Mathematical Logic
Propositional Logic - Tableaux*

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Tableaux

- Early work by Beth and Hintikka (around 1955). Later refined and popularised by Raymond Smullyan:
  - R.M. Smullyan. First-order Logic. Springer-Verlag, 1968.
- Modern expositions include:
  - M. Fitting. First-order Logic and Automated Theorem Proving. 2nd edition. Springer-Verlag, 1996.
  - M. DAgostino, D. Gabbay, R. H¨ahnle, and J. Posegga (eds.). Handbook of Tableau Methods. Kluwer, 1999.
  - R. H¨ahnle. Tableaux and Related Methods. In: A. Robinson and A. Voronkov (eds.), Handbook of Automated Reasoning, Elsevier Science and MIT Press, 2001.
  - Proceedings of the yearly Tableaux conference:

    http://i12www.ira.uka.de/TABLEAUX/

How does it work?

The tableau method is a method for proving, in a mechanical manner, that a given set of formulas is not satisfiable. In particular, this allows us to perform automated *deduction*:

Given : set of premises Γ and conclusion φ

Task : prove $\Gamma \models \varphi$

How? show $\Gamma \cup \neg \varphi$ is not satisfiable (which is equivalent), i.e. add the complement of the conclusion to the premises and derive a contradiction (refutation procedure)

Reduce Logical Consequence to (un)Satisfiability

Theorem

$\Gamma \models \varphi$ if and only if $\Gamma \cup \{\neg \varphi\}$ is unsatisfiable

Proof.

$\Rightarrow$ Suppose that $\Gamma \models \varphi$, this means that every interpretation I that satisfies Γ, it does satisfy φ, and therefore $I \not\models \neg \varphi$. This implies that there is no interpretations that satisfies together Γ and $\neg \varphi$.

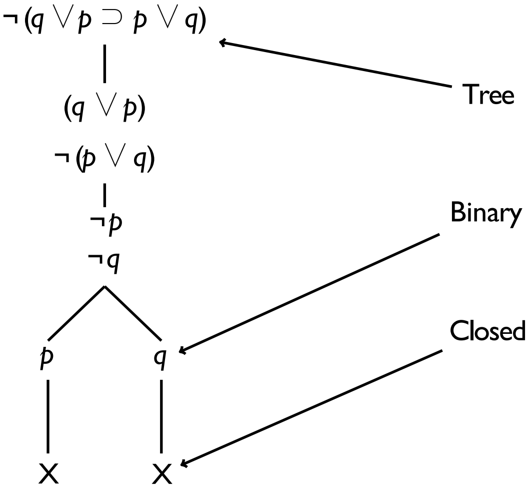
$\Leftarrow$ Suppose that $I \models \Gamma$, let us prove that $I \models \varphi$, Since $\Gamma \cup \{\neg \varphi\}$ is not satisfiable, then $I \not\models \neg \varphi$ and therefore $I \models \varphi$.

$\square$

Constructing Tableau Proofs

- **Data structure**: a proof is represented as a tableau - i.e., a binary tree - the nodes of which are labelled with formulas.
- **Start**: we start by putting the premises and the negated conclusion into the root of an otherwise empty tableau.
- **Expansion**: we apply expansion rules to the formulas on the tree, thereby adding new formulas and splitting branches.
- **Closure**: we close branches that are obviously contradictory.
- **Success**: a proof is successful iff we can close all branches.

An example


$\neg(q \vee p \supset p \vee q)$
Tree
$(q \vee p)$
$\neg(p \vee q)$
Binary
$\neg p$
$\neg q$
Closed
p
q
X
X

Expansion Rules of Propositional Tableau

α rules

$$\frac{\varphi \wedge \psi}{\begin{array}{c}\varphi\\ \psi\end{array}} \qquad \frac{\neg(\varphi \vee \psi)}{\begin{array}{c}\neg\varphi\\ \neg\psi\end{array}} \qquad \frac{\neg(\varphi \supset \psi)}{\begin{array}{c}\varphi\\ \neg\psi\end{array}}$$

$\neg\neg$-Elimination

$$\frac{\neg\neg\varphi}{\varphi}$$

β rules

$$\frac{\varphi \vee \psi}{\varphi \mid \psi} \qquad \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \mid \neg\psi} \qquad \frac{\varphi \supset \psi}{\neg\varphi \mid \psi}$$

Branch Closure

$$\frac{\begin{array}{c}\varphi\\ \neg\varphi\end{array}}{X}$$

Note: These are the standard ("Smullyan-style") tableau rules.

