all concepts.

Abox - Concept realization

Consider T and A from the previous example.

T	A
Undergraduate $\sqsubseteq \neg$ Teach	Master(Chen)
Bachelor $\equiv$ Student $\sqcap$ Undergraduate	PhD(Enzo)
Master $\equiv$ Student $\sqcap \neg$ Undergraduate	Assistant(Rui)
PhD $\equiv$ Master $\sqcap$ Research	
Assistant $\equiv$ PhD $\sqcap$ Teach	

Given the instance Rui, and the concept set {Student, PhD, Assistant} find the most specific concept C such that A $\models$ C(Rui)

Rui is in the extension of all the concepts above.

The following chain of subsumptions holds: Assistant $\sqsubseteq$ PhD $\sqsubseteq$ Student

Therefore, the most specific concept is Assistant.

CWA vs OWA

Closed world Assumption CWA (in Databases): anything which is not explicitly asserted <u>is false</u>

Open World Assumption OWA (in Abox): anything which is not explicitly asserted (positive or negative) <u>is unknown</u>

NOTE: a Database has/is <u>one</u> model. Query answering is model checking.

NOTE: an ABox has <u>a set of</u> models. Query answering is satisfiability.

NOTE: In ABoxes, like in databases, we use CWA.