

Mathematical Logics

Propositional Logic - Introduction*

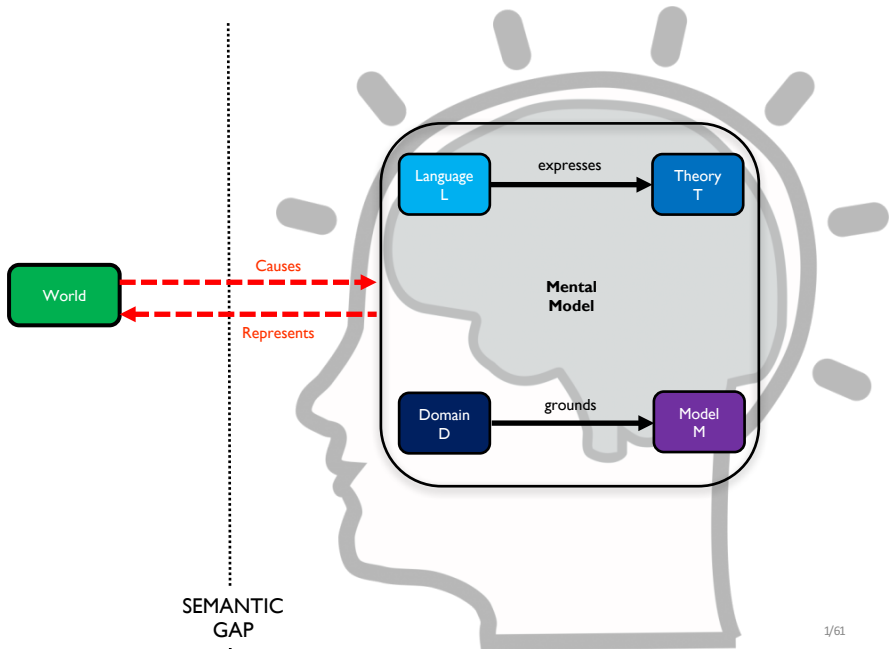
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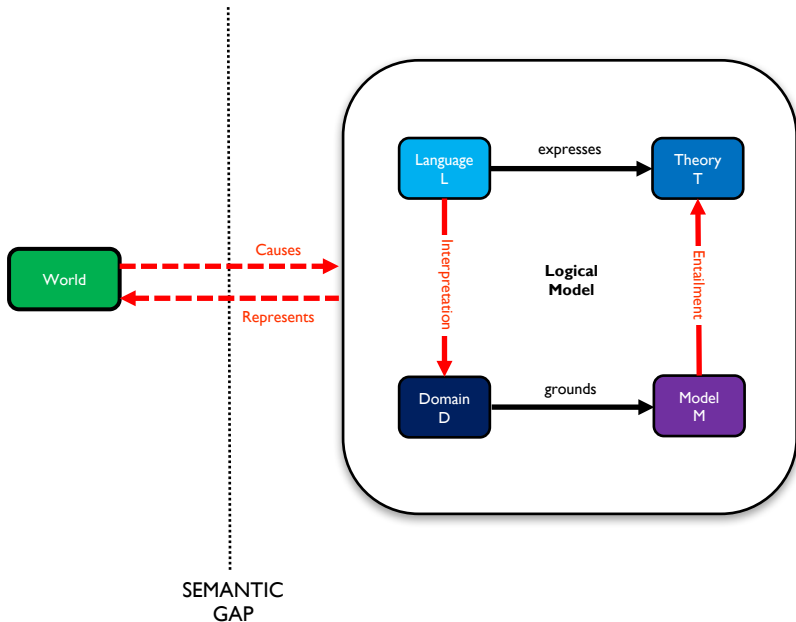


**Originally by Luciano Serafini and Chiara Ghidini
Modified by Fausto Giunchiglia and Mattia Fumagalli*

Mental Model



Logical Model



Logical Model

World

Logical
Model

Language
L

Domain
D

Theory
T

Model
M



MONKEY			
/	A	N	G
M	T #1	Ba #2	Ba #3
...

