Mathematical Logic - 2017

Exercises: MODAL LOGIC

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Introduction

- We want to model situations like this one:
  1. "Fausto is always happy" circumstances"
  2. "Fausto is happy under certain

- In PL/ClassL we could have: HappyFausto

- In modal logic we have:
  1. $\Box$ HappyFausto
  2. $\Diamond$ HappyFausto

As we will see, this is captured through the notion of "possible words" and of "accessibility relation"

Syntax

- We extend PL with two logical modal operators:

  □ (box) and ◊ (diamond)

  □P : “Box P” or “necessarily P” or “P is necessary true”

  ◊P : “Diamond P” or “possibly P” or “P is possible”

  Note that we define □P = ¬◊¬P, i.e. □ is a primitive symbol

- The grammar is extended as follows:

  <Atomic Formula> ::= A | B | ... | P | Q | ... | ⊥ | ⊤ |

  <wff> ::= <Atomic Formula> | ¬<wff> | <wff> ∧ <wff> | <wff> ∨ <wff> |

  <wff> → <wff> | <wff> ↔ <wff> | □ <wff> | ◊ <wff>

Syntax

Say whether the following strings of symbols are well formed modal formulas on $P = \{p, q\}$

1. $\Box \to p$
2. $\Box p \to p$
3. $\Box p \to \Box\Box p$
4. $\Box\Diamond q \wedge \bot\Diamond$
5. $\Box p \to \Diamond p$
6. $\Diamond\top$
7. $p \to \Box\Diamond p$

Well formed formulas: 2., 3., 5., 6. and 7.

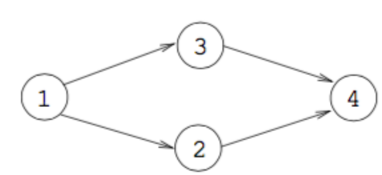
Syntax

Let the kripke frame $\mathcal{F} = (W, R)$ *given by*

$$\mathcal{F} = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (2, 4), (3, 4)\})$$

Depict the labeled graph corresponding to $\mathcal{F}$.

Solution.



Semantics

A basic frame (or simply a frame) is an algebraic structure

$$\mathcal{F} = \langle W, R \rangle$$

where $R \subseteq W \times W$.
An interpretation $\mathcal{I}$ (or assignment) of a modal language in a frame $\mathcal{F}$, is a function

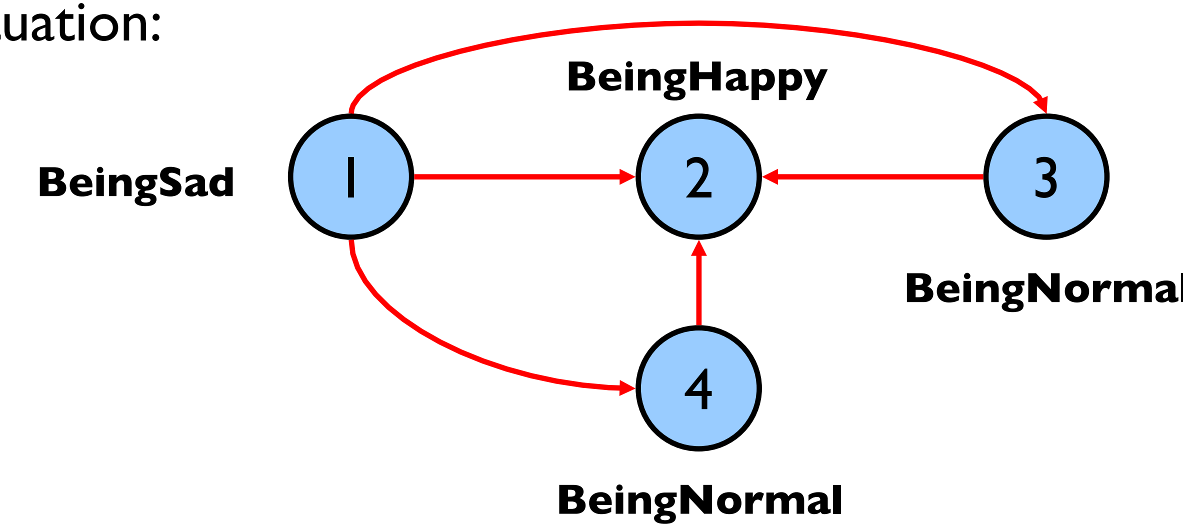
$$\mathcal{I} : P \to 2^W$$

Intuitively $w \in \mathcal{I}(p)$ means that p is true in w, or that w is of type p.
A model $\mathcal{M}$ is a pair $\langle \mathit{frame}, \mathit{interpretation} \rangle$. I.e.:

$$\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$$

Semantics

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <1, 3>, <1, 4>, <3, 2>, <4, 2>}