We omit the rules for $\equiv$. We rewrite $\varphi \equiv \psi$ as $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$

Smullyans Uniform Notation

Two types of formulas: conjunctive (α) and disjunctive (β):

α	α_1	α_2
$\varphi \wedge \psi$	φ	ψ
$\neg(\varphi \vee \psi)$	$\neg\varphi$	$\neg\psi$
$\neg(\varphi \supset \psi)$	φ	$\neg\psi$

β	β_1	β_2
$\varphi \vee \psi$	φ	ψ
$\neg(\varphi \wedge \psi)$	$\neg\varphi$	$\neg\psi$
$\varphi \supset \psi$	$\neg\varphi$	ψ

We can now state α and β rules as follows:

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad\qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

Note: α rules are also called deterministic rules. β rules are also called splitting rules.

An example

$$\neg(q \vee p \supset p \vee q)$$

$$(q \vee p)$$

$$\neg(p \vee q)$$

$$\neg p$$

$$\neg q$$

$$\begin{array}{ccc} & / \quad \backslash & \\ p & & q \\ \times & & \times \end{array}$$

Some definitions for tableaux

Definition (type-*alpha* and type-β formulae)

- Formulae of the form $\varphi \wedge \psi$, $\neg(\varphi \vee \psi)$, and $\neg(\varphi \supset \psi)$ are called type-α formulae.
- Formulae of the form $\varphi \vee \psi$, $\neg(\varphi \wedge \psi)$, and $\varphi \supset \psi$ are called type-β formulae

Note: type-*alpha* formulae are the ones where we use α rules. type-β formulae are the ones where we use β rules.

Definition (Closed branch)

A closed branch is a branch which contains a formula and its negation.

Definition (Open branch)

An open branch is a branch which is not closed

Definition (Closed tableaux)

A tableaux is closed if all its branches are closed.

Definition (Derivation $\Gamma \vdash \varphi$)

Let φ and Γ be a propositional formula and a finite set of propositional formulae, respectively. We write $\Gamma \vdash \varphi$ to say that there exists a closed tableau for $\Gamma \cup \{\neg \varphi\}$

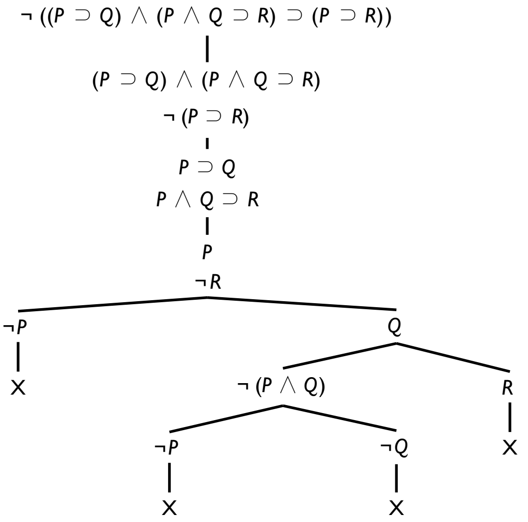
Tableaux and satisfiability

- A tableau for Γ attempts to build a propositional interpretation for Γ. If the tableaux is closed, it means that no model exist.
- We can use tableaux to check if a formula is satisfiable.

Exercise

Check whether the formula $\neg((P \supset Q) \wedge (P \wedge Q \supset R) \supset (P \supset R))$ is satisfiable

Solution



The tableau is closed and the formula is not satisfiable.

