Mathematical Logics
Modal Logic: Introduction*

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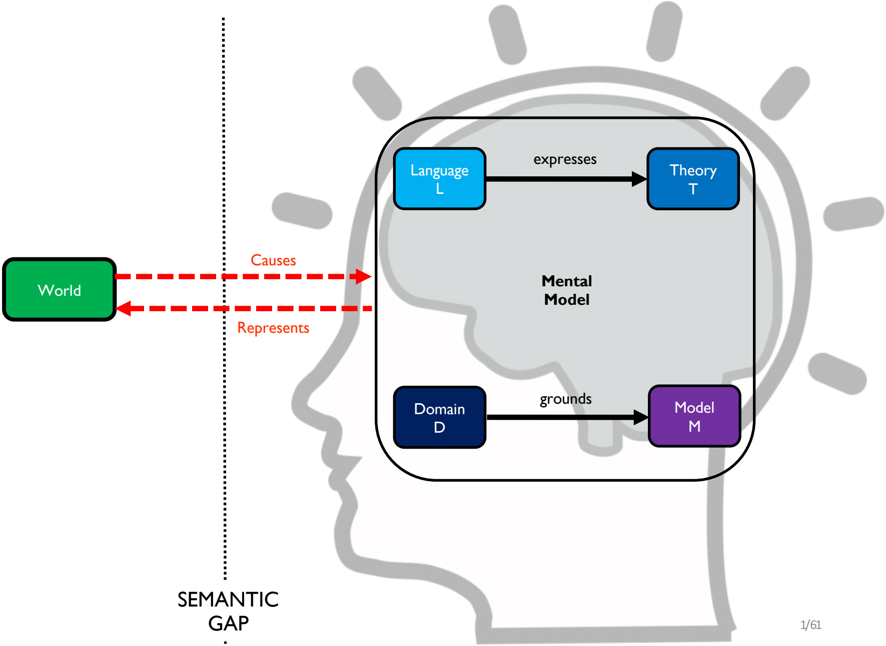
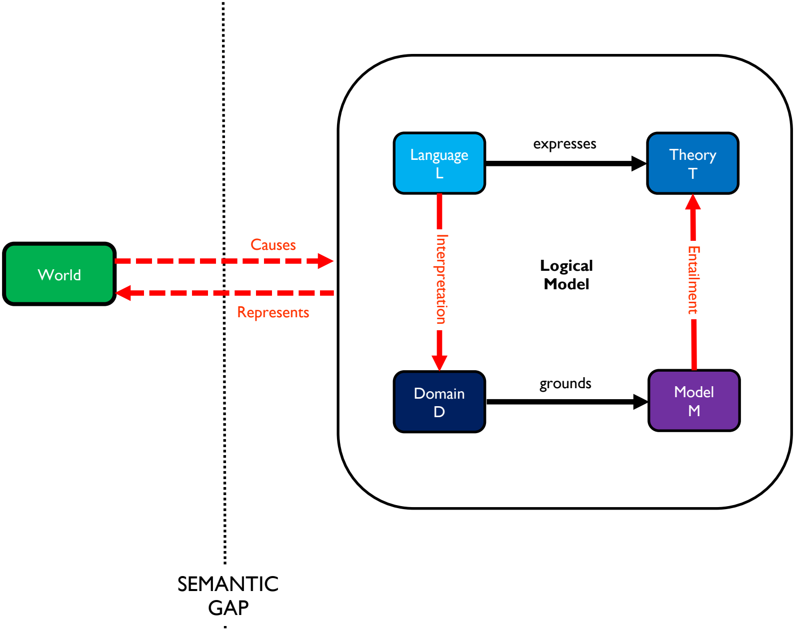
Mental Model