  I(BeingHappy) = {2} I(BeingSad) = {1} I(BeingNormal) = {3, 4}

Satisfiability of modal formulas

Truth is relative to a world, so we define that relation of $\models$ between a world in a model and a formula (NOTE: wRw' can be read as "w' is accessible from w via R")

$M, w \models p$ iff $w \in I(p)$

$M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$

$M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$

$M, w \models \varphi \supset \psi$ iff $M, w \models \varphi$ $\Rightarrow$ *implies* $M, w \models \psi$

$M, w \models \varphi \equiv \psi$ iff $M, w \models \varphi$ iff $M, w \models \psi$

$M, w \models \neg\varphi$ iff not $M, w \models \varphi$

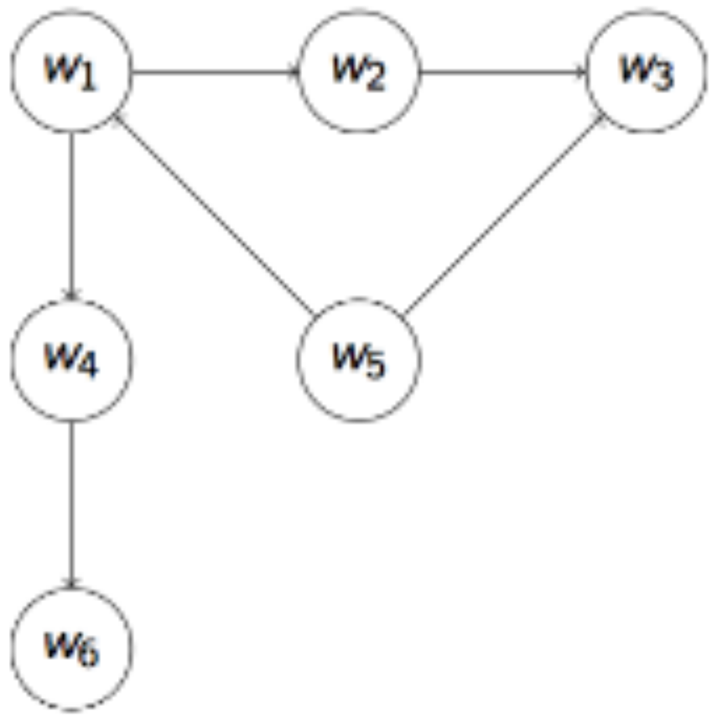
$M, w \models \Box\varphi$ iff for all w' s.t. wRw', $M, w' \models \varphi$

$M, w \models \Diamond\varphi$ iff there is a w' s.t. wRw' and $M, w' \models \varphi$

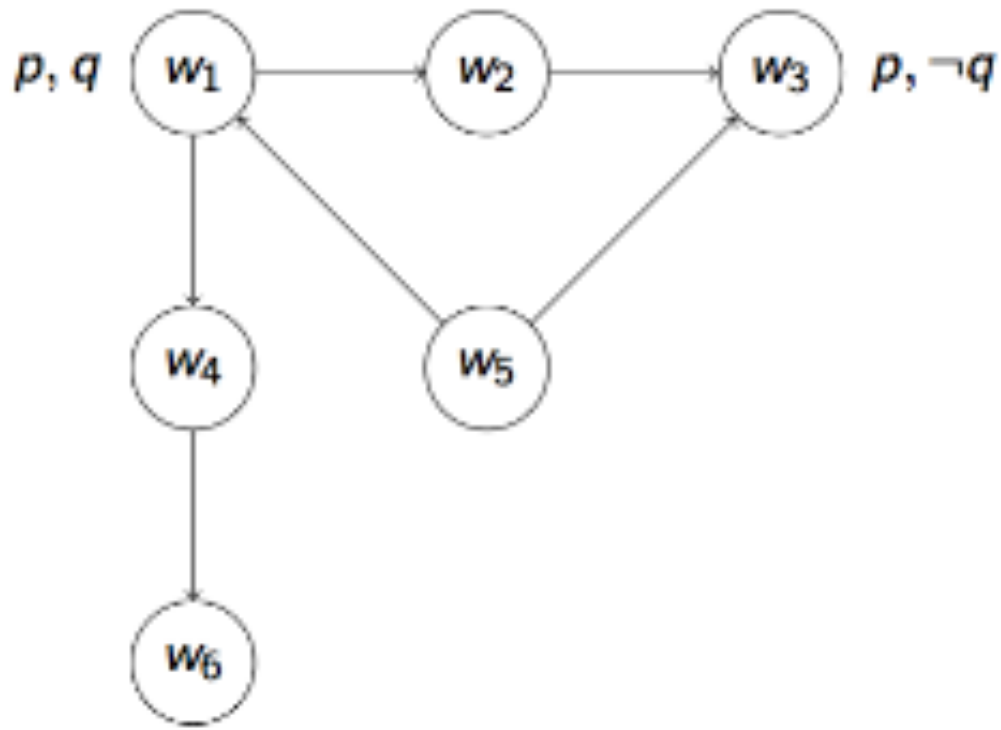
φ is globally satisfied in a model M, in symbols, $M \models \varphi$ if

