



Mathematical Logic - 2015

Propositional Logic: exercises

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Propositional logic language

Propositional alphabet:

- Logical symbols: \neg , \wedge , \vee , \rightarrow , and \leftrightarrow
- Non logical symbols A set Ω of symbols called propositional variables
- Separator symbols “(” and “)”
- “Meta-symbols”, i.e. \models , \top or \perp

Definition (Well formed formulas):

- Every $P \in \Omega$ is an atomic formula
- Every atomic formula is a formula
- If A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, e $A \leftrightarrow B$ are formulas

Symbols in PL

Which of the following symbols are used in PL?

$\sqcap \neg \top \vee \equiv \sqcup \subseteq \rightarrow \leftrightarrow \perp \wedge \vDash$

$\sqcap \neg \top \vee \equiv \sqcup \subseteq \rightarrow \leftrightarrow \perp \wedge \vDash$

Well formed formulas

Which of the following are well formed formulas?

$(\wedge P Q)$

no

$(P \neg \neg)$

no

$(P + P)$

no

$\neg \neg \neg \neg \neg \neg \neg P$

yes

$((P \rightarrow Q) \rightarrow (Q \rightarrow P))$

yes

$\neg \neg \neg \neg \neg \neg P \neg \rightarrow Q$

no

$((P \wedge Q) \rightarrow P)$

yes

From NL to PL

How to formalise natural language sentences?

- It is the case that P:

P

- It is not the case that P:

$\neg P$

- P and Q. P but Q. Although P, Q:

$(P \wedge Q)$

- P or Q:

$(P \vee Q)$

- P if and only if Q:

$(P \leftrightarrow Q)$

- If P, then Q:

$(P \rightarrow Q)$

- P if Q:

$(Q \rightarrow P)$

From NL to PL

- Q only if P:

$$(Q \rightarrow P)$$

- P just in case Q:

$$(P \leftrightarrow Q)$$

- Not P or not Q:

$$\neg P \vee \neg Q$$

- It is not the case that both P and Q:

$$\neg(P \wedge Q)$$

- Both not P and not Q:

$$(\neg P \wedge \neg Q)$$

- P is necessary for Q:

$$(Q \rightarrow P)$$

- P is sufficient for Q:

$$(P \rightarrow Q)$$

From NL to PL

- P is both necessary and sufficient for Q:

$$(P \leftrightarrow Q)$$

- P unless Q:

$$(P \vee Q)$$

- Among P and Q, only P:

$$(P \wedge \neg Q)$$

- Among P and Q, not P:

$$(\neg P \wedge Q)$$

- Only one among P and Q:

$$(\neg P \wedge Q) \vee (P \wedge \neg Q)$$

- At most one among P and Q:

$$\neg(P \wedge Q)$$

- At least one among P and Q:

$$(P \vee Q)$$

Formalizing NL

Let's consider a propositional language where p means "*Paola is happy*", q means "*Paola paints a picture*", and r means "*Renzo is happy*". Formalize the following sentences:

- "if Paola is happy and paints a picture then Renzo isn't happy"

$$(p \wedge q) \rightarrow \neg r$$

- "if Paola is happy, then she paints a picture"

$$p \rightarrow q$$

- "Paola is happy only if she paints a picture"

$$\neg(p \wedge \neg q) \text{ which is equivalent to } p \rightarrow q \text{ !!!}$$

Formalizing NL

Exercise

Let A = "Angelo comes to the party", B = "Bruno comes to the party", C = "Carlo comes to the party", and D = "Davide comes to the party".

Formalize the following sentences:

- 1 "If Davide comes to the party then Bruno and Carlo come too"
- 2 "Carlo comes to the party only if Angelo and Bruno do not come"
- 3 "If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"
- 4 "Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"
- 5 "A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"
- 6 "Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"

Formalizing NL

"If Davide comes to the party then Bruno and Carlo come too"

$$D \rightarrow (B \wedge C)$$

"Carlo comes to the party only if Angelo and Bruno do not come"

$$C \rightarrow (\neg A \wedge \neg B)$$

"If Davide comes to the party, then, if Carlo doesn't come then Angelo comes"

$$D \rightarrow (\neg C \rightarrow A)$$

"Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come"

$$(C \rightarrow \neg D) \wedge (D \rightarrow \neg B)$$

"A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes"

$$A \rightarrow (\neg B \wedge \neg C \rightarrow D)$$

"Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes"

$$(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \rightarrow C))$$

Truth valuation

- A **truth valuation** on a propositional language L is a mapping v assigning to each formula A of L a truth value $v(A)$, given the domain $D = \{T, F\}$
- $v(A) = T$ or F according to the modeler, with A atomic
- $v(\neg A) = T$ iff $v(A) = F$
- $v(A \wedge B) = T$ iff $v(A) = T$ and $v(B) = T$
- $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$
- $v(\perp) = F$ (since $\perp =_{df} P \wedge \neg P$)
- $v(\top) = T$ (since $\top =_{df} \neg \perp$)

Truth valuation

- ❑ Two formulas F and G are logically equivalent (denoted with $F \leftrightarrow G$) if for each interpretation \mathcal{I} , $\mathcal{I}(F) = \mathcal{I}(G)$.
- ❑ Let F and G be formulas. G is a **logical consequence** of F (denoted with $F \models G$) if each interpretation satisfying F satisfies also G .
- ❑ Let F be a formula:
 - ❑ F is valid if every interpretation satisfies F
 - ❑ F is satisfiable if F is satisfied by some interpretation
 - ❑ F is unsatisfiable if there isn't any interpretation satisfying F

Truth valuation and Truth Tables

- ❑ A **truth valuation** on a PL language L is a mapping v that assigns to each formula P of L a truth value $v(P)$.
- ❑ A **truth table** is composed of **one column for each input variable** and **one (or more) final column for all of the possible results of the logical operation** that the table is meant to represent. Each row of the truth table therefore contains one possible assignment of the input variables, and the result of the operation for those values.

	VARIABLES		LOGICAL OPERATION
	A	B	$A \wedge B$
POSSIBLE ASSIGNMENTS	T	T	T
	T	F	F
	F	T	F
	F	F	F

Example

□ Calculate the Truth Table of the following formulas:

(1) $A \wedge B$;

(2) $A \vee B$;

(3) $A \leftrightarrow B$.

		VARIABLES		(1)	(2)	(3)
		A	B	$A \wedge B$	$A \vee B$	$A \leftrightarrow B$
POSSIBLE ASSIGNMENTS	T	T	T	T	T	
	T	F	F	T	F	
	F	T	F	T	F	
	F	F	F	F	T	

Provide the models for the propositions

□ A truth valuation v is a model for a proposition P iff $v(P) = \text{true}$

□ List the models for the following formulas:

1. $A \wedge \neg B$

2. $(A \wedge B) \vee (B \wedge C)$

3. $(A \vee B) \rightarrow C$

4. $(\neg A \leftrightarrow B) \leftrightarrow C$

MODEL



A	B	$A \wedge \neg B$
T	T	F
T	F	T
F	T	F
F	F	F

Truth Tables Example (1)

Use the truth tables method to determine whether $(p \rightarrow q) \vee (p \rightarrow \neg q)$ is valid.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The formula is valid since it is satisfied by every interpretation.