Logical Model



Logical Model

World

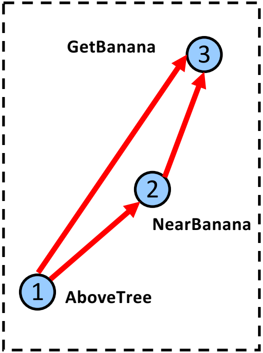
Logical Model

Language L

Domain D

Theory T

Model M



L = "*AboveTree, NearBanana, GetBanana 1, 2,* ∧, ∨, ¬, →, □, ◊, …"

T = "{<1, 2>, <1, 3>, <2, 3>}"

D: {1, 2, 3}

I: "I(AboveTree) = 1, I(NearBanana) = 2, I(GetBanana) = 3"

M,1 ⊨ *AboveTree*, **M,2** ⊨ □*GetBanana*, …

SEMANTIC GAP

TestBooks and Readings

- *Hughes, G. E., and M.J. Cresswell (1996) A New Introduction to Modal Logic. Routledge.*
  Introductory textbook. Provides an historic perspective and a lot of explanations.
- *Blackburn, Patrick, Maarten de Rijke, and Yde Venema (2001) Modal Logic. Cambridge Univ. Press*
  More modern approach. It focuses on the formalisation of frames and structures.
- *Chellas, B. F. (1980) Modal Logic: An Introduction. Cambridge Univ. Press*

  The focus is on the axiomatization of the modal operators $\Box$ and $\Diamond$

Origins of modal logics

- (Modern modal logic) Developed in the early twentieth century,
- **Clarence Irving Lewis**, thought that Russell's description of the truth-functional conditional operator as material implication (i.e, $A \supset B$ is true if either A is false or B is true) was misleading.
  He suggested to define a new form of implication called strict implication which literally can be seen like this

  it is not possible that A is true and B is false (1)

- He proposed to formalise (1) as

$$\neg\Diamond (A \wedge \neg B) \qquad (2)$$

Origins of modal logics - ctn'd

The novelties in $\neg\Diamond(A \wedge \neg B)$ are:

- A modal operator $\Diamond$ for representing the fact that a statement is *possibly true* (*impossible, necessary, . . .*)
- The fact that the truth value of $\neg\Diamond(A \wedge \neg B)$ is not a function of the truth values of A and B as it refers to a set of *possible situations* (lately called possible worlds) in which you have to consider the truth of A and B.

What is Modality?

- A modality is an expression that is used to *qualify* the truth of a judgement (or, in other words, an operator that expresses a "mode" in which a proposition is true)
- It can be seen as an operator that takes a proposition and returns a more complex proposition.

Proposition	Modal Expression
John drives a Ferrari	John *is able to* drive a Ferrari
Everybody pays taxes	It is *obligatory* that everybody pays taxes

- Modalities are expressed in natural language through **modal verbs** such as *can/could, may/might, must, will/would*, and *shall/should.*

What is Modality?

- In logic modalities are formalized using an operator such as $\Box$ ($\Diamond$) that can be applied to a formula φ to obtain another formula $\Box\varphi$ ($\Diamond\varphi$).
- The truth value of $\Box\varphi$ is not a function of the truth value of φ.

Example

- The fact that John is able to drive a Ferrari may be true independently from the fact that John is actually driving a Ferrari.
- The fact that it is *obligatory* that everybody pays taxes is typically true, and this is independent from the fact that everybody actually pays taxes.

Note: $\neg$ is not a modal operator since the truth value of $\neg\varphi$ is a function of the truth value of φ.

Modalities

- A modality is an expression that is used to *qualify* the truth of a judgement.
- Historically, the first modalities formalized with modal logic were the so called alethic modalities i.e.,
  1. it is possible that a certain proposition holds, usually denoted with $\Diamond\varphi$
  2. it is necessary that a certain proposition holds, usually denoted with $\Box\varphi$
- Afterwards a number of modal logics for different "*qualifications*" have been studied. The most common are. . .