$L = \text{"C, A, B, N, R, G, } \wedge, \vee, \neg, \rightarrow, \dots\text{"}$

$T = \text{"G* } \rightarrow (A \vee N)\text{"}$

$D: \{\#1, \#2, \#3\}$

$I: \text{"I(A) = \#1, I(N) = \#2, I(G) = \#3"}\text{"}$

$M: \text{"\#1, \#2, \#3*"}\text{"}$

$M \models A$

$M \models N$

$M \models A \vee N$

SEMANTIC
GAP

*Where G informally means "Monkey gets banana"

*Where #3 stands for "Monkey actually gets Banana".

Propositional logic - Intuition

- Propositional logic is the logic of **propositions**
- a proposition can be **true** or **false** in the state of the world. the
- same proposition can be expressed in different ways.

E.g.

- “B. Obama is drinking a bier”
- “The U.S.A. president is drinking a bier”, and
- “B. Obama si sta facendo una birra”

express the same proposition.

- The language of propositional logic allows us to express propositions.

Propositional logic language

Definition (Propositional alphabet)

Logical symbols $\neg, \wedge, \vee, \supset, \text{ and } \equiv$

Non logical symbols A set \mathcal{P} of symbols called **propositional variables**

Separator symbols “(” and “)”

Definition (Well formed formulas (or simply formulas))

- every $\mathcal{P} \in \mathcal{P}$ is an **atomic formula**
- every atomic formula is a **formula**
- if A and B are formulas then $\neg A, A \wedge B, A \vee B, A \supset B, \text{ e } A \equiv B$ are **formulas**

Example ((non) formulas)

Formulas	Non formulas
$P \rightarrow Q$	$P Q$
$P \rightarrow (Q \rightarrow R)$	$(P \rightarrow \wedge ((Q \rightarrow R)$
$P \wedge Q \rightarrow R$	$P \wedge Q \rightarrow \neg R \neg$

Problem

How do we read the formula $P \wedge Q \rightarrow R$?

The formula $P \wedge Q \rightarrow R$ can be read in two ways:

- 1 $(P \wedge Q) \rightarrow R$
- 2 $P \wedge (Q \rightarrow R)$

Symbol priority

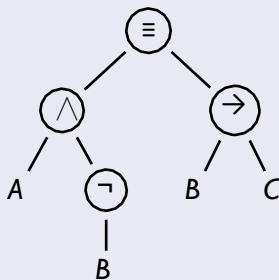
\neg has higher priority, then \wedge , \vee , \rightarrow and \equiv . Parenthesis can be used around formulas to stress or change the priority.

Symbol	Priority
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\equiv	5

Formulas as trees

A formula can be seen as a tree. Leaf nodes are associated to propositional variables, while intermediate (non-leaf) nodes are associated to connectives.

For instance the formula $(A \wedge \neg B) \equiv (B \rightarrow C)$ can be represented as the tree



Definition

(Proper) Subformula

- A is a **subformula** of itself
- A and B are **subformulas** of $A \wedge B$, $A \vee B$, $A \supset B$, $A \equiv B$ A is a
- subformula of $\neg A$
- if A is a subformula of B and B is a subformula of C , then A is a subformula of C .
- A is a **proper subformula** of B if A is a subformula of B and A is different from B .

Remark

The subformulas of a formula represented as a tree correspond to all the different subtrees of the tree associated to the formula, one for each node.

