Mathematical Logic

Propositional Logic and First Order Logic*

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Finite domain

If we are interested in representing facts on a finite domain that contains n elements we can use the following theorem:

Theorem

The formula

$$\varphi_{|\Delta|=n} = \exists x_1,\dots,x_n \left(\bigwedge_{i \neq j=1}^{n} x_i \neq x_j \wedge \forall x \left(\bigvee_{i=1}^{n} x_i = x \right) \right)$$

is true in $I = \langle \Delta^I, {}^I \rangle$ if and only if $|\Delta^I| = n$, i.e., the cardinality of Δ is equal to n, i.e., Δ^I contains exactly n elements.

Finite domain

Proof.

We show that if $I \models \varphi_n$ then $|\Delta^I| = n$

1. If $I \models \varphi_n$ then there are $d_1, \ldots, d_n \in \Delta^I$ s.t.
2. $I \models \bigwedge_{i \neq j=1}^{n} x_i \neq x_j \wedge \forall x (\bigvee_{i=1}^{n} x_i = x)\ [a[x_1 := d_1, \ldots, x_n := d_n]]$
3. $I \models \bigwedge_{i \neq j=1}^{n} x_i \neq x_j\ [a[x_1 := d_1, \ldots, x_n := d_n]]$
4. From 3 we have that for all $1 \leq i \neq j \leq n$,
   $I \models x_i \neq x_j\ [a[x_i := d_i, x_j = d_j]]$
5. this implies that $d_i \neq d_j$ for $1 \leq i \neq j \leq n$.
6. since $d_1, \ldots, d_n \in \Delta^I$, we have that $|\Delta^I| \geq n$.
7. from 2 we have $I \models \forall x\ (\bigvee_{i=1}^{n} x_i = x)\ [a[x_1 := d_1, \ldots, x_n := d_n]]$
8. ths implies that for any $d \in \Delta^I$,
   $I \models (\bigvee_{i=1}^{n} x_i = x)\ [a[x_1 := d_1, \ldots, x_n := d_n, x := d]]$
9. which implies that for some i, $I \models x_i = x[a[x_i := d_i, x = d]]$, i.e., $d_i = d$ for some $1 \leq i \leq n$.
10. Since this is true for all $d \in \Delta^I$, then $|\Delta^I| \leq n$.

Finite domain, with names for every element

Unique Name Assumption (UNA)

Is the assumption under which the language contains a name for each element of the domain, i.e., the language contains the constant $c_1, \ldots, c_n$, and each constant is the name of one and only one domain element.

Theorem

The formula

$$\varphi_{\Delta=\{c_1,\ldots,c_n\}} = \left(\bigwedge_{i \neq j=1}^{n} c_i \neq c_j \wedge \forall x \left(\bigvee_{i=1}^{n} c_i = x \right) \right)$$

$\varphi_{\Delta=\{c_1,\ldots,c_n\}}$ *is also called Unique Name Assumption.*

Proof.

The proof is similar to the one of the previous theorem. Try it by exercise.

Finite domain - Grounding

Under the hypothesis of finite domain with a constant name for every elements, First order formulas can be propositionalized, aka grounded as follows:

$$\varphi_{\Delta=\{c_1,\ldots,c_n\}} \models \forall x \varphi(x) \equiv \varphi(c_1) \wedge \ldots \wedge \varphi(c_n) \quad (1)$$

$$\varphi_{\Delta=\{c_1,\ldots,c_n\}} \models \exists x \varphi(x) \equiv \varphi(c_1) \wedge \ldots \wedge \varphi(c_n) \quad (2)$$

Generalizing:

$$\varphi_{\Delta=\{c_1,\ldots,c_n\}} \models \forall x_1 \ldots x_k \varphi(x_1,\ldots,x_k) \equiv \bigwedge_{c_{i_1},\ldots,c_{i_k} \in \{c_1,\ldots,c_n\}} \varphi(c_{i_1},\ldots,c_{i_k}) \quad (3)$$

$$\varphi_{\Delta=\{c_1,\ldots,c_n\}} \models \exists x_1 \ldots x_k \varphi(x_1,\ldots,x_k) \equiv \bigvee_{c_{i_1},\ldots,c_{i_k} \in \{c_1,\ldots,c_n\}} \varphi(c_{i_1},\ldots,c_{i_k}) \quad (4)$$

Finite predicate extension

The assumption that states that a predicate P is true only for a finite set of objects for which the language contains a name, can be formalized by the following formulas:

$$\forall x(P(x) \equiv x = c_1 \lor \ldots \lor x = c_n)$$

Example

- The days of the week are: Monday, Tuesday, . . . , Sunday.

  $$\forall x(WeekDay(x) \equiv x = \text{Mon} \lor x = \text{Tue} \lor \ldots \lor x = \text{Sun})$$ The

- WorkingDays Monday, Tuesday, . . . , Friday:

  $$\forall x(WorkingDay(x) \equiv x = \text{Mon} \lor x = \text{Tue} \lor \ldots \lor x = \text{Fri})$$

Infinite domain

Is it possible to write a (set of) formula(s) that are satisfied only by an interpretation with infinite domain

Theorem

Let $\varphi_{inf\text{-}dom}$ be the formula:

$$\varphi_{inf\text{-}dom} = \forall x \neg R(x, x) \wedge \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \supset R(x,z)) \wedge \forall x \exists y R(x,y)$$

If $I \models \varphi_{inf\text{-}dom}$ then $|\Delta^I| = \infty$.

Observe that:

- $\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \supset R(x,z))$ represents the fact that R is interpreted in a transitive relation
- $\forall x \neg R(x, x)$ represents the fact that R is anti-reflexive

Infinite domain

Proof.

- By definition there is a $d_0 \in \Delta^I$.

  Since I $\models \forall x \exists y R(x, y)$, there must be a $d_1 \in \Delta^I$ such that $\langle d_0, d_1 \rangle \in R^I$. For the same reason there must be a $d_2 \in \Delta^I$, such that $\langle d_1, d_2 \rangle \in R^I$. And so on . . . . This means that there must be an infinite sequence $d_0, d_1, d_2, \ldots$ such that $\langle d_i, d_{i+1} \rangle$, for every i
- ≥ 0.

  Since I $\models \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \supset R(x,z))$, then for all
- $i < j$, $\langle d_i, d_j \rangle \in R^I$.

  suppose, by contradiction, that $|\Delta^I| = k$ for some finite number k .
- This means there is an i, j with $0 \leq i < j \leq k + 1$ such that $d_i = d_j$.

  The fact that $\langle d_i, d_j \rangle \in R^I$ implies that $\langle d_i, d_j \rangle \in R^I$. But this contradicts the fact that I $\models \forall x \neg R(x, x)$.

□