Modalities

Modality	Symbol	Expression Symbolised
Alethic	$\Box\varphi$	it is *necessary* that φ
	$\Diamond\varphi$	it is *possible* that φ
Deontic	$O\varphi$	it is *obligatory* that φ
	$P\varphi$	it is *permitted* that φ
	$F\varphi$	it is *forbidden* that φ
Temporal	$G\varphi$	it will *always* be the case that φ
	$F\varphi$	it will *eventually* be the case that φ
Epistemic	$B_a\varphi$	agent a *believes* that φ
	$K_a\varphi$	agent a *knows* that φ
Contextual	$ist(c, \varphi)$	φ is *true in the context c*
Dynamic	$[\alpha]\varphi$	φ must be true after the execution of program α
	$(\alpha)\varphi$	φ can be true after the execution of program α
Computational	$AX\varphi$	φ is true for every immediate successor state
	$AG\varphi$	φ is true for every successor state
	$AF\varphi$	φ will eventually be true in all the possible evolutions
	$A\varphi U\vartheta$	φ is true until ϑ becomes true
	$EX\varphi$	φ is true in at least one immediate successor state

Modal logics & relational structures

- Historically, modal logics were developed in order to formalise the different modalities that qualify the truth of a formula;
- Modern modal logics have a different goal. They are motivated by the study of relational structures.

□efinition (Relational structure)

A relational structure is a tuple

$\langle W, R_{a_1}, \ldots, R_{a_n} \rangle$

where $R_{a_i} \subseteq W \times \ldots \times W$

- each $w \in W$ is called, point (world, state, time instant, situation, . . .)
- each R_{a_i} is called accessibility relation (or simply relation)

Alternative notation $\langle W, R_a \rangle_{a \in A}$

The importance of relational structures

- In Computer Science, Artificial Intelligence and Knowledge Representation there are many examples of relational structures:
  - Graphs and labelled graphs;
  - Ontologies;
  - Finite state machines;
  - Computation paths; . ..
- Modal logics allow us to predicate on properties of relational structures.
  - Loop detection;
  - Reachability of a (set of) node(s);
  - Properties of a relation such as Transitivity, Reflexivity, ...

Examples of Relational structures

- Strict partial order (SPO)

  $\langle W, < \rangle$ $<$ is transitive and irreflexive[1]

- Strict linear order

  $\langle W, < \rangle$ (SPO) + for each $v \neq w \in W$, $v < w$ or $w < v$

- Partial order (PO)

  $\langle W, \leq \rangle$ $\leq$ is transitive, reflexive, and antisymmetric

- Linear order

  $\langle W, \leq \rangle$ (PO) + for each $v, w \in W$, $v \leq w$ or $w \leq v$

- Labeled transition system (LTS)

  $\langle W, R_a \rangle_{a \in A}$ and $R_a \subseteq W \times W$

- XML document

  $\langle W, R_l \rangle_{l \in L}$, W contains the components of an XML document and L is the set of labels that appear in the document

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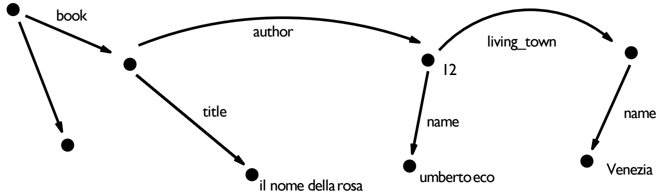
[1] Antisymmetry follows.