Example

The subformulas of $(p \rightarrow (q \vee r)) \rightarrow (p \wedge \neg p)$ are

$$(p \rightarrow (q \vee r)) \rightarrow (p \wedge \neg p)$$

$$(p \rightarrow (q \vee r))$$

$$p \wedge \neg p$$

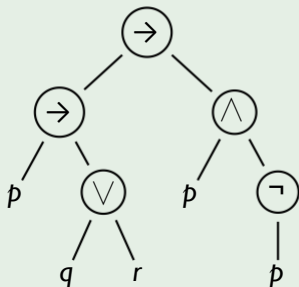
$$p$$

$$\neg p$$

$$q \vee r$$

$$q$$

$$r$$



Proposition

Every formula has a finite number of subformulas

Interpretation of Propositional Logic

Definition (Interpretation)

A **Propositional interpretation** is a function $I : P \rightarrow \{\text{True}, \text{False}\}$

Remark

If $|P|$ is the cardinality of P , then there are $2^{|P|}$ different interpretations, i.e. all the different subsets of P . If $|P|$ is finite then there is a finite number of interpretations.

Remark

A propositional interpretation can be thought as a subset S of P , and I is the characteristic function of S , i.e., $A \in S$ iff $I(A) = \text{True}$.

Interpretation of Propositional Logic

Example

	p	q	r	Set theoretic representation
I1	True	True	True	$\{p, q, r\}$
I2	True	True	False	$\{p, q\}$
I3	True	False	True	$\{p, r\}$
I4	True	False	False	$\{p\}$
I5	False	True	True	$\{q, r\}$
I6	False	True	False	$\{q\}$
I7	False	False	True	$\{r\}$
I8	False	False	False	$\{\}$

Satisfiability of a propositional formula

Definition (I satisfies a formula, $I \models A$)

A formula A is **true in/satisfied by** an interpretation I , in symbols $I \models A$, according to the following inductive definition:

- If $P \in \mathcal{P}$, $I \models P$ if $I(P) = \text{True}$.
- $I \models \neg A$ if not $I \models A$ (also written $I \models A$)
- $I \models A \wedge B$ if, $I \models A$ and $I \models B$
- $I \models A \vee B$ if, $I \models A$ or $I \models B$
- $I \models A \rightarrow B$ if, when $I \models A$ then $I \models B$
- $I \models A \equiv B$ if, $I \models A$ iff $I \models B$

Satisfiability of a propositional formula

Example (interpretation)

Let $P = \{P, Q\}$.

$I(P) = \text{True}$ and $I(Q) = \text{False}$ can be also expressed with

$I = \{P\}$.

Example (Satisfiability)

Let $I = \{P\}$. Check if $I \models (P \wedge Q) \vee (R \rightarrow S)$:

Replace each occurrence of each primitive propositions of the formula with the truth value assigned by I , and apply the definition for connectives.

$$(\text{True} \wedge \text{False}) \vee (\text{False} \rightarrow \text{False}) \quad (1)$$

$$\text{False} \vee \text{True} \quad (2)$$

$$\text{True} \quad (3)$$

Satisfiability of a propositional formula

Proposition

*If for any propositional variable P appearing in a formula A ,
 $I(P) = I'(P)$, then $I \models A$ iff $I' \models A$*

Lazy evaluation algorithm (1/2)

$(A = p)$	<pre>check($I \models p$) if $I(p) = \text{true}$ then return YES else return NO</pre>
$(A = B \wedge C)$	<pre>check($I \models B \wedge C$) if check($I \models B$) then return check($I \models C$) else return NO</pre>
$(A = B \vee C)$	<pre>check($I \models B \vee C$) if check($I \models B$) then return YES else return check($I \models C$)</pre>

Lazy evaluation algorithm (2/2)

$(A = B \rightarrow C)$

```
check( $I \models B \rightarrow C$ ) if  
  check( $I \models B$ )  
    then return check( $I \models C$ )  
  else return YES
```

$(A = B \equiv C)$

```
check( $I \models B \equiv C$ ) if  
  check( $I \models B$ )  
    then return check( $I \models C$ )  
  else return not(check( $I \models C$ ))
```

Formalizing English Sentences

Exercise

Let's consider a propositional language where p means "Paola is happy", q means "Paola paints a picture", and r means "Renzo is happy".