$M, w \models \varphi$ for all $w \in W$

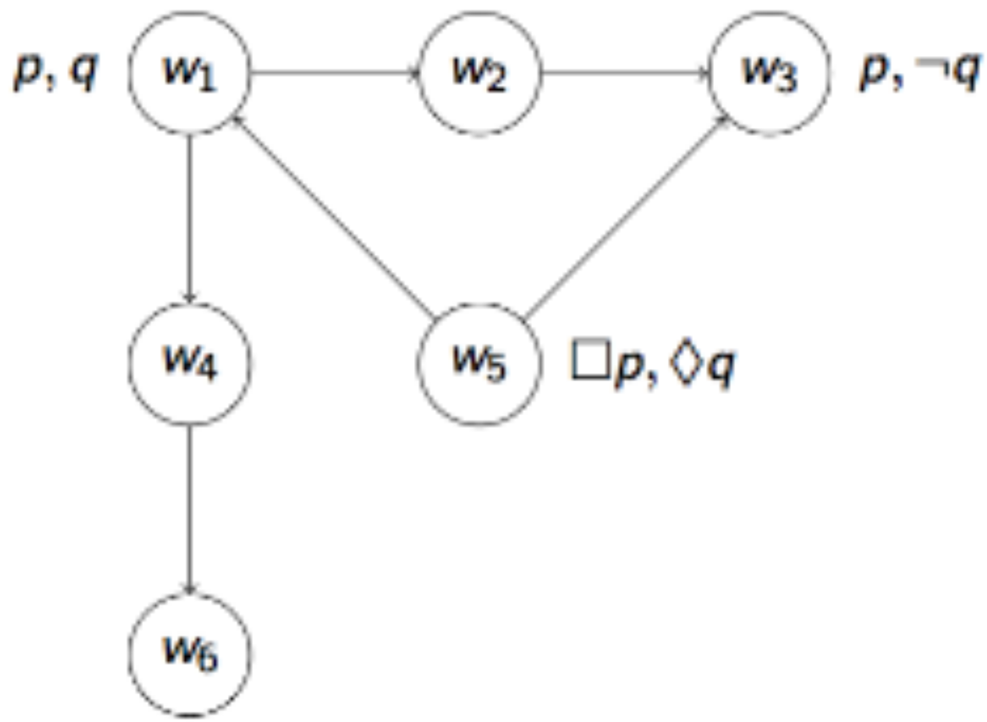
Satisfiability examples



Satisfiability examples

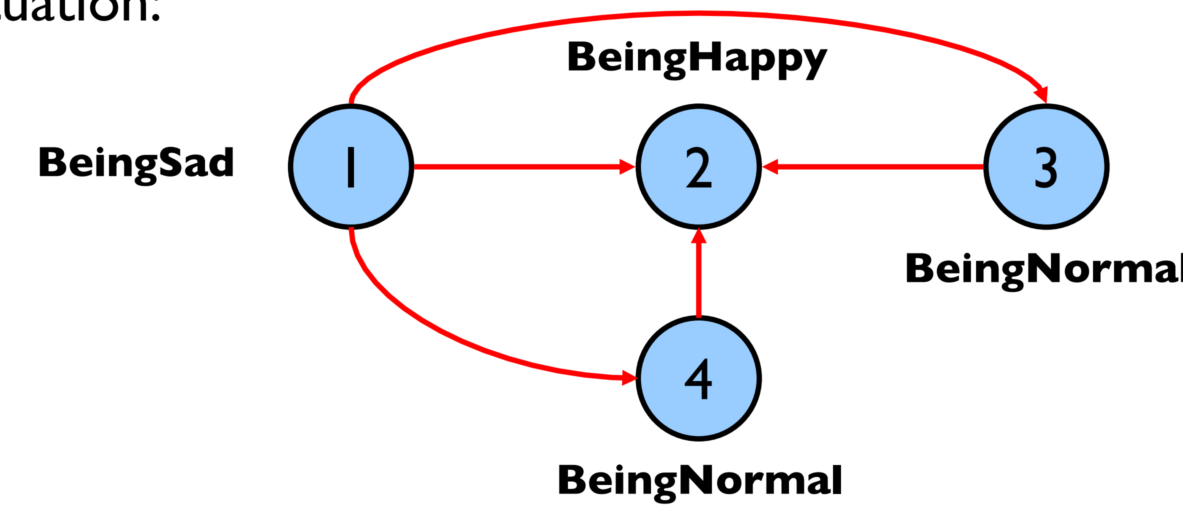


Satisfiability examples



Semantics: Kripke Model

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <1, 3>, <1, 4>, <3, 2>, <4, 2>}

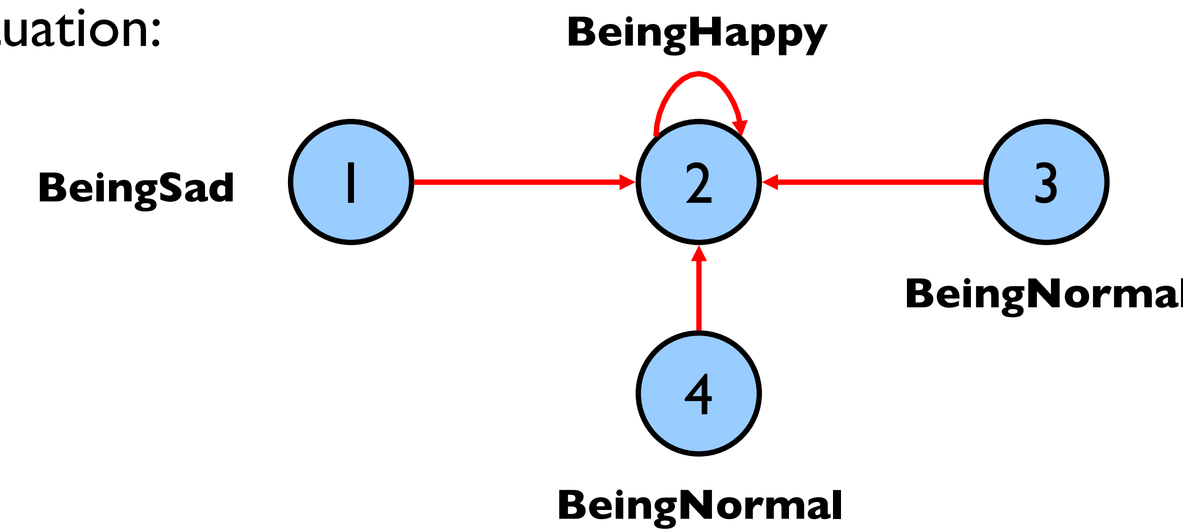
  I(BeingHappy) = {2} I(BeingSad) = {1} I(BeingNeutral) = {3, 4}

  $M, 2 \vDash BeingHappy$ $M, 2 \vDash \neg BeingSad$

  $M, 4 \vDash \Box BeingHappy$ $M, 1 \vDash \Diamond BeingHappy$ $M, 1 \vDash \neg\Diamond BeingSad$

Satisfiability

- Consider the following situation:



- M = <W, R, I>

  W = {1, 2, 3, 4}

  R = {<1, 2>, <2, 2>, <3, 2>, <4, 2>}

  I(BeingHappy) = {2} I(BeingSad) = {1} I(BeingNormal) = {3, 4}

  Does □BeingHappy is (globally) satisifable?

  M, w ⊨ □BeingHappy for all w ∈ W, therefore □BeingHappy is satisfiable in M.