XML document as a relational stucture



Relational structures in FOL

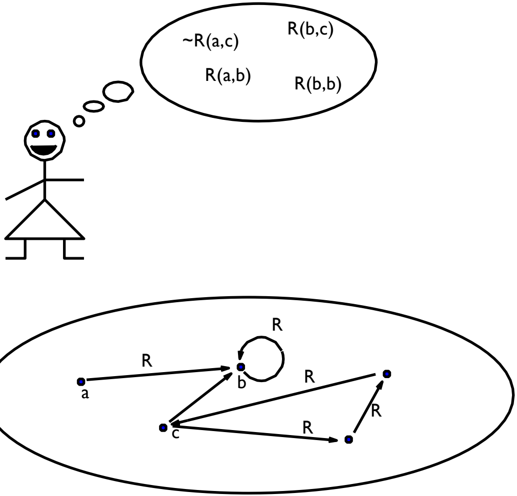
- Relational structures can be investigated in FOL;
- The language must contain at least a binary relation R, and we can formalize the properties of a relational structure using formulae such as
  - $\forall x R(x, x)$ (R is reflexive)
  - $\forall x \exists y R(x, y)$ (R is serial)
  - $\forall xy (R(x, y) \supset R(y, x))$ (R is symmetric)
  - . . .
- So, why do we need modal logics?

Relational structures in first order and modal logic

- In First Order Logic we describes a relational structure from an external point of view, (and our description is not relative to a particular point).
- Modal logics describe relational structures from an internal point of view, rather than from the top perspective
- A formula has a meaning in a point $w \in W$ of a structure

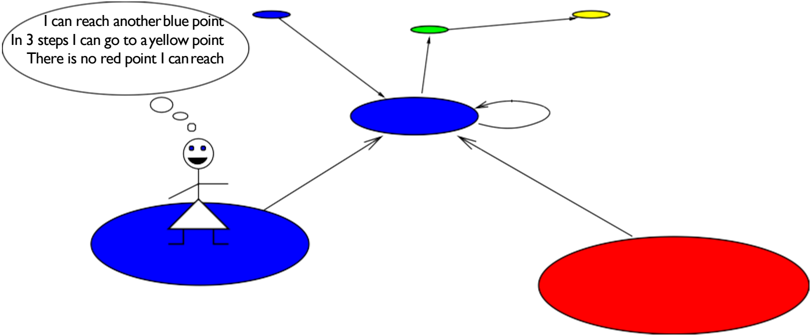
Relational structures in first order and modal logic

In first order logic, relational structures are described from the top point of view. each point of *W* and the re-lation *R* can benamed.



Relational structures in first order and modal logic

In modal logics, relational structures are described from an internal perspective there is no way to mention points of W and the relation R.



An example: seriality

Let us assume to have a strict linear serial order.

- In first order logic I can observe an infinite sequence of points;
- in modal logic I know that I can always move to the next point (that is, from the point where I am I can always see (and move to) a successor point).

The Language of a basic modal logic

If P is a set of primitive proposition, the set of formulas of the basic modal logic is defined as follows:

- each $p \in$ P is a formula (atomic formula);
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$, $A \supset B$ and $A \equiv B$ are formulas
- if A is a formula $\Box A$ and $\Diamond A$ are formulas.

Intuitive interpretation of the basic modal logic

The formula $\Box\varphi$ can be intuitively interpreted in many ways

- φ is necessarily true (classical modal logic)
- φ is known/believed to be true (epistemic logic)
- φ is provable in a theory (provability logic)
- φ will be always true (temporal logic)
- . . .

In all these cases $\Diamond\varphi$ is interpreted as $\neg\Box\neg\varphi$.

In other words, $\Diamond\varphi$, stands for $\neg\varphi$ is not necessarily true, that is, φ is possibly true.

Semantics for the basic modal logic

A basic frame (or simply a frame) is an algebraic structure

$$F = \langle W, R \rangle$$

where $R \subseteq W \times W$.
An interpretation I (or assignment) of a modal language in a frame F, is a function

$$I : P \rightarrow 2^W$$

Intuitively $w \in I(p)$ means that p is true in w, or that w is of type p.
A model M is a pair (*frame, interpretation*). I.e.:

$$M = \langle F, I \rangle$$

Satisfiability of modal formulas

Truth is relative to a world, so we define that relation of $\models$ between a world in a model and a formula