Formalize the following sentences:

- 1 "if Paola is happy and paints a picture then Renzo isn't happy"

$$p \wedge q \rightarrow \neg r$$

- 2 "if Paola is happy, then she paints a picture"

$$p \rightarrow q$$

- 3 "Paola is happy only if she paints a picture"

$$\neg(p \wedge \neg q) \text{ which is equivalent to } p \rightarrow q \quad !!!$$

The precision of formal languages avoid the ambiguities of natural languages.

Valid, Satisfiable, and Unsatisfiable formulas

Definition

A formula A is

Valid if **for all interpretations** I , $I \models A$

Satisfiable if **there is an interpretation** I s.t., $I \models A$

Unsatisfiable if **for no interpretations** I , $I \models A$

Proposition

$A \text{ Valid} \rightarrow A \text{ satisfiable} \leftrightarrow A \text{ not unsatisfiable}$

$A \text{ unsatisfiable} \leftrightarrow A \text{ not satisfiable} \rightarrow A \text{ not Valid}$

Valid, Satisfiable, and Unsatisfiable formulas

Proposition

<i>if A is</i>	<i>then $\neg A$ is</i>
<i>Valid</i>	<i>Unsatisfiable</i>
<i>Satisfiable</i>	<i>not Valid</i>
<i>not Valid</i>	<i>Satisfiable</i>
<i>Unsatisfiable</i>	<i>Valid</i>

Checking Validity and (un)satisfiability of a formula

Truth Table

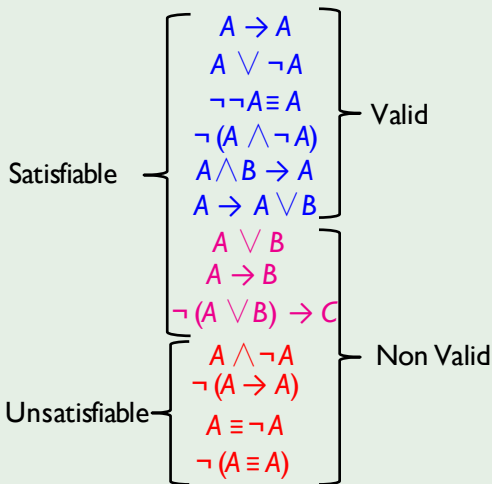
Checking (un)satisfiability and validity of a formula A can be done by enumerating all the interpretations which are relevant for S , and for each interpretation I check if $I \models A$.

Example (of truth table)

A	B	C	$A \rightarrow (B \vee \neg C)$
true	true	true	true
true	true	false	true
true	false	true	false
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

Valid, Satisfiable, and Unsatisfiable formulas

Example



Prove that the **blue formulas** are valid, that the **magenta formulas** are satisfiable but not valid, and that the **red formulas** are unsatisfiable.

Valid, Satisfiable, and Unsatisfiable sets of formulas

Definition

A set of formulas Γ is

Valid if for all interpretations I , $I \models A$ for all formulas $A \in \Gamma$

Satisfiable if there is an interpretation I , $I \models A$ for all $A \in \Gamma$

Unsatisfiable if for no interpretation I , s.t. $I \models A$ for all $A \in \Gamma$

Proposition

For any *finite set* of formulas Γ , (i.e., $\Gamma = \{A_1, \dots, A_n\}$ for some $n \geq 1$), Γ is valid (resp. satisfiable and unsatisfiable) if and only if $A_1 \wedge \dots \wedge A_n$ (resp. satisfiable and unsatisfiable).

Truth Tables: Example

Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?

Recall some definitions

- Let F be a formula:
 - F is **valid** if every interpretation satisfies F
 - F is **satisfiable** if F is satisfied by some interpretation
 - F is **unsatisfiable** if there isn't any interpretation satisfying F

Truth Tables: Example (2)

Use the truth tables method to determine whether $(p \rightarrow q) \vee (p \rightarrow \neg q)$ is valid.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The formula is valid since it is satisfied by every interpretation.