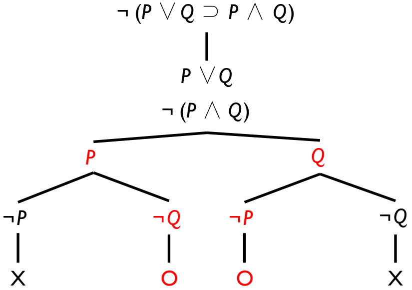
Using the tableau to build interpretations.

For each open branch in the tableau, and for each propositional atom p in the formula we define

$$I(p) = \begin{cases} \text{True} & \text{if } p \text{ belongs to the branch,} \\ \text{False} & \text{if } \neg p \text{ belongs to the branch.} \end{cases}$$

If neither p nor $\neg p$ belong to the branch we can define $I(p)$ in an arbitrary way.

Models for $\neg(P \lor Q \supset P \land Q)$



Two models:

- $I(P) = $ True, $I(Q) = $ False
- $I(P) = $ False, $I(Q) = $ True

Double-check with the truth tables!

P	Q	$P \vee Q$	$P \wedge Q$	$P \vee Q \supset P \wedge Q$	$\neg (P \vee Q \supset P \wedge Q)$
T	T	T	T	T	F
F	F	F	F	T	F
T	F	T	F	F	T
F	T	T	F	F	T

Double-check with the truth tables!

P	Q	$P \vee Q$	$P \wedge Q$	$P \vee Q \supset P \wedge Q$	$\neg (P \vee Q \supset P \wedge Q)$
T	T	T	T	T	F
F	F	F	F	T	F
T	F	T	F	F	T
F	T	T	F	F	T

Termination

Assuming we analyze each formula at most once, we have:

Theorem (Termination)

For any propositional tableau, after a finite number of steps no more expansion rules will be applicable.

Hint for proof: This must be so, because each rule results in ever shorter formulas.

Note: Importantly, termination will *not* hold in the first-order case.

Preliminary definition

Definition

A literal is an atomic formula p or the negation $\neg p$ of an atomic formula.

Termination

Hint of proof:

Base case Assume that we have a literal formula. Then it is a propositional variable or a negation of a propositional variable and no expansion rules are applicable.

Inductive step Assume that the theorem holds for any formula with at most n connectives and prove it with a formula ϑ with $n + 1$ connectives.
Three cases:

- ϑ is a type-α formula (of the form $\varphi \wedge \psi$, $\neg(\varphi \vee \psi)$, or $\neg(\varphi \supset \psi)$)

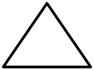
  We have to apply an α-rule

$$\begin{array}{c} \vartheta \\ | \\ \alpha 1 \\ \alpha 2 \end{array}$$

and we mark the formula ϑ as analysed once.

Since $\alpha 1$ and $\alpha 2$ contain less connectives than ϑ we can apply the inductive hypothesis and say that we can build a propositional tableau such that each formula is analyzed at most once and after a finite number of steps no more expansion rules will be applicable.

$\alpha 1, \alpha 2$



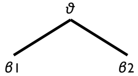
We concatenate the two trees and the proof is done.

Termination

Three cases:

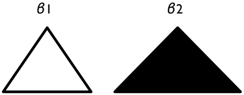
- ϑ is a type-β formula (of the form $\varphi \vee \psi$, $\neg(\varphi \wedge \psi)$, or $\varphi \supset \psi$)

We have to apply a β-rule

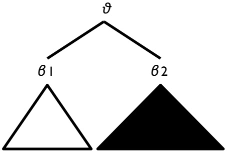


and we mark the formula ϑ as analyzed once.

Since $\beta 1$ and $\beta 2$ contain less connectives than ϑ we can apply the inductive hypothesis and say that we can build two propositional tableaux, one for $\beta 1$ and one for $\beta 2$ such that each formula is analyzed at most once and after a finite number of steps no more expansion rules will be applicable.



We concatenate the 3 trees and the proof is done.



Termination

- ϑ is of the form $\neg\neg\varphi$.