$$M, w \models p \text{ iff } w \in I(p)$$

$$M, w \models \varphi \wedge \psi \text{ iff } M, w \models \varphi \text{ and } M, w \models \psi$$

$$M, w \models \varphi \vee \psi \text{ iff } M, w \models \varphi \text{ or } M, w \models \psi$$

$$M, w \models \varphi \supset \psi \text{ iff } M, w \models \varphi \implies \textit{implies } M, w \models \psi$$

$$M, w \models \varphi \equiv \psi \text{ iff } M, w \models \varphi \text{ iff } M, w \models \psi$$

$$M, w \models \neg\varphi \text{ iff not } M, w \models \varphi$$

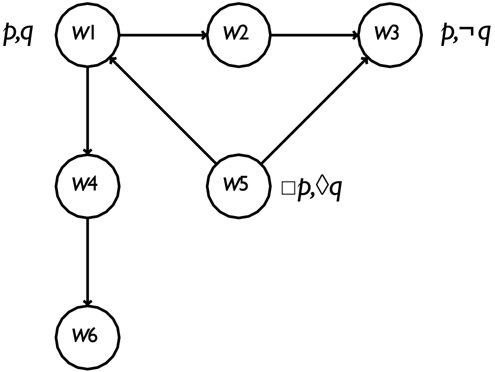
$$M, w \models \Box\varphi \text{ iff for all } w' \text{ s.t. } wRw', M, w' \models \varphi$$

$$M, w \models \Diamond\varphi \text{ iff there is a } w' \text{ s.t. } wRw' \text{ and } M, w' \models \varphi$$

φ is globally satisfied in a model M, in symbols, $M \models \varphi$ if

$$M, w \models \varphi \quad \text{for all } w \in W$$

Satisfiability example



Validity relation on frames

A formula φ is valid in a world w of a frame F, in symbols
$F, w \vDash \varphi$ iff

$$M, w \vDash \varphi \text{ for all } I \text{ with } M = \langle F, I \rangle$$

A formula φ is valid in a frame F, in symbols $F \vDash \varphi$ iff

$$F, w \vDash \varphi \text{ for all } w \in W$$

If C is a class of frames, then a formula φ is valid in the class of frames C, in symbols $\vDash_C \varphi$ iff

$$F \vDash \varphi \text{ for all } F \in C$$

A formula φ is valid, in symbols $\vDash \varphi$ iff

$$F \vDash \varphi \text{ for all models frames } F$$

Logical consequence

- φ is a local logical consequence of Γ, in symbols $\Gamma \vDash \varphi$, if for every model $M = \langle F, I \rangle$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

- φ is a local logical consequence of Γ in a class of frames C, in symbols $\Gamma \vDash_C \varphi$ if for avery model $M = \langle F, I \rangle$ with $F \in C$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

Hilbert axioms for normal modal logic

A1 $\varphi \supset (\psi \supset \varphi)$

A2 $(\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset$

A3 $\vartheta))$ $(\neg\psi \supset \neg\varphi) \supset ((\neg\psi \supset \varphi) \supset \varphi)$

MP $\dfrac{\varphi \quad \varphi \supset \psi}{\psi}$

K $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$

Nec $\dfrac{\varphi}{\Box\varphi}$ the necessitation rule

The above set of axioms and rules is called **K**, and every modal logic with a validity relation closed under the rules of **K** is a Normal Modal Logic.

Remark on Nec

Notice that **Nec** rule is not the same as

$$\varphi \supset \Box\varphi \qquad (3)$$

indeed formula (3) is not valid.

Assignment Find a model in which (3) is false

Satisfiability – exercises

Exercise

Show that each of the following formulas is not valid by constructing a frame $F = (W, R)$ that contains a world that does not satisfy them.