Truth Tables: Example (3)

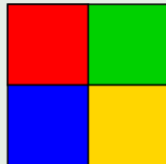
Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with F) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	F
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying F , thus F is satisfiable.

Example (The colored blanket)

- $P = \{B, R, Y, G\}$
- the intuitive interpretation of B (R , Y , and G) is that **the blanket is completely blue** (red, yellow and green)









Exercise










Find all the interpretations that, according to the intuitive interpretation given above, represent a possible situation. Consider the three cases in which

- 1 the blanket is composed of exactly 4 pieces, and yellow, red, blue and green are the only allowed colors;
- 2 the blanket can be composed of any number of pieces (at least 1), and yellow, red, blue and green are the only allowed colors;
- 3 the blanket can be composed of any number of pieces and there can be other colors.











Exercise (Solution)

- 1
- $I_1 = \{B\}$ corresponding to 
 - $I_2 = \{Y\}$ corresponding to 
 - $I_3 = \{R\}$ corresponding to 
 - $I_4 = \{G\}$ corresponding to 
 - $I_5 = \emptyset$ corresponding to any blanket that is not monochrome, e.g.   , ...
 - $I_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
 - ...

Exercise (Solution)

- ②
 - $I_1 = \{B\}$ corresponding to any blue blankets, no matter its shape, e.g. , , and 
 - $I_2 = \{Y\}$ corresponding to any yellow blankets, no matter its shape, e.g. , , and 
 - ...
 - $I_5 = \emptyset$ corresponds to any blanket which is not monochrome no matter of its shape, e.g. , , and 
 - $I_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
 - ...

Exercise (Solution)

- ③
 - $I_1 = \{B\}$ corresponding to any blue blankets, no matter its shape, e.g. , , and 
 - $I_2 = \{Y\}$ corresponding to any yellow blankets, no matter its shape, e.g. , , and 
 - ...
 - $I_5 = \emptyset$ corresponds to any blanket which is neither completely blue, red, yellow, nor green, no matter of its shape, e.g., , 
, and 
 - $I_6 = \{R, B\}$ does not correspond to any blanket, since a blanket cannot be both completely blue and red. More in general all the interpretations that satisfies more than one proposition do not correspond to any real situation.
 - ...

Logical consequence

Definition (Logical consequence)

A formula A is a logical consequence of a set of formulas Γ , in symbols

$$\Gamma \models A$$

Iff for any interpretation I that satisfies all the formulas in Γ , I satisfies A ,

Example (Logical consequence)

- $p \models p \vee q$
- $q \vee p \models p \vee q$
- $p \vee q, p \rightarrow r, q \rightarrow r \models r$
- $p \rightarrow q, p \models q$
- $p, \neg p \models q$

Proving Logical consequence in a direct manner

Example

Proof of $p \models p \vee q$ Suppose that $I \models p$, then by definition $I \models p \vee q$.

Proof of $q \vee p \models p \vee q$ Suppose that $I \models q \vee p$, then either $I \models q$ or $I \models p$. In both cases we have that $I \models p \vee q$.

Proof of $p \vee q, p \rightarrow r, q \rightarrow r \models r$ Suppose that $I \models p \vee q$ and $I \models p \rightarrow r$ and $I \models q \rightarrow r$. Then either $I \models p$ or $I \models q$. In the first case, since $I \models p \rightarrow r$, then $I \models r$, In the second case, since $I \models q \rightarrow r$, then $I \models r$.

Proof of $p, \neg p \models q$ Suppose that $I \models \neg p$, then not $I \models p$, which implies that there is no I such that $I \models p$ and $I \models \neg p$. This implies that all the interpretations that satisfy p and $\neg p$ (actually none) satisfy also q .