Expressing properties structure

formula true at w	property of w
$\Diamond \top$	w has a successor point
$\Diamond\Diamond \top$	w has a successor point with a successor point
$\underbrace{\Diamond \ldots \Diamond}_{n} \top$	there is a path of length n starting at w
$\Box \bot$	w does not have any successor point
$\Box\Box \bot$	every successor of w does not have a successor point
$\underbrace{\Box \ldots \Box}_{n} \bot$	every path starting form w has length less then n

Expressing properties structure

formula true at *w*	**property of *w***
$\Diamond p$	*w* has a successor point which is *p*
$\Diamond\Diamond p$	*w* has a successor point with a successor point which is *p*
$\underbrace{\Diamond \ldots \Diamond}_{n} p$	there is a path of length *n* starting at *w* and ending at a point which is *p*
$\Box p$	every successor of *w* are *p*
$\Box\Box p$	all the successors of the successors of *w* are *p*
$\underbrace{\Box \ldots \Box}_{n} p$	all the paths of length *n* starting form *w* ends in a point which is *p*

Expressing properties structure

$W = \{1,2,3\}$, $R = \{(1,2),(2,2),(2,3)\}$, $\mathcal{I}(p) = \{1,2\}$.

Draw it as a labelled graph and then verify which of the following holds:

1. $\mathcal{M}, 1 \models p$

2. $\mathcal{M}, 2 \models \Diamond p$

3. $\mathcal{M}, 3 \models \Box p$

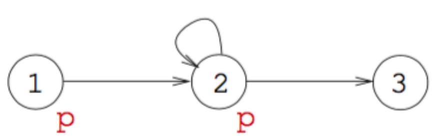
4. $\mathcal{M}, 1 \models \Box\Box p$

5. $\mathcal{M}, 1 \models \Box\Diamond p$

6. $\mathcal{M}, 1 \models \Diamond\neg p$

7. $\mathcal{M}, 2 \models \Diamond\neg p$

Solution.



1. ,2. ,3. ,5. ,7.

Expressing properties structure

1. (a) For each point s in the following model, give a modal formula that is only true at s:

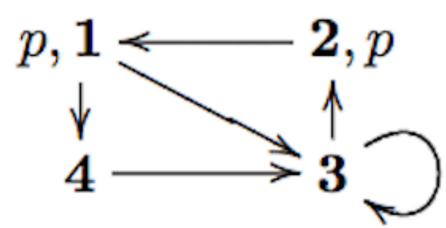
$$\begin{tikzcd} \mathbf{1} \arrow[r] & \mathbf{2} \arrow[d] \\ \mathbf{4} \arrow[u] \arrow[r] & \mathbf{3} \end{tikzcd}$$

Use modal formulas involving only "true" ($\top$) and "false" ($\bot$).

1(a) The most remarkable point in the model is 3: an end-point, that satisfies $\Box\bot$. Points 1, 2 and 4 differ in their access to it: 2 uniquely satisfies $\Diamond\top \wedge \Box\Box\bot$. The "dirty solution" for world 1: it is the only world to see world 2: $\Diamond(\Diamond\top \wedge \Box\Box\bot)$. The still dirtier solution for world 4: conjoin the negations of the definitions for 1, 2 and 3.

Expressing properties structure

2. (a) Determine in which states of the following model the modal formula $\Diamond\Box\Diamond p$ is true:



2(a) Compute a table for truth of all sub-formulas in all worlds:

p:	1, 2	$\Diamond p$:	2, 3
$\Box\Diamond p$:	3, 4	$\Diamond\Box\Diamond p$:	1, 3, 4

Validity relation on frames

A formula φ is valid in a world w of a frame F, in symbols $F, w \vDash \varphi$ iff

$$M, w \vDash \varphi \quad \text{for all } I \text{ with } M = \langle F, I \rangle$$

A formula φ is valid in a frame F, in symbols $F \vDash \varphi$ iff

$$F, w \vDash \varphi \quad \text{for all } w \in W$$

If C is a class of frames, then a formula φ is valid in the class of frames C, in symbols $\vDash_C \varphi$ iff