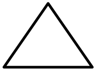
  We have to apply the $\neg\neg$-Elimination rule

$$\begin{array}{c} \neg\neg\varphi \\ | \\ \varphi \end{array}$$

and we mark the formula $\neg\neg\varphi$ as analyzed once.

Since φ contains less connectives than $\neg\neg\varphi$ we can apply the inductive hypothesis and say that we can build a propositional tableaux for it such that each formula is analyzed at most once and after a finite number of steps no more expansion rules will be applicable.

φ



We concatenate the 2 trees and the proof is done.

Soundness and Completeness

To actually believe that the tableau method is a valid decision procedure we have to prove:

Theorem (Soundness)

If $\Gamma \vdash \varphi$ *then* $\Gamma \models \varphi$

Theorem (Completeness)

If $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$

Remember: We write $\Gamma \vdash \varphi$ to say that there exists a closed tableau for $\Gamma \cup \{\neg \varphi\}$.

A last definition - Fairness

Definition (Fairness)

We call a propositional tableau fair if every non-literal of a branch gets eventually analysed on this branch.

Decidability

The proof of Soundness and Completeness confirms the decidability of propositional logic:

Theorem (Decidability)

The tableau method is a decision procedure for classical propositional logic.

Proof. To check validity of φ, develop a tableau for $\neg\varphi$. Because of termination, we will eventually get a tableau that is either (1) closed or (2) that has a branch that cannot be closed.

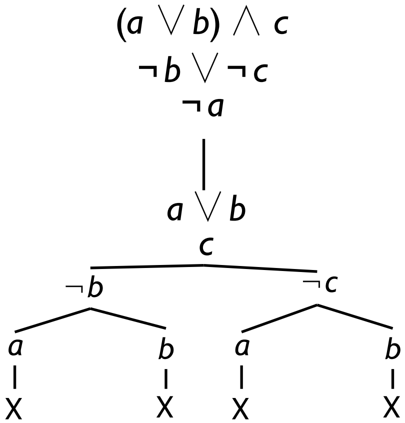
- In case (1), the formula φ must be valid (soundness).
- In case (2), the branch that cannot be closed shows that $\neg\varphi$ is satisfiable (see completeness proof), i.e. φ cannot be valid.

This terminates the proof.

Exercise

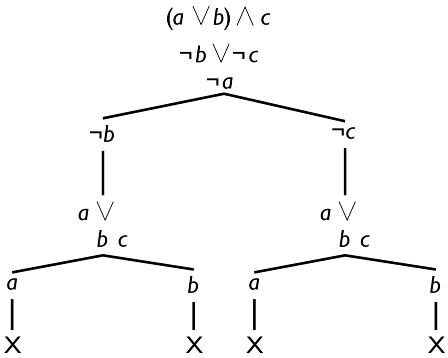
Exercise

Build a tableau for $\{(a \vee b) \wedge c, \neg b \vee \neg c, \neg a\}$



Another solution

What happens if we first expand the disjunction and then the conjunction?



Expanding β rules creates new branches. Then α rules may need to be expanded in all of them.

Strategies of expansion

- Using the "wrong" policy (e.g., expanding disjunctions first) leads to an increase of *size* of the tableau, which leads to an increase of *time*;
- yet, unsatisfiability is still proved if set is unsatisfiable;
- this is not the case for other logics, where applying the wrong policy may inhibit proving unsatisfiability of some unsatisfiable sets.

Finding Short Proofs

- It is an open problem to find an efficient algorithm to decide in all cases which rule to use next in order to derive the shortest possible proof.
- However, as a rough guideline always apply any applicable *non-branching rules* first. In some cases, these may turn out to be redundant, but they will never cause an exponential blow-up of the proof.

Efficiency

- Are analytic tableaus an efficient method of checking whether a formula is a tautology?
- Remember: using the truth-tables to check a formula involving *n* propositional atoms requires filling in 2^n rows (exponential = very bad).
- Are tableaux any better?
- In the worst case no, but if we are lucky we may skip some of the 2^n rows !!!