Proof of $(p \wedge q) \vee (\neg p \wedge \neg q) \models p \equiv q$ Left as an exercise

Proof of $(p \rightarrow q) \models \neg p \vee q$ Left as an exercise

Proving Logical consequence using the truth tables

Use the truth tables method to determine whether $p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of $\neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	F	F	T

Definition

Logical Equivalence Two formulas F and G are **logically equivalent** (denoted with $F \equiv G$) if for each interpretation I , $I(F) = I(G)$.

Truth Tables: Example (5)

Use the truth tables method to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \rightarrow (q \wedge \neg q)$	$\neg p$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Properties of propositional logical consequence

Proposition

If Γ and Σ are two sets of propositional formulas and A and B two formulas, then the following properties hold:

Reflexivity $\{A\} \models A$

Monotonicity *If $\Gamma \models A$ then $\Gamma \cup \Sigma \models A$*

Cut *If $\Gamma \models A$ and $\Sigma \cup \{A\} \models B$ then $\Gamma \cup \Sigma \models B$*

Compactness *If $\Gamma \models A$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$, such that $\Gamma_0 \models A$*

Deduction theorem *If $\Gamma, A \models B$ then $\Gamma \models A \rightarrow B$*

Refutation principle $\Gamma \models A$ *iff* $\Gamma \cup \{\neg A\}$ *is unsatisfiable*

Reflexivity $\{A\} \models A$.

PROOF: For all I if $I \models A$, then $I \models A$.

Monotonicity If $\Gamma \models A$ then $\Gamma \cup \Sigma \models A$

PROOF: For all I if $I \models \Gamma \cup \Sigma$, then $I \models \Gamma$, by hypothesis ($\Gamma \models A$) we can infer that $I \models A$, and therefore that $\Gamma \cup \Sigma \models A$

Cut If $\Gamma \models A$ and $\Sigma \cup \{A\} \models B$ then $\Gamma \cup \Sigma \models B$.

PROOF: For all I , if $I \models \Gamma \cup \Sigma$, then $I \models \Gamma$ and $I \models \Sigma$. The hypothesis $\Gamma \models A$ implies that $I \models A$. Since $I \models \Sigma$, then $I \models \Sigma \cup \{A\}$.

The hypothesis $\Sigma \cup \{A\} \models B$, implies that $I \models B$.

We can therefore conclude that $\Gamma \cup \Sigma \models B$.

Compactness If $\Gamma \models A$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$, such that $\Gamma_0 \models A$.

PROOF: Let P_A be the primitive propositions occurring in A . Let I_1, \dots, I_n (with $n \leq 2^{|P_A|}$), be all the interpretations of the language P_A that do not satisfy A . Since $\Gamma \models A$, then there should be I'_1, \dots, I'_n interpretations of the language of Γ , which are extensions of I_1, \dots, I_n , and such that $I'_k \models \Gamma_k$ for some $\Gamma_k \in \Gamma$. k

Let $\Gamma_0 = \{\Gamma_1, \dots, \Gamma_k\}$. Then $\Gamma_0 \models A$. Indeed if $I \models \Gamma_0$ then I is an extension of an interpretation J of P_A that satisfies A , and therefore $I \models A$.

Deduction theorem If $\Gamma, A \models B$ then $\Gamma \models A \rightarrow B$

PROOF: Suppose that $I \models \Gamma$. If $I \models A$, then $I \models A \rightarrow B$.

If instead $I \not\models A$, then by the hypothesis $\Gamma, A \models B$, implies that $I \models B$, which implies that $I \models B$. We can therefore conclude that $I \models A \rightarrow B$.

Refutation principle $\Gamma \models A$ iff $\Gamma \cup \{\neg A\}$ is unsatisfiable

PROOF:

(\Rightarrow) Suppose by contradiction that $\Gamma \cup \{\neg A\}$ is satisfiable. This implies that there is an interpretation I such that $I \models \Gamma$ and $I \models \neg A$, i.e., $I \not\models A$. This contradicts that fact that for all interpretations that satisfies Γ , they satisfy A

(\Leftarrow) Let $I \models \Gamma$, then by the fact that $\Gamma \cup \{\neg A\}$ is unsatisfiable, we have that $I \not\models \neg A$, and therefore $I \models A$. We can conclude that $\Gamma \models A$.