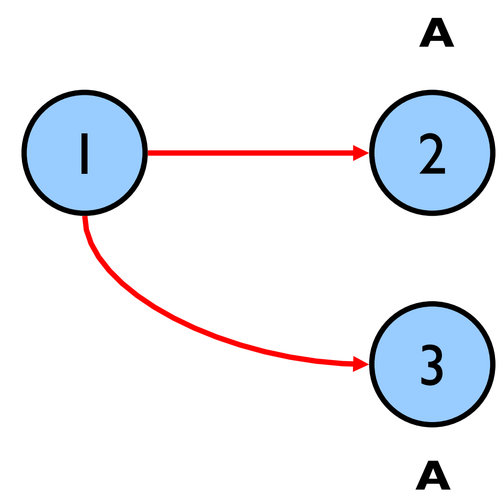
$F \vDash \varphi$ for all $F \in$ C

A formula φ is valid, in symbols $\vDash \varphi$ iff

$$F \vDash \varphi \quad \text{for all models frames } F$$

Validity

- Prove that P: $\Box A \rightarrow \Diamond A$ is valid



- In all models M = <W, R, I>,

  (1) $\Box A$ means that for every $w \in W$ such that wRw' then $M, w' \vDash A$

  (2) $\Diamond A$ means that for some $w \in W$ such that wRw' then $M, w' \vDash A$

  It is clear that if (1) then (2) in the example

  (as we will see this is valid in serial frames)

Kinds of frames

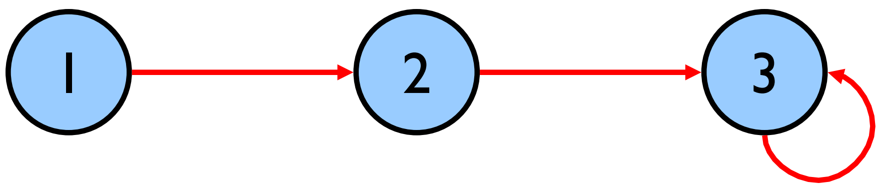
- Given the frame F = <W, R>, the relation R is said to be:

  - **Serial** iff for every $w \in W$, there exists $w' \in W$ s.t. wRw'
  - **Reflexive** iff for every $w \in W$, wRw
  - **Symmetric** iff for every $w, w' \in W$, if wRw' then $w'Rw$
  - **Transitive** iff for every $w, w', w'' \in W$, if wRw' and $w'Rw''$ then wRw''

- We call a frame <W, R> serial, reflexive, symmetric or transitive according to the properties of the relation R

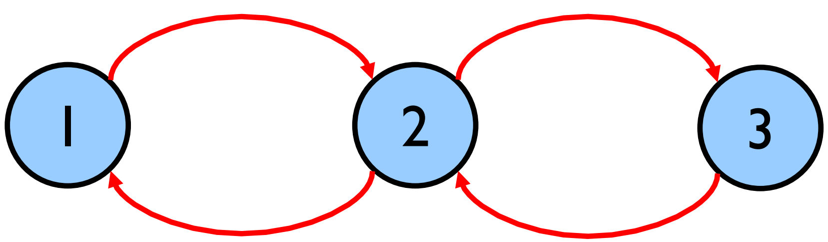
Kinds of frames

- Serial: for every w $\in$ W, there exists w’ $\in$ W s.t. wRw’

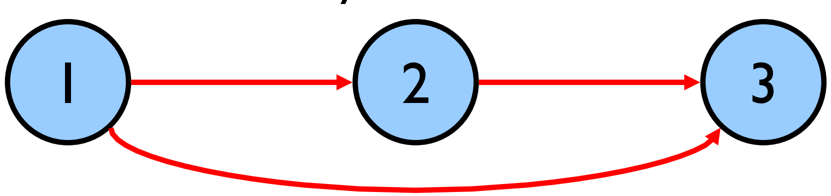


- Reflexive: for every w $\in$ W, wRw

- Symmetric: for every w, w’ $\in$ W, if wRw’ then w’Rw



- Transitive: for every w, w’, w” $\in$ W, if wRw’ and w’Rw” then wRw”



Modal logic exercise

Given the Kripke model M = <W, R, I> with:

W = {1, 2, 3}   R = {<1, 2>, <2, 1>, <1, 3>, <3, 3>}   I(A) = {1, 2} and I(B) = {2, 3}

- **Say whether the frame <W, R> is serial, reflexive, symmetric or transitive.**
  - It is serial.
- **Is M, 1 ⊨ ◊(A ∧ B)? Provide a proof for your response.**
  - Yes, because A ∧ B is true in 2 and 2 is accessible from 1.
- **Is □A satisfiable in M? Provide a proof for your response.**
  - We should have that M, w ⊨ □A for all worlds w. This means that for all worlds w there is a w’ such that wRw’ and M, w’ ⊨ A.
  - For w = 1 we have 1R3 and M, 3 ⊨ ¬A. Therefore the response is NO.