1. $\Box\bot$
2. $\Diamond\varphi \supset \Box\varphi$
3. $\Diamond\Box\varphi \supset \Box\Diamond\varphi$

Multi-Modal Logics

All the definitions given for basic modal logic can be generalized in the case in which we have n $\Box$-operators $\Box_1, \ldots, \Box_n$ (and also $\Diamond_1, \ldots, \Diamond_n$), which are interpreted in the frame

$$F = (W, R_1, \ldots R_n)$$

Every $\Box_i$ and $\Diamond_i$ is interpreted w.r.t. the relation R_i .

A logic with n modal operators is called Multi-Modal.

Multi-Modal logics are often used to model Multi-Agent systems where modality $\Box_i$ is used to express the fact that "agent i knows (believes) that φ".

Exercises

Exercise

Let $F = (W, R_1, \ldots, R_n)$ be a frame for the modal language with n modal operator $\Box_1, \ldots, \Box_n$. Show that the following properties holds:

1. $F \vDash \mathbf{K}_i$ (where $\mathbf{K}_i$ is obtained by replacing $\Box$ with $\Box_i$ in the axiom $\mathbf{K}$)
2. If $R_i \subseteq R_j$ then $F \vDash \Diamond_i\varphi \supset \Diamond_j\varphi$
3. If $R_i \subseteq R_j$ then $F \vDash \Box_j\varphi \supset \Box_i\varphi$
4. $F \nvDash \Box_i p \supset \Box_j p$ for any primitive proposition p
5. If $R_i \subseteq R_j \circ R_k$, then[a] $F \vDash \Diamond_i\varphi \supset \Diamond_j\Diamond_k\varphi$

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[a] Given two binary relations R and S on the set W, $R \circ S = \{(v, u) | (v, w) \in R \text{ and } (w, u) \in S\}$

Other exercises

Exercise

Prove that the following formulae are valid:

- $\models \Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$
- $\models \Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$
- $\models \neg\Diamond\varphi \equiv \Box\neg\varphi$
- $\neg\Box\Diamond\Diamond\Box\Box\Diamond\Box\varphi \equiv \Diamond\Box\Box\Diamond\Diamond\Box\Diamond\neg\varphi$ (i.e., pushing in $\neg$ changes $\Box$ into $\Diamond$ and $\Diamond$ into $\Box$)

Suggestion: keep in mind the analogy $\Box/\forall$ and $\Diamond/\exists$.

Exercise

Exercise

Consider the frame $F = (W, R)$ with

- $W = \{0, 1, \ldots n - 1\}$
- $R = \{(0, 1), (1, 2), \ldots, (n - 1, 0)\}$

Show that the following formulas are valid in F

1. $\Box\varphi \equiv \Diamond\varphi$
2. $\varphi \equiv \underbrace{\Box \ldots \Box}_{n}\varphi$

Answers also the following questions:

3. can you explain which property of the frame R is formalized by formula 1 and 2?
4. Can you imagine another frame F', different from F that satisfies formulas 1 and 2?

Expressing properties on structures

formula true at w	property of w
$\Diamond \top$	w has a successor point
$\Diamond\Diamond \top$	w has a successor point with a successor point
$\underbrace{\Diamond \ldots \Diamond}_{n} \top$	there is a path of length n starting at w
$\Box \bot$	w does not have any successor point
$\Box\Box \bot$	every successor of w does not have a successor point
$\underbrace{\Box \ldots \Box}_{n} \bot$	every path starting form w has length less then n

Expressing properties on structures

formula true at *w*	**property of *w***
$\Diamond p$	*w* has a successor point which is *p*
$\Diamond\Diamond p$	*w* has a successor point with a successor point which is *p*
$\underbrace{\Diamond \ldots \Diamond}_{n} p$	there is a path of length *n* starting at *w* and ending at a point which is *p*
$\Box p$	every successor of *w* are *p*
$\Box\Box p$	all the successors of the successors of *w* are *p*
$\underbrace{\Box \ldots \Box}_{n} p$	all the paths of length *n* starting form *w* ends in a point which is *p*