Definition (Propositional theory)

A theory is a set of formulas closed under the logical consequence relation. I.e. T is a theory iff $T \models A$ implies that $A \in T$

Example (Of theory)

- T_1 is the set of valid formulas $\{A \mid A \text{ is valid}\}$
- T_2 is the set of formulas which are true in the interpretation $I = \{P, Q, R\}$
- T_3 is the set of formulas which are true in the set of interpretations $\{I_1, I_2, I_3\}$
- T_4 is the set of all formulas

Show that T_1 , T_2 , T_3 and T_4 are theories

Example (Of non theory)

- N_1 is the set $\{A, A \rightarrow B, C\}$
- N_2 is the set $\{A, A \rightarrow B, B, C\}$
- N_3 is the set of all formulas containing P

Show that N_1 , N_2 and N_3 are not theories

Remark

A propositional theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas, which are infinite) (e.g., $A \rightarrow A$ for all formulas A)

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $A \in T$, $\Omega \models A$.

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Definition (Logical closure)

For any set Γ , $cl(\Gamma) = \{A \mid \Gamma \models A\}$

Proposition (Logical closure)

For any set Γ , the logical closure of Γ , $cl(\Gamma)$ is a theory

Proposition

Γ is a set of axioms for $cl(\Gamma)$.

Compact representation of knowledge

The axiomatization of a theory is a compact way to represent a set of interpretations, and thus to represent a set of possible (acceptable) world states. In other words is a way to **represent all the knowledge we have** of the real world.

minimality

The axioms of a theory constitute the basic knowledge, and all the *generable knowledge* is obtained by logical consequence. An important feature of a set of axioms, is that they are minimal, i.e., no axioms can be derived from the others.

Example

Pam_Attends_Logic_Course

John_is_a_PhD_Student

$Pam_Attends_Logic_Course \rightarrow Pam_is_a_Ms_Student \vee Pam_is_a_PhD_Student$

$Pam_is_a_Ms_Student \rightarrow \neg Pam_is_a_Ba_Student$

$Pam_is_a_PhD_Student \rightarrow \neg Pam_is_a_Ba_Student$

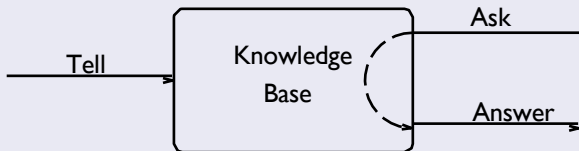
$\neg (John_is_a_PhD_Student \wedge John_is_a_Ba_Student)$

The axioms above constitute the basic knowledge about the people that attend logic course. The facts $\neg Pam_is_a_Ba_Student$ and $\neg John_is_a_Ba_Student$ don't need to be added to this basic knowledge, as they can be derived via logical consequence.

Logic based systems

A logic-based system for representing and reasoning about knowledge is composed by a **Knowledge base** and a **Reasoning system**. A knowledge base consists of a finite collection of formulas in a logical language. The main task of the knowledge base is to answer queries which are submitted to it by means of a **Reasoning system**.

Logic based system for knowledge representation



Tell: this action incorporates the new knowledge encoded in an axiom (formula). This allows to build a KB.

Ask: allows to query what is known, i.e., whether a formula ϕ is a logical consequences of the axioms contained in the KB ($KB \models \phi$)

Proposition

Given a set of interpretations S , the set of formulas A which are satisfied by all the interpretations in S is a theory. i.e.

$$T_S = \{A \mid I \models A \text{ for all } I \in S\}$$

is a theory.

Knowledge representation problem

Given a set of interpretations S which correspond to **admissible situations** find a set of axioms Ω